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SPECTRAL INSTABILITY OF PEAKONS IN THE *b*-FAMILY OF THE CAMASSA–HOLM EQUATIONS*

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Abstract. We prove spectral instability of peakons in the b-family of Camassa–Holm equations that includes the integrable cases of $b = 2$ and $b = 3$. We start with a linearized operator defined on functions in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ and extend it to a linearized operator defined on weaker functions in $L^2(\mathbb{R})$. For $b \neq \frac{5}{2}$, the spectrum of the linearized operator in $L^2(\mathbb{R})$ is proved to cover a closed vertical strip of the complex plane. For $b = \frac{5}{2}$, the strip shrinks to the imaginary axis, but an additional pair of real eigenvalues exists due to projections to the peakon and its spatial translation. The spectral instability results agree with the linear instability results in the case of the Camassa–Holm equation for $b = 2$.

Key words. spectral stability, nonlinear wave, Camassa–Holm equation, peakon, traveling wave solutions

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1. Introduction.

We consider the following b-family of Camassa–Holm equations (which we call b-CH):

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}. \tag{1}$$

The family generalizes the classical cases of the Camassa–Holm (CH) equation for $b = 2$ and the Degasperis–Procesi (DP) equation for $b = 3$.

The CH equation has first appeared in the study of bi-Hamiltonian structure of the Korteweg–de Vries (KdV) equation [13]. It was later introduced by Camassa and Holm [2] in hydrodynamical applications as a model for unidirectional wave propagation on shallow water. The hydrodynamical relevance of the CH equation as a model for shallow water waves was discussed in [3,5,21].

The DP equation can also be regarded as a model for nonlinear shallow water dynamics with its asymptotic accuracy equal to the CH equation [10]. The b-CH family of equations was introduced in [9,11] by using transformations of the integrable KdV equation within the same asymptotic accuracy.

One of the most intriguing properties of the b-CH equations is the occurrence of wave breaking when the solutions stay bounded, but their gradients develop singularities in a finite time. Related to the wave breaking is the existence of peaked traveling waves called *peakons*. The exact peakon solution is given by

$$u(x,t) = ce^{-|x-ct|} \qquad x \in \mathbb{R}. \tag{2}$$

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This solution is related to the reformulation of the b-CH equation (1) in the weaker form

$$u_t + uu_x + \frac{1}{4}\phi' * \left[bu^2 + (3-b)u_x^2\right] = 0, \tag{3}$$

where $*$ denotes convolution and $\phi(x) = e^{-|x|}$ is Green's function satisfying $(1-\partial_x^2)\phi = 2\delta_0$ with δ_0 being the Dirac delta distribution centered at $x = 0$.

According to (2), the exact peakon solution in the form $u(x,t) = c\phi(x - ct)$ satisfies the integral equation (3). We can assume without loss of generality that $c = 1$ because the scaling transformation $u(x,t) \to au(x, at)$ with arbitrary $a \in \mathbb{R}$ leaves (3) invariant. It can then be checked directly that $\phi(x) = e^{-|x|}$ satisfies the integral equation

$$-\phi + \frac{1}{2}\phi^2 + \frac{1}{4}\phi * \left[b\phi^2 + (3-b)(\phi')^2\right] = 0 \tag{4}$$

piecewisely on both sides from the peak at $x = 0$. The integral equation (4) arises after substituting the traveling wave reduction to the b-CH equation (3) and integrating in x with zero conditions at infinity.

Stability of peakons has been considered in the literature. Numerical simulations in [18, 19] showed that the peakons of the b-CH equation are likely to be unstable for $b < 1$. This conjecture was recently illustrated in [4] with the analysis of the linearized operator at the peakon solution and additional numerical experiments. In the case of $b < -1$, the numerical results of [18, 19] suggested that arbitrary initial data moves to the left and asymptotically separates out into a number of smooth time-independent solitary waves. Smooth time-independent solitary wave solutions were shown to be orbitally stable for $b < -1$ in [20]. Smooth traveling solitary wave solutions in a regularized version of the b-CH equation were shown to be orbitally stable for $b = 2$ in [8] and for $b = 3$ in [22].

For $b > 1$, numerical simulations in [18, 19] showed that arbitrary initial data asymptotically resolves into a number of peakons. Orbital stability of peakons in the energy space $H^1(\mathbb{R})$ was shown for the CH equation ($b = 2$) in [6, 7] by using conservation of two energy integrals. This method was extended in [23], where the authors showed orbital stability of peakons for the DP equation ($b = 3$) in $L^2(\mathbb{R})$. Since solutions of the initial-value problem for the b-CH equation with $b > 1$ are ill-posed in $H^s(\mathbb{R})$ for $s < \frac{3}{2}$ [17] (and in $H^{\frac{3}{2}}(\mathbb{R})$ for $1 < b \leq 3$ [16]) due to the lack of continuous dependence and norm inflation, smooth solutions to the CH and DP equations were considered in [7] and [23] close to the peakons in the energy space.

The largest class of initial data for which the initial-value problem is well-posed for the b-CH equation is given by $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ [24]. It was recently shown in [26] for the CH equation ($b = 2$) that, although the peakons are orbitally stable in $H^1(\mathbb{R})$, they are unstable with respect to perturbations in $W^{1,\infty}(\mathbb{R})$ in the sense that the $W^{1,\infty}$ norm of the peaked perturbations may grow in time and may reach infinity in a finite time leading to the wave breaking of the solution. This analysis was performed by using the method of characteristics in the nonlinear evolution of the CH equation.

Previous studies of stability avoid the question of the linearized stability of peakons because it was believed that "the nonlinearity plays a dominant role rather than being a higher-order correction" and that "the passage from the linear to the nonlinear theory is not an easy task, and may even be false" [6]. The first study of the linearized evolution of peaked perturbations in [26] gave valid evidence to this concern since it was found that the H^1 norm of the peaked perturbations grows within the linearized approximation exponentially as $e^{\frac{1}{2}t}$, while it does not grow in the full nonlinear evolution. It was recently clarified in [25] in the setting of peaked periodic waves of the CH equation ($b = 2$) that both the growth of the H^1 norm of perturbations in the linearized approximation and its boundedness in the full nonlinear evolution of the CH equation are related to the same two energy integrals.

These preliminary results raised an open question of whether the stability of the peakons can be understood from the spectral stability theory, where the instability of traveling waves follows from the presence of the spectrum of a linearized operator in the right half plane of the complex plane. *The main purpose of this work is to give a definitive answer to this question with rigorous analysis of the spectral instability of peakons in the b-CH equation for any b.*

We now explain the main results and organization of this paper.

Since the local well-posedness of the initial-value problem for the b-CH equation (3) holds in the space $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ which includes peakons and their peaked perturbations [24, 26], we first introduce the linearized operator acting on functions in this space. However, this space is restrictive for the spectral stability theory; hence we use projections to the peakon and its spatial translation in order to extend the linearized operator in $L^2(\mathbb{R})$ with a suitable defined domain, similarly to the recent studies of peaked periodic waves in the reduced Ostrovsky equation [14, 15]. Transformation of the linearized operator relies on Lemmas 2.3 and 2.5.

We then analyze the spectrum of the linearized operator in $L^2(\mathbb{R})$. The main result given by Theorem 2.9 states that, for $b \neq \frac{5}{2}$, the spectrum covers a closed vertical strip of the complex plane. The half-width of the strip is exactly $\frac{1}{2}$ for $b = 2$ which coincides with the exponential growth $e^{\frac{1}{2}t}$ of the H^1 norm of the perturbations obtained in [26]. We also note that the half-width of the strip for $b < \frac{5}{2}$ agrees with the observation made in Remark 3.6 in [4] in their analysis of the differential operator linearized at the peakon of the b-CH equation.

For $b = \frac{5}{2}$, the strip shrinks to the imaginary axis, but we show in Corollary 2.6 that the projections to the peakon and its spatial translation also grow exponentially according to a system of two first-order differential equations. With the growth of perturbations for $b \neq \frac{5}{2}$ in Corollary 2.10, these results suggest that *the peakons of the b-CH equation* (3) *are linearly unstable in* $L^2(\mathbb{R})$ *for every b.*

Section 2 explains the derivation of the linearized operator acting on functions in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ and its extension to $L^2(\mathbb{R})$ with a suitable defined domain, after which the main result is formulated. Section 3 gives the proof of the main result with the analysis of spectral properties of the linearized operator in $L^2(\mathbb{R})$. Section 4 explains how the time evolution of the linearized equation is related to the spectral properties of the linearized operator in $L^2(\mathbb{R})$. Section 5 concludes the paper with a summary and discussion of further directions.

2. Linearized evolution.

We recall from [26] that if $u \in C([0,T), H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}))$ is a weak solution to the b-CH equation (3) such that $u(t, \cdot + \xi(t)) \in C^1(-\infty, 0) \cap C^1(0, \infty)$ for $t \in [0,T)$, then the single peak at $x = \xi(t)$ moves along the local characteristic curve with $\xi'(t) = u(t, \xi(t))$. Therefore, we decompose the solution near the peakon $\phi(x) = e^{-|x|}$ traveling with the unit speed into the following sum:

$$u(t,x) = \phi(x - t - a(t)) + v(t, x - t - a(t)), \tag{5}$$

where $v(t,\cdot) \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ is the peaked perturbation such that $v(t,\cdot) \in C^1(-\infty, 0) \cap C^1(0, \infty)$ and $a(t)$ is the deviation of the peak position of the perturbed peakon from its unperturbed position satisfying $a'(t) = v(t,0)$. Substituting (5) into (3) and using the stationary equation (4) yield the following linearized equation:

$$v_t = (1-\phi)v_\xi + (v_0 - v)\phi' - \frac{1}{2}\phi' * [b\phi v + (3-b)\phi' v_\xi]\,, \tag{6}$$

where $v_0(t) := v(t,0)$ and $\xi := x - t - a(t)$. By using an elementary identity proven in [26],

$$\phi' * (\phi' v') = \phi * (\phi' v) - \phi' * (\phi v) + 2(v_0 - v)\phi' \quad \forall v \in H^1(\mathbb{R}),$$

the linearized equation (6) can be cast in the equivalent form:

$$v_t = (1-\phi)v_\xi + (b-2)(v_0 - v)\phi' + Q(v), \tag{7}$$

where

$$Q(v) := \frac{1}{2}(b-3)\phi * (\phi' v) - \frac{1}{2}(2b-3)\phi' * (\phi v). \tag{8}$$

Using another elementary identity from [26],

$$\phi * (\phi' v) + \phi' * (\phi v) + 2\phi v_{-1} = 0 \quad \forall v \in H^1(\mathbb{R}), \quad v_{-1}(\xi) := \int_0^\xi v(\xi')d\xi,$$

$Q(v)$ can be rewritten into the following two equivalent forms:

$$Q(v) = \frac{3}{2}(b-2)\phi * (\phi' v) + (2b-3)\phi v_{-1} = -\frac{3}{2}(b-2)\phi' * (\phi v) + (3-b)\phi v_{-1}. \tag{9}$$

The following lemma ensures compactness of the linear operator Q in $L^2(\mathbb{R})$.

LEMMA 2.1. *The linear operator $Q : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is compact.*

Proof. Each term in either (8) or (9) can be written as an integral operator of the form

$$\int_{-\infty}^{\infty} K(\xi,\xi')v(\xi')\,d\xi'$$

for some kernel $K \in L^2(\mathbb{R}^2)$. As such, each of those terms defines a Hilbert–Schmidt integral operator, known to be compact (see [27], p. 262). To be more specific, $\phi*(\phi' v)$ corresponds to the kernel

$$K_1 = -\mathrm{sgn}(\xi')\, e^{-|\xi-\xi'|-|\xi'|},$$

while ϕv_{-1} corresponds to

$$K_2 = \begin{cases} \mathrm{sgn}(\xi)\, e^{-|\xi|} & \text{for } \ 0 \le |\xi'| \le |\xi|, \\ 0 & \text{otherwise,} \end{cases}$$

for which we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |K_1(\xi,\xi')|^2 d\xi d\xi' = \int_{-\infty}^{\infty} \left(|\xi| + \frac{1}{2}\right) e^{-2|\xi|} d\xi = 1$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |K_2(\xi,\xi')|^2 d\xi d\xi' = \int_{-\infty}^{\infty} |\xi| e^{-2|\xi|} d\xi = \frac{1}{2}.$$

Hence, $K \in L^2(\mathbb{R}^2)$, and Q is the compact Hilbert–Schmidt operator in $L^2(\mathbb{R})$. ◻

The linearized equation (7) with $Q(v)$ given by either (8) or (9) is well defined in $v(t,\cdot) \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. This can be shown by using either the local well-posedness theory [24] or the method of characteristic curves [26]. The linearized evolution depends on the value $v_0(t) = v(t,0)$ which is well defined due to Sobolev embedding of $H^1(\mathbb{R})$ into the space of bounded and continuous functions.

Next, we extend the linearized equation to the larger space $\mathrm{Dom}(L) \subset L^2(\mathbb{R})$ associated with the linearized operator

$$L := (1-\phi)\partial_\xi + (2-b)\phi' + Q. \tag{10}$$

Since Q is a compact operator in $L^2(\mathbb{R})$ by Lemma 2.1 and $\phi \in W^{1,\infty}(\mathbb{R})$, the domain of L in $L^2(\mathbb{R})$ is defined by

$$\mathrm{Dom}(L) = \left\{ v \in L^2(\mathbb{R}) : \quad (1-\phi)v' \in L^2(\mathbb{R}) \right\}. \tag{11}$$

It follows from the bound $\|(1-\phi)v'\|_{L^2} \le \|v'\|_{L^2}$ that $H^1(\mathbb{R})$ is continuously embedded into $\mathrm{Dom}(L)$. However, $H^1(\mathbb{R})$ is not equivalent to $\mathrm{Dom}(L)$ because $\phi' \in \mathrm{Dom}(L)$ but $\phi' \notin H^1(\mathbb{R})$ since $\xi\delta_0(\xi) \in L^2(\mathbb{R})$ but $\phi \notin C^1(\mathbb{R})$. Generally, functions in $\mathrm{Dom}(L)$ do not have to be continuous across the peak at $\xi = 0$; therefore, v_0 may not be defined if $v \in \mathrm{Dom}(L)$ but $v \notin H^1(\mathbb{R})$.

In order to extend the linearized equation (7) in $\mathrm{Dom}(L)$, we are going to use another equivalent reformulation of the linearized equation (7) for $v(t,\cdot) \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. This is described in Lemma 2.3 after the proof of the following elementary property of L.

Lemma 2.2. *For $L : \mathrm{Dom}(L) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$, it is true for every b that*

$$L\phi = (2-b)\phi' \text{ and } L\phi' = 0. \tag{12}$$

Proof. By using the first expression in (9), we obtain for every $\xi \neq 0$

$$Q(\phi) = \frac{3}{2}(b-2)\phi * (\phi\phi') + (2b-3)\phi \int_0^\xi \phi(\xi')d\xi' = (1-b)(\phi' - \phi\phi')$$

and

$$Q(\phi') = \frac{3}{2}(b-2)\phi * (\phi'\phi') + (2b-3)\phi \int_0^\xi \phi'(\xi')d\xi' = -\phi + (b-1)\phi^2,$$

which yields (12) after substituting into (10). Note that $\phi, \phi' \in \mathrm{Dom}(L)$ and $\xi\delta_0(\xi) = 0$ in $L^2(\mathbb{R})$. ∎

Lemma 2.3. *Consider the class of functions in*

$$X := C(\mathbb{R}, H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})) \cap C^1(\mathbb{R}, L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})).$$

Then, $v \in X$ is a solution to the linearized equation (7) if and only if $\tilde{v} := v - v_0\phi \in X$ satisfying $\tilde{v}(t,0) = 0$ is a solution of the linearized equation

$$\tilde{v}_t = L\tilde{v} - \frac{3}{2}(b-2)\langle \phi\phi', \tilde{v}\rangle \phi, \tag{13}$$

where the inner product $\langle\cdot,\cdot\rangle$ is defined in $L^2(\mathbb{R})$.

Proof. Substituting $v(t,\xi) = \tilde{v}(t,\xi) + v_0(t)\phi(\xi)$ into (7) yields

$$\tilde{v}_t + v_0'(t)\phi = L\tilde{v} + v_0 L\phi + (b-2)v_0\phi'.$$

The last two terms cancel out due to the first identity (12) in Lemma 2.2. On the other hand, taking the limit $\xi \to 0$ into (7) in the class of functions $v \in X$ yields

$$v_0'(t) = \lim_{\xi\to 0} Q(v)(\xi) = \frac{3}{2}(b-2)\langle \phi\phi', v\rangle,$$

where either representation in (9) can be used together with the spatial symmetry of ϕ. Since $\langle \phi\phi', \phi\rangle = 0$, it follows that $\langle \phi\phi', v\rangle = \langle \phi\phi', \tilde{v}\rangle$ so that the two equations above yield (13). Since $v_0(t) = v(t,0)$, it is true that $\tilde{v}(t,0) = 0$. This constraint is preserved in the time evolution of (13) since $\lim_{\xi\to 0} L\tilde{v} = \frac{3}{2}(b-2)\langle \phi\phi', \tilde{v}\rangle$ for $\tilde{v} \in X$.

The proof in the opposite direction from (13) to (7) is identical. ∎

The equivalent evolution equation (13) is still defined for $\tilde{v}(t,\cdot) \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. However, the right-hand side is now well defined if $\tilde{v}(t,\cdot) \in \mathrm{Dom}(L) \subset L^2(\mathbb{R})$. This enables us to define linear stability of the peakons as follows.

Definition 2.4. *The peakon solution $u(t,x) = \phi(x-t)$ of the b-CH equation* (3) *is said to be linearly stable if, for every $\tilde{v}_0 \in \mathrm{Dom}(L) \subset L^2(\mathbb{R})$, there exists a positive constant C and a unique solution $\tilde{v} \in C(\mathbb{R}, \mathrm{Dom}(L))$ to the linearized equation* (13) *with $\tilde{v}(0,\xi) = \tilde{v}_0(\xi)$ such that*

$$\|\tilde{v}(t,\cdot)\|_{L^2} \le C\|v_0\|_{L^2}, \quad t > 0.$$

Otherwise, it is said to be linearly unstable.

In order to prove that the peakons are linearly unstable in the sense of Definition 2.4 for every b, we split the linearized equation (13) in $\mathrm{Dom}(L) \subset L^2(\mathbb{R})$ into two parts, where one is defined by the linearized operator L in (10)–(11) and the other one is defined by a system of two first-order differential equations. This task is achieved with the secondary decomposition described in the following lemma.

Lemma 2.5. *Consider the class of functions in*

$$Y := C(\mathbb{R}, \mathrm{Dom}(L)) \cap C^1(\mathbb{R}, L^2(\mathbb{R})).$$

Then, $\tilde{v} \in Y$ is a solution to the linearized equation (13) *if $w := \tilde{v} - \alpha\phi - \beta\phi' \in Y$ is a solution of the linearized equation*

$$\frac{dw}{dt} = Lw, \tag{14}$$

with α and β satisfying the system

$$\frac{d\alpha}{dt} = (2-b)\beta + \frac{3}{2}(2-b)\langle \phi\phi', w\rangle, \quad \frac{d\beta}{dt} = (2-b)\alpha. \tag{15}$$

Proof. Substituting $\tilde{v}(t,\xi) = \alpha(t)\phi(\xi) + \beta(t)\phi'(\xi) + w(t,\xi)$ into (13) yields

$$\alpha'(t)\phi + \beta'(t)\phi' + w_t = (2-b)\alpha\phi' + Lw + (2-b)\beta\phi + \frac{3}{2}(2-b)\langle \phi\phi', w\rangle\phi,$$

where we have used $\langle \phi, (\phi')^2\rangle = \frac{2}{3}$, $\langle \phi\phi', \phi\rangle = 0$, and the identities (12) in Lemma 2.2. Separating ϕ, ϕ', and the rest yields (14) and (15). ∎

COROLLARY 2.6. *The peakon is linearly unstable in the sense of Definition* 2.4 *for every* $b \neq 2$.

Proof. Setting $w = 0$ in (14) and (15) gives the second-order homogeneous system

$$\frac{d\alpha}{dt} = (2-b)\beta, \quad \frac{d\beta}{dt} = (2-b)\alpha,$$

where the linear instability with the exponential growth $e^{|2-b|t}$ exists for every $b \neq 2$. ∎

Remark 2.7. In order to ensure the uniqueness of the decomposition $\tilde{v}(t,\xi) = \alpha(t)\phi(\xi) + \beta(t)\phi'(\xi) + w(t,\xi)$, we compute the adjoint operator $L^* : \mathrm{Dom}(L) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ with respect to $\langle \cdot, \cdot \rangle$:

$$L^* v := (\phi - 1) v_\xi + (3-b)\phi' v + \frac{1}{2}(b-3)\phi'(\phi * v) + \frac{1}{2}(2b-3)\phi(\phi' * v). \tag{16}$$

With straightforward computations, we obtain

$$\begin{aligned}
&L^* 1 = 0,\\
&L^* \mathrm{sgn} = 3(b-2)\phi^2,\\
&L^* \phi^2 = 2(b-3)\phi\phi' + \frac{8}{3}(3-b)\phi^2\phi',\\
&L^*(\phi\phi') = (b-4)\phi^2 + \frac{8}{3}(3-b)\phi^3,
\end{aligned}$$

where $\mathrm{sgn}(\xi) = 1$ if $\xi > 0$ and $\mathrm{sgn}(\xi) = -1$ if $\xi < 0$.

If $b = 2$, the constraints $\langle 1, w \rangle$ and $\langle \mathrm{sgn}, w \rangle$ are preserved in the time evolution of $w \in Y \cap C(\mathbb{R}, L^1(\mathbb{R}))$. One can uniquely define $\alpha(t)$ and $\beta(t)$ by the orthogonality conditions $\langle 1, w \rangle = 0$ and $\langle \mathrm{sgn}, w \rangle = 0$ so that $\alpha = \frac{1}{2}\langle 1, \tilde{v} \rangle$ and $\beta = -\frac{1}{2}\langle \mathrm{sgn}, \tilde{v} \rangle$, where we have used $\langle 1, \phi \rangle = -\langle \mathrm{sgn}, \phi' \rangle = 2$ and $\langle 1, \phi' \rangle = \langle \mathrm{sgn}, \phi \rangle = 0$.

If $b = 3$, the constraints $\langle \phi^2, w \rangle$ and $\langle \mathrm{sgn} + 3\phi\phi', w \rangle$ are preserved in the time evolution of $w \in Y \cap C(\mathbb{R}, L^1(\mathbb{R}))$. One can uniquely define $\alpha(t)$ and $\beta(t)$ by the orthogonality conditions $\langle \phi^2, w \rangle = \langle \mathrm{sgn} + 3\phi\phi', w \rangle = 0$ since $\langle \phi^2, \phi' \rangle = \langle \mathrm{sgn} + 3\phi\phi', \phi \rangle = 0$.

In Appendix A, we derive an antisymmetric function v_b that satisfies $L^* v_b = 0$ and that is bounded at infinity and locally square integrable for every $b > \frac{5}{2}$. This also provides a way to uniquely define $\alpha(t)$ and $\beta(t)$ by the orthogonality conditions $\langle 1, w \rangle = \langle v_b, w \rangle = 0$ since $\langle 1, \phi' \rangle = \langle v_b, \phi \rangle = 0$.

If $b \leq \frac{5}{2}$ and $b \neq 2$, the decomposition $\tilde{v}(t,\xi) = \alpha(t)\phi(\xi) + \beta(t)\phi'(\xi) + w(t,\xi)$ is not uniquely defined so that the proof of Lemma 2.5 in the opposite direction from (14) and (15) to (13) is incomplete. For instance, $w(t,\xi) = \phi(\xi) + (2-b)t\phi'(\xi)$ is a valid solution of the linearized equation (14) due to Lemma 2.2, although it is redundant because system (15) gives $\alpha(t) = -1$ and $\beta(t) = -(2-b)t$ that generates $\tilde{v}(t,\xi) = 0$ as a solution of (13).

With the transformations of Lemmas 2.3 and 2.5, we have reduced the linearized evolution (7) defined in X to the linearized evolution (14) defined in Y. As a result, the spectral properties of the operator L in (10)–(11) determine the linear stability of the peakons in addition to the result of Corollary 2.6. The operator L is defined according to the following standard definition (see Definition 6.1.9 in [1]).

DEFINITION 2.8. *Let* A *be a linear operator on a Banach space* X *with* $\mathrm{Dom}(A) \subset X$. *The complex plane* $\mathbb{C}$ *is decomposed into the following two sets:*

1. *the resolvent set*

$$\rho(A) = \{\lambda \in \mathbb{C}: \ \operatorname{Ker}(A-\lambda I) = \{0\}, \ \operatorname{Ran}(A-\lambda I) = X, \\ (A-\lambda I)^{-1} : X \to X \text{ is bounded}\};$$

2. *the spectrum*

$$\sigma(A) = \mathbb{C} \setminus \rho(A),$$

which is further decomposed into the following three disjoint sets:

(a) *the point spectrum*

$$\sigma_{\rm p}(A) = \{\lambda \in \sigma(A): \ \operatorname{Ker}(A-\lambda I) \neq \{0\}\},$$

(b) *the residual spectrum*

$$\sigma_{\rm r}(A) = \{\lambda \in \sigma(A): \ \operatorname{Ker}(A-\lambda I) = \{0\}, \ \operatorname{Ran}(A-\lambda I) \neq X\},$$

(c) *the continuous spectrum*

$$\sigma_{\rm c}(A) = \{\lambda \in \sigma(A): \ \operatorname{Ker}(A-\lambda I) = \{0\}, \ \operatorname{Ran}(A-\lambda I) = X, \\ (A-\lambda I)^{-1} : X \to X \text{ is unbounded}\}.$$

The following theorem represents the main result of this paper.

THEOREM 2.9. *The spectrum of the linear operator L defined by* (10)–(11) *is given by*

$$\sigma(L) = \left\{\lambda \in \mathbb{C}: \ |\,\mathsf{Re}(\lambda)\,| \leq \left|\frac{5}{2} - b\right|\right\}.$$

Moreover, the point spectrum is located for $0 < |\,\mathsf{Re}(\lambda)\,| < \frac{5}{2} - b$ if $b < \frac{5}{2}$, and the residual spectrum is located for $0 < |\,\mathsf{Re}(\lambda)\,| < b - \frac{5}{2}$ if $b > \frac{5}{2}$, whereas the continuous spectrum is located for $\mathsf{Re}(\lambda) = 0$, except $\lambda = 0$, and $\mathsf{Re}(\lambda) = \pm|\frac{5}{2} - b|$ in both cases. Additionally, $\lambda = 0$ is the eigenvalue of the point spectrum of algebraic multiplicity 2 embedded into the continuous spectrum for every b.

COROLLARY 2.10. *The peakon is linearly unstable in the sense of Definition 2.4 for every $b \neq \frac{5}{2}$.*

Proof. If $\lambda_0 \in (0, \frac{5}{2} - b)$ is a real eigenvalue for the point spectrum of L for $b < \frac{5}{2}$ and $w_0 \in \operatorname{Dom}(L) \subset L^2(\mathbb{R})$ is the corresponding eigenfunction of L, then the linearized equation (14) has the exact solution $w(t,\xi) = e^{\lambda_0 t} w_0(\xi)$ with the exponential growth of $\|w(t,\cdot)\|_{L^2}$.

If $\lambda_0 \in (0, b - \frac{5}{2})$ is a real eigenvalue for the residual spectrum of L for $b > \frac{5}{2}$, then λ_0 is the eigenvalue for the point spectrum of L^* since $\sigma_r(L) \subseteq \sigma_p(L^*)$ by Lemma 6.2.6 in [1]. Let $w_0 \in \operatorname{Dom}(L^*) \subset L^2(\mathbb{R})$ be an eigenfunction of L^* for the eigenvalue λ_0. Since $\operatorname{Dom}(L^*) = \operatorname{Dom}(L)$, we consider the decomposition $w(t,\xi) = a(t) w_0(\xi) + \tilde{w}(t,\xi)$, where $a(t)$ is uniquely determined by the orthogonality condition $\langle w_0, \tilde{w}(t,\cdot)\rangle = 0$. Both $a(t)$ and $\tilde{w}(t,\xi)$ are found from

$$\frac{da}{dt} w_0 + \frac{d\tilde{w}}{dt} = aLw_0 + L\tilde{w}.$$

Projecting to w_0 yields $\frac{da}{dt} = \lambda_0 a$ with the exponential growth of $a(t)$ if $\lambda_0 > 0$. This also gives the exponential growth of $\|w(t,\cdot)\|_{L^2}$ due to the orthogonality $\langle w_0, \tilde{w}(t,\cdot)\rangle = 0$.

In both cases, the linear evolution of $w \in Y$ grows exponentially in the L^2 norm if $b \neq \frac{5}{2}$. By Definition 2.4, the peakon is linearly unstable in Y. ∎

Remark 2.11. Since linear instabilities of Corollaries 2.6 and 2.10 vanish at different values of b, the peakon is linearly unstable in Y for every b.

Remark 2.12. For each exponentially growing solution $w \in Y$ of the linearized equation (14), one can find the unique solution of the second-order system (15) which grows either exponentially with the same rate if $|b-2| > |\frac{5}{2} - b|$ (or $b > \frac{9}{4}$) (when the unstable eigenvalue $|b-2|$ is outside the strip in Theorem 2.9) or exponentially times polynomially if $|b-2| \le |\frac{5}{2} - b|$ (or $b \le \frac{9}{4}$) (when the unstable eigenvalue $|b-2|$ is inside the strip in Theorem 2.9).

3. Spectrum of L.

We decompose L given by (10)–(11) as $L = L_0 + Q$, where $L_0 : \mathrm{Dom}(L) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is given by

$$L_0 := (1-\phi)\partial_\xi + (2-b)\phi' \tag{17}$$

and $Q : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is the compact operator by Lemma 2.1. By Theorem 1 in [15], if the intersections $\sigma_p(L_0) \cap \rho(L)$ and $\sigma_p(L) \cap \rho(L_0)$ are empty, then $\sigma(L) = \sigma(L_0)$. The proof of Theorem 2.9 is achieved by computing the spectrum of L_0, the point spectrum L, and the residual spectrum of L.

3.1. Spectrum of L_0.

The spectrum of L_0 is described by the following theorem.

THEOREM 3.1. *The spectrum of the linear operator* $L_0 : \mathrm{Dom}(L) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ *defined by* (17) *is given by*

$$\sigma(L_0) = \left\{ \lambda \in \mathbb{C} : \ |\,\mathsf{Re}(\lambda)\,| \le \left| \frac{5}{2} - b \right| \right\}.$$

Moreover, the point spectrum is located for $0 < |\,\mathsf{Re}(\lambda)\,| < \frac{5}{2} - b$ *if* $b < \frac{5}{2}$, *and the residual spectrum is located for* $0 < |\,\mathsf{Re}(\lambda)\,| < b - \frac{5}{2}$ *if* $b > \frac{5}{2}$, *whereas the continuous spectrum is located for* $\mathsf{Re}(\lambda) = 0$ *and* $\mathsf{Re}(\lambda) = \pm|\frac{5}{2} - b|$ *in both cases.*

Proof. Given simplicity of the definition of L_0, the proof is obtained by computing the point, residual, and continuous spectrum of L_0 explicitly.

Point spectrum of L_0. We solve the differential equation

$$(1-\phi)\frac{dv}{d\xi} + (2-b)\phi' v = \lambda v, \quad \xi \in \mathbb{R}. \tag{18}$$

The differential equation (18) is solved separately for $\xi > 0$ and $\xi < 0$ with the following general solution:

$$v(\xi) = \begin{cases} v_+ e^{\lambda\xi}(1-e^{-\xi})^{2+\lambda-b}, & \xi > 0, \\ v_- e^{\lambda\xi}(1-e^{\xi})^{2-\lambda-b}, & \xi < 0, \end{cases} \tag{19}$$

where v_+ and v_- are arbitrary constants.

The differential equation (18) has the following symmetry: if $\lambda = \lambda_0$ is an eigenvalue with the eigenfunction $v = v_0(\xi)$, then $\lambda = -\lambda_0$ is an eigenvalue with the eigenfunction $v = v_0(-\xi)$. Therefore, it is sufficient to consider the case of $\mathsf{Re}(\lambda) \ge 0$.

Since $v(\xi) \sim v_+ e^{\lambda\xi}$ as $\xi \to +\infty$, then $v \in L^2(\mathbb{R})$ for $\mathsf{Re}(\lambda) \ge 0$ is satisfied by the only choice of $v_+ = 0$. Since $v(\xi) \sim v_- e^{\lambda\xi}$ as $\xi \to -\infty$, then $v \in L^2(\mathbb{R})$ is satisfied for arbitrary v_- if and only if $\mathsf{Re}(\lambda) > 0$. Since $v(\xi) \sim v_- |\xi|^{2-b-\lambda}$ as $\xi \to 0^-$, then $v \in L^2(\mathbb{R})$ is satisfied for arbitrary v_- if and only if $\mathsf{Re}(\lambda) + b - 2 < \frac{1}{2}$, that is, for

$\mathsf{Re}(\lambda) < \frac{5}{2} - b$. In all other cases, we have to set $v_- = 0$ for $v \in L^2(\mathbb{R})$ so that the zero function is the only solution of (18) in $L^2(\mathbb{R})$. Summarizing and using the symmetry above, $\sigma_p(L_0)$ exists if $b < \frac{5}{2}$ and is located for $0 < |\,\mathsf{Re}(\lambda)\,| < \frac{5}{2} - b$.

Residual spectrum of L_0. By Lemma 6.2.6 in [1], if $\sigma_p(L_0)$ is an empty set, then $\sigma_r(L_0) = \sigma_p(L_0^*)$, where

$$L_0^* = -\partial_\xi(1-\phi) + (2-b)\phi' = -(1-\phi)\partial_\xi + (3-b)\phi'$$

is the adjoint operator to L_0 in $L^2(\mathbb{R})$. The differential equation

$$-(1-\phi)\frac{dv}{d\xi} + (3-b)\phi' v = \lambda v, \quad \xi \in \mathbb{R}, \tag{20}$$

becomes (18) after the transformation: $\lambda \mapsto -\lambda$ and $b - 2 \mapsto 3 - b$. Therefore, we obtain the following general solution by applying this transformation to (19):

$$v(\xi) = \begin{cases} v_+ e^{-\lambda\xi}(1-e^{-\xi})^{b-\lambda-3}, & \xi > 0, \\ v_- e^{-\lambda\xi}(1-e^{\xi})^{b+\lambda-3}, & \xi < 0, \end{cases} \tag{21}$$

where v_+ and v_- are arbitrary constants. Proceeding similarly with the limits $\xi \to \pm\infty$ and $\xi \to 0^\pm$ shows that nonzero solutions of (20) exist in $L^2(\mathbb{R})$ if and only if $v_- = 0$, $b > \frac{5}{2}$, and $0 < |\,\mathsf{Re}(\lambda)\,| < \frac{5}{2} - b$. This yields $\sigma_r(L_0)$ since $\sigma_p(L_0)$ is an empty set for $b > \frac{5}{2}$.

Resolvent set of L_0. Let us consider the resolvent equation

$$L_0 v - \lambda v = f, \tag{22}$$

where $f \in L^2(\mathbb{R})$ is arbitrary and $\mathsf{Re}(\lambda) \geq 0$ is assumed without loss of generality. We multiply both sides of (22) by $\bar{v}$ and integrate over $\mathbb{R}$. Using the definition of L_0 given in (17), one finds

$$\left\langle ((1-\phi)v)', v \right\rangle + (3-b)\left\langle \phi' v, v\right\rangle - \lambda \|v\|^2 = \langle f, v\rangle . \tag{23}$$

By integration by parts, since $\lim_{\xi\to\pm\infty} v(\xi) = 0$ for $v \in \mathrm{Dom}(L)$, we have that

$$\left\langle ((1-\phi)v)', v \right\rangle = -\left\langle v, ((1-\phi)v)' \right\rangle - \left\langle \phi' v, v\right\rangle ;$$

thus

$$\mathsf{Re}(\langle ((1-\phi)v)', v\rangle) = -\frac{1}{2}\left\langle \phi' v, \bar{v}\right\rangle . \tag{24}$$

Taking the real part of (23), multiplying by -1, and using (24), we get

$$\left(b - \frac{5}{2}\right)\left\langle \phi' v, v\right\rangle + \mathsf{Re}(\lambda)\,\|v\|^2 = -\,\mathsf{Re}(\langle f, v\rangle)\,, \tag{25}$$

where

$$-\|v\|^2 \leq \langle \phi' v, v\rangle \leq \|v\|^2. \tag{26}$$

Using the upper bound of (26) in (25) in the case $b \leq \frac{5}{2}$, we find that

$$\left(\mathsf{Re}(\lambda) + b - \frac{5}{2}\right)\|v\|^2 \leq |\mathsf{Re}(\langle f, v\rangle)| \leq \|f\|\|v\|;$$

where the Cauchy–Schwarz inequality has been used. Hence, for every $\mathsf{Re}(\lambda) > \frac{5}{2} - b$, there exists C_λ such that $\|v\| \leq C_\lambda \|f\|$ so that this λ belongs to $\rho(L_0)$.

In a very similar way, using the lower bound of (26) in (25) in the case $b \geq \frac{5}{2}$, we find that

$$\left(\mathsf{Re}(\lambda) - b + \frac{5}{2}\right) \|v\|^2 \leq |\mathsf{Re}(\langle f, v \rangle)| \leq \|f\| \|v\|,$$

hence $\mathsf{Re}(\lambda) > b - \frac{5}{2}$ belongs to $\rho(L_0)$.

Continuous spectrum of L_0. By Theorem 4 in [12, p. 1438], if L_0 is a differential operator on $\mathbb{R} = (-\infty, 0) \cup (0, \infty)$ and $L_0^\pm$ are restrictions of L_0 on $(-\infty, 0)$ and $(0, \infty)$, then $\sigma_c(L_0) = \sigma_c(L_0^+) \cup \sigma_c(L_0^-)$. Therefore, we can represent the resolvent equation (22) separately for $\xi > 0$ and $\xi < 0$.

For $\xi > 0$, we use the transformation $(0, \infty) \ni \xi \mapsto z \in \mathbb{R}$ by $z = \log(e^\xi - 1)$. Then, $\mathfrak{v}(z) := v(\xi)$ satisfies the resolvent equation

$$\mathfrak{L}_0^+ \mathfrak{v} - \lambda \mathfrak{v} = \mathfrak{f},$$

where $\mathfrak{L}_0^+ := \partial_z + (b - 2)(1 + e^z)^{-1}$ and $\mathfrak{f}(z) := f(\xi)$. Since

$$\int_0^\infty v(\xi)^2 d\xi = \int_{-\infty}^0 e^z \frac{\mathfrak{v}(z)^2 dz}{1 + e^z} + \int_0^\infty \frac{\mathfrak{v}(z)^2 dz}{1 + e^{-z}},$$

$\sigma_c(L_0^+)$ is the union of $\sigma_c(\mathfrak{L}_0^+)$ in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-; e^z)$, where e^z is the exponential weighted L^2 space. Since $\mathfrak{L}_0^+ = \partial_z + e^{-z}(b - 2)(1 + e^{-z})^{-1}$ and $e^{-z}(1 + e^{-z})^{-1}$ is bounded and decaying exponentially to 0 as $z \to +\infty$, $\sigma_c(\mathfrak{L}_0^+)$ in $L^2(\mathbb{R}_+)$ is located at $\mathsf{Re}(\lambda) = 0$. Since $\mathfrak{L}_0^+ = \partial_z + (b-2) - (b-2)e^z(1 + e^z)^{-1}$ and $e^z(1 + e^z)^{-1}$ is bounded and decaying exponentially to 0 as $z \to -\infty$, $\sigma_c(\mathfrak{L}_0^+)$ in $L^2(\mathbb{R}_-; e^z)$ is located at $\mathsf{Re}(\lambda) = b - \frac{5}{2}$.

Similarly, for $\xi < 0$, we use the transformation $(-\infty, 0) \ni \xi \mapsto z \in \mathbb{R}$ by $z = -\log(e^{-\xi} - 1)$ and obtain the resolvent equation for $\mathfrak{v}(z) := v(\xi)$:

$$\mathfrak{L}_0^- \mathfrak{v} - \lambda \mathfrak{v} = \mathfrak{f},$$

where $\mathfrak{L}_0^- := \partial_z + (2 - b)(1 + e^{-z})^{-1}$ and $\mathfrak{f}(z) := f(\xi)$. Since

$$\int_{-\infty}^0 v(\xi)^2 d\xi = \int_{-\infty}^0 \frac{\mathfrak{v}(z)^2 dz}{1 + e^z} + \int_0^\infty e^{-z} \frac{\mathfrak{v}(z)^2 dz}{1 + e^{-z}},$$

$\sigma_c(L_0^-)$ is the union of $\sigma_c(\mathfrak{L}_0^-)$ in $L^2(\mathbb{R}_-)$ and $L^2(\mathbb{R}_+; e^{-z})$. Similar to the previous arguments, the continuous spectrum of $\mathfrak{L}_0^- = \partial_z + e^z(2 - b)(1 + e^z)^{-1}$ in $L^2(\mathbb{R}_-)$ is located at $\mathsf{Re}(\lambda) = 0$, and the continuous spectrum of $\mathfrak{L}_0^- = \partial_z + (2 - b) - (2 - b)e^{-z}(1 + e^{-z})^{-1}$ in $L^2(\mathbb{R}_+; e^{-z})$ is located at $\mathsf{Re}(\lambda) = \frac{5}{2} - b$.

By Theorem 4 in [12], $\sigma_c(L_0)$ is located for $\mathsf{Re}(\lambda) = 0$ and $\mathsf{Re}(\lambda) = \pm|\frac{5}{2} - b|$. ◻

3.2. Point spectrum of L. For the point spectrum of L, we consider the spectral problem

$$Lv - \lambda v = 0, \quad v \in \mathrm{Dom}(L) \subset L^2(\mathbb{R}). \tag{27}$$

By Lemma 2.2, 0 is always a double eigenvalue associated with the two-dimensional invariant subspace $\{\phi, \phi'\}$. If $b \neq 2$, the double zero eigenvalue is defective with only one linearly independent eigenfunction. For $b = 2$, the double zero eigenvalue is semi-simple. In what follows, we are looking for other solutions of the spectral problem (27).

Remark 3.2. The double zero eigenvalue of L is related to the secondary decomposition $\tilde{v} = \alpha\phi + \beta\phi' + w \in Y$ in Lemma 2.5. By Remark 2.7, the decomposition is not unique in general. It is therefore unclear how to set up a constrained subspace of $L^2(\mathbb{R})$ so that the double zero eigenvalue of L is eliminated by the constraints.

The following lemma characterizes the point spectrum of L.

LEMMA 3.3. *In addition to the double zero eigenvalue that exists for every* b, *the linear operator* $L : \mathrm{Dom}(L) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ *defined by* (10) *admits the point spectrum for* $0 < |\,\mathsf{Re}(\lambda)\,| < \frac{5}{2} - b$ *if* $b < \frac{5}{2}$.

Remark 3.4. Some ideas of the proof of Lemma 3.3 are similar to what was done recently in [4]. However, the space of functions was different, and the direct linearization of the b-CH equation was used in [4] without appealing to the decomposition (5). In addition, we find all the solutions for the spectral problem (27) explicitly.

Proof. The spectral problem (27) has the same symmetry as the differential equation (18): if $\lambda = \lambda_0$ is an eigenvalue with the eigenfunction $v = v_0(\xi)$, then $\lambda = -\lambda_0$ is an eigenvalue with the eigenfunction $v = v_0(-\xi)$. Therefore, it is sufficient to consider the case $\mathsf{Re}(\lambda) \geq 0$.

Applying the operator $1 - \partial_\xi^2$ to (27) separately for $\xi < 0$ and $\xi > 0$ yields the following differential equation:

$$\lambda(v - v'') = (1 - \phi)(v' - v''') - b\phi'(v - v''). \tag{28}$$

Indeed, for $\xi < 0$, we compute from the representation (9)

$$\begin{aligned}&(1 - \partial_\xi^2)\phi * (\phi' v)\\ &\quad = (1 - \partial_\xi^2)\left[\int_{-\infty}^{\xi} e^{-\xi+2\eta} v(\eta)d\eta + \int_{\xi}^{0} e^{\xi} v(\eta)d\eta + \int_{0}^{\infty} e^{\xi-2\eta} v(\eta)d\eta\right] = 2e^{\xi} v\end{aligned}$$

and

$$(1 - \partial_\xi^2)\phi v_{-1} = (1 - \partial_\xi^2)e^{\xi}\int_0^{\xi} v(\eta)d\eta = -2e^{\xi}v - e^{\xi}v',$$

which yields (28) for $\xi < 0$ after adding all terms. Computations for $\xi > 0$ are similar.

Let $m := v - v''$. The differential equation (28) becomes the first-order equation

$$(1 - \phi)\frac{dm}{d\xi} - b\phi' m = \lambda m, \tag{29}$$

which coincides with the differential equation (18) after replacing b by $b-2$. Therefore, the exact solution is given by the corresponding transformation of the solution (19), namely,

$$m(\xi) = \left\{\begin{array}{ll} m_+ e^{\lambda\xi}(1 - e^{-\xi})^{\lambda - b}, & \quad \xi > 0,\\ m_- e^{\lambda\xi}(1 - e^{\xi})^{-\lambda - b}, & \quad \xi < 0, \end{array}\right. \tag{30}$$

where m_+ and m_- are arbitrary constants. The eigenfunction v is recovered from solutions of the second-order equation

$$v(\xi) - v''(\xi) = \left\{\begin{array}{ll} m_+ e^{\lambda\xi}(1 - e^{-\xi})^{\lambda - b}, & \quad \xi > 0,\\ m_- e^{\lambda\xi}(1 - e^{\xi})^{-\lambda - b}, & \quad \xi < 0. \end{array}\right. \tag{31}$$

If $v \in L^2(\mathbb{R})$, then $m \in H^{-2}(\mathbb{R})$, the dual space of $H^2(\mathbb{R})$. Since $m(\xi) \sim m_+ e^{\lambda\xi}$ as $\xi \to +\infty$, then $v \in L^2(\mathbb{R})$ for $\mathrm{Re}(\lambda) \geq 0$ exists with the only choice $m_+ = 0$ and $v(\xi) = c_+ e^{-\xi}$, where c_+ is arbitrary. Similarly, $m(\xi) \sim m_- e^{\lambda\xi}$ as $\xi \to -\infty$. If $\mathrm{Re}(\lambda) = 0$, then $v \in L^2(\mathbb{R})$ exists with the only choice $m_- = 0$ and $v(\xi) = c_- e^{\xi}$, where c_- is arbitrary. In this case, $v = c_1\phi + c_2\phi'$ with $c_1 \pm c_2 = c_\mp$, which corresponds to the double eigenvalue $\lambda = 0$ in $\sigma_p(L)$.

It remains to consider the case $m_+ = 0$, $m_- \neq 0$, and $\mathrm{Re}(\lambda) > 0$. With the normalization $m_- = 1$, the solution of (31) for $\xi < 0$ can be written in the form

$$v(\xi) = e^{\lambda\xi} f(\xi)(1 - e^{\xi})^{2-\lambda-b}, \quad \xi < 0, \tag{32}$$

where $f(\xi)$ satisfies the second-order differential equation

$$\begin{aligned}&(1-e^{\xi})^2(f'' + 2(2-b)f' + (b-1)(b-3)f)\\ &+(\lambda+b-2)(1-e^{\xi})(2f' + (3-2b)f) + (\lambda+b-2)(\lambda+b-1)f = -1.\end{aligned} \tag{33}$$

The homogeneous part of the second-order (33) with the regular singular point $\xi = 0$ is associated with the indicial equation

$$\sigma^2 + (2\lambda + 2b - 5)\sigma + (\lambda + b - 2)(\lambda + b - 1) = 0$$

for power solutions $f(\xi) \sim \xi^{\sigma}$. If $\lambda + b \neq \{1, 2\}$, then 0 is not among the roots of the indicial equation, whereas if $\lambda + b = \{1, 2\}$, then 0 is a simple root of the indicial equation. By the Frobenius theory (see, e.g., Chapter 4 in [28]), there exists a particular solution to the differential equation (33) with the following behavior near the regular singular point $\xi = 0$:

$$f_p(\xi) \sim \begin{cases} 1 + \mathcal{O}(|\xi|), & \lambda + b \neq \{1,2\}, \\ \log|\xi| + \mathcal{O}(|\xi|\log|\xi|), & \lambda + b = \{1,2\} \end{cases} \quad \text{as} \quad \xi \to 0^-, \tag{34}$$

which yields the corresponding behavior of $v_p(\xi)$ from (32),

$$v_p(\xi) \sim \begin{cases} |\xi|^{2-b-\lambda}, & \lambda + b \neq \{1,2\}, \\ \log|\xi|, & \lambda + b = 2, \\ |\xi|\log|\xi|, & \lambda + b = 1 \end{cases} \quad \text{as} \quad \xi \to 0^-. \tag{35}$$

Hence, (31) has a solution $v \in L^2(\mathbb{R})$ if and only if $\mathrm{Re}(\lambda) + b - 2 < \frac{1}{2}$ for $0 < \mathrm{Re}(\lambda) < \frac{5}{2} - b$.

Summarizing and using the symmetry above, $\sigma_p(L)$ exists if $b < \frac{5}{2}$ and is located for $0 < |\,\mathrm{Re}(\lambda)\,| < \frac{5}{2} - b$. ∎

Remark 3.5. In the case of CH equation ($b = 2$), there exists the exact solution of the differential equation (31) with $\lambda \neq \{-1, 0, 1\}$ in the form

$$v(\xi) = \frac{1}{\lambda(1-\lambda^2)} \begin{cases} m_+(\lambda + e^{-\xi})(e^{\xi} - 1)^{\lambda}, & \xi > 0, \\ m_-(\lambda - e^{\xi})(e^{-\xi} - 1)^{-\lambda}, & \xi < 0. \end{cases} \tag{36}$$

If $m_- \neq 0$, then $v(\xi) \sim e^{\lambda\xi}$ as $\xi \to -\infty$ and $v(\xi) \sim |\xi|^{-\lambda}$ as $\xi \to 0^-$ so that $v \in L^2(\mathbb{R})$ if $0 < \mathrm{Re}(\lambda) < \frac{1}{2}$.

3.3. Residual spectrum of *L*. By Lemma 6.2.6 in [1], $\sigma_r(L) \subseteq \sigma_p(L^*)$. For the point spectrum of L^*, we consider the spectral problem

$$L^* v - \lambda v = 0, \quad v \in \mathrm{Dom}(L) \subset L^2(\mathbb{R}), \tag{37}$$

where L^* is defined in (16). The following lemma describes the point spectrum of L^*.

LEMMA 3.6. *The linear operator* $L^* : \mathrm{Dom}(L) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ *defined by* (16) *has a nonempty point spectrum only if* $b > \frac{5}{2}$, *in which case it is located for* $0 < |\,\mathsf{Re}(\lambda)\,| < b - \frac{5}{2}$ *and* $\lambda = 0$.

Proof. It is again sufficient to consider the case of $\mathsf{Re}(\lambda) \geq 0$. Like in Appendix A, by making the substitution $v = (1 - \partial_\xi^2)k$ and integrating by parts under the assumption that k and k' are bounded and continuous functions, we obtain

$$(1-\phi)k''' + (b-3)\phi' k'' + (2(b-1)\phi - 1)k' = \lambda(k - k_{\xi\xi}). \tag{38}$$

The operator $(1 - \partial_\xi^2)$ can be factored out for both cases $\xi > 0$ and $\xi < 0$ as follows:

$$(1 - \partial_\xi^2)\left[(\phi - 1)k' + (1-b)\phi' k - \lambda k\right] = 0. \tag{39}$$

The first-order equation

$$(\phi - 1)k' + (1-b)\phi' k = \lambda k \tag{40}$$

follows from (29) after the transformation: $\lambda \mapsto -\lambda$ and $b \mapsto 1 - b$. Therefore, we obtain the following general solution by applying this transformation to (30):

$$k(\xi) = \begin{cases} k_+ e^{-\lambda\xi}(1 - e^{-\xi})^{b-\lambda-1}, & \xi > 0, \\ k_- e^{-\lambda\xi}(1 - e^{\xi})^{b+\lambda-1}, & \xi < 0, \end{cases} \tag{41}$$

where k_+ and k_- are arbitrary constants. Proceeding similarly with the limits $\xi \to \pm\infty$ and $\xi \to 0^\pm$ for $\mathsf{Re}(\lambda) > 0$ shows that the corresponding nonzero function $v = (1 - \partial_\xi^2)k$ is in $L^2(\mathbb{R})$ only if $k_- = 0$ and $b - \mathsf{Re}(\lambda) - 1 > 3/2$, which gives $0 < \mathsf{Re}(\lambda) < b - \frac{5}{2}$ if $b > \frac{5}{2}$.

In general, it follows from (39) that

$$(\phi - 1)k' + (1-b)\phi' k - \lambda k = \begin{cases} K_+ e^{-\xi} + M_+ e^{\xi}, & \xi > 0, \\ K_- e^{\xi} + M_- e^{-\xi}, & \xi < 0, \end{cases} \tag{42}$$

where $K_\pm$ and $M_\pm$ are arbitrary constants. We will show for $\lambda \neq 0$ that these constants must be set to zero so that the general equation (42) is reduced to (40).

The coefficients $M_\pm$ give unbounded solutions which can be normalized to be $k(\xi) \sim e^{|\xi|}$ if $\lambda \neq 1$ and $k(\xi) \sim \xi e^{|\xi|}$ if $\lambda = 1$. Since $v = (1 - \partial_\xi^2)k$, we obtain from (38) for $\lambda \neq 1$ that

$$\lim_{|\xi|\to\infty}(v' + \lambda v) = \lim_{|\xi|\to\infty} e^{-|\xi|}\left(-k''' - (b-3)\mathrm{sgn}(\xi)k'' + 2(b-1)k'\right) = b. \tag{43}$$

This implies for $b \neq 0$ and $\lambda \neq 0$ that $v(\xi) \to b/\lambda$ as $|\xi| \to \infty$; hence $v \notin L^2(\mathbb{R})$. For $b \neq 0$ and $\lambda = 1$ (the exceptional case), the limit (43) is unbounded; hence $v \notin L^2(\mathbb{R})$. For $b = 0$, (38) is satisfied with $k = e^{|\xi|}$ for which $v \equiv 0$. Hence, for $\lambda \neq 0$, we have to set $M_+ = M_- = 0$ to ensure that $v \in L^2(\mathbb{R})$.

It follows from (42) with $M_+ = M_- = 0$ that k and k' are continuous across $\xi = 0$ if and only if

$$\begin{cases} K_\pm = (\pm(b-1) - \lambda)k(0), \\ \mp K_\pm = (\pm(b-2) - \lambda)k'(0) + (1-b)k(0), \end{cases}$$

where we have used $\phi'(0^\pm) = \mp 1$ and $\phi(0^\pm) = 1$. If $\lambda \neq 0$, this system yields $K_+ = K_- = 0$, and hence solution (41) is the only suitable solution of the differential equation (38) with $\lambda \neq 0$ such that k and k' are bounded and continuous and $v = (1-\partial_\xi^2)k$ is in $L^2(\mathbb{R})$.

It remains to consider the case of $\lambda = 0$. Solutions of the differential equation (38) are analyzed in Appendix A. By combining the symmetric solution $k_{\rm sym}$ with the constant solution, we construct the decaying solution for $\lambda = 0$:

$$k = (1-\phi)^{b-1} - 1. \tag{44}$$

This solution gives $v = (1-\partial_\xi^2)k$ in $\mathrm{Dom}(L) \subset L^2(\mathbb{R})$ provided that $b > \frac{5}{2}$. In addition to the constant solution and the exponentially decaying solution (44), there exists the third, exponentially growing solution of the third-order equation (38) with $\lambda = 0$. However, as follows from (42) and (43) for $\lambda = 0$, the third solution diverges like $k(\xi) \sim e^{|\xi|}$ and gives either the divergent solution $v(\xi) \sim b\xi$ as $|\xi| \to \infty$ for $b \neq 0$ or the zero solution $v \equiv 0$ if $b = 0$. We conclude that the decaying solution (44) is the only element of $\mathrm{Ker}(L^*)$, and it exists only if $b > \frac{5}{2}$. ◻

3.4. Proof of Theorem 2.9. We check assumptions of Theorem 1 in [15] that the intersections $\sigma_p(L_0) \cap \rho(L)$ and $\sigma_p(L) \cap \rho(L_0)$ are empty.

For $b < \frac{5}{2}$, $\sigma_p(L_0)$ in Theorem 3.1 consists of the bands $0 < |\,\mathsf{Re}(\lambda)\,| < \frac{5}{2} - b$, whereas $\sigma_p(L)$ in Lemma 3.3 consists of the same bands and an additional double zero eigenvalue. Since the resolvent set $\rho(L_0)$ in Theorem 3.1 consists of the bands $|\,\mathsf{Re}(\lambda)\,| > \frac{5}{2} - b$, $\sigma_p(L) \cap \rho(L_0)$ is an empty set; Since $\sigma_p(L_0) \subset \sigma_p(L)$, $\sigma_p(L_0) \cap \rho(L)$ is also an empty set.

For $b \geq \frac{5}{2}$, $\sigma_p(L_0)$ in Theorem 3.1 is an empty set, hence $\sigma_p(L_0) \cap \rho(L)$ is an empty set, whereas $\sigma_p(L) = \{0\}$ does not belong to the resolvent set of L_0 in the bands $|\,\mathsf{Re}(\lambda)\,| > b - \frac{5}{2}$; hence $\sigma_p(L) \cap \rho(L_0)$ is also an empty set.

Since Q is a compact operator in $L^2(\mathbb{R})$ by Lemma 2.1, Theorem 1 in [15] suggests that $\sigma(L) = \sigma(L_0)$. The proof of Theorem 2.9 repeats the statement of Theorem 3.1 with additional information about the double zero eigenvalue.

4. Time evolution of the linearized system.

Due to the difference in the spectral properties of the linearized operator L in $L^2(\mathbb{R})$ for $b < \frac{5}{2}$ and $b > \frac{5}{2}$ in Theorem 2.9, the growth of the L^2 norm of the peaked perturbations is different due to the point spectrum for $b < \frac{5}{2}$ and due to the residual spectrum for $b > \frac{5}{2}$, as follows from the proof of Corollary 2.10. We illustrate this difference here by studying exact solutions to the initial-value problem

$$\left\{ \begin{array}{l} v_t = (1-\phi)v_\xi + (2-b)\phi' v, \qquad t > 0, \\ v|_{t=0} = \mathfrak{v}_0, \end{array} \right. \tag{45}$$

where $\mathfrak{v}_0 \in \mathrm{Dom}(L)$. The initial-value problem coincides with the linearized equation for the truncated operator L_0. By Theorem 3.1, the unstable spectrum of the operator L_0 is the point spectrum for $b < \frac{5}{2}$ and the residual spectrum for $b > \frac{5}{2}$.

The initial-value problem (45) can be solved exactly by using the method of characteristics. The following proposition gives the bounds obtained from the exact solution.

PROPOSITION 4.1. *For every $\mathfrak{v}_0 \in \mathrm{Dom}(L)$, the initial-value problem* (45) *admits the unique solution $v \in C(\mathbb{R}, \mathrm{Dom}(L))$ satisfying the following properties:*

- *If $b = \frac{5}{2}$, then $\|v(t,\cdot)\|_{L^2} = \|\mathfrak{v}_0\|_{L^2}$.*
- *If $b > \frac{5}{2}$, then $\|v(t,\cdot)\|_{L^2(-\infty,0)} \leq \|\mathfrak{v}_0\|_{L^2(-\infty,0)}$, and there is a positive constant C_0 that depends on $\mathfrak{v}_0$ such that $\|v(t,\cdot)\|_{L^2(0,\infty)} \geq C_0(e^t - 1)^{b-\frac{5}{2}}$.*

- *If* $b < \frac{5}{2}$, *then* $\|v(t,\cdot)\|_{L^2(0,\infty)} \leq \|\mathfrak{v}_0\|_{L^2(0,\infty)}$, *and there exists* $\mathfrak{v}_0 \in \mathrm{Dom}(L)$ *such that* $\|v(t,\cdot)\|_{L^2(-\infty,0)} = e^{\lambda_0 t}\|\mathfrak{v}_0\|_{L^2(-\infty,0)}$ *with* $\lambda_0 \in (0, \frac{5}{2} - b)$.

Proof. By using the method of characteristics, we introduce the characteristic curves $\xi = X(t,s)$ satisfying $X_t = \phi(X) - 1$ with $X|_{t=0} = s$. The exact solution for $X(t,s)$ is readily available (see, e.g., [26]):

$$X(t,s) = \begin{cases} \log\left[1 + (e^s - 1)e^{-t}\right], & s > 0, \\ -\log\left[1 + (e^{-s} - 1)e^{t}\right], & s < 0. \end{cases} \tag{46}$$

Along the characteristic curves, the function $V(t,s) := v(t, \xi = X(t,s))$ satisfies

$$\begin{cases} V_t = (2-b)\phi'(X(t,s))V, & t > 0, \\ V|_{t=0} = \mathfrak{v}_0(s), \end{cases} \tag{47}$$

which is uniquely solved separately for $s > 0$ and $s < 0$ by

$$V(t,s) = \mathfrak{v}_0(s)\left[\frac{\partial X}{\partial s}\right]^{2-b} = \begin{cases} \mathfrak{v}_0(s)\left[1 + (e^t - 1)e^{-s}\right]^{b-2}, & s > 0, \\ \mathfrak{v}_0(s)\left[1 + (e^{-t} - 1)e^{s}\right]^{b-2}, & s < 0. \end{cases} \tag{48}$$

By the chain rule, we obtain

$$\|v(t,\cdot)\|^2_{L^2(0,\infty)} = \int_0^\infty |\mathfrak{v}_0(s)|^2 \left[1 + (e^t - 1)e^{-s}\right]^{2b-5} ds \tag{49}$$

and

$$\|v(t,\cdot)\|^2_{L^2(-\infty,0)} = \int_{-\infty}^0 |\mathfrak{v}_0(s)|^2 \left[1 + (e^{-t} - 1)e^{s}\right]^{2b-5} ds. \tag{50}$$

If $b = \frac{5}{2}$, the linear evolution is L^2-preserving in time for both $s > 0$ and $s < 0$.
If $b > \frac{5}{2}$, then it follows from (49) and (50) that

$$\|v(t,\cdot)\|^2_{L^2(0,\infty)} \geq (e^t - 1)^{2b-5} \int_0^\infty |\mathfrak{v}_0(s)|^2 e^{-(2b-5)s} ds,$$

and

$$\|v(t,\cdot)\|^2_{L^2(-\infty,0)} \leq \int_{-\infty}^0 |\mathfrak{v}_0(s)|^2 ds.$$

This proves that the L^2 norm of the perturbation grows exponentially in time on the right side of the peak at $\xi = 0$, while that on the left side of the peak stays bounded in time.

If $b < \frac{5}{2}$, then it follows from (49) and (50) that

$$\|v(t,\cdot)\|^2_{L^2(0,\infty)} \leq \int_0^\infty |\mathfrak{v}_0(s)|^2 ds$$

so that the L^2 norm of the perturbation stays bounded in time on the right side of the peak at $\xi = 0$. If $\mathfrak{v}_0$ is the eigenfunction of the point spectrum of L_0 for the eigenvalue $\lambda_0 \in (0, \frac{5}{2} - b)$, which follows from (19), that is,

$$\mathfrak{v}_0(s) = \frac{e^{\lambda_0 s}}{(1 - e^s)^{\lambda_0 + b - 2}}, \quad s < 0, \tag{51}$$

then after elementary transformations, we obtain for $s < 0$

$$V(t,s) = \mathfrak{v}_0(s)\left[1 + (e^{-t} - 1)e^{s}\right]^{b-2} = \frac{e^{\lambda_0 t + \lambda_0 X(t,s)}}{(1 - e^{X(t,s)})^{\lambda_0 + b - 2}} = \mathfrak{v}_0(X(t,s))e^{\lambda_0 t}$$

so that $\|v(t,\cdot)\|_{L^2(-\infty,0)} = \|\mathfrak{v}_0\|_{L^2(-\infty,0)} e^{\lambda_0 t}$ grows exponentially in time. ∎

Remark 4.2. The results of Proposition 4.1 are clearly related to the spectral properties of L_0 in Theorem 3.1. In more details, we emphasize the following:

- If $b = \frac{5}{2}$, the preservation of the L^2 norm of the solution $v \in C(\mathbb{R}, \mathrm{Dom}(L))$ is related to $\sigma(L_0) = \{i\mathbb{R}\}$.
- If $b > \frac{5}{2}$, the growth rate $e^{(b-\frac{5}{2})t}$ of the L^2 norm of the solution $v \in C(\mathbb{R}, \mathrm{Dom}(L))$ agrees with the width of the unstable strip at $0 < |\,\mathrm{Re}(\lambda)\,| \leq b - \frac{5}{2}$.
- If $b < \frac{5}{2}$, the instability is obtained from the eigenfunction (51) corresponding to the eigenvalue of the point spectrum in the strip at $0 < |\,\mathrm{Re}(\lambda)\,| < \frac{5}{2} - b$.

Remark 4.3. For $b < \frac{5}{2}$, it may first seem from (50) that the L^2 norm on the left side of the peak stays bounded with the formal limit

$$\lim_{t\to+\infty} \|v(t,\cdot)\|^2_{L^2(-\infty,0)} = \int_{-\infty}^{0} |\mathfrak{v}_0(s)|^2 (1-e^s)^{2b-5} ds. \tag{52}$$

However, the limit may not exist if $\mathfrak{v}_0(0) \neq 0$. For the eigenfunction in (51), the integral of $|\mathfrak{v}_0(s)|^2(1-e^s)^{2b-5}$ in (52) diverges due to the weak singularity of $\mathfrak{v}_0(s)$ as $s \to 0^-$, which is allowed in $\mathrm{Dom}(L) \subset L^2(\mathbb{R})$. On the other hand, if the initial condition $\mathfrak{v}_0$ provides convergence of the integral of $|\mathfrak{v}_0(s)|^2(1-e^s)^{2b-5}$ in (52), then the L^2 norm of the solution $v \in C(\mathbb{R}, \mathrm{Dom}(L))$ is bounded for all times.

Remark 4.4. Exact solutions for the initial-value problem associated with the full linearized operator L can be obtained by the method of characteristics if $b = 2$ [26]. The growth of perturbations in H^1 norm was obtained in [26] by explicit computations. Since perturbations were considered in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ with the additional condition $\mathfrak{v}_0(0) = 0$, the L^2 norm of such perturbations does not grow in time, and the eigenfunctions (51) are excluded from the choice of the initial condition.

5. Conclusion.

We have resolved an open question of whether the stability of the peakons in the b-CH equation can be understood from the spectral stability theory. We have proven that the spectrum of a linearized operator at the traveling peakons includes points in the right half plane of the complex plane for every b. This implies the exponential growth of perturbations to the peakons in the linearized evolution.

Let us conclude by mentioning open problems which would follow from this research. Since multipeakon solutions are available in the b-CH equation, it is interesting to study how the instability of single peakons affects interactions between multipeakons. Other peakon-bearing equations are also widely studied, including the cubic versions and the coupled versions of the b-CH equations. It is worth studying if similar spectral instability holds for the peakons in other models.

Appendix A. Antisymmetric solution v_b to $L^*v = 0$ in the case $b > \frac{5}{2}$.

We consider the equation

$$L^* v = 0,$$

where L^* is defined in (16). We make the substitution $v = k - k_{\xi\xi}$ and assume that k and k_ξ are bounded and continuous functions. By using integration by parts for the last two terms in (16) applied to $k_{\xi\xi}$, we obtain the following differential equation:

$$(1-\phi)k_{\xi\xi\xi} + (b-3)\phi' k_{\xi\xi} + (2(b-1)\phi - 1)k_\xi = 0.$$

For $\xi > 0$ and $\xi < 0$ separately, there exist three linearly independent solutions of the third-order differential equations. Two bounded solutions are available in the closed form

$$k_1 = 1, \quad k_2 = (1-\phi)^{b-1},$$

whereas the third unbounded solution k_3 is not available in a closed form.

By using $k_2(\xi)$, we construct two particular solutions on the entire line:

$$k_{\mathrm{sym}} = (1-\phi)^{b-1}, \quad k_{\mathrm{anti}} = \mathrm{sgn}(1-\phi)^{b-1},$$

where $\mathrm{sgn}(\xi) = 1$ if $\xi > 0$ and $\mathrm{sgn}(\xi) = -1$ if $\xi < 0$. The function k_{anti} and its first derivative are continuous if $b \geq 2$. Applying $(1-\partial_\xi^2)$ to k_{anti}, we compute the antisymmetric solution v_b to $L^* v = 0$:

$$v_b = \mathrm{sgn}(1-\phi)^{b-3}\left(b(b-2)\phi^2 + (3-b)\phi - 1\right).$$

The solution v_b is bounded at infinity and locally square integrable if $b > \frac{5}{2}$.

There exists also the symmetric solution to $L^* v = 0$ obtained by applying $(1-\partial_\xi^2)$ to k_{sym}. The solution is not used because $L^* v = 0$ is also satisfied by the symmetric, bounded, and continuous function $v = 1$.

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