Existence and stability of Klein–Gordon breathers in the small-amplitude limit

D.E. Pelinovsky, T. Penati, and S. Paleari

Abstract We consider a discrete Klein–Gordon (dKG) equation on $\mathbb{Z}^d$ in the limit of the discrete nonlinear Schrödinger (dNLS) equation, for which small-amplitude breathers have precise scaling with respect to the small coupling strength ϵ. By using the classical Lyapunov–Schmidt method, we show existence and linear stability of the KG breather from existence and linear stability of the corresponding dNLS soliton. Nonlinear stability, for an exponentially long time scale of the order $O(\exp(\epsilon^{-1}))$, is obtained via the normal form technique, together with higher order approximations of the KG breather through perturbations of the corresponding dNLS soliton.

1 Introduction

Nonlinear oscillators with weak linear couplings on the d-dimensional cubic lattice are described by the discrete Klein–Gordon (dKG) equation

$$\ddot{u}_n + V'(u_n) = \epsilon(\Delta u)_n, \quad n \in \mathbb{Z}^d, \tag{1}$$

where $\epsilon > 0$ is the small coupling strength, Δ is the discrete Laplacian operator on $\ell^2(\mathbb{Z}^d)$, and $V(u)$ is a nonlinear potential for each oscillator. The total energy of the nonlinear oscillators conserves in time t and is given by the Hamiltonian function

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$$H(u) = \frac{1}{2}\sum_{n\in\mathbb{Z}^d} \dot{u}_n^2 + \frac{\epsilon}{2}\sum_{n\in\mathbb{Z}^d}\sum_{|k-n|=1}(u_k - u_n)^2 + \sum_{n\in\mathbb{Z}^d} V(u_n). \tag{2}$$

For illustrative purposes, we deal with a hard anharmonic potential in the form

$$V(u) = \frac{1}{2}u^2 + \frac{1}{2+2p}u^{2+2p}, \tag{3}$$

where $p \in \mathbb{N}$ is assumed for analyticity of the vector field. There exists the unique global solution $u(t) \in C^2(\mathbb{R}, \ell^2(\mathbb{Z}^d))$ to the Cauchy problem for the dKG equation (1) with (3) equipped with the initial datum $(u, \dot{u}) \in \ell^2(\mathbb{Z}^d)\times\ell^2(\mathbb{Z}^d)$, where u stands for $\{u_n\}_{n\in\mathbb{Z}^d}$. Because our main results are formulated for small initial datum $(u, \dot{u})$, most of the results are applicable for general anharmonic potentials expanded as

$$V(u) = \frac{1}{2}u^2 + \alpha_p u^{2+2p} + O(u^{4+2p}) \quad \text{as} \quad u \to 0, \tag{4}$$

if $\alpha_p \neq 0$. The general anharmonic potential V is classified as *soft* if $\alpha_p < 0$ and *hard* if $\alpha_p > 0$.

Discrete breathers are time-periodic solutions localized on the lattice. Such solutions can be constructed asymptotically by exploring the two opposite limit: the anti-continuum limit $\epsilon \to 0$ of weak coupling between the oscillators [17] and the continuum limit $\epsilon \to \infty$ of strong coupling [4]. Compared to these asymptotic approximations, we explore here a different limit of the dKG equation to the discrete nonlinear Schrödinger (dNLS) equation, where the weak coupling between the oscillators is combined together with small amplitudes of each oscillator. To be precise, we assume the scaling

$$u_n = \epsilon^{1/2p}\tilde{u}_n, \quad n \in \mathbb{Z}^d \tag{5}$$

and rewrite the dKG equation (1) with the potential (3) in the perturbed form:

$$\ddot{u}_n + u_n + \epsilon u_n^{1+2p} = \epsilon(\Delta u)_n, \quad n \in \mathbb{Z}^d, \tag{6}$$

where the tilde notations have been dropped. By using a formal expansion $u_n(t) = a_n(\epsilon t)e^{it} + \bar{a}_n(\epsilon t)e^{-it} + O(\epsilon)$, the following dNLS equation for the complex amplitudes is derived from the requirement that the correction term $O(\epsilon)$ remains bounded in $\ell^2(\mathbb{Z}^d)$ on the time scale of $O(\epsilon^{-1})$:

$$2ia_n' + \gamma_p|a_n|^{2p}a_n = (\Delta a)_n, \quad n \in \mathbb{Z}^d, \tag{7}$$

where the prime denotes the derivative with respect to the slow time variable $\tau = \epsilon t$ and the numerical coefficient γ_p is given by

$$\gamma_p = \binom{1+2p}{1+p} = \frac{(2p+1)!}{p!(p+1)!}. \tag{8}$$

The asymptotic relation between the dKG equation (6) and the dNLS equation (7) was observed first in [25] and was made rigorous by using two equivalent analytical methods in our previous work [23].

Discrete breathers of the dKG equation (1) are approximated by discrete solitons (standing localized waves) of the dNLS equation (7) in the form $a_n(\tau) = A_n e^{-\frac{i}{2}\Omega\tau}$, where the time-independent amplitudes satisfies the stationary dNLS equation

$$\Omega A_n + \gamma_p |A_n|^{2p} A_n = (\Delta A)_n, \quad n \in \mathbb{Z}^d. \tag{9}$$

The elementary staggering transformation

$$A_n = (-1)^n \tilde{A}_n, \quad \Omega = -4d - \tilde{\Omega} \tag{10}$$

relates the defocusing version (9) to the focusing version

$$(\Delta\tilde{A})_n + \gamma_p |\tilde{A}_n|^{2p} \tilde{A}_n = \tilde{\Omega}\tilde{A}_n, \quad n \in \mathbb{Z}^d. \tag{11}$$

In the recent past, existence and stability of discrete solitons in the focusing version (11) has been studied in many details depending on the exponent p and the dimension d. Various approximations of discrete solitons of the dNLS equation (11) are described in [8]. Let us review some relevant results on this subject.

The stationary dNLS equation (11) is the Euler–Lagrange equation of the constrained variational problem

$$\mathcal{E}_\nu = \inf_{a \in \ell^2(\mathbb{Z}^d)} \{E(a) : \quad N(a) = \nu\}, \tag{12}$$

where

$$E(a) = \sum_{n\in\mathbb{Z}^d} \sum_{|k-n|=1} |a_k - a_n|^2 - \frac{1}{p+1} \sum_{n\in\mathbb{Z}^d} |a_n|^{2p+2} \tag{13}$$

is the conserved energy, $N(a) = \sum_{n\in\mathbb{Z}^d} |a_n|^2$ is the conserved mass, and $\nu > 0$ is fixed. The existence of a ground state as a minimizer of the constrained variational problem (12) was proven in Theorem 2.1 in [26] for every $\mathcal{E}_\nu < 0$. By Theorem 3.1 in [26], if $p < \frac{2}{d}$, the ground state exists for every $\nu > 0$, however, if $p \geqslant \frac{2}{d}$, there exists an excitation threshold $\nu_d > 0$ and the ground state only exists for $\nu > \nu_d$.

Variational and numerical approximations for $d = 1$ were employed to analyze the structure of discrete solitons of the stationary dNLS equation (11) near the critical case $p = 2$ [11, 18]. It was shown for single-pulse solitons that although the dependence $\Omega \mapsto \nu$ is monotone for $p = 1$, it becomes non-monotone for $p \gtrapprox 1.5$ covering the whole range $\nu > 0$ for $p < 2$ and featuring the excitation threshold for $p \geqslant 2$. Further analytical estimates on the excitation threshold in the stationary dNLS equation were developed in [7, 9, 10].

Spectral stability of discrete solitons in the dNLS equation (7) was analyzed in the limit $\Omega \to \infty$, which can be recast as the anti-continuum limit of the dNLS equation. It was shown for $d = 1$ in [22, 24] (see Section 4.3.3 in [21]) that the single-pulse solitons are stable in the limit $\Omega \to \infty$ for every $p \in \mathbb{N}$. Asymptotic stability of single-pulse solitons for $d = 1$ and $p \geqslant 3$ was also proven in the same limit in [1] after similar asymptotical stability results were obtained for small solitons of the dNLS equation in the presence of a localized potential [6, 16].

Spectral and orbital stability of single-pulse discrete solitons in the dNLS equation (7) is determined by the monotonicity of the dependence $\Omega \mapsto \nu$ according to the Vakhitov–Kolokolov criterion [13, 21]. It was shown in [15] that this criterion is related to a similar energy criterion for spectral stability of discrete breathers in the dKG equation (1). If ω is a frequency of the discrete breathers and H is the value of their energy, then the monotonicity of the dependence $\omega \mapsto H$ is related to the monotonicity of the dependence $\Omega \mapsto \nu$ in the dNLS limit. Further results on the energy criterion for spectral stability of discrete breathers in the dKG equation (1) are given in [12, 27]. In spite of many convincing numerical evidences, the orbital stability of single-pulse discrete breathers is still out of reach in the energy methods.

The purpose of this paper is to make precise the correspondence between existence and linear stability of discrete breathers in the dKG equation (1) and discrete solitons in the dNLS equation (7). This work clarifies applications mentioned in Section 4 of our previous paper [23]. We show how the Lyapunov–Schmidt reduction method can be employed equally well to study existence and linear stability of small-amplitude discrete breathers near the point of their bifurcation from the dNLS limit under reasonable assumptions on existence and linear stability of the discrete solitons of the dNLS equation. We also show how normal form methods (see [2, 3, 5, 19, 20]), combined with the Lyapunov-Schmidt reduction, are implemented to provide higher order approximation of the discrete breathers, in the same dNLS limit. These results represent a considerable improvement with respect to the corresponding results in [19]. Long-time nonlinear stability of small-amplitude discrete breathers then follows, assuming the discrete soliton of the stationary dNLS equation (11) is a ground state of the variational problem (12).

The remainder of this paper consists of three sections. Section 2 proves the existence of discrete breathers obtained via the Lyapunov–Schmidt decomposition. Section 3 describes the linear stability results obtained by the extension of the same technique. Section 4 gives the normal form arguments towards the long-time nonlinear stability of small-amplitude breathers.

2 Existence via Lyapunov-Schmidt decomposition

Breathers are T-periodic solutions of the dKG equation (1) localized on the lattice. One can consider such strong solutions of the dKG equation (1) in the space $u(t) \in H^2_{\rm per}([0,T];\ell^2(\mathbb{Z}^d))$. By scaling the time variable as $\tau = \omega t$ with $\omega = 2\pi/T$, it is convenient to consider 2π-periodic solutions $U(\tau) \in H^2_{\rm per}([-\pi,\pi];\ell^2(\mathbb{Z}^d))$ with parameter ω, such that $u(t) = U(\omega t)$. Breather solutions can be equivalently represented by the Fourier series

$$U(\tau) = \sum_{m\in\mathbb{Z}} A^{(m)} e^{im\tau}. \tag{14}$$

Since U is real, the complex-valued Fourier coefficients satisfy the constraints:

$$A^{(m)} = \overline{A^{(-m)}}, \quad m \in \mathbb{Z}. \tag{15}$$

If the periodic solution has zero initial velocity, i.e., $U'(0) = 0$, then it follows from reversibility of the dKG equation (1) in time[1] that the periodic solution is even in time, which implies

$$A^{(m)} = A^{(-m)}, \quad m \in \mathbb{Z}. \tag{16}$$

[1] Given a solution $\gamma := \{u(t), \dot u(t)\}$ to the dKG equation (1), another solution is $\tilde\gamma = \{\tilde u(t) = u(-t), \widetilde{\dot u}(t) = -\dot u(-t)\}$. If γ is a periodic solution with initial zero velocity, then the same is true for $\tilde\gamma$, and since the two solutions have the same initial configuration $u(0) = \tilde u(0)$, they are solutions of the same Cauchy problem, hence they coincide.

As a consequence of the two symmetries, the Fourier coefficients are real, hence the representation (14) becomes Fourier cosine series with real-valued coefficients:

$$U(\tau) = A^{(0)} + 2\sum_{m\in\mathbb{N}} A^{(m)} \cos(m\tau). \tag{17}$$

After the scaling transformation (5), breather solutions to the scaled dKG equation (6) satisfy the following boundary-value problem:

$$\omega^2 U'' + U + \epsilon U^{1+2p} = \epsilon \Delta U, \quad U \in H^2_{\rm per}([-\pi,\pi]; \ell^2(\mathbb{Z}^d)). \tag{18}$$

The existence problem can be rewritten in real-valued Fourier coefficients as

$$(1 - m^2\omega^2) A^{(m)} + \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} U^{1+2p}(\tau) e^{-im\tau} d\tau = \epsilon \Delta A^{(m)}, \quad m \in \mathbb{N}_0. \tag{19}$$

At $\epsilon = 0$, bifurcation of breathers is expected at $\omega_m = 1/m$, $m \in \mathbb{N}$, from which the lowest bifurcation value $\omega_1 = 1$ gives a branch of fundamental (single-period) breathers. If the solution branch $\omega(\epsilon)$ and $\{A^{(m)}(\epsilon)\}_{m\in\mathbb{N}} \in \ell^{2,2}(\mathbb{Z}; \ell^2(\mathbb{Z}^d))$ is parameterized by ϵ, then we are looking for the branch of fundamental breathers to satisfy the limiting conditions:

$$\lim_{\epsilon\to 0} \omega(\epsilon) = 1, \quad \lim_{\epsilon\to 0} A^{(1)}(\epsilon) \neq 0, \quad \text{and} \quad \lim_{\epsilon\to 0} A^{(m)}(\epsilon) = 0, \quad m \neq 1. \tag{20}$$

The limiting conditions (20) are not sufficient for persistence argument. In order to define uniquely a continuation of the solution branch in ϵ, we consider the stationary dNLS equation in the form:

$$\Omega\mathcal{A} + \gamma_p |\mathcal{A}|^{2p}\mathcal{A} = \Delta\mathcal{A}, \quad \mathcal{A} \in \ell^2(\mathbb{Z}^d), \tag{21}$$

where Ω is parameter and γ_p is a numerical coefficient given by (8). We restrict consideration to the case of dNLS solitons given by real $\mathcal{A}$, for which we introduce the Jacobian operator for the stationary dNLS equation (21) at $\mathcal{A}$:

$$J_\Omega := \Omega + (1+2p)\gamma_p \mathcal{A}^{2p} - \Delta. \tag{22}$$

Since $\sigma(\Delta) = [-4d, 0]$ in $\ell^2(\mathbb{Z}^d)$ and $\mathcal{A} \in \ell^2(\mathbb{Z}^d)$ is expected to decay exponentially at infinity, we need to consider Ω in $\mathbb{R}\backslash[-4d, 0]$.

Remark 1 Since the discrete solitons in the focusing stationary dNLS equation (11) exist for $\tilde{\Omega} > 0$ [26], the staggering transformation (10) suggests that the discrete solitons in the defocusing stationary dNLS equation (21) exist for $\Omega < -4d$.

Assuming existence of a dNLS soliton $\mathcal{A}$ in the stationary dNLS equation (21) for some $\Omega \in \mathbb{R}\backslash[-4d, 0]$ and invertibility of J_Ω at this $\mathcal{A}$ in (22), we will prove existence and uniqueness of the branch $\omega(\epsilon)$ and $\{A^{(m)}(\epsilon)\}_{m\in\mathbb{N}} \in \ell^{2,2}(\mathbb{Z}; \ell^2(\mathbb{Z}^d))$ of fundamental breathers satisfying the limiting conditions:

$$\lim_{\epsilon\to 0} \frac{\omega(\epsilon) - 1}{\epsilon} = -\frac{1}{2}\Omega, \qquad \lim_{\epsilon\to 0} A^{(m)}(\epsilon) = \begin{cases} \mathcal{A}, & m = 1, \\ 0, & m \neq 1. \end{cases} \tag{23}$$

The following theorem gives the existence and uniqueness result for breathers.

Theorem 1 *Fix $p \in \mathbb{N}$. Assume the existence of real $\mathcal{A} \in \ell^2(\mathbb{Z}^d)$ in the stationary dNLS equation (21) for some $\Omega \in \mathbb{R}\backslash[-4d, 0]$ such that the Jacobian operator J_Ω at this $\mathcal{A}$ in (22) has trivial null space in $\ell^2(\mathbb{Z}^d)$. There exists $\epsilon_0 > 0$ and $C_0 > 0$ such that the breather equation (19) for every $\epsilon \in (0, \epsilon_0)$ admits a unique C^ω solution branch $\omega(\epsilon) \in \mathbb{R}$ and $\{A^{(m)}(\epsilon)\}_{m\in\mathbb{N}} \in \ell^{2,2}(\mathbb{Z}; \ell^2(\mathbb{Z}^d))$ satisfying the bounds*

$$\left|\omega(\epsilon) - 1 + \frac{1}{2}\epsilon\Omega\right| \leqslant C_0\epsilon^2 \tag{24}$$

and

$$\|A^{(0)}\|_{\ell^2(\mathbb{Z}^d)} + \|A^{(1)} - \mathcal{A}\|_{\ell^2(\mathbb{Z}^d)} + \sum_{m\geqslant 2} \|A^{(m)}\|_{\ell^2(\mathbb{Z}^d)} \leqslant C_0\epsilon, \tag{25}$$

for every $\epsilon \in (0, \epsilon_0)$.

Remark 2 In order to explain the relevance of the stationary dNLS equation (21), we set $\omega^2 = 1 - \epsilon\Omega$, where Ω is fixed independently of ϵ, and rewrite equation (19) for $m = 1$ after dividing it by ϵ. This procedure yields the bifurcation equation:

$$\Omega A^{(1)}(\epsilon) + \frac{1}{2\pi}\int_{-\pi}^{\pi} U^{1+2p}(\tau, \epsilon)e^{-i\tau}d\tau = \Delta A^{(1)}(\epsilon),$$

where $U(\tau, \epsilon)$ is given by the Fourier series (14) with amplitudes $\{A^{(m)}(\epsilon)\}_{m\in\mathbb{Z}}$ satisfying symmetries (15) and (16). Formally, at the leading order (20), we have:

$$\Omega A^{(1)}(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[A^{(1)}(0)e^{i\tau} + \overline{A^{(1)}(0)}e^{-i\tau}\right]^{1+2p} e^{-i\tau}d\tau = \Delta A^{(1)}(0)$$

Expanding

$$\left[A^{(1)}(0)e^{i\tau} + \overline{A^{(1)}(0)}e^{-i\tau}\right]^{1+2p} = \sum_{k=0}^{1+2p}\binom{1+2p}{k}\left(A^{(1)}(0)\right)^k\left(\overline{A^{(1)}(0)}\right)^{1+2p-k} e^{i(2k-2p-1)\tau}$$

and evaluating the integral at the only nonzero term for $k = p + 1$ yields the limiting dNLS equation (21) with $\mathcal{A} = A^{(1)}(0)$.

Proof In order to solve the breather equation (19) as $\epsilon \to 0$ near the limiting solution (20), we proceed with the classical Lyapunov-Schmidt decomposition (see for example [4, 25]). We introduce the Hilbert spaces

$$X_2 := H^2_{\text{per}}([-\pi, \pi]; \ell^2(\mathbb{Z}^d)), \qquad X_0 := L^2_{\text{per}}([-\pi, \pi]; \ell^2(\mathbb{Z}^d))$$

and the dual spaces under the Fourier series (14):

$$\hat{X}_2 := \ell^{2,2}(\mathbb{Z}; \ell^2(\mathbb{Z}^d)), \qquad \hat{X}_0 := \ell^2(\mathbb{Z}; \ell^2(\mathbb{Z}^d)).$$

The breather solution U is an element of X_2, which is uniquely identified by the sequence A in $\hat{X}_2$. In other words, a solution is given by a sequence of Fourier coefficients $\{A^{(m)}\}_{m\in\mathbb{Z}}$ in $\ell^{2,2}(\mathbb{Z})$, where each Fourier coefficient $A^{(m)}$ is a complex sequence $A^{(m)} = \{A^{(m)}_n\}_{n\in\mathbb{Z}^d}$ in $\ell^2(\mathbb{Z}^d)$. The Sobolev norm in space $\hat{X}_2$ is given by

$$\|A\|_{\hat{X}_2} = \left(\sum_{m\in\mathbb{Z}} (1+|m|^2)^2 \left\| A^{(m)} \right\|^2_{\ell^2(\mathbb{Z}^d)} \right)^{1/2} .$$

Let us introduce also the linear operator $L_\omega : X_2 \to X_0$, which is given in Fourier space by $\hat{L}_\omega : \hat{X}_2 \to \hat{X}_0$:

$$\left(\hat{L}_\omega A\right)^{(m)} = (1 - m^2\omega^2)A^{(m)}, \quad m \in \mathbb{Z}. \tag{26}$$

We define the linear subspace $V_2 = \mathrm{span}(\{e_n e^{i\tau}\}_{n\in\mathbb{Z}^d}, \{e_n e^{-i\tau}\}_{n\in\mathbb{Z}^d})$ as the kernel of $L_{\omega=1}$ in X_2 and W_2 its orthogonal complement in $X_2 = V_2 \oplus W_2$. In the Fourier space, we set $\hat{V}_2$ as a kernel of $\hat{L}_{\omega=1}$ in $\hat{X}_2$ and $\hat{W}_2$ its orthogonal complement in $\hat{X}_2 = \hat{V}_2 \oplus \hat{W}_2$. In a similar way, we introduce the range subspace W_0 for the operator $L_{\omega=1}$, which is a subspace X_0 whose codimension is equal to the dimension of V_2, so that $X_0 = V_0 \oplus W_0$, and similarly $\hat{X}_0 = \hat{V}_0 \oplus \hat{W}_0$. Any element of $\hat{X}_2$ can be decomposed into

$$A = A^\sharp + A^\flat , \qquad A^\sharp \in \hat{V}_2 , \qquad A^\flat \in \hat{W}_2 . \tag{27}$$

The breather equation (19) can be written in the abstract form:

$$F(A,\omega,\epsilon) := \hat{L}_\omega A + \epsilon N(A) - \epsilon\Delta A = 0 , \tag{28}$$

where $N(A)$ is the nonlinear term. If $p \in \mathbb{N}$, then the nonlinear map $F(A,\omega,\epsilon) : \hat{X}_2\times\mathbb{R}\times\mathbb{R} \to \hat{X}_0$ is C^ω in its variables. The nonlinear equation (28) is projected onto $\hat{V}_0$ and $\hat{W}_0$, thus yields the following two equations:

$$\Pi_{\hat{V}_0} F(A^\sharp + A^\flat,\omega,\epsilon) = 0 , \qquad\qquad \Pi_{\hat{W}_0} F(A^\sharp + A^\flat,\omega,\epsilon) = 0 . \tag{29}$$

The former one is known as the kernel equation and the latter one is known as the range equation. We shall solve the range equation for small ϵ assuming that $|\omega - 1| = O(\epsilon)$ by using the implicit function theorem.

Exploiting the fact that $\hat{V}_0$ and $\hat{W}_0$ are invariant under Δ and that $\hat{L}_{\omega=1}A^\sharp = 0$ by definition, the range equation in (29) takes the form

$$\left(\hat{L}_\omega - \epsilon\Delta\right)A^\flat + \epsilon\Pi_{\hat{W}_0} N(A^\sharp + A^\flat) = 0 . \tag{30}$$

The perturbed linear operator $\hat{L}_\omega - \epsilon\Delta$ can be inverted on $\hat{W}_0$ for ϵ small enough if $|\omega - 1| = O(\epsilon)$. Indeed, first write using Neumann series

$$\left(\hat{L}_\omega - \epsilon\Delta\right)^{-1} = \left[\left(\epsilon\hat{L}_\omega^{-1}\Delta\right)^k\right]\hat{L}_\omega^{-1} , \tag{31}$$

where $\hat{L}_\omega^{-1}$ is well defined on $\hat{W}_0$ thanks to the diagonal form:

$$(\hat{L}_\omega^{-1}A)^{(m)} = \frac{1}{1 - m^2\omega^2} A^{(m)} , \qquad m \neq \pm 1 .$$

Let us introduce a parametrization of ω by

$$\omega^2 = 1 - \epsilon\Omega, \tag{32}$$

where Ω is fixed independently of ϵ. It follows by elementary computation that there exists $\epsilon_*(\Omega)$ that only depends on Ω such that for every $\epsilon \in (0, \epsilon_*(\Omega))$,

$$|1 - m^2\omega^2| = |1 - m^2(1 - \epsilon\Omega)| > \frac{1}{2}(1 + m^2), \quad \forall m \in \mathbb{Z}\backslash\{-1, 1\},$$

thus obtaining the estimate

$$\|\hat{L}_\omega^{-1}\|_{\hat{W}_0 \to \hat{W}_2} \leqslant 2,$$

and consequently

$$\|\epsilon\hat{L}_\omega^{-1}\Delta\|_{\hat{W}_0 \to \hat{W}_2} \leqslant 8d\epsilon.$$

By Neumann formula (31) there exists $\epsilon_0 := \min\{\epsilon^*(\Omega), (8d)^{-1}\}$ and $C_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$,

$$\|(\hat{L}_\omega - \epsilon\Delta)^{-1}\|_{\hat{W}_0 \to \hat{W}_2} \leqslant C_0. \tag{33}$$

Since X_2 is a Banach algebra with respect to multiplication and $\hat{X}_2$ is a Banach algebra with respect to convolution, the nonlinear term $N(A)$ in (30) is closed in $\hat{X}_2$. By writing the range equation as the fixed-point equation for $A^\flat$:

$$A^\flat = -\epsilon\left(\hat{L}_\omega - \epsilon\Delta\right)^{-1}\Pi_{\hat{W}_0} N(A^\sharp + A^\flat) \tag{34}$$

and using the implicit function theorem thanks to the parametrization (32) and the uniform bound (33), we conclude that for every $\epsilon \in (0, \epsilon_0)$, $\Omega \in \mathbb{R}$, and $A^\sharp \in \hat{V}_2 \subset \hat{X}_2$, there exists a unique solution $A^\flat \in \hat{W}_2 \subset \hat{X}_2$ to the fixed-point equation (34) such that the mapping $(A^\sharp, \Omega, \epsilon) \to A^\flat$ is C^ω and the solution is as small as $O(\epsilon)$ thanks to the leading order approximation

$$A^\flat = -\epsilon\hat{L}_\omega^{-1}\Pi_{\hat{W}_0} N(A^\sharp) + O(\epsilon^2), \tag{35}$$

which provides the bound

$$\|A^\flat\|_{\hat{X}_2} \leqslant C\epsilon, \tag{36}$$

for some ϵ-independent C.

Inserting the parametrization (32) and the mapping $(A^\sharp, \Omega, \epsilon) \to A^\flat$ into the kernel equation in (29) and dividing by ϵ, we obtain

$$\Omega A^\sharp - \Delta A^\sharp + \Pi_{\hat{V}_0} N(A^\sharp + A^\flat(A^\sharp, \Omega, \epsilon)) = 0.$$

Thanks to the computations in Remark 2 and the bound (36), one can rewrite the kernel equation explicitly in terms of the real-valued amplitude $A^{(1)}$ as follows:

$$f(A^{(1)}, \Omega, \epsilon) := \Omega A^{(1)} - \Delta A^{(1)} + \gamma_p A^{(1)}|A^{(1)}|^{2p} + \epsilon R(A^{(1)}, \Omega, \epsilon) = 0, \tag{37}$$

where $R(A^{(1)}, \Omega, \epsilon) : \ell^2(\mathbb{Z}^d) \times \mathbb{R} \times \mathbb{R} \to \ell^2(\mathbb{Z}^d)$ is C^ω and bounded as $\epsilon \to 0$ thanks to the bound (36). Thanks to the assumptions of the theorem, $\mathcal{A} \in \ell^2(\mathbb{Z}^d)$ is a root of

$$f(\mathcal{A}, \Omega, 0) = 0 \tag{38}$$

and

$$D_{A^{(1)}} f(\mathcal{A}, \Omega, 0) = J_\Omega \tag{39}$$

is a bounded and invertible operator on $\ell^2(\mathbb{Z}^d)$. By the implicit function theorem, there exists $\epsilon_1 < \epsilon_0$ such that for every $\epsilon \in (0, \epsilon_1)$ and $\Omega \in \mathbb{R}$ for which $\mathcal{A} \in \ell^2(\mathbb{Z}^d)$ exists in (38) and J_Ω is invertible in (39), there exists a unique solution $A^{(1)} \in \ell^2(\mathbb{Z}^d)$ to the kernel equation (37) such that the mapping $(\Omega, \epsilon) \to A^{(1)}$ is C^ω and the solution satisfies the bound

$$\|A^{(1)} - \mathcal{A}\|_{\ell^2(\mathbb{Z}^d)} \leqslant C\epsilon, \tag{40}$$

for some ϵ-independent C. Combining (36) and (40) with the decompositions (27) and (32) yields bounds (24) and (25). □

3 Stability via Lyapunov-Schmidt decomposition

Linearizing $u(t) = U(\tau) + w(t)$ of the dKG equation (6) at the breather solution $U(\tau) \in H^2_{\rm per}([-\pi, \pi]; \ell^2(\mathbb{Z}^d))$ with $\tau = \omega t$ yields the linearized dKG equation:

$$\ddot{w} + w + \epsilon(1 + 2p)U^{2p} w = \epsilon \Delta w. \tag{41}$$

By Floquet theorem, every solution of the 2π-periodic linear equation (41) can be represented in the form $w(t) = W(\tau)e^{\lambda t}$, where $\lambda \in \mathbb{C}$ is the spectral parameter and $W(\tau) \in H^2_{\rm per}([-\pi, \pi]; \ell^2(\mathbb{Z}^d))$ is an eigenfunction of the spectral problem:

$$\omega^2 W'' + 2\lambda\omega W' + \lambda^2 W + W + \epsilon(1 + 2p)U^{2p} W = \epsilon \Delta W. \tag{42}$$

Let us represent $W(\tau) \in H^2_{\rm per}([-\pi, \pi]; \ell^2(\mathbb{Z}^d))$ by the Fourier series:

$$W(\tau) = \sum_{m \in \mathbb{Z}} B^{(m)} e^{im\tau}. \tag{43}$$

With the help of (14) and (43), the spectral problem (42) is rewritten in Fourier coefficients as

$$\left[1 + (\lambda + im\omega)^2\right] B^{(m)} + \frac{\epsilon(1 + 2p)}{2\pi} \int_{-\pi}^{\pi} U^{2p}(\tau) W(\tau) e^{-im\tau} d\tau = \epsilon \Delta B^{(m)}. \tag{44}$$

No symmetry reductions exist generally for the Fourier coefficients $\{B^{(m)}\}_{m \in \mathbb{Z}}$.

At $\epsilon = 0$ and $\omega = 1$, the spectral problem (44) admits a double set of eigenvalues λ defined by

$$\Sigma_\pm := \{i(\pm 1 - m), \quad m \in \mathbb{Z}\}, \tag{45}$$

where $\Sigma_+ = \Sigma_-$ and each eigenvalue has infinite multiplicity due to the lattice $\mathbb{Z}^d$. In terms of the Floquet multipliers

$$\mu := e^{\lambda T} = e^{2\pi\lambda/\omega}, \tag{46}$$

all eigenvalues at $\epsilon = 0$ and $\omega = 1$ correspond to the same Floquet multiplier $\mu = 1$.

Remark 3 The degeneracy of the Floquet multiplier μ in (46) is understood in terms of the following symmetry for the spectral problem (44). Fix $k \in \mathbb{Z}$ and apply transformation

$$\lambda = ik + \tilde{\lambda}, \quad m = -k + \tilde{m}, \quad B^{(m)} = \tilde{B}^{(\tilde{m})}.$$

The eigenvalue-eigenvector pair $\left(\tilde{\lambda}, \{\tilde{B}^{(\tilde{m})}\}_{\tilde{m}\in\mathbb{Z}}\right)$ satisfies the same spectral problem (44) but in tilde variables. Therefore, the spectral problem (44) near every nonzero point $\lambda \in \Sigma_\pm$ repeats its behavior near $\lambda = 0$. It is hence sufficient to consider the spectral problem (44) near $\lambda = 0$.

Let us review the spectral stability problem for the dNLS equation (7). The dNLS soliton $a(\tau) = e^{-\frac{i}{2}\Omega\tau}\mathcal{A}$ is defined by solutions of the stationary dNLS equation (21) with real $\mathcal{A} \in \ell^2(\mathbb{Z}^d)$. Linearizing with the expansion $a(\tau) = e^{-\frac{i}{2}\Omega\tau}[\mathcal{A} + b(\tau)]$ yields the linearized dNLS equation:

$$2ib' + \left(\Omega - \Delta + \gamma_p(p+1)\mathcal{A}^{2p}\right)b + \gamma_p p\mathcal{A}^{2p}\bar{b} = 0. \tag{47}$$

Separating variables by $b(\tau) = [b_+ + ib_-]\,e^{\Lambda\tau}$ and $\bar{\varphi}(\tau) = [b_+ - ib_-]\,e^{\Lambda\tau}$, where $\Lambda \in \mathbb{C}$ is the spectral parameter and $(b_+, b_-) \in \ell^2(\mathbb{Z}^d) \times \ell^2(\mathbb{Z}^d)$ is an eigenfunction, yields the spectral problem:

$$\begin{bmatrix} 0 & -(\Omega - \Delta + \gamma_p\mathcal{A}^{2p}) \\ \Omega - \Delta + \gamma_p(1+2p)\mathcal{A}^{2p} & 0 \end{bmatrix}\begin{bmatrix} b_+ \\ b_- \end{bmatrix} = 2\Lambda\begin{bmatrix} b_+ \\ b_- \end{bmatrix}. \tag{48}$$

The spectral problem (48) can be written in the Hamiltonian form $\mathcal{J}\mathcal{H}''(\mathcal{A})\mathbf{f} = 2\Lambda\mathbf{f}$, where $\mathbf{f} = (b_+, b_-)^T$,

$$\mathcal{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{H}''(\mathcal{A}) = \begin{bmatrix} \Omega - \Delta + \gamma_p(1+2p)\mathcal{A}^{2p} & 0 \\ 0 & \Omega - \Delta + \gamma_p\mathcal{A}^{2p} \end{bmatrix}.$$

The first diagonal entry in $\mathcal{H}''(\mathcal{A})$ coincides with the Jacobian operator (22) for the stationary dNLS equation (21).

Remark 4 Since $\mathcal{H}''(\mathcal{A})$ and $\Omega - \Delta$ are bounded operators in $\ell^2(\mathbb{Z}^d)$, whereas $\Omega \in \mathbb{R}\backslash[-4d, 0]$ and $\mathcal{A}^{2p}$ decays exponentially at infinity, the operator $(\Omega - \Delta)^{-1}\mathcal{A}^{2p}$ is a compact (Hilbert–Schmidt) operator. As a result, $\sigma_c(\mathcal{H}''(\mathcal{A})) = [\Omega, \Omega + 4d]$ and $\sigma_d(\mathcal{H}''(\mathcal{A}))$ consists of finitely many eigenvalues of finite multiplicities, where σ_c and σ_d denotes the absolutely continuous and discrete spectra of the self-adjoint operator $\mathcal{H}''(\mathcal{A})$ in the Hilbert space $\ell^2(\mathbb{Z}^d)$.

It follows from Remark 4 that if $\Omega < -4d$ (see Remark 1), there exist finitely many positive eigenvalues of $\sigma_d(\mathcal{H}''(\mathcal{A}))$, whereas if $\Omega > 0$, there exist finitely many negative eigenvalues of $\sigma_d(\mathcal{H}''(\mathcal{A}))$. In either case, the stability theory in linear Hamiltonian systems [13, 21] is applied to conclude that there exist finitely many eigenvalues Λ with $\mathrm{Re}(\Lambda) \neq 0$ in the spectral problem (48). The continuous spectrum of $\mathcal{J}\mathcal{H}''(\mathcal{A})$ coincides with the purely continuous spectrum of $\mathcal{J}\mathcal{H}''(0)$ and is located on

$$\sigma_c(\mathcal{J}\mathcal{H}''(\mathcal{A})) = \{i[\Omega, \Omega + 4d]\} \cup \{-i[\Omega, \Omega + 4d]\}. \tag{49}$$

The following theorem guarantees the persistence of simple isolated eigenvalues of the spectral problem (48) in the spectral problem (44) near $\lambda = 0$.

Theorem 2 *Under the assumption of Theorem 1, assume that $\Lambda \in \mathbb{C}$ is a simple isolated eigenvalue of the spectral problem (48) such that $2\Lambda \notin \sigma_c(\mathcal{J}\mathcal{H}''(\mathcal{A}))$ and $(b_+, b_-) \in \ell^2(\mathbb{Z}^d) \times \ell^2(\mathbb{Z}^d)$. There exists $\epsilon_0 > 0$ and $C_0 > 0$ such that the spectral problem (44) for every $\epsilon \in (0, \epsilon_0)$ admits a unique C^ω branch of the eigenvalue–eigenvector pair with $\lambda(\epsilon) \in \mathbb{C}$ and $\{B^{(m)}(\epsilon)\}_{m\in\mathbb{N}} \in \ell^{2,2}(\mathbb{Z}; \ell^2(\mathbb{Z}^d))$ satisfying*

$$|\lambda(\epsilon) - \epsilon\Lambda| \leqslant C_0\epsilon^2, \tag{50}$$

$$\|B^{(1)} - b_+ - ib_-\|_{\ell^2(\mathbb{Z}^d)} + \|B^{(-1)} - b_+ + ib_-\|_{\ell^2(\mathbb{Z}^d)} \leqslant C_0\epsilon, \tag{51}$$

and

$$\|B^{(0)}\|_{\ell^2(\mathbb{Z}^d)} + \sum_{m\geqslant 2} \|B^{(m)}\|_{\ell^2(\mathbb{Z}^d)} \leqslant C_0\epsilon, \tag{52}$$

for every $\epsilon \in (0, \epsilon_0)$.

Proof We adopt the same Hilbert spaces as those used in the proof of Theorem 1. Any element of $\hat{X}_2$ can be decomposed into

$$B = B^\sharp + B^\flat, \qquad B^\sharp \in \hat{V}_2, \qquad B^\flat \in \hat{W}_2. \tag{53}$$

We assume that $\omega(\epsilon)$ and $\{A^{(m)}(\epsilon)\}_{m\in\mathbb{Z}}$ are given by Theorem 1 with the error bounds (24) and (25). Let us introduce the linear operator $\hat{M}_{\lambda,\omega} : \hat{X}_2 \to \hat{X}_0$:

$$\left(\hat{M}_{\lambda,\omega} B\right)^{(m)} = \left[1 + (\lambda + im\omega)^2\right] B^{(m)}, \quad m \in \mathbb{Z}. \tag{54}$$

The spectral problem (44) for Fourier coefficients can be written in the abstract form:

$$F(B, \lambda, \epsilon) := \hat{M}_{\lambda,\omega(\epsilon)} B + \epsilon S(A(\epsilon), B) - \epsilon\Delta B = 0, \tag{55}$$

where $S(A(\epsilon), B)$ is the linear map on B obtained from the nonlinear term $N(A)$. Since $p \in \mathbb{N}$, the map $F(B, \lambda, \epsilon) : \hat{X}_2 \times \mathbb{C} \times \mathbb{R} \to \hat{X}_0$ is C^ω in its arguments. Projecting equation (55) onto $\hat{V}_0$ and $\hat{W}_0$ yields the following range and kernel equations:

$$\Pi_{\hat{V}_0} F(B^\sharp + B^\flat, \lambda, \epsilon) = 0, \qquad \Pi_{\hat{W}_0} F(B^\sharp + B^\flat, \lambda, \epsilon) = 0\,. \tag{56}$$

The range equation in system (56) can be solved in the same way as the range equation in system (29). By using the implicit function theorem, for every $\epsilon \in (0, \epsilon_0)$, $\Lambda \in \mathbb{C}$, and $B^\sharp \in \hat{V}_2 \subset \hat{X}_2$, there exists a unique solution $B^\flat \in \hat{W}_2 \subset \hat{X}_2$ of the range equation $\Pi_{\hat{V}_0} F(B^\sharp + B^\flat, \epsilon\Lambda, \epsilon) = 0$ such that the mapping $(B^\sharp, \Lambda, \epsilon) \to B^\flat$ is C^ω and the solution is as small as $O(\epsilon)$ thanks to the bound

$$\|B^\flat\|_{\hat{X}_2} \leqslant C\epsilon, \tag{57}$$

for some ϵ-independent C.

Inserting $\omega = 1 - \frac{1}{2}\epsilon\Omega + O(\epsilon^2)$, $\lambda = \epsilon\Lambda$, and the C^ω mapping $(B^\sharp, \Lambda, \epsilon) \to B^\flat$ into the kernel equation in system (56) and dividing by ϵ, we obtain the following system of two equations on the two amplitudes $(B^{(1)}, B^{(-1)})$:

$$(\Omega \pm 2i\Lambda)B^{(\pm 1)} - \Delta B^{(\pm 1)} + \gamma_p \mathcal{A}^{2p} \left[(p+1)B^{(\pm 1)} + pB^{(-1)}\right] \\ + \epsilon R^{(\pm 1)}(B^{(1)}, B^{(-1)}, \Lambda, \epsilon) = 0, \quad (58)$$

where $R^{(\pm 1)}(B^{(1)}, B^{(-1)}, \Lambda, \epsilon) : \ell^2(\mathbb{Z}^d) \times \ell^2(\mathbb{Z}^d) \times \mathbb{R} \times \mathbb{R} \to \ell^2(\mathbb{Z}^d)$ is a linear map on $(B^{(1)}, B^{(-1)})$ with C^ω coefficients which are bounded as $\epsilon \to 0$ thanks to the bound (57). In the derivation of numerical coefficients in (58), we have used the following explicit computation:

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\mathcal{A}e^{i\tau} + \mathcal{A}e^{-i\tau}\right]^{2p} \left[B^{(1)}e^{i\tau} + B^{(-1)}e^{-i\tau}\right] e^{\mp i\tau} d\tau \\ = &\sum_{k=0}^{2p} \binom{2p}{k} \mathcal{A}^{2p} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[B^{(1)}e^{i(2k-2p+1\mp 1)\tau} + B^{(-1)}e^{i(2k-2p-1\mp 1)\tau}\right] d\tau \\ = &\frac{p+1}{2p+1}\gamma_p \mathcal{A}^{2p} B^{(\pm 1)} + \frac{p}{2p+1}\gamma_p \mathcal{A}^{2p} B^{(-1)}. \end{aligned}$$

At $\epsilon = 0$, the system (58) becomes the spectral problem (48) in variables $B^{(\pm 1)} = b_+ \pm i b_-$. It is assume that Λ is a simple isolated eigenvalue in the spectral problem (48) with $2\Lambda \notin \sigma_c(\mathcal{J}\mathcal{H}''(\mathcal{A}))$ and a related eigenvector $(b_+, b_-) \in \ell^2(\mathbb{Z}^d) \times \ell^2(\mathbb{Z}^d)$. For $\epsilon \neq 0$, the eigenvalue Λ becomes the characteristic root of the linear system (58). By the analytic perturbation theory for closed linear operators (see Theorem 1.7 in Chapter VII on p. 368 in [14]), simple characteristic roots and the associated eigenvectors are continued in ϵ as C^ω functions. This completes justification of the bounds (50) and (51). □

Remark 5 If $2\Lambda \in i\mathbb{R}\backslash\sigma_c(\mathcal{J}\mathcal{H}''(\mathcal{A}))$, the bound (50) is not sufficient to guarantee that the eigenvalue λ remains on $i\mathbb{R}$.

In order to obtain a definite prediction that the simple isolated eigenvalue $\Lambda \in i\mathbb{R}$ of the spectral problem (48) persist as a simple isolated eigenvalue $\lambda \in i\mathbb{R}$ of the spectral problem (44), we use the Krein signature theory for linearized Hamiltonian systems. Consider the linearized dKG equation (41) and define

$$k(w) := i \sum_{n\in\mathbb{Z}^d} w_n \dot{\bar{w}}_n - \bar{w}_n \dot{w}_n. \quad (59)$$

It is straightforward to verify that $k(w)$ is independent of t. Let us represent the eigenvalue-eigenvector pair by $w(t) = W(\tau)e^{\lambda t}$ with $\lambda \in \mathbb{C}$ and $W(\tau) \in H^2_{\rm per}([-\pi,\pi];\ell^2(\mathbb{Z}^d))$. Then, $k(w) = K(W,\lambda)e^{(\lambda+\bar{\lambda})t}$ with

$$K(W,\lambda) := i\omega \sum_{n\in\mathbb{Z}^d} \left(W_n \bar{W}_n' - \bar{W}_n W_n'\right) - i(\lambda - \bar{\lambda}) \sum_{n\in\mathbb{Z}^d} |W_n|^2. \quad (60)$$

The following lemma reproduces the main result of the Krein theory.

Lemma 1 *Let $\lambda \in \mathbb{C}$ be a simple isolated eigenvalue in the spectral problem (42) with the eigenvector $W(\tau) \in H^2_{\rm per}([-\pi,\pi];\ell^2(\mathbb{Z}^d))$. Then, $K(W,\lambda) = 0$ if $\mathrm{Re}(\lambda) \neq 0$ and $K(W,\lambda) \neq 0$ if $\lambda \in i\mathbb{R}\backslash\{0\}$.*

Proof The spectral problem (42) can be formulated in the Hamiltonian form $JH''(U)\mathbf{f} = \lambda\mathbf{f}$, where $\mathbf{f} = (W, Q)$, $J^* = -J = J^{-1}$, and $H''(U)$ is self-adjoint in $L^2_{\rm per}([-\pi,\pi];\ell^2(\mathbb{Z}^d))$. Since $Q = \lambda W + \omega W'$, we note that

$$\lambda K(W,\lambda) = i\langle H''(U)\mathbf{f}, \mathbf{f}\rangle = i\langle \mathbf{f}, H''(U)\mathbf{f}\rangle = -\bar{\lambda} K(W,\lambda),$$

so that if $\mathrm{Re}(\lambda) \neq 0$ then $K(W,\lambda) = 0$. If $\lambda \in i\mathbb{R}\backslash\{0\}$ is a simple isolated eigenvalue, then we claim that $K(W,\lambda) \neq 0$. Indeed, if we assume $K(W,\lambda) = 0$, then there exists a generalized eigenvector from solution of the nonhomogeneous equation

$$JH''(U)\mathbf{g} = \lambda\mathbf{g} + \mathbf{f},$$

since the condition of the Fredholm alternative theorem is satisfied:

$$\langle J^{-1}\mathbf{f}, \mathbf{f}\rangle = \lambda^{-1}\langle H''(U)\mathbf{f}, \mathbf{f}\rangle = -iK(W,\lambda) = 0.$$

Therefore, λ is at least a double eigenvalue in contradiction with the assumption that λ is simple. Therefore, $K(W,\lambda) \neq 0$. □

Equipped with Lemma 1, we can now prove an analogue of Theorem 2 about persistence of simple isolated eigenvalues on $i\mathbb{R}$.

Theorem 3 *Under the assumption of Theorem 1, assume that $\Lambda \in i\mathbb{R}\backslash\{0\}$ is a simple isolated eigenvalue of the spectral problem (48) with $(b_+, b_-) \in \ell^2(\mathbb{Z}^d)\times\ell^2(\mathbb{Z}^d)$. There exists $\epsilon_0 > 0$ and $C_0 > 0$ such that the spectral problem (44) for every $\epsilon \in (0,\epsilon_0)$ admits a unique C^ω branch of the eigenvalue–eigenvector pair with $\lambda(\epsilon) \in i\mathbb{R}$ and $\{B^{(m)}(\epsilon)\}_{m\in\mathbb{N}} \in \ell^{2,2}(\mathbb{Z};\ell^2(\mathbb{Z}^d))$ satisfying (50), (51), and (52).*

Proof By Remark 5, we only need to prove that $\lambda(\epsilon) = \epsilon\Lambda + O(\epsilon^2)$ remains on $i\mathbb{R}$. By smoothness of the branch of eigenvalue-eigenvectors in ϵ, we can compute the limit $\epsilon \to 0$ for the Krein quantity $K(W,\lambda)$ in (60). We obtain

$$\lim_{\epsilon\to 0} K(W,\lambda) = 2\|B^{(1)}\|^2_{\ell^2(\mathbb{Z}^d)} - 2\|B^{(-1)}\|^2_{\ell^2(\mathbb{Z}^d)} = 4i\langle b_-, b_+\rangle_{\ell^2(\mathbb{Z}^d)} - 4i\langle b_+, b_-\rangle_{\ell^2(\mathbb{Z}^d)},$$

which is the Krein quantity for the spectral problem (48). Since $\Lambda \in i\mathbb{R}\backslash\{0\}$ is simple and isolated, the Krein quantity for the spectral problem (48) enjoys the same properties as in Lemma 1. In particular, it is real and nonzero. By continuity in ϵ, $K(W,\lambda)$ is nonzero for every $\epsilon \in (0,\epsilon_0)$, so that by Lemma 1, the eigenvalue $\lambda(\epsilon) = \epsilon\Lambda + O(\epsilon^2)$ of the spectral problem (42) satisfies $\mathrm{Re}(\lambda) = 0$. □

Remark 6 Theorem 2 and 3 imply that the spectral stability of dNLS solitons is transferred to the spectral stability of dKG breathers if bifurcations of new isolated eigenvalues from the continuous spectrum in (49) do not result in the appearance of new eigenvalues with $\mathrm{Re}(\lambda) \neq 0$ in the spectral problem (48). Such arguments follow from the Krein theory [21]. In the anti-continuum limit of the dNLS equation (7), one can find conditions excluding bifurcations of new isolated eigenvalues from the continuous spectrum of the spectral problem (48) [24].

4 Long-time nonlinear stability via resonant normal forms

The resonant normal form we consider here is based on the scheme already illustrated in [2, 5], which is suitable for infinite dimensional Hamiltonian systems and can be implemented by working at the level of either the Hamiltonian fields (as we decide to do, following [2]) or the Hamiltonian function (as in [5]).

In what follows we first present a result according to which the Hamiltonian of our problem can be put into a resonant normal form up to an exponentially small remainder. The truncated normal form represents a generalized dNLS equation in the same spirit as in [20]. We then give a theorem about the existence of a breather for the dKG equation, exponentially close to discrete soliton of the normal form; we stress here that such an estimate is a significant improvement with respect to the one obtained in [19] where the two objects were proven to be only order one close in the small parameter. As a last step, under additional hypothesis that the dNLS soliton is a minimizer in the variational problem (12) we state a stability result for the discrete breathers on an exponentially long time scale. The proofs of the above mentioned results are illustrated respectively in Subsections 4.3, 4.4 and 4.5.

4.1 Setting, preliminaries and normal form result

We consider the Hamiltonian corresponding to the scaled model (6)

$$H = \frac{1}{2}\sum_{j\in\mathbb{Z}^d}\left(u_j^2+v_j^2\right) + \frac{\epsilon}{2p+2}\sum_{j\in\mathbb{Z}^d} u_j^{2p+2} + \frac{\epsilon}{2}\sum_{j\in\mathbb{Z}^d}\sum_{|j-h|=1}\left(u_j-u_h\right)^2, \tag{61}$$

where $v_j = \dot{u}_j$. The Hamiltonian (61) can be obtained scaling both the variables $(u_n, \dot{u}_n)$ according to (5), and the original Hamiltonian original energy (2) by $\epsilon^{-\frac{1}{p}}$. In the following, (61) will be considered as a nearly integrable Hamiltonian system

$$H = G + F\,, \qquad G := \frac{1}{2}\sum_{j\in\mathbb{Z}^d}\left(u_j^2+v_j^2\right), \qquad F := H - G = O(\epsilon)\,, \tag{62}$$

where G is an integrable Hamiltonian and F is a perturbation of order $O(\epsilon)$.

We need some notations (we refer to Section 5 of [2] for further details). We consider $z := (u, v)$ in the complexified phase space $\mathcal{P} = \ell^2(\mathbb{C})\times\ell^2(\mathbb{C})$ with the usual ℓ^2 norm, which makes it Hilbert with the usual inner product. Given $0 < R < 1$ and $0 < \mathfrak{d} \leqslant \frac{1}{4}$, we restrict to a ball around the origin $B_{R,\mathfrak{d}} := \{z \in \mathcal{P} \quad s.t. \quad \|z\| < R(1-\mathfrak{d})\}$. To deal with complex valued functions g and Hamiltonian vector fields X_g on such a generic ball, we make use of the supremum norm

$$\mathcal{N}_{\mathfrak{d}}(g) := \sup_{z\in B_{R,\mathfrak{d}}} |g(z)|\,, \qquad \mathcal{N}_{\mathfrak{d}}^{\nabla}(g) := \frac{1}{R}\sup_{z\in B_{R,\mathfrak{d}}} \left\|X_g(z)\right\|\,. \tag{63}$$

Our aim is to construct a normal form K admitting a second conserved quantity G

$$H = K + \mathcal{P}\,, \qquad \{K, G\} = 0\,;$$

this additional conserved quantity, which correspond to the ℓ^2 norm, corresponds to the invariance under the rotation symmetry, given by the periodic flow Φ^t_G of the Hamiltonian field X_G. The normal form K is thus a generalized dNLS model (see also [20]); given the smallness of $\mathcal{P}$, G turns out to be an approximated conserved quantity for H, whose variation can be kept bounded on exponentially long times.

Theorem 4 *For any positive* $\mathfrak{d} \leqslant 1/4$*, any dimension* $d \geqslant 1$ *and any* $R < 1$*, there exists* $\epsilon^*(\mathfrak{d}, d, R)$ *such that, for* $\epsilon < \epsilon^*$ *there exists a canonical change of coordinates* $T_\mathcal{X}$ *mapping*

$$B_{R,2\mathfrak{d}} \subset T_\mathcal{X}\big(B_{R,\mathfrak{d}}\big) \subset B_{R,0} \qquad\qquad B_{R,3\mathfrak{d}} \subset T_\mathcal{X}\big(B_{R,2\mathfrak{d}}\big) \subset B_{R,\mathfrak{d}} \tag{64}$$

which puts the Hamiltonian (61) *into the resonant normal form*

$$H = G + Z + \mathcal{P}\,, \qquad \{G, Z\} = 0\,, \qquad N^{\nabla}_{\mathfrak{d}}(\mathcal{P}) \leqslant \mu \exp\left(-\frac{1}{\mu}\right), \tag{65}$$

where $\mu := \frac{12e\pi\epsilon}{\mathfrak{d}} = \mathcal{O}(\epsilon)$*. Moreover, for any initial datum* $z_0 \in B_{R,3\mathfrak{d}}$*, there exists a positive constant* C *such that the variations of* G *and* Z *are bounded as follows*

$$\begin{aligned} |G(z(t)) - G(z_0)| &< C\mu N_0(G)\,, \\ |Z(z(t)) - Z(z_0)| &< C\mu N_0(F)\,, \end{aligned} \qquad |t| \leqslant T^* := \exp\left(\frac{1}{\mu}\right). \tag{66}$$

The construction is based on the linear operator $T_\mathcal{X}$ associated to a generating sequence $\{\mathcal{X}_s\}_{s=1}^r$, where $\mathcal{X}_s = \mathcal{O}(\epsilon^s)$, which acts recursively on G and F as follows

$$\begin{aligned} T_\mathcal{X} G &= \sum_{r\geqslant 0} G_r\,, \qquad G_0 := G\,, \qquad\qquad\quad G_r := \sum_{l=1}^{r} \frac{l}{r}\{\mathcal{X}_l, G_{r-l}\}\,, \\ T_\mathcal{X} F &= \sum_{r\geqslant 0} F_r\,, \qquad F_0 := 0\,, \quad F_1 := F\,, \quad F_r := \sum_{l=1}^{r-1} \frac{l}{r-1}\{\mathcal{X}_l, F_{r-l}\}\,. \end{aligned} \tag{67}$$

In the above recursive definition, it coherently turns out that $F_r = \mathcal{O}(\epsilon^r)$. Such a linear operator also provides the close-to-the-identity nonlinear transformation

$$T_\mathcal{X} z = z + \sum_{r\geqslant 1} z_r\,, \qquad z_r = \sum_{l=1}^{r} \frac{l}{r}\{\mathcal{X}_l, z\}_{r-l}\,. \tag{68}$$

The generating sequence $\mathcal{X}$, and the corresponding transformation $T_\mathcal{X}$, will be determined in order to put the Hamiltonian in resonant normal form up to order $\mathcal{O}(\epsilon^r)$

$$H^{(r)} = T_\mathcal{X} H = G + Z + \mathcal{R}^{(r+1)}\,, \qquad \{G, Z\} = 0\,, \qquad \mathcal{R}^{(r+1)} = \mathcal{O}(\epsilon^{r+1})\,. \tag{69}$$

Thus $\mathcal{X} = \{\mathcal{X}_s\}$ and the normal form terms $Z = \sum_{s=1}^r Z_s$ have to satisfy

$$\{G, \mathcal{X}_s\} + Z_s = \Psi_s\,, \qquad 1 \leqslant s \leqslant r\,, \tag{70}$$

where $\mathcal{X}_s$, Z_s and Ψ_s are all homogeneous terms of order ϵ^s, with

$$\Psi_1 = F_1 = F\,, \qquad\qquad \Psi_s := \frac{1}{s}F_s + \sum_{l=1}^{s-1}\frac{l}{s}\{\mathcal{X}_l, Z_{s-l}\}\ . \tag{71}$$

At first order $r = 1$, we obtain again equation (21) as leading order approximation of the dKG breather. Indeed we have to put into normal form the initial perturbation $\Psi_1 := F_1$. The first normal form term Z_1 represents its average, and it turns out that at first order the Hamiltonian $K^{(1)}$ can be given by the corresponding dNLS model

$$K^{(1)} = \sum_j |\psi_j|^2 + \frac{\epsilon}{p+1}\sum_j |\psi_j|^{2p+2} + \epsilon \sum_{|j-h|=1} |\psi_j - \psi_h|^2\,, \tag{72}$$

once complex coordinates are introduced

$$u_j = \psi_j + i\overline{\psi_j} = \frac{1}{\sqrt{2}}(\zeta_j + i\eta_j)\,, \quad \Rightarrow \quad \psi_j = \zeta_j/\sqrt{2}\,, \qquad i\eta_j = \overline{\zeta_j}\,, \tag{73}$$

so that the quadratic part of $K^{(1)}$ reads $\sum_j |\zeta_j|^2 + \epsilon\sum_{|j-h|=1}|\zeta_j - \zeta_h|^2$. To average the nonlinearity one follows the same calculations already used in the Remark 2

$$\frac{1}{2\pi}\int_0^{2\pi} u_j^{2p+2}\circ\Phi_g^t dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2^{p+1}}\Big(\zeta_j e^{it} + i\eta_j e^{-it}\Big)^{2p+2} dt = \Gamma_p|\zeta_j|^{2p+2}\,,$$

with $\Gamma_p := \frac{1}{2^p}\gamma_p$; thus that the nonlinear term reads $\frac{\Gamma_p}{2(p+1)}\sum_j|\zeta_j|^{2p+2}$, and its standard shape is recovered introducing the complex variable (73) which allows to rescale the prefactor 2^{-p}. Discrete solitons of (72) with frequency close to one

$$\psi_j = \mathcal{A}e^{i(1-\frac{\epsilon}{2}\Omega)t} \tag{74}$$

are then extremizer of $\mathcal{Z}_1 := \epsilon^{-1}Z_1$ constrained to constant values of the norm $G = \nu$, thus providing again (21) with $\Omega = \Omega(\nu)$.

4.2 High order approximation and nonlinear stability results

Let us consider $K := H - \mathcal{P}$ in (65) and its equations

$$\dot{z} = X_K(z)\,, \qquad\qquad K = G + Z\,, \qquad\qquad \{Z, G\} = 0\ . \tag{75}$$

To generalize the discrete soliton approximation, we rewrite the ansatz (74) as

$$\zeta_{\rm ds} = \mathfrak{A}e^{i(1-\frac{1}{2}\epsilon\Omega)t} \tag{76}$$

where $\mathfrak{A}$ is the real amplitude of the soliton[2], which is assumed to be small enough to belong to the domain of validity of the normal form (65); once inserted in (75), it provides the equation

[2] notice the use of the gothic font instead of the calligraphic one to distinguish between the objects of the generalized dNLS – given by the higher order normal form – to those of the standard dNLS

for $\mathfrak{A}$

$$f := f_0 + \epsilon f_1 = 0\,, \qquad \begin{cases} f_0 & := \Omega\mathfrak{A} + \gamma_p|\mathfrak{A}|^{2p}\mathfrak{A} - \Delta\mathfrak{A}\,, \\ f_1 & := \epsilon^{-2}X_Z(\mathfrak{A})\,; \end{cases} \tag{77}$$

where f_0 gives the standard dNLS equation (21), while f_1 is the perturbation due to the normal form steps $r \geqslant 2$; we recall that, due to $\{G, Z\} = 0$, X_Z is equivariant under the action of the symmetry $e^{i\theta}$

$$X_Z\left(\mathfrak{A}e^{i(1-\frac{1}{2}\epsilon\Omega)t}\right) = X_Z(\mathfrak{A})e^{i(1-\frac{1}{2}\epsilon\Omega)t}\,.$$

The next statement represent the higher order version of Theorem 1, under the same assumption on $\mathcal{A}$ and J_Ω: it claims the existence of the breather for the Klein-Gordon close to the discrete soliton of the normal form K.

Theorem 5 *Let $\mathcal{A}$ be a solution of* (21) *with J_Ω of* (22) *invertible in $\ell^2(\mathbb{Z}^d, \mathbb{R})$. Then:*

1. *there exists $\epsilon_1^* < \epsilon^*$ such that for any $0 < \epsilon < \epsilon_1^*$ there exists a unique solution $\mathfrak{A}(\Omega, \epsilon)$ of* (77)*, analytic in ϵ. Moreover, the following estimates hold true*

$$\|\mathfrak{A} - \mathcal{A}\|_{\ell^2} \leqslant C\epsilon\,, \qquad \sup \frac{\left\|(J_{\Omega,\epsilon} - J_\Omega)(z)\right\|_{\ell^2(\mathbb{Z}^d;\mathbb{R})}}{\|z\|_{\ell^2(\mathbb{Z}^d;\mathbb{R})}} \leqslant C\epsilon\,, \tag{78}$$

where $J_{\Omega,\epsilon} := D_{\mathfrak{A}}f(\mathfrak{A}(\Omega, \epsilon), \Omega, \epsilon)$ is the differential of f evaluated at $\mathfrak{A}(\Omega, \epsilon)$.

2. *Let*

$$\zeta_{br}(\tau) = \sum_m \mathfrak{A}^{(m)}e^{im\tau}\,, \qquad \tau := \omega t\,, \tag{79}$$

be the Fourier expansion of the breather of $\dot{z} = X_H(z)$. Then, there exists positive $\epsilon_2^* < \epsilon_1^*$ such that for every $0 < \epsilon < \epsilon_2^*$ the breather (79) *admits a unique analytic solution branch $\omega(\epsilon)$ and $\{\mathfrak{A}^{(m)}(\epsilon)\} \in \ell^{2,2}(\mathbb{Z}; \ell^2(\mathbb{Z}^d))$ satisfying the bounds*

$$|\omega(\epsilon) - 1 + \frac{\epsilon}{2}\Omega| \leqslant C\epsilon^2\,, \tag{80}$$

$$\|\mathfrak{A}^{(0)}\|_{\ell^2(\mathbb{Z}^d)} + \|\mathfrak{A}^{(1)} - \mathfrak{A}\|_{\ell^2(\mathbb{Z}^d)} + \sum_{m\geqslant 2}\|\mathfrak{A}^{(m)}\|_{\ell^2(\mathbb{Z}^d)} \leqslant C\exp\left(-\frac{c}{\epsilon}\right)\,. \tag{81}$$

3. *Let $z_{ds}(t) = T_\mathcal{X}^{-1}(\zeta_{ds})$ and $z_{br}(t) = T_\mathcal{X}^{-1}(\zeta_{br})$ be the discrete soliton and the discrete breather solutions in the original coordinates, and T_{ds} and T_{br} the corresponding periods; then it holds true*

$$\sup_{|t|\leqslant\max\{T_{ds},T_{br}\}} \|z_{ds}(t) - z_{br}(t)\| \leqslant C\exp\left(-\frac{c}{\epsilon}\right)\,. \tag{82}$$

Remark 7 In Theorems 5 and 6, c and C are suitable constants independent of ϵ.

We now assume a stronger condition than the invertibility of the Jacobian operator J_Ω; we require $\mathcal{A}$ to be a nondegenerate extremizer for $\mathcal{Z}_1$ constrained to constant values of the norm G. Under this assumption, which implies invertibility of J_Ω, it follows that for ϵ sufficiently small also the discrete soliton $\mathfrak{A}$ obtained in Proposition 4.1 is a nondegenerate extremizer for $\mathcal{Z} := \epsilon^{-1}Z$ constrained to the sphere $\mathcal{S} := \{G(z) = \nu\}$, with ν sufficiently small (as required by the normal form construction). As a consequence, $\mathfrak{A}$ is an orbitally stable periodic orbits (see

[3, 19, 26]) for the normal form $K = G + Z$: we are going to show that $\mathfrak{A}$ is an approximate periodic orbit for the full system $H = G + K + P$ which is orbitally stable for exponentially long times and that the same kind of stability holds true for the Klein-Gordon breathers.

Let us introduce with $\bar{\mathfrak{A}} := \{\mathfrak{A}^{(m)}\}$ and denote with $\mathsf{O}(\bar{\mathfrak{A}})$ the closed curve described by the Klein-Gordon breather

$$\mathsf{O}(\bar{\mathfrak{A}}) := \{z_{\rm br}(t),\ t \in [0,T]\} \qquad\qquad \mathsf{O} := T_{\mathcal{X}}^{-1}\mathsf{O}(\bar{\mathfrak{A}})\ .$$

The next Theorem provides the orbital stability of $\mathsf{O}(\bar{\mathfrak{A}})$:

Theorem 6 *Let $z_0 \in B_{R,3\mathfrak{d}}$ with $R < 1$. Then $\forall\ 0 < \mu \ll 1$, $\exists\ 0 < \delta \ll 1$ such that*

$$\inf_{w\in\mathsf{O}} \|z_0 - w\| < \delta \quad \Rightarrow \quad \inf_{w\in\mathsf{O}} \left\|\Phi_H^t(z_0) - w\right\| < \mu\,, \quad |t| < \exp\left(\frac{c}{\epsilon}\right)\,. \tag{83}$$

4.3 Proof of Theorem 4 (Normal Form Theorem)

We give a sketch of the proof, which would be long and technical if all the details were included. The estimates here included can be obtained by following [5, 2].

Recursive estimates, which are the most technical aspect of the whole construction, need estimates on the initial size of the perturbation F and its vector field X_F. We thus introduce the main quantities E and ω_1 providing the initial estimates

$$N_0(F) \leqslant E := \epsilon\left[4dR^2 + \frac{1}{2p+2}R^{2p+2}\right]\,, \qquad\qquad N_0^{\nabla}(F) \leqslant \omega_1 := 2\epsilon\left[C_d + R^{2p}\right]\ .$$

Remark 8 The magnitudes of E and ω_1 are coherent: since $E = O(\epsilon R^2)$, then its differential, divided by R according to the definition of $N_{\mathfrak{d}}^{\nabla}(\cdot)$ in (63), has to be $O(\epsilon)$.

In order to solve (70) we average along the periodic flow Φ_G^t of period 2π, as claimed by the following Lemma (for the proof, see [3]):

Lemma 2 *The homological equation* (70)*, i.e. $\{G, \mathcal{X}\} = \Psi(z) - Z(z)$, is solved by*

$$Z(z) = \frac{1}{2\pi}\int_0^{2\pi} \Psi\circ\Phi_G^t(z)dt\,, \qquad\qquad \mathcal{X}(z) = \frac{1}{2\pi}\int_0^{2\pi} t\left[(\Psi - Z)\circ\Phi_G^t\right](z)dt\ ;$$

for any $\mathfrak{d}$ it satisfies the following estimates

$$\begin{aligned} &N_{\mathfrak{d}}(Z) \leqslant N_{\mathfrak{d}}(\Psi)\,, &&N_{\mathfrak{d}}^{\nabla}(Z) \leqslant N_{\mathfrak{d}}^{\nabla}(\Psi)\,,\\ &N_{\mathfrak{d}}(\mathcal{X}) \leqslant 2\pi N_{\mathfrak{d}}(\Psi)\,, &&N_{\mathfrak{d}}^{\nabla}(\mathcal{X}) \leqslant 2\pi N_{\mathfrak{d}}^{\nabla}(\Psi)\ . \end{aligned} \tag{84}$$

At first order, Lemma 2 immediately provides the estimates

$$N_{\mathfrak{d}}^{\nabla}(Z_1) \leqslant \omega_1\,, \qquad\qquad N_{\mathfrak{d}}^{\nabla}(\mathcal{X}_1) \leqslant \phi := 2\pi\omega_1\,, \tag{85}$$

which introduce the main perturbation parameter $\phi = O(\epsilon)$ of the normal form scheme. Let now the arbitrary integer $r \geqslant 1$ be the order of the normal form construction, i.e. the number of generating functions $\mathcal{X}_s$ in the generating sequence $\mathcal{X} = \{\mathcal{X}_s\}_{s=1}^r$ and thus the number of homological equations (70) to be solved. The first important result gives the bounds for the quantities involved in (70)

Lemma 3 *Let* $\mathfrak{d}_s = \frac{s\mathfrak{d}}{r}$, *with* $\mathfrak{d} \leqslant \frac{1}{4}$. *Then, for any* $1 \leqslant s \leqslant r$ *it holds true*

$$N^{\nabla}_{\mathfrak{d}_{s-1}}(\Theta) \leqslant \frac{\omega_1}{s}\left(\frac{6r\phi}{\mathfrak{d}}\right)^{s-1} \quad \forall\Theta \in \{F_s, \Psi_s, Z_s\}\,, \quad N^{\nabla}_{\mathfrak{d}_{s-1}}(\mathcal{X}_s) \leqslant \frac{\phi}{s}\left(\frac{6r\phi}{\mathfrak{d}}\right)^{s-1}. \tag{86}$$

The above Lemma, and in particular the last of (86), allows to control the deformation of functions and vector fields under the canonical transformation; indeed, let

$$\begin{aligned} T_{\mathcal{X}} f &= \sum_{r\geqslant 0} f_r\,, \qquad & f_r &:= \sum_{j=1}^{r} \frac{j}{r} L_{\mathcal{X}_j} f_{r-j}\,, \qquad & f_0 &= f\,, \\ T_{\mathcal{X}} g &= \sum_{r\geqslant 1} g_r\,, \qquad & g_r &:= \sum_{j=1}^{r-1} \frac{j}{r-1} L_{\mathcal{X}_j} g_{r-j}\,, \qquad & g_1 &= g\,, \end{aligned} \tag{87}$$

then the following holds true

Lemma 4 *Let us introduce* $M_1 < M_2 < 1$

$$M_1(\phi, r) := \frac{\phi(e+3r)}{\mathfrak{d}}\,, \qquad M_2(\phi, r) := \frac{\phi(2e+3r)}{\mathfrak{d}}\,. \tag{88}$$

Then, for any $\mathfrak{d} \leqslant \frac{1}{4}$ *and any* $r \geqslant 1$ *the following bounds hold true*

$$\begin{aligned} N_{\mathfrak{d}}(z_r) &\leqslant M_1^{r-1}\tilde{G}_1\,, \qquad & N_{\mathfrak{d}}(f_r) &\leqslant M_1^{r-1}\tilde{B}_1\,, \qquad & N_{\mathfrak{d}}(g_r) &\leqslant M_1^{r-2}\tilde{\Gamma}_2\,, \\ & & N^{\nabla}_{\mathfrak{d}}(f_r) &\leqslant M_2^{r-1}\bar{B}_1\,, & N^{\nabla}_{\mathfrak{d}}(g_r) &\leqslant M_1^{r-2}\bar{\Gamma}_2\,, \end{aligned} \tag{89}$$

together with

$$\tilde{G}_1 := \frac{R\phi}{\mathfrak{d}}, \quad \tilde{B}_1 := \frac{\phi}{\mathfrak{d}}N_0(f), \quad \tilde{\Gamma}_2 := \frac{\phi}{\mathfrak{d}}N_0(g), \quad \bar{B}_1 := \frac{2\phi}{\mathfrak{d}}N^{\nabla}_0(f), \quad \bar{\Gamma}_2 := \frac{2\phi}{\mathfrak{d}}N^{\nabla}_0(g),$$

Given the above estimates, we can obtain the inclusions (64); indeed, we have

$$N_{\mathfrak{d}}(T_{\mathcal{X}}z - z) \leqslant \sum_{r\geqslant 1} N_{\mathfrak{d}}(z_r) \leqslant \frac{\tilde{G}_1}{1-M_1} < 2\tilde{G}_1 = \phi\left(\frac{2R}{\mathfrak{d}}\right),$$

provided we ask for $M_1 < \frac{1}{2}$; thus the deformation is $O(R\epsilon)$. The remainder $\mathcal{R}^{(r+1)}$ in (69), at an arbitrary step r, is given by

$$\mathcal{R}^{(r+1)} = \sum_{s\geqslant r+1} G_s + \sum_{s\geqslant r+1} F_s\,;$$

by exploiting (89) and the initial estimates, the following bounds hold true

$$N^{\nabla}_{\mathfrak{d}}(\Theta) \leqslant \left(\frac{2\phi}{\mathfrak{d}}\right)M_2^{s-1},\ \forall\Theta \in \{G_s, F_s\} \quad \Rightarrow \quad N^{\nabla}_{\mathfrak{d}}(\mathcal{R}^{(r+1)}) \leqslant \left(\frac{2\phi}{\mathfrak{d}}\right)\frac{M_2^r}{1-M_2}\,. \tag{90}$$

The exponential estimate (65) is derived from (90) by expanding M_2^r as

$$M_2^r = \left(\frac{6\phi}{\mathfrak{d}}\right)^r r^r \left(1+\frac{1}{r}\right)^r < e\left(\frac{6\phi}{\mathfrak{d}}\right)^r r^r \,, \tag{91}$$

and optimizing the number of normal form steps $r = r_{opt} := \lfloor \frac{\mathfrak{d}}{6e\phi} \rfloor$; in this way r is related ϵ. Finally, the variation of G along the generic orbit in the neighbourhood of the origin is obtained combining the estimate of the Poisson bracket

$$|G(t) - G(0)| \leqslant \int_0^t |\{G, \mathcal{R}^{(r+1)}\}(z(s))| ds \leqslant |t| N_{\mathfrak{d}}(G) \left(\frac{4e\phi}{\mathfrak{d}^2}\right) \exp\left[-\left(\frac{\mathfrak{d}}{6e\phi}\right)\right] ,$$

with the bound on the deformation $|G(z) - G(z)| = |G(z) - G(T_\chi(z))| = |(T_\chi G - G)(z)|$, and exploiting the fact that G coincides with the norm on the phase space. In a similar way one can control the variation of Z.

4.4 Proof of Theorem 5 (high order approximation)

The proof of point *(1)* is an easy application of the Implicit Function Theorem (I.F.T.), based on the main assumption that J_Ω is invertible in the subspace of real square-summable sequences. Indeed $\mathfrak{A} = \mathcal{A}$ is the "unperturbed" solution of equation (21) $f(\mathcal{A}, \Omega, 0) = f_0(\mathcal{A}, \Omega) = 0$, and the Jacobian of f evaluated at $(\mathcal{A}, \epsilon = 0)$ is given by $D_{\mathfrak{A}} f(\mathcal{A}, \Omega, 0) = J_\Omega$, which is invertible. The first of (78) is standard, while the second can be easily obtained from

$$D_{\mathfrak{A}} f(\mathfrak{A}) = D_{\mathfrak{A}} f_0(\mathfrak{A}) + O(\epsilon) = D_{\mathfrak{A}} f_0(\mathcal{A}) + D^2_{\mathfrak{A}} f_0(\mathcal{A})(\mathfrak{A} - \mathcal{A}) + O(\epsilon) = J_\Omega + O(\epsilon) \,.$$

Once proved the existence of $\mathfrak{A}$, discrete soliton of K, we follow the same strategy used for Theorem (1) in order to prove the existence of an analytic branch of Klein-Gordon breathers[3] $\bar{\mathfrak{A}}$ exponentially close to $\mathfrak{A}(\Omega, \epsilon)$. Let us consider the Hamilton equation for (69) restricting to the variable ζ only (recall $i\eta = \bar{\zeta}$)

$$\omega \partial_\tau \zeta = i\zeta + \partial_\eta Z + \partial_\eta \mathcal{R}^{(r+1)} \,;$$

by inserting the Fourier expansion (79) with real coefficients $\mathfrak{A}^{(m)}$, we get the equation

$$\hat{L}_\omega \bar{\mathfrak{A}} + \partial_\eta Z(\bar{\mathfrak{A}}) + \partial_\eta \mathcal{R}^{(r+1)}(\bar{\mathfrak{A}}) = 0 \,, \qquad \left(\hat{L}_\omega \bar{\mathfrak{A}}\right)^{(m)} := i(1 - m\omega)\mathfrak{A}^{(m)} \,,$$

since the Hamilton equations are first order in time. The Kernel $\hat{V}_2$ of the linear operator $\hat{L}_{\omega=1}$ is given only by $e^{i\tau}$, all the other harmonics belonging to the Range $\hat{W}_2$. We decompose $\bar{\mathfrak{A}} = \bar{\mathfrak{A}}^\sharp + \bar{\mathfrak{A}}^\flat$ and project the equations on $\hat{V}_0$ and $\hat{W}_0$. The great difference with respect to the proof of Theorem 1, is that the resonant normal form construction, performed averaging with respect to the periodic flow $e^{i\tau}$, provides a natural decomposition of the term $\partial_\eta Z(\bar{\mathfrak{A}}^\sharp)$, so that one has for free

[3] We again "identify" solitons and breathers with thier amplitude(s).

$$\Pi_{\hat{W}_0}\partial_\eta Z(\bar{\mathfrak{A}}^\sharp) = 0\,, \qquad\qquad \Pi_{\hat{V}_0}\partial_\eta Z(\bar{\mathfrak{A}}^\sharp) = \partial_\eta Z(\bar{\mathfrak{A}}^\sharp)\,. \tag{92}$$

This remarkable property allows to write the Range equation as

$$\hat{L}_\omega \bar{\mathfrak{A}}^\flat + \Big[\Pi_{\hat{W}_0}\partial_\eta Z(\bar{\mathfrak{A}}^\flat + \bar{\mathfrak{A}}^\sharp) - \Pi_{\hat{W}_0}\partial_\eta Z(\bar{\mathfrak{A}}^\sharp)\Big] + \Pi_{\hat{W}_0}\partial_\eta \mathcal{R}^{(r+1)}(\bar{\mathfrak{A}}^\flat + \bar{\mathfrak{A}}^\sharp) = 0\,,$$

where the term in square brackets is at least linear in $\bar{\mathfrak{A}}^\flat$, $O(\epsilon\bar{\mathfrak{A}}^\flat)$, hence a small perturbation with respect to $\hat{L}_{\omega=1}$. The usual leading order approximation provided by the I.F.T. gives, for some constant C independent of ϵ,

$$\bar{\mathfrak{A}}^\flat \approx -\left(\hat{L}_\omega\right)^{-1}\Pi_{\hat{W}_0}\partial_\eta \mathcal{R}^{(r+1)}(\bar{\mathfrak{A}}^\flat) \qquad \Rightarrow \qquad \left\|\bar{\mathfrak{A}}^\flat\right\| \leqslant C\epsilon^{r+1}\,. \tag{93}$$

The Kernel equation, after dividing by ϵ, takes the form

$$\Omega\bar{\mathfrak{A}}^\sharp - \Delta\bar{\mathfrak{A}}^\sharp + \Gamma_p|\bar{\mathfrak{A}}^\sharp|^{2p}\bar{\mathfrak{A}}^\sharp + \sum_{s=2}^{r}\epsilon^{s-1}\Pi_{\hat{V}_0}\partial_\eta \mathcal{Z}_s\left(\bar{\mathfrak{A}}^\sharp\right) + O(\epsilon^r) = 0\,,$$

where we have introduced the scaled functions $\mathcal{Z}_s := \epsilon^{-s}Z_s$. Notice that in the small remainder we included not only the smallness of the vector field $\partial_\eta \mathcal{R}^{(r+1)}$, but also

$$\Pi_{\hat{V}_0}\partial_\eta \mathcal{Z}_s(\bar{\mathfrak{A}}^\flat + \bar{\mathfrak{A}}^\sharp) - \Pi_{\hat{V}_0}\partial_\eta \mathcal{Z}_s(\bar{\mathfrak{A}}^\sharp) = \Pi_{\hat{V}_0}\partial_\eta \mathcal{Z}_s(\bar{\mathfrak{A}}^\flat + \bar{\mathfrak{A}}^\sharp) - \partial_\eta \mathcal{Z}_s(\bar{\mathfrak{A}}^\sharp) = O(\bar{\mathfrak{A}}^\flat)\,,$$

which is of order $O\big(\epsilon^{r+1}\big)$ for any $1 \leqslant s \leqslant r$, because of (93). The Kernel equation now turns out to be a perturbation of order $O(\epsilon^r)$ of

$$f(\bar{\mathfrak{A}}^\sharp, \Omega, \epsilon) := \Omega\bar{\mathfrak{A}}^\sharp - \Delta\bar{\mathfrak{A}}^\sharp + \Gamma_p|\bar{\mathfrak{A}}^\sharp|^{2p}\bar{\mathfrak{A}}^\sharp + \sum_{s=2}^{r}\epsilon^{s-1}\Pi_{\hat{V}_0}\partial_\eta \mathcal{Z}_s\left(\bar{\mathfrak{A}}^\sharp\right) = 0\,, \tag{94}$$

since the last term in the sum of $\mathcal{Z}_s$ is of order $O(\epsilon^{r-1})$. Equation (94) admits the solution $\bar{\mathfrak{A}} = \bar{\mathfrak{A}}(\Omega, \epsilon)$ given by Proposition 4.1., and since $J_{\Omega,\epsilon}$ is invertible for ϵ sufficiently small, a fixed point argument (see for example Appendix of [4]) can be used to conclude the proof; estimate (80) is standard, while estimate (81) follows once the generic step r is replaced with the optimal choice $r = r_{opt}(\epsilon)$.

In order to prove point *(3)* we exploit the fact that the two periods, $T_{\rm ds}$ and $T_{\rm br}$, are both approximately equal to 2π, because of (80). Hence

$$\|\zeta_{\rm ds}(t) - \zeta_{\rm br}(t)\| \leqslant \sum_{m\neq 1}\left\|\mathfrak{A}^{(m)}\right\| + \left\|\mathfrak{A}^{(1)}e^{i\omega t} - \mathfrak{A}e^{i(1-\frac{\epsilon}{2}\Omega)t}\right\| \;;$$

moreover, on a time interval of order $O(1) = \max\{T_{\rm ds}, T_{\rm br}\}$, we have

$$\left\|\mathfrak{A}^{(1)}e^{i\omega t} - \mathfrak{A}e^{i(1-\frac{\epsilon}{2}\Omega)t}\right\| \leqslant \left\|\mathfrak{A}^{(1)}e^{O(\epsilon^2)t} - \mathfrak{A}\right\| \leqslant C\left\|\mathfrak{A}^{(1)} - \mathfrak{A}\right\|\,.$$

The estimates holds true also in the original coordinates, exploiting the Lipschitz continuity of the canonical transformation T_χ^{-1}, with a Lipschitz constant $L = O(1)$.

4.5 Proof of Theorem 6 (exponentially long time stability)

We collect the main geometrical ideas already exploited in [3, 19], omitting most of the details that the interested reader can find in the quoted papers.

We first have to prove the orbital stability of the discrete soliton $\mathfrak{A}$, interpreted as approximate breather solution of the Klein-Gordon model. We denote with $\mathsf{O}(\mathfrak{A})$ the closed orbit described by the discrete soliton profile $\mathfrak{A}$ during its periodic evolution

$$\mathsf{O}(\mathfrak{A}) := \{e^{i(1-\frac{\epsilon}{2}\Omega)t}\mathfrak{A},\ t \in [0, T_{\rm ds}]\} .$$

We consider a tubolar neighbourhood $\mathcal{W}_0$ of $\mathsf{O}(\mathfrak{A})$, in the transformed coordinates which give the original Hamiltonian the normal form (65). Any point $z \in \mathcal{W}_0$ can be represented with a local set of coordinates

$$z = (\varphi, E, v) \in \mathbb{R} \times \mathbb{R} \times V_\xi ,$$

where ξ is the projection of z on $\mathsf{O}(\mathfrak{A})$ and V_ξ is the orthogonal complement to the symmetry field $X_G(\xi)$ in the tangent space $T_\xi\mathcal{S} = V_\xi \oplus X_G(\xi)$. The three coordinates represent, respectively: the scalar coordinate φ is the tangential displacement along the field X_G, the scalar coordinate E is the displacement in the direction ∇G orthogonal to the surface $\mathcal{S}$ and the vector valued coordinate v is the tangential displacement in the directions of V_ξ. In order to measure the orbital distance of a generic point z from $\mathsf{O}(\mathfrak{A})$ we need to control only the directions transversal to the orbits, hence E and v. The main point is that E is related to the variation of G while v is related to the variation of $\mathcal{Z}$: the first is obvious, while the second holds because we have asked $\mathfrak{A}$ to be a nondegenerate extremizer of $\mathcal{Z}$ on $\mathcal{S}$, hence locally we have

$$\|v\|^2 = \|z - \xi\|^2 \leqslant \frac{1}{C}|\mathcal{Z}(z) - \mathcal{Z}(\xi)| ,$$

where C is a constant depending on $\mathcal{Z}''(\xi)$. Let us consider an initial datum $z_0 \in \mathcal{W}_0$ and its piece of orbit $\Phi^t_H(z_0) \cap \mathcal{W}_0$; for any point on this curve we have

$$\begin{aligned}\inf_{w\in\mathsf{O}(\mathfrak{A})\cap\mathcal{W}_0} \left\|\Phi^t_H(z_0) - w\right\| &\leqslant c_1|E(t)| + c_2\,\|v(t)\| \leqslant \\ &\leqslant C\sqrt{|G(\Phi^t_H(z_0)) - G(\mathfrak{A})| + |\mathcal{Z}(\Phi^t_H(z_0)) - \mathcal{Z}(\mathfrak{A})|} .\end{aligned}$$

If z_0 is taken in a suitable domain where (66) hold true, then the two terms in the square root can be bounded by

$$|G(\Phi^t_H(z_0)) - G(\mathfrak{A})| \leqslant |G(\Phi^t_H(z_0)) - G(z_0)| + |G(z_0) - G(\mathfrak{A})| \tag{95}$$

$$|\mathcal{Z}(\Phi^t_H(z_0)) - \mathcal{Z}(\mathfrak{A})| \leqslant |\mathcal{Z}(\Phi^t_H(z_0)) - |\mathcal{Z}(z_0)| + |\mathcal{Z}(z_0) - \mathcal{Z}(\mathfrak{A})| ; \tag{96}$$

the first right hand terms are exactly controlled by (66) on exponentially long times, while the second right hand terms are controlled by the initial distance from the orbit. This allows to get (83) for the discrete soliton $\mathfrak{A}$ in the normal form coordinates. The stability result then is transferred to the Klein-Gordon breather exploiting the exponentially small distance between the two orbits, as stated by (82). Finally, the same stability result holds also in the original coordinates, as claimed by (91), since $T_\mathcal{X}^{-1}$ is Lipschitz with a Lipschitz constant $L = O(1)$.

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