SIAM J. MATH. ANAL.
Vol. 51, No. 4, pp. 2850–2883

CONVERGENCE OF PETVIASHVILI'S METHOD NEAR PERIODIC WAVES IN THE FRACTIONAL KORTEWEG–DE VRIES EQUATION*

UYEN LE† AND DMITRY E. PELINOVSKY‡

Abstract. Petviashvili's method has been successfully used for approximating solitary waves in nonlinear evolution equations. It was discovered empirically that the method may fail when approximating periodic waves. We consider the case study of the fractional Korteweg–de Vries equation and explain divergence of Petviashvili's method from unstable eigenvalues of the generalized eigenvalue problem. We also show that a simple modification of the iterative method after the mean value shift results in the unconditional convergence of Petviashvili's method. The results are illustrated numerically for the classical Korteweg–de Vries and Benjamin–Ono equations.

Key words. fractional Korteweg–de Vries equation, traveling periodic waves, Petviashvili's method, convergence analysis

AMS subject classifications. 35Q53, 35P30, 37K50, 37K55, 65J15

DOI. 10.1137/18M1215050

1. Introduction.

A robust iterative method for approximating solitary waves was proposed by V. I. Petviashvili in 1976 [37]. Since then, it has become a popular numerical toolbox [40] with many recent generalizations in [29, 30] and in [1, 2, 3].

In the context of Euler equations for water waves, Petviashvili's iterative method turns out to be very useful for computing solitary gravity waves [14, 19]. However, it has been found empirically that the iterative algorithm does not converge for periodic waves; hence suitable generalizations were proposed in the case of infinite [20] and finite [15] depths. The work [20] explores the generalization of Petviashvili's method for nonpower nonlinearities proposed originally in [29]. The work [15] relies on an iteration-dependent shift of the field variable to enforce positivity of the periodic wave, after which the classical Petviashvili method can be employed.

In a setting of fractional Korteweg–de Vries (KdV) and extended Boussinesq equations, another modification of the Petviashvili method was proposed in [4, 18], where an iteration-independent shift of the field variable was computed from the underlying equation. Numerical results in [4] illustrated convergence of the Petviashvili method for the periodic waves after the shift.

The main purpose of this work is to explain analytically the failure of the classical Petviashvili method for approximating periodic waves and to prove convergence of the same method after a suitable shift of the field variable. We consider the toy problem given by the fractional KdV equation with a quadratic nonlinearity, which is a simplified model arising from the Euler equations in the shallow limit [8]. The

*Received by the editors September 19, 2018; accepted for publication (in revised form) April 23, 2019; published electronically July 9, 2019.
https://doi.org/10.1137/18M1215050

Funding: The work of the second author was supported by the state task program in the sphere of scientific activity of the Ministry of Education and Science of the Russian Federation (5.5176.2017/8.9) and by the President of Russian Federation for leading scientific schools (grant NSH-2685.2018.5).

†Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, L8S 4K1, Canada (leu@mcmaster.ca).

‡Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, L8S 4K1, Canada, and Department of Applied Mathematics, Nizhny Novgorod State Technical University, 603950 Nizhny Novgorod, Russia (dmpeli@math.mcmaster.ca).

fractional KdV equation is taken in the normalized form

$$u_t + 2uu_x + (D_\alpha u)_x = 0, \tag{1.1}$$

where D_α is a fractional derivative operator defined by its Fourier symbol

$$\widehat{D_\alpha u}(\xi) = -|\xi|^\alpha \hat{u}(\xi), \quad \xi \in \mathbb{R}.$$

The case $\alpha = 2$ corresponds to the classical KdV equation, whereas the case $\alpha = 1$ corresponds to the integrable BO (Benjamin–Ono) equation. Henceforth, we assume that $\alpha > 0$.

Global existence in the fractional KdV equation (1.1) for the initial data in the energy space $H^{\alpha/2}$ was proven in [31] for $\alpha > 1/2$ and for $\alpha = 1/2$ and small data. More recently, local existence for the initial data in H^s was shown for $\alpha > 0$ and $s > 3/2 - 5\alpha/4$ in [33].

Existence and stability of periodic waves in the fractional KdV equation (1.1) were analyzed by using perturbative [27], variational [9, 11, 25], and fixed-point [10] methods. For the classical KdV and BO equations, stability of periodic waves was also proven in [6]. These results, especially perturbation expansions in the limit of small wave amplitudes, are also useful in our analysis of convergence of iterative methods near the periodic waves.

Periodic traveling waves are solutions of the fractional KdV equation (1.1) in the form $u(x,t) = \psi(x-ct)$, where ψ is a periodic function in its argument and $c > 0$ is the speed parameter for the wave traveling to the right. Without loss of generality, due to scaling and translation invariance of the fractional KdV equation (1.1), we scale the period of ψ to 2π and translate ψ to become an even function of its argument. Due to the Galilean invariance, integration of the nonlinear equation for ψ is performed with zero integration constant. All together, the wave profile ψ is a 2π-periodic even solution to the following boundary-value problem:

$$(c - D_\alpha)\psi = \psi^2, \quad \psi \in H^\alpha_{\rm per}(-\pi, \pi). \tag{1.2}$$

We say that the periodic wave has a *single-lobe* profile if there exist only one maximum and minimum of ψ on the period. For uniqueness of solutions, we place the maximum of ψ at $x = 0$ and the minimum of ψ at $x = \pm\pi$.

In addition to the waves traveling to the right, the fractional KdV equation (1.1) also has periodic traveling waves in the form $u(x,t) = \phi(x+ct)$, where $c > 0$ is the speed parameter for the wave traveling to the left and ϕ is a 2π-periodic even solution to the following boundary-value problem:

$$(c + D_\alpha)\phi + \phi^2 = 0, \quad \phi \in H^\alpha_{\rm per}(-\pi, \pi). \tag{1.3}$$

A very simple formula connects the right-propagating waves with the left-propagating waves:

$$\phi(x) = -c + \psi(x). \tag{1.4}$$

The wave profile ϕ is a solution to the boundary-value problem (1.3) with some $c > 0$ if and only if ψ is a solution to the boundary-value problem (1.2) with the same $c > 0$. Section 2 gathers some results on existence of solutions to the boundary-value problems (1.2) and (1.3).

Remark 1.1. Although most of the previous works (see, e.g., [6, 10, 11, 25, 27]) are devoted to right-propagating waves with profile ψ, there are no a priori reasons to prefer these waves over the left-propagating waves with profile ϕ (see, e.g., [9]). Perturbative expansions for waves of small amplitudes are more easily developed for the left-propagating waves with profile ϕ since they arise in the local bifurcation theory from linearization of the zero equilibrium (see Theorem 2.1 below). On the other hand, the proof of positivity of the wave profile ψ is developed more easily from the boundary-value problem (1.2) (see Theorem 2.2 below).

Let us now explain how Petviashvili's iterative methods can be employed in order to approximate solutions to the boundary-value problems (1.2) and (1.3) numerically. In fact, the most interesting interplay between convergent and divergent iterations arises in the context of the boundary-value problem (1.3).

Suppose that $\phi \in H^\alpha_{\rm per}(-\pi,\pi)$ is a *single-lobe periodic solution* to the boundary-value problem (1.3) for some $c > 0$. The classical Petviashvili method for approximating ϕ is defined as follows: Consider $\mathcal{L}_{c,\alpha} := -c - D_\alpha$ as a linear operator in $L^2_{\rm per}(-\pi,\pi)$ with the domain $H^\alpha_{\rm per}(-\pi,\pi)$ and define the Petviashvili quotient:

$$M(w) := \frac{\langle \mathcal{L}_{c,\alpha} w, w\rangle}{\langle w^2, w\rangle}, \quad w \in H^\alpha_{\rm per}(-\pi,\pi), \tag{1.5}$$

where $\langle \cdot,\cdot\rangle$ denotes the standard inner product in $L^2_{\rm per}(-\pi,\pi)$. For $c \notin \{1, 2^\alpha, 3^\alpha, \dots\}$, for which the linear operator $\mathcal{L}_{c,\alpha} : H^\alpha_{\rm per}(-\pi,\pi) \to L^2_{\rm per}(-\pi,\pi)$ is invertible, and for any suitable initial guess $w_0 \in H^\alpha_{\rm per}(-\pi,\pi)$, define a sequence $\{w_n\}_{n\in\mathbb{N}}$ in $H^\alpha_{\rm per}(-\pi,\pi)$ by the iterative rule

$$w_{n+1} = T_{c,\alpha}(w_n) := [M(w_n)]^2 \mathcal{L}^{-1}_{c,\alpha}(w_n^2), \quad n \in \mathbb{N}. \tag{1.6}$$

Here we have selected the quadratic exponent of $M(w_n)$ so that $T_{c,\alpha}(w)$ is a homogeneous power function in w of degree *zero*. This ensures the fastest convergence rate of the iterative method (1.6) near a solution of the nonlinear equation (1.3) [36].

As is well understood since the first proof of convergence in [36] (see also the follow-up works in [2, 3, 12, 17, 29]), convergence of the iterative method is analyzed from contraction of the linearized operator at the fixed point $\phi \in H^\alpha_{\rm per}(-\pi,\pi)$ of $T_{c,\alpha}$. By Lemma 1.2 in [36], the set of fixed points of $T_{c,\alpha}$ coincides with the set of solutions to the boundary-value problem (1.3). Contraction of the corresponding linearized operator is defined by the spectrum of the generalized eigenvalue problem

$$\mathcal{H}_{c,\alpha} v = \lambda \mathcal{L}_{c,\alpha} v, \quad v \in H^\alpha_{\rm per}(-\pi,\pi), \tag{1.7}$$

where

$$\mathcal{H}_{c,\alpha} := -c - D_\alpha - 2\phi \tag{1.8}$$

is the associated linearized operator in $L^2_{\rm per}(-\pi,\pi)$ with the domain $H^\alpha_{\rm per}(-\pi,\pi)$. Note that $\mathcal{H}_{c,\alpha}$ is the Jacobian operator for the boundary-value problem (1.3), which also plays the crucial role in the stability analysis of the traveling periodic waves [6, 9, 25, 27].

Section 3 presents the main result on convergence of the iterative method (1.6). Here and in what follows, the following critical values of α are important:

$$\alpha_0 := \frac{\log 3}{\log 2} - 1, \qquad \alpha_1 := \frac{\log 5}{\log 2} - 1, \tag{1.9}$$

where $1/2 < \alpha_0 < 1 < \alpha_1 < 2$. The proof of the main result is achieved by the count of unstable eigenvalues in the generalized eigenvalue problem (1.7) and by perturbative arguments.

THEOREM 1.1. *For every $c > 1$ and $\alpha \in (\alpha_0, 2]$, there exists a single-lobe solution $\phi \in H^\alpha_{\rm per}(-\pi,\pi)$ to the boundary-value problem* (1.3). *There is $c_0 > 1$ such that for every $c \in (1, c_0)$ this solution is an unstable fixed point of the iterative method* (1.6) *for $\alpha \in (\alpha_0, \alpha_1)$ and an asymptotically stable fixed point (up to a translation) for $\alpha \in (\alpha_1, 2]$. If $c > 2^\alpha$, this unique solution is an unstable fixed point of the iterative method* (1.6) *for $\alpha \in (\alpha_0, 2]$.*

Remark 1.2. The single-lobe solution to the boundary-value problem (1.3) exists also for $\alpha < \alpha_0$ but is located for $c < 1$ in the small-amplitude limit.

Remark 1.3. The constraint $\alpha \leq 2$ is necessary in order to apply results of [25] on existence of single-lobe solution ϕ and the nondegeneracy of the kernel of $\mathcal{H}_{c,\alpha}$ at ϕ. The periodic wave ϕ may develop oscillations for $\alpha > 2$ and sufficiently large c, in which case methods of [25] are not applicable.

Remark 1.4. Theorem 1.1 implies that the iterative method (1.6) diverges from ϕ for the classical BO equation with $\alpha = 1$. Although the iterative method (1.6) converges to ϕ for the classical KdV equation with $\alpha = 2$ for $c \in (1, c_0)$, we show numerically that it diverges from ϕ for $c > c_0$ with $c_0 \approx 2.3$. Instabilities of the iterative method (1.6) are explained by the unstable eigenvalues of the generalized eigenvalue problem (1.7).

As is suggested by Theorem 1.1, the iterative method (1.6) is unsuccessful in approximating the solution ϕ to the boundary-value problem (1.3). On the other hand, we can develop a similar method for the solution ψ of the equivalent boundary-value problem (1.2), which is related to ϕ by the transformation (1.4). By setting $\tilde{\mathcal{L}}_{c,\alpha} := c - D_\alpha$, we denote

$$\tilde{M}(w) := \frac{\langle \tilde{\mathcal{L}}_{c,\alpha} w, w\rangle}{\langle w^2, w\rangle}, \quad w \in H^\alpha_{\rm per}(-\pi,\pi), \tag{1.10}$$

and define a sequence $\{w_n\}_{n\in\mathbb{N}}$ in $H^\alpha_{\rm per}(-\pi,\pi)$ for any suitable initial guess $w_0 \in H^\alpha_{\rm per}(-\pi,\pi)$ by the iterative rule

$$w_{n+1} = \tilde{T}_{c,\alpha}(w_n) := \left[\tilde{M}(w_n)\right]^2 \tilde{\mathcal{L}}^{-1}_{c,\alpha}(w_n^2), \quad n \in \mathbb{N}. \tag{1.11}$$

Contraction of the linearized operator of the iterative rule (1.11) is defined by the spectrum of the generalized eigenvalue problem

$$\tilde{\mathcal{H}}_{c,\alpha} v = \lambda \tilde{\mathcal{L}}_{c,\alpha} v, \quad v \in H^\alpha_{\rm per}(-\pi,\pi), \tag{1.12}$$

where the new Jacobian operator for the boundary-value problem (1.2) is identical to the Jacobian operator (1.8) of the boundary-value problem (1.3):

$$\tilde{\mathcal{H}}_{c,\alpha} := c - D_\alpha - 2\psi = -c - D_\alpha - 2(-c+\psi) = \mathcal{H}_{c,\alpha}, \tag{1.13}$$

where the transformation (1.4) has been used.

Section 4 presents the main result on convergence of the iterative method (1.11). In addition to the count of unstable eigenvalues in the generalized eigenvalue problem (1.12), we use here positivity of the wave profile ψ for solutions to the boundary-value problem (1.2).

THEOREM 1.2. *For every $c > 1$ and $\alpha \in (\alpha_0, 2]$, there exists a single-lobe solution $\psi \in H^\alpha_{\rm per}(-\pi,\pi)$ to the boundary-value problem* (1.2) *such that $\psi(x) > 0$ for every $x \in [-\pi,\pi]$. This solution is an asymptotically stable (up to a translation) fixed point of the iterative method* (1.11) *for every $c > 1$ and $\alpha \in (\alpha_0, 2]$.*

Remark 1.5. The unconditional convergence of the iterative method (1.11) compared to the iterative method (1.6) has a well-known physical interpretation. The phase velocity of the linear waves of the fractional KdV equation (1.1) on the zero background is strictly negative; hence the traveling wave $u(x,t) = \phi(x+ct)$ propagating to the left is in resonance with the linear waves. On the other hand, the traveling wave on the constant background $b := -c < 0$ propagates to the right and avoids resonances with the linear waves on the background $b < 0$, which still have negative phase velocity.

Remark 1.6. The new iterative method (1.11) can be considered as a modification of the classical Petviashvili method (1.6) after the shift of the field variable proposed in [4]. The modified algorithm consists of three steps. In the first step, the constant value b is found from the constant solution of the stationary problem (1.3). Solving $cb + b^2 = 0$ for nonzero b yields $b = -c$. In the second step, the change of variables $\phi = b + \psi$ transforms the original problem (1.3) to the new problem (1.2), which is confirmed from the transformation formula (1.4) since $b = -c$. Finally, the third step is the iterative method for the transformed problem (1.2), which is defined by the new iterative operator $\tilde{T}_{c,\alpha}$ in (1.11).

Remark 1.7. In the case of solitary waves, the boundary-value problem (1.3) for ϕ and $c > 0$ admits no solutions, and the iterative method (1.6) cannot be defined since $\mathcal{L}_{c,\alpha}$ is not invertible in $L^2(\mathbb{R})$ for $c > 0$. On the other hand, the boundary-value problem (1.2) for ψ and $c > 0$ admits solitary wave solutions, and the iterative method (1.11) is well-defined to approximate this solution, as shown numerically in [18].

Remark 1.8. Setting the integration constant to zero in the boundary-value problem (1.2) and the simple transformation (1.4) between the boundary-value problems (1.2) and (1.3) works only for Galilean-invariant models such as the fractional KdV equation with the quadratic nonlinearity. For the case of non-Galilean-invariant models with nonquadratic nonlinearity, many properties established in this work will become less explicit and can only be verified numerically. We refer the reader to the computational algorithm implemented in [4] for the fractional KdV equation with general power nonlinearity. The idea in [4] was also to transform the boundary-value problem with a shift of the field variable and then to use Petviashvili's method for nonhomogeneous nonlinearity. While convergence was clearly observed in the numerical experiments, it is harder to prove the convergence result analytically.

2. Periodic waves of the fractional KdV equation. We collect together some results on existence of periodic wave solutions to the boundary-value problems (1.2) and (1.3). Some of the previous results have been improved, and we specify explicitly where the improvement has been made. Section 2.1 presents results on the small-amplitude limit of the periodic waves with profile ϕ. Sections 2.2 and 2.3 collect explicit expressions for the periodic waves in the classical KdV and BO equations, respectively. Section 2.4 gives results on the positivity of the wave profile ψ.

2.1. Small-amplitude limit of the periodic waves. The following result reports on existence of the periodic wave ϕ of the boundary-value problem (1.3) in the small-amplitude limit. The small-amplitude periodic waves bifurcate from the constant zero solution to the boundary-value problem (1.3). The construction of the small-amplitude periodic waves is nearly identical to Lemma 2.1 in [27] subject to the following two changes: First, the constant of integration is set to zero thanks to the Galilean invariance, while in [27] the constant was carried as an additional (redundant) parameter of the problem. Second, the speed c is used as the main parameter of the periodic solution while the period is set to 2π, whereas in [27] c was set to 1 and the period was taken as the main parameter of the periodic solution.

Although the formal computations of the periodic waves in the small-amplitude limit hold for every $\alpha > 0$, the justification of the perturbative expansions requires $\alpha > 1/2$, for which $H^{\alpha}_{\rm per}(-\pi,\pi)$ is a Banach algebra with respect to multiplication with a continuous embedding into $L^{\infty}_{\rm per}(-\pi,\pi)$. A typical justification of the perturbative expansions is based on the method of Lyapunov–Schmidt reductions which requires smoothness of the nonlinear mappings. This smoothness is guaranteed in $H^{\alpha}_{\rm per}(-\pi,\pi)$ with $\alpha > 1/2$. Since refinement to $\alpha \in (0,1/2)$ is not important for the subject of our work, we leave the restriction $\alpha > 1/2$ in the same way it was used in Theorem A.1 in [27].

THEOREM 2.1. *For every $\alpha > \alpha_0$, there exists $c_0 > 1$ such that for every $c \in (1, c_0)$ there exists a locally unique single-lobe solution ϕ of the boundary-value problem* (1.3) *with the global maximum at $x = 0$. The wave profile ϕ and the wave speed c are real-analytic functions of the wave amplitude a satisfying the following Stokes expansions:*

$$\phi_{a,\alpha}(x) = a\cos(x) + a^2\phi_2(x) + a^3\phi_3(x) + a^4\phi_4(x) + \mathcal{O}(a^5) \tag{2.1}$$

and

$$c_{a,\alpha} = 1 + c_2 a^2 + c_4 a^4 + \mathcal{O}(a^6), \tag{2.2}$$

where the α-dependent correction terms $\{\phi_2,\phi_3,\phi_4\}$ and $\{c_2,c_4\}$ are defined in (2.3)–(2.7) *below.*

Proof. We give algorithmic computations of the higher-order coefficients to the periodic wave by using the classical Stokes expansions

$$\phi(x) = \sum_{k=1}^{\infty} a^k \phi_k(x), \quad c = 1 + \sum_{k=1}^{\infty} c_{2k} a^{2k}.$$

The correction terms satisfy recursively

$$\left\{ \begin{array}{ll} \mathcal{O}(a): & (1+D_\alpha)\phi_1 = 0, \\ \mathcal{O}(a^2): & (1+D_\alpha)\phi_2 + \phi_1^2 = 0, \\ \mathcal{O}(a^3): & (1+D_\alpha)\phi_3 + c_2\phi_1 + 2\phi_1\phi_2 = 0, \\ \mathcal{O}(a^4): & (1+D_\alpha)\phi_4 + c_2\phi_2 + 2\phi_1\phi_3 + \phi_2^2 = 0, \\ \mathcal{O}(a^5): & (1+D_\alpha)\phi_5 + c_2\phi_3 + c_4\phi_1 + 2\phi_1\phi_4 + 2\phi_2\phi_3 = 0. \end{array} \right.$$

For the single-lobe wave profile ϕ with the global maximum at $x = 0$, we select uniquely $\phi_1(x) = \cos(x)$ since $\mathrm{Ker}_{\rm even}(1 + D_\alpha) = \mathrm{span}\{\cos(\cdot)\}$ in the space of even functions in $L^2_{\rm per}(-\pi,\pi)$. In order to select uniquely all other corrections to the Stokes expansion (2.1), we require the corrections terms $\{\phi_k\}_{k\geq 2}$ to be orthogonal to ϕ_1 in $L^2_{\rm per}(-\pi,\pi)$.

Solving the inhomogeneous equation at $\mathcal{O}(a^2)$ yields the following exact solution in $H^{\alpha}_{\rm per}(-\pi,\pi)$:

$$\phi_2(x) = -\frac{1}{2} + \frac{1}{2(2^\alpha - 1)}\cos(2x). \tag{2.3}$$

The inhomogeneous equation at $\mathcal{O}(a^3)$ admits a solution $\phi_3 \in H^\alpha_{\rm per}(-\pi,\pi)$ if and only if the right-hand side is orthogonal to ϕ_1, which selects uniquely the correction c_2 by

$$c_2 = 1 - \frac{1}{2(2^\alpha - 1)}. \tag{2.4}$$

After the resonant term is removed, the inhomogeneous equation at $\mathcal{O}(a^3)$ yields the exact solution in $H^\alpha_{\rm per}(-\pi,\pi)$:

$$\phi_3(x) = \frac{1}{2(2^\alpha-1)(3^\alpha-1)} \cos(3x). \tag{2.5}$$

By continuing the algorithm, we find the exact solution of the inhomogeneous equation at $\mathcal{O}(a^4)$ in $H^\alpha_{\rm per}(-\pi,\pi)$:

$$\begin{aligned} \phi_4(x) = {} & \frac{1}{4} - \frac{1}{4(2^\alpha-1)} - \frac{1}{8(2^\alpha-1)^2} + \frac{1}{4(2^\alpha-1)^2}\left[\frac{2}{3^\alpha-1} - \frac{1}{2^\alpha-1}\right]\cos(2x) \\ & + \frac{1}{8(2^\alpha-1)(4^\alpha-1)}\left[\frac{4}{3^\alpha-1} + \frac{1}{2^\alpha-1}\right]\cos(4x). \end{aligned} \tag{2.6}$$

Finally, the inhomogeneous equation at $\mathcal{O}(a^5)$ admits a solution $\phi_5 \in H^\alpha_{\rm per}(-\pi,\pi)$ if and only if the right-hand side is orthogonal to ϕ_1, which selects uniquely the correction c_4 by

$$c_4 = -\frac{1}{2} + \frac{1}{2(2^\alpha-1)} + \frac{1}{4(2^\alpha-1)^2} + \frac{1}{4(2^\alpha-1)^3} - \frac{3}{4(2^\alpha-1)^2(3^\alpha-1)}. \tag{2.7}$$

Note that $c_2 > 0$ if $\alpha > \alpha_0 := \log 3/\log 2 - 1$, which implies that the small-amplitude periodic wave with profile ϕ exists in the boundary-value problem (1.3) for $c > 1$ and $\alpha > \alpha_0$. The periodic wave has a global maximum at $x = 0$ for small a since $x = 0$ is the only maximum of $\phi_1(x) = \cos(x)$, whereas $\phi'(0) = 0$ and $\phi''(0) = -a + \mathcal{O}(a^2) < 0$.

Justification of the existence, uniqueness, and analyticity of the Stokes expansions (2.1) and (2.2) is performed with the method of Lyapunov–Schmidt reductions for $\alpha > 1/2$; see Lemma 2.1 and Theorem A.1 in [27]. Since $\alpha_0 > 1/2$, the justification procedure applies for every $\alpha > \alpha_0$. ◻

Remark 2.1. If $\alpha < \alpha_0$, then $c_2 < 0$, so that the small-amplitude periodic wave exists for $c \in (c_0, 1)$ with some $c_0 < 1$. The critical value α_0 can also be seen in the expansion of the wave period T (for fixed $c = 1$) with respect to the wave amplitude a in Lemma 2.1 of [27].

Remark 2.2. Variational results on existence of finite-amplitude periodic waves in the boundary-value problem (1.2) are obtained in Proposition 2.1 of [25] in the energy space $H^{\alpha/2}_{\rm per}(-\pi,\pi)$ for $\alpha \in (1/3, 2]$. It is shown that there exists a local minimizer of energy for fixed momentum and mass for every $c > 0$; however, it is overlooked in [25] that the local minimizer may coincide with the nonzero constant solution $\psi_c(x) = c$ for all $x \in [-\pi,\pi]$ to the same boundary-value problem (1.2). The same problem is present in Proposition 12 in [9]. We fix this problem in Lemma 2.3 and Remark 2.5 below.

For further reference, we prove the following technical result. For notational convenience, we omit parameters a and α when we refer to the periodic wave profile ϕ which solves the boundary-value problem (1.3) for some $c > 1$.

LEMMA 2.1. *There exists $c_0 > 0$ such that for every $c \in (1, c_0)$, the periodic wave ϕ defined in Theorem* 2.1 *satisfies*

$$\int_{-\pi}^{\pi} \phi^3 dx \begin{array}{ll} < 0, & \alpha > \alpha_0, \\ > 0, & \alpha < \alpha_0, \end{array} \Bigg\} \tag{2.8}$$

and

$$\int_{-\pi}^{\pi} \phi(\phi')^2 dx \begin{array}{ll} < 0, & \alpha > \alpha_1, \\ > 0, & \alpha < \alpha_1, \end{array} \Bigg\} \tag{2.9}$$

where α_0 and α_1 are given by (1.9).

Proof. By using Stokes expansions (2.1), we compute

$$\int_{-\pi}^{\pi} \phi^3 dx = \frac{3\pi a^4}{4(2^\alpha - 1)}(3 - 2^{\alpha+1}) + \mathcal{O}(a^6)$$

and

$$\int_{-\pi}^{\pi} \phi(\phi')^2 dx = \frac{\pi a^4}{4(2^\alpha - 1)}(5 - 2^{\alpha+1}) + \mathcal{O}(a^6),$$

from which (2.8) and (2.9) follows thanks to the definition (1.9). ◻

Remark 2.3. Since the Fourier basis $\{e^{inx}\}_{n\in\mathbb{Z}}$ in $L^2_{\rm per}(-\pi, \pi)$ diagonalizes $\mathcal{L}_{c,\alpha}$, the spectrum of $\mathcal{L}_{c,\alpha}$ in $L^2_{\rm per}(-\pi, \pi)$ is obtained for every $c \in \mathbb{R}$ and $\alpha > 0$ as follows:

$$\sigma(\mathcal{L}_{c,\alpha}) = \{-c + |n|^\alpha, \ \ n \in \mathbb{Z}\}. \tag{2.10}$$

The following lemma clarifies the number and multiplicity of negative and zero eigenvalues of the Jacobian operator $\mathcal{H}_{c,\alpha}$ in $L^2_{\rm per}(-\pi, \pi)$, where the expression for $\mathcal{H}_{c,\alpha}$ is given by (1.8).

LEMMA 2.2. *For every $\alpha > \alpha_0$, there exists $c_0 > 1$ such that for every $c \in (1, c_0)$, $\sigma(\mathcal{H}_{c,\alpha})$ in $L^2_{\rm per}(-\pi, \pi)$ consists of one simple negative eigenvalue, a simple zero eigenvalue, and a countable sequence of positive eigenvalues bounded away from zero.*

Proof. Note that $\sigma(\mathcal{H}_{c,\alpha})$ in $L^2_{\rm per}(-\pi, \pi)$ is purely discrete for every $c > 1$, thanks to the compactness of $[-\pi, \pi]$ and boundedness of $\phi \in L^\infty_{\rm per}(-\pi, \pi)$. For $c = 1$, $\mathcal{H}_{c=1,\alpha}$ coincides with $\mathcal{L}_{c=1,\alpha}$; hence it follows from (2.10) that $\sigma(\mathcal{H}_{c=1,\alpha})$ has a simple negative eigenvalue, a double zero eigenvalue, and a countable sequence of positive eigenvalues bounded away from zero.

Since $\mathcal{H}_{c,\alpha} - \mathcal{L}_{c,\alpha} = -2\phi$ is a bounded perturbation and (ϕ, c) depend analytically on a, the analytic perturbation theory (Theorem VII.1.7 in [28]) guarantees continuity of eigenvalues for $c > 1$ close to their limiting values as $c \to 1$. Therefore, the proof is achieved if we can show that the double zero eigenvalue of $\mathcal{H}_{c,\alpha}$ in $L^2_{\rm per}(-\pi, \pi)$ splits as $c > 1$ into a simple zero eigenvalue and a simple positive eigenvalue.

Since $\mathrm{Ker}(\mathcal{H}_{c=1,\alpha}) = \mathrm{span}\{\cos(\cdot), \sin(\cdot)\}$ and $\mathcal{H}_{c,\alpha}\phi' = 0$ for every $c > 1$ with odd ϕ, the zero eigenvalue associated with the subspace $\mathrm{Ker}_{\rm odd}(\mathcal{H}_{c=1,\alpha}) = \mathrm{span}\{\sin(\cdot)\}$ persists for $c > 1$. It remains to check the shift of the zero eigenvalue associated with the subspace $\mathrm{Ker}_{\rm even}(\mathcal{H}_{c=1,\alpha}) = \mathrm{span}\{\cos(\cdot)\}$. Hence, we expand $\mathcal{H}_{c,\alpha}$ in powers of a by using (2.1),

$$\mathcal{H}_{c,\alpha} = -1 - D_\alpha - 2a\cos(x) - \frac{a^2}{2^\alpha - 1}\left[\cos(2x) - \frac{1}{2}\right] + \mathcal{O}(a^3), \tag{2.11}$$

and look for solutions $(\lambda, v) \in \mathbb{R} \times H^\alpha_{\rm per}(-\pi,\pi)$ of the eigenvalue problem $\mathcal{H}_{c,\alpha} v = \lambda v$ near $(\lambda, v) = (0, \cos(\cdot))$ by using the expansions

$$\left\{ \begin{array}{l} v(x) = \cos(x) + a v_1(x) + a^2 v_2(x) + \mathcal{O}(a^3), \\ \lambda = a\lambda_1 + a^2 \lambda_2 + \mathcal{O}(a^3). \end{array} \right.$$

The correction terms in $H^\alpha_{\rm per}(-\pi,\pi)$ satisfy recursively

$$\left\{ \begin{array}{ll} \mathcal{O}(a): & (1+D_\alpha) v_1 + 1 + \cos(2x) + \lambda_1 \cos(x) = 0, \\ \mathcal{O}(a^2): & (1+D_\alpha) v_2 + 2\cos(x) v_1 + \frac{1}{2^\alpha - 1}\left[\cos(2x) - \frac{1}{2}\right]\cos(x) + \lambda_2 \cos(x) = 0. \end{array} \right.$$

In order to determine them uniquely, we impose orthogonality conditions of $\{v_k\}_{k\geq 1}$ to $\cos(\cdot)$ in $L^2_{\rm per}(-\pi,\pi)$. The linear inhomogeneous equation at $\mathcal{O}(a)$ admits a solution $v_1 \in H^\alpha_{\rm per}(-\pi,\pi)$ if and only if $\lambda_1 = 0$, after which the solution is found explicitly:

$$v_1(x) = \frac{1}{2^\alpha - 1}\cos(2x) - 1.$$

The linear inhomogeneous equation at $\mathcal{O}(a^2)$ admits a solution $v_2 \in H^\alpha_{\rm per}(-\pi,\pi)$ if and only if $\lambda_2 = 2c_2$, where c_2 is defined by (2.4). Since $c_2 > 0$ if $\alpha > \alpha_0$, the small positive eigenvalue $\lambda = 2c_2 a^2 + \mathcal{O}(a^3)$ bifurcates from the zero eigenvalue as $c > 1$. A functional-analytic setup for justification of perturbative expansions can be found in [27] (see also [24]) for $\alpha > 1/2$, which is met since $\alpha_0 > 1/2$. ◻

Remark 2.4. By using variational methods, it was shown in Proposition 3.1 and Lemma 3.3 of [25] that $\ker(\mathcal{H}_{c,\alpha}) = {\rm span}\{\phi'\}$ is one-dimensional, the zero eigenvalue is the lowest eigenvalue in the subspace of odd functions in $L^2_{\rm per}(-\pi,\pi)$, and $\sigma(\mathcal{H}_{c,\alpha})$ has either one or two negative eigenvalues for every $c > 1$ and $\alpha \in (1/3, 2]$. By Lemma 2.2 above, $\sigma(\mathcal{H}_{c,\alpha})$ has only one simple negative eigenvalue for $\alpha > \alpha_0$ if $c > 1$.

The following lemma shows that the small-amplitude single-lobe solutions of Theorem 2.1 agree with the variational characterization of single-lobe solutions as minimizers of energy subject to fixed momentum and mass in Proposition 2.1 of [25] for $c > 1$.

LEMMA 2.3. *Let $\psi = c_{a,\alpha} + \phi_{a,\alpha}$ be the locally unique single-lobe solution of the boundary-value problem* (1.2) *for $\alpha > \alpha_0$ and $c > 1$ defined by Theorem* 2.1. *Then this solution is a local minimizer of the energy*

$$E(u) = -\frac{1}{2}\int_{-\pi}^{\pi} u(D_\alpha u) dx - \frac{1}{3}\int_{-\pi}^{\pi} u^3 dx, \tag{2.12}$$

subject to the fixed momentum $P(u) = \frac{1}{2}\int_{-\pi}^{\pi} u^2 dx$ and mass $M(u) = \int_{-\pi}^{\pi} u dx$.

Proof. The Euler–Lagrange equation associated with the action functional

$$\Lambda_{c,b}(u) := E(u) + cP(u) + bM(u) \tag{2.13}$$

is given by the equation

$$(c - D_\alpha)\psi - \psi^2 + b = 0, \tag{2.14}$$

which coincides with equation (1.2) for $b = 0$. With the transformation

$$\psi(x) = \frac{1}{2}(c - \sqrt{c^2 + 4b}) + \tilde{\psi}(x),$$

the Euler–Lagrange equation (2.14) transforms to the form

$$(\tilde{c} - D_\alpha)\tilde{\psi} - \tilde{\psi}^2 = 0,$$

with $\tilde{c} := \sqrt{c^2 + 4b}$. By expansion (2.2) in Theorem 2.1, we have the following Stokes expansion for the new speed:

$$\tilde{c} = 1 + c_2 a^2 + \mathcal{O}(a^4),$$

from which parameter a is defined in terms of (c, b) near $(1, 0)$ by

$$c_2 a^2 = c - 1 + 2b + \mathcal{O}((c-1)^2 + b^2). \tag{2.15}$$

A single-lobe periodic solution of the Euler–Lagrange equation (2.14) for (c, b) near $(1, 0)$ is defined by expansion (2.1) in Theorem 2.1 as follows:

$$\psi(x) = 1 - b + a\cos(x) + a^2(c_2 + \phi_2(x)) + \mathcal{O}(a^3 + b^2). \tag{2.16}$$

This single-lobe periodic solution is a critical point of the action functional $\Lambda_{c,b}(u)$ in (2.13); hence we denote it as $\psi_{c,b}(x)$. The Hessian operator of the action functional $\Lambda_{c,b}(u)$ at the critical point $u = \psi_{c,b}$ is given by $\Lambda''_{c,b}(\psi_{c,b}) = c - D_\alpha - 2\psi_{c,b}$, which coincides with $\mathcal{H}_{c,\alpha} = \tilde{\mathcal{H}}_{c,\alpha}$ for $b = 0$. By Lemma 2.2, $\sigma(\mathcal{H}_{c,\alpha})$ consists of one simple negative eigenvalue, a simple zero eigenvalue, and the rest of its spectrum is bounded away from zero. The zero eigenvalue persists with respect to b since $\Lambda''_{c,b}(\psi_{c,b})\psi'_{c,b}(x) = 0$ for every $b \in \mathbb{R}$. If b is small, the negative and positive eigenvalues of $\mathcal{H}_{c,\alpha}$ persist as negative and positive eigenvalues of $\Lambda''_{c,b}(\psi_{c,b})$.

When the critical point $\psi_{c,b}$ of $\Lambda_{c,b}(u)$ is considered as a critical point of $E(u)$ subject to the fixed $P(u)$ and $M(u)$, the space $L^2_{\rm per}(-\pi, \pi)$ is constrained by two orthogonality conditions,

$$\langle \psi_{c,b}, v \rangle = 0, \quad \langle 1, v \rangle = 0, \tag{2.17}$$

imposed on the perturbation $v \in H^\alpha_{\rm per}(-\pi, \pi)$ to the periodic wave $\psi_{c,b} \in H^\alpha_{\rm per}(-\pi, \pi)$. By Theorem 4.1 in [35], the number of negative eigenvalues of $\Lambda''_{c,b}(\psi_{c,b})$ in $L^2_{\rm per}(-\pi, \pi)$ is reduced under the constraints (2.17) by the number of negative eigenvalues of the matrix

$$\begin{bmatrix} \langle [\Lambda''_{c,b}(\psi_{c,b})]^{-1}\psi_{c,b}, \psi_{c,b}\rangle & \langle [\Lambda''_{c,b}(\psi_{c,b})]^{-1}1, \psi_{c,b}\rangle \\ \langle [\Lambda''_{c,b}(\psi_{c,b})]^{-1}\psi_{c,b}, 1\rangle & \langle [\Lambda''_{c,b}(\psi_{c,b})]^{-1}1, 1\rangle \end{bmatrix} = -\begin{bmatrix} \frac{\partial \mathcal{P}_{c,b}}{\partial c} & \frac{\partial \mathcal{P}_{c,b}}{\partial b} \\ \frac{\partial \mathcal{M}_{c,b}}{\partial c} & \frac{\partial \mathcal{M}_{c,b}}{\partial b} \end{bmatrix},$$

where we denote $\mathcal{P}_{c,b} := P(\psi_{c,b})$, $\mathcal{M}_{c,b} := M(\psi_{c,b})$ and have used the derivative equations

$$\Lambda''_{c,b}(\psi_{c,b})\partial_c\psi_{c,b} = -\psi_{c,b}, \quad \Lambda''_{c,b}(\psi_{c,b})\partial_b\psi_{c,b} = -1.$$

Thanks to the expansion (2.16), for (c, b) near $(1, 0)$ we compute

$$\mathcal{P}_{c,b} = \pi\left[1 - 2b + a^2\left(2c_2 - \frac{1}{2}\right) + \mathcal{O}(a^4 + b^2)\right]$$

and

$$\mathcal{M}_{c,b} = 2\pi\left[1 - b + a^2\left(c_2 - \frac{1}{2}\right) + \mathcal{O}(a^4 + b^2)\right],$$

from which we obtain

$$\left[\begin{array}{cc} \frac{\partial \mathcal{P}_{c,b}}{\partial c} & \frac{\partial \mathcal{P}_{c,b}}{\partial b} \\ \frac{\partial \mathcal{M}_{c,b}}{\partial c} & \frac{\partial \mathcal{M}_{c,b}}{\partial b} \end{array}\right] = \frac{\pi}{c_2}\left[\begin{array}{cc} \frac{3}{2} - \frac{1}{2^\alpha - 1} & 1 - \frac{1}{2^\alpha - 1} \\ 1 - \frac{1}{2^\alpha - 1} & -\frac{1}{2^\alpha - 1} \end{array}\right],$$

where the chain rule with the expansion (2.15) has been used. Since the determinant of this matrix is negative and equal to $-\frac{\pi^2}{c_2}$, there exist exactly one positive and one negative eigenvalue. Hence, by Theorem 4.1 in [35], the number of negative eigenvalues of $\Lambda''_{c,b}(\psi_{c,b})$ in $L^2_{\rm per}(-\pi,\pi)$ is reduced under the constraints (2.17) by one. This implies that $\psi_{c,b}$ is a local constrained minimizer of $E(u)$ subject to the fixed $P(u)$ and $M(u)$ for (c,b) near $(1,0)$. When $b=0$, $\psi_{c,b=0}$ coincides with $\psi = c_{a,\alpha} + \phi_{a,\alpha}$ in the assertion of the lemma. ∎

Remark 2.5. Another periodic wave solution of the boundary-value problem (1.2) is the constant wave $\psi_c(x) = c$. It is also a critical point of the action functional $\Lambda_{c,b=0}(u)$ with the Hessian operator $\Lambda''_{c,b=0}(\psi_c) = -c - D_\alpha = \mathcal{L}_{c,\alpha}$. Thanks to the Fourier basis in (2.10), $\Lambda''_{c,b=0}(\psi_c)$ has only one simple negative eigenvalue for $c \in (0,1)$ and three or more negative eigenvalues for $c > 1$. The constraints of fixed momentum $P(u)$ and fixed mass $M(u)$ impose only one orthogonality condition $\langle 1, v\rangle = 0$ since $\psi_c(x) = c$. Computing $\mathcal{P}_c := P(\psi_c) = \pi c^2$ shows that the constraint removes exactly one negative eigenvalue of $\Lambda''_{c,b=0}(\psi_c)$. Hence, the constant wave ψ_c is a local constrained minimizer of $E(u)$ subject to fixed $P(u)$ and $M(u)$ for $c \in (0,1)$, but it is a saddle point of $E(u)$ for $c > 1$.

The following lemma gives the *isospectrality* result for the linearized operator $\mathcal{H}_{c,\alpha}$ for all $c > 1$.

LEMMA 2.4. *For every $c > 1$ and $\alpha \in (\alpha_0, 2]$, $\sigma(\mathcal{H}_{c,\alpha})$ in $L^2_{\rm per}(-\pi,\pi)$ consists of one simple negative eigenvalue, a simple zero eigenvalue, and a countable sequence of positive eigenvalues bounded away from zero.*

Proof. By Proposition 2.1 of [25] (corrected thanks to the results of Lemma 2.3 and Remark 2.5), the single-lobe solution ψ of the boundary-value problem (1.2) exists as a constrained minimizer of $E(u)$ for fixed $P(u)$ and $M(u)$ for every $c > 1$ and $\alpha \in (\alpha_0, 2]$, and the solution is a C^1 function of c for $c > 1$. The single-lobe solution ϕ of the boundary-value problem (1.3) is related to ψ by the transformation (1.4) so that $\mathcal{H}_{c,\alpha} = \tilde{\mathcal{H}}_{c,\alpha}$.

By Proposition 3.1 of [25], the kernel of $\mathcal{H}_{c,\alpha}$ at the single-lobe solution $\phi \in H^\alpha_{\rm per}(-\pi,\pi)$ is simple with $\ker(\mathcal{H}_{c,\alpha}) = {\rm span}\{\phi'\}$ for every $c > 1$ and $\alpha \in (\alpha_0, 2]$. The number of negative eigenvalues of $\mathcal{H}_{c,\alpha}$ may change in the parameter continuations in c if and only if the eigenvalues pass through zero. By Lemma 2.2, $\sigma(\mathcal{H}_{c,\alpha})$ at the single-lobe solution ϕ in $L^2_{\rm per}(-\pi,\pi)$ consists of one simple negative eigenvalue, a simple zero eigenvalue, and a countable sequence of positive eigenvalues bounded away from zero for $c > 1$ near $c = 1$ and $\alpha > \alpha_0$. By Lemma 2.3, the small-amplitude periodic wave in Theorem 2.1 coincides with the constrained minimizers of Proposition 2.1 of [25]. By the continuity argument and Proposition 3.1 of [25], the isospectrality of $\mathcal{H}_{c,\alpha}$ holds for every $c > 1$ and $\alpha \in (\alpha_0, 2]$. ∎

Remark 2.6. For the KdV case with $\alpha = 2$, a different homotopy argument for the proof of isospectrality of $\sigma(\mathcal{H}_{c,\alpha})$ can be developed (see, e.g., [26]), based on the classical results on the nondegeneracy of the energy-to-period function in [38] and [22]. For the BO case with $\alpha = 1$, explicit computations based on complex analysis techniques were developed much earlier in [5].

2.2. Periodic waves in the KdV equation. For the KdV equation (see, e.g., Proposition 4.1 in [23]), the solution ϕ to the boundary-value problem (1.3) with $\alpha = 2$ is given by

$$\phi(x) = \frac{2K(k)^2}{\pi^2}\left[1 - 2k^2 - \sqrt{1-k^2+k^4} + 3k^2 \mathrm{cn}^2\left(\frac{K(k)}{\pi}x; k\right)\right], \tag{2.18}$$

where cn is the Jacobi elliptic function, $K(k)$ is a complete elliptic integral of the first kind, and $k \in (0,1)$ is the elliptic modulus that parameterizes the wave speed c by

$$c = \frac{4K(k)^2}{\pi^2}\sqrt{1-k^2+k^4}. \tag{2.19}$$

The small-amplitude expansions (2.1)–(2.2) are recovered from (2.18)–(2.19) with the wave amplitude $a := 3k^2/4 + \mathcal{O}(k^4)$ as $k \to 0$.

We prove that the map $(0,1) \ni k \mapsto c \in (1,\infty)$ is strictly increasing; hence the explicit solution (2.18)–(2.19) exists for every $c > 1$ (see also [6]). We also extend the inequalities (2.8) and (2.9) with $\alpha = 2$ for every $c > 1$.

LEMMA 2.5. *The map $(0,1) \ni k \mapsto c \in (1,\infty)$ for the solution (2.18)–(2.19) is strictly increasing. In addition, for every $c > 1$, we have*

$$\int_{-\pi}^{\pi} \phi^3 dx < 0, \quad \int_{-\pi}^{\pi} \phi(\phi')^2 dx < 0. \tag{2.20}$$

Proof. We have $\phi = 0$ and $c = 1$ at $k = 0$. Thanks to the smoothness of ϕ and c in k, it holds from (2.19) by explicit differentiation that

$$\frac{\pi^2\sqrt{1-k^2+k^4}}{4K(k)}\frac{dc}{dk} = 2(1-k^2+k^4)\frac{dK(k)}{dk} - k(1-2k^2)K(k).$$

By using the differential relation,

$$\frac{dK(k)}{dk} = \frac{E(k) - (1-k^2)K(k)}{k(1-k^2)},$$

the previous expression can be reduced to the form

$$\frac{\pi^2 k(1-k^2)\sqrt{1-k^2+k^4}}{4K(k)}\frac{dc}{dk} = 2(1-k^2+k^4)E(k) - (2-3k^2+k^4)K(k) =: I(k),$$

where $E(k)$ is a complete elliptic integral of the second kind and $I(k)$ is introduced for convenience. Note that $I(0) = 0$. We claim that the map $(0,1) \ni k \mapsto I$ is strictly increasing. Indeed, by using the differential relation

$$\frac{dE(k)}{dk} = \frac{E(k) - K(k)}{k},$$

we obtain after straightforward computations

$$\frac{dI(k)}{dk} = 5k\left[(1-k^2)K(k) - (1-2k^2)E(k)\right] > 0,$$

where the last inequality follows from the fact that $K(k) > E(k)$ for every $k \in (0,1)$. Since $I(0) = 0$, we have $I(k) > 0$ for every $k \in (0,1)$, which implies that $\frac{dc}{dk} > 0$ for every $k \in (0,1)$.

Let us now prove the inequalities (2.20) for every $c > 1$. Since ϕ and c are smooth in k, we differentiate the nonlinear equation in the boundary-value problem (1.3) with $\alpha = 2$ in k and obtain

$$[c + D_{\alpha=2} + 2\phi]\,\frac{\partial \phi}{\partial k} + \frac{dc}{dk}\phi = 0.$$

Multiplying this equation by ϕ and integrating on $[-\pi, \pi]$ imply that

$$\int_{-\pi}^{\pi} \phi^2 \frac{\partial \phi}{\partial k} dx = -\frac{dc}{dk}\int_{-\pi}^{\pi} \phi^2 dx,$$

where we have used the facts that $D_{\alpha=2}$ is self-adjoint in $L^2_{\rm per}(-\pi,\pi)$ and $\phi, \partial_a \phi \in H^{\alpha=2}_{\rm per}(-\pi,\pi)$. Since $\frac{dc}{dk} > 0$ for every $k \in (0,1)$, the map $k \mapsto \int_{-\pi}^{\pi}\phi^3 dx$ is strictly decreasing with $\int_{-\pi}^{\pi}\phi^3 dx = 0$ at $k = 0$. Therefore, $\int_{-\pi}^{\pi}\phi^3 dx < 0$ for $k \in (0,1)$ by the continuity argument in k.

Finally, the inequality $\int_{-\pi}^{\pi}\phi(\phi')^2 dx < 0$ for every $c > 1$ follows from the boundary-value problem (1.3) with $\alpha = 2$:

$$\int_{-\pi}^{\pi} \phi(\phi')^2 dx = -\frac{1}{c}\left[\int_{-\pi}^{\pi}(\phi')^2\phi'' dx + \int_{-\pi}^{\pi}\phi^2(\phi')^2 dx\right],$$

where the first term on the right-hand side is zero, thanks to the smoothness of ϕ. ∎

2.3. Periodic waves in the BO equation. For the BO equation (see, e.g., [32]), the solution ϕ to the boundary-value problem (1.3) with $\alpha = 1$ is given by

$$\phi(x) = \frac{\cosh\gamma\cos x - 1}{\sinh\gamma(\cosh\gamma - \cos x)}, \quad c = \coth\gamma. \tag{2.21}$$

The small-amplitude expansions (2.1)–(2.2) are recovered from (2.21) with the wave amplitude $a := 2e^{-\gamma} + \mathcal{O}(e^{-3\gamma})$ as $\gamma \to \infty$. It follows from the simple expression $c = \coth\gamma$ that the map $(0,\infty) \ni \gamma \mapsto c \in (1,\infty)$ is strictly decreasing, and hence the explicit solution (2.21) exists for every $c > 1$.

Let us now show that the inequalities (2.8) and (2.9) with $\alpha = 1$ hold for every $c > 1$, that is,

$$\int_{-\pi}^{\pi}\phi^3 dx < 0, \quad \int_{-\pi}^{\pi}\phi(\phi')^2 dx > 0. \tag{2.22}$$

Indeed, by using the explicit formula (2.21) and symbolic computations with Wolfram's *Mathematica*, we obtain

$$\int_{-\pi}^{\pi}\phi^3 dx = -\pi(c-1)^2(2c+1)$$

and

$$\int_{-\pi}^{\pi}\phi(\phi')^2 dx = \frac{1}{4}\pi(c^2-1)^2,$$

from which the inequalities (2.22) hold for every $c > 1$.

2.4. Positivity of the periodic waves. The following result states that the single-lobe wave profile ψ in the boundary-value problem (1.2) for every $c > 1$ and $\alpha \in (\alpha_0, 2]$ is positive and satisfies $\psi(x) \geq \psi(\pm\pi) > 0$ for every $x \in [-\pi, \pi]$. The result has not appeared in the literature; e.g., a remark in the proof of Proposition 2.1 in [25] states that a periodic solution need not be positive everywhere. On the other hand, positivity of the Fourier coefficients in the Fourier series for the periodic wave ψ is proven in Theorem 3.5 of [10] for every $\alpha > 1/2$ and for sufficiently large periods (which is equivalent to $c > 1$ at the 2π-period).

Our proof has similarities to the work of [39] on the second-order differential equations. However, the existence of constant solutions is eliminated in [39] by the space-dependent coefficients in the boundary-value problem. For the problem (1.2), we have to use the Leray–Schauder index to single out single-lobe periodic solutions from the constant solutions.

THEOREM 2.2. *For every $c > 1$ and $\alpha \in (\alpha_0, 2]$, there exists a single-lobe solution ψ of the boundary-value problem* (1.2) *such that $\psi(x) > 0$ for every $x \in [-\pi, \pi]$.*

Proof. For $c \in (1, c_0)$ with some $c_0 > 1$, the assertion follows from Theorem 2.1 thanks to the transformation (1.4) and smallness of a in the Stokes expansion (2.1). In order to prove the same for every $c > 1$, we use Krasnoselskii's fixed-point theorem in a positive cone and a homotopy argument with the Leray–Schauder index to trace a branch of the single-lobe positive periodic solution in c. The proof is divided into five steps.

Step 1. The Green function for $(c - D_\alpha)$. Let $G_{c,\alpha} \in L^2_{\text{per}}(-\pi, \pi)$ denote the Green function for the positive operator $(c - D_\alpha)$. It is obtained from the solution

$$\varphi(x) = \int_{-\pi}^{\pi} G_{c,\alpha}(x - s) h(s) ds$$

of the following linear inhomogeneous equation:

$$(c - D_\alpha)\varphi = h, \quad h \in L^2_{\text{per}}(-\pi, \pi). \tag{2.23}$$

By Fourier series, the solution for G is available in Fourier series form,

$$G_{c,\alpha}(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{e^{inx}}{c + |n|^\alpha}, \tag{2.24}$$

from which it follows that $G_{c,\alpha} \in L^2_{\text{per}}(-\pi, \pi)$ if $\alpha > 1/2$ but $G_{c,\alpha}(0) = \infty$ if $\alpha \leq 1$. It is proven in [34] for $\alpha \in (0, 1)$ (and the proof is extended for $\alpha \in [1, 2]$; see [7]) that there is a positive (c, α)-dependent constant $m_{c,\alpha}$ such that

$$G_{c,\alpha}(x) \geq m_{c,\alpha}, \quad x \in [-\pi, \pi]. \tag{2.25}$$

In addition, for $\alpha > 1/2$, there exists a positive (c, α)-dependent constant $M_{c,\alpha}$ such that

$$\|G_{c,\alpha}\|_{L^2_{\text{per}}} \leq M_{c,\alpha}. \tag{2.26}$$

Step 2. Nonlinear operator $A_{c,\alpha}$ in a positive cone $P_{c,\alpha}$. Let us consider a positive cone in the space of $L^2_{\text{per}}(-\pi, \pi)$-functions defined by

$$P_{c,\alpha} := \left\{ \psi \in L^2_{\text{per}}(-\pi, \pi) : \quad \psi(x) \geq \frac{m_{c,\alpha}}{M_{c,\alpha}} \|\psi\|_{L^2_{\text{per}}}, \;\; x \in [-\pi, \pi] \right\}. \tag{2.27}$$

Define the following nonlinear operator $A_{c,\alpha}(\psi) : L^2_{\rm per}(-\pi,\pi) \mapsto L^2_{\rm per}(-\pi,\pi)$ for any $c > 0$:

$$A_{c,\alpha}(\psi) := (c - D_\alpha)^{-1}\psi^2 \quad \Rightarrow \quad A_{c,\alpha}(\psi)(x) = \int_{-\pi}^{\pi} G_{c,\alpha}(x-s)\psi(s)^2 ds. \tag{2.28}$$

The operator $A_{c,\alpha}$ is bounded and continuous in $L^2_{\rm per}(-\pi,\pi)$ thanks to the generalized Young inequality and the bound (2.26):

$$\|A_{c,\alpha}(\psi)\|_{L^2_{\rm per}} \leq \|G_{c,\alpha}\|_{L^2_{\rm per}} \|\psi^2\|_{L^1_{\rm per}} \leq M_{c,\alpha}\|\psi\|^2_{L^2_{\rm per}}. \tag{2.29}$$

Moreover, $A_{c,\alpha}$ is compact because it is the limit of compact operators $A^{(N)}_{c,\alpha}$ given by the first $2N+1$ Fourier coefficients. Indeed, we have

$$\begin{aligned}\|A_{c,\alpha}(\psi) - A^{(N)}_{c,\alpha}(\psi)\|^2_{L^2_{\rm per}} &= \frac{1}{2\pi}\sum_{|n|>N} \frac{|(\psi^2)_n|^2}{(c+|n|^\alpha)^2} \leq \frac{1}{2\pi}\|(\psi^2)_n\|^2_{\ell^\infty} \sum_{|n|>N}\frac{1}{(c+|n|^\alpha)^2} \\ &\leq \frac{1}{2\pi}\|\psi^2\|^2_{L^1_{\rm per}} \sum_{|n|>N}\frac{1}{(c+|n|^\alpha)^2} = \frac{1}{2\pi}\|\psi\|^4_{L^2_{\rm per}}\sum_{|n|>N}\frac{1}{(c+|n|^\alpha)^2},\end{aligned}$$

where the numerical series converges for every $\alpha > 1/2$. Therefore, for every $\psi \in L^2_{\rm per}(-\pi,\pi)$,

$$\lim_{N\to\infty} \|A_{c,\alpha}(\psi) - A^{(N)}_{c,\alpha}(\psi)\|_{L^2_{\rm per}} = 0,$$

so that $A_{c,\alpha}$ maps bounded sets in $L^2_{\rm per}(-\pi,\pi)$ to precompact sets in $L^2_{\rm per}(-\pi,\pi)$. By using positivity of the Green function in (2.25), we confirm that the operator $A_{c,\alpha}(\psi)$ is closed in $P_{c,\alpha} \subset L^2_{\rm per}(-\pi,\pi)$:

$$A_{c,\alpha}(\psi)(x) \geq m_{c,\alpha}\|\psi\|^2_{L^2_{\rm per}} \geq \frac{m_{c,\alpha}}{M_{c,\alpha}}\|A_{c,\alpha}(\psi)\|_{L^2_{\rm per}}. \tag{2.30}$$

A fixed point ψ of $A_{c,\alpha}(\psi)$ in $P_{c,\alpha} \subset L^2_{\rm per}(-\pi,\pi)$ corresponds to the positive function ψ such that $\psi(x) > 0$ for every $x \in [-\pi,\pi]$.

Step 3. Existence of a fixed point in the positive cone $P_{c,\alpha}$. Let

$$B_r := \{\psi \in L^2_{\rm per}(-\pi,\pi) : \ \ \|\psi\|_{L^2_{\rm per}} < r\}$$

be a ball of radius r in $L^2_{\rm per}(-\pi,\pi)$. The existence of a fixed point of $A_{c,\alpha}(\psi)$ in $P_{c,\alpha} \cap (\bar{B}_{r_+}\backslash B_{r_-})$ for some $0 < r_- < r_+ < \infty$ follows from Krasnoselskii's fixed-point theorem (see, e.g., Corollary 20.1 in [16]) if there exist r_- and r_+ such that

$$\|A_{c,\alpha}(\psi)\|_{L^2_{\rm per}} < \|\psi\|_{L^2_{\rm per}}, \quad \psi \in P_{c,\alpha} \cap \partial B_{r_-}, \tag{2.31}$$

and

$$\|A_{c,\alpha}(\psi)\|_{L^2_{\rm per}} > \|\psi\|_{L^2_{\rm per}}, \quad \psi \in P_{c,\alpha} \cap \partial B_{r_+}. \tag{2.32}$$

Bound (2.31) follows from (2.29) with $M_{c,\alpha} r_- < 1$. Bound (2.32) follows from (2.30) with $\sqrt{2\pi} m_{c,\alpha} r_+ > 1$; hence the two radii satisfy the constraints

$$0 < r_- < \frac{1}{M_{c,\alpha}} \leq \frac{1}{\sqrt{2\pi}m_{c,\alpha}} < r_+ < \infty, \tag{2.33}$$

where $\sqrt{2\pi}m_{c,\alpha} \leq M_{c,\alpha}$ follows from (2.25). Hence, there exists a fixed point of $A_{c,\alpha}(\psi)$ in $P_{c,\alpha} \cap (\bar{B}_{r_+}\backslash B_{r_-})$.

Step 4. Regularity of the fixed point. We use bootstrapping arguments similar to those used in the proof of Proposition 2.1 in [25] and show that the fixed point of $A_{c,\alpha}$ in $L^2_{\rm per}(-\pi,\pi)$ belongs to $H^\alpha_{\rm per}(-\pi,\pi)$; thus ψ is a positive solution of the boundary-value problem (1.2). Indeed, if $\psi \in L^4_{\rm per}(-\pi,\pi)$, then $\psi \in H^\alpha_{\rm per}(-\pi,\pi)$ thanks to the estimate

$$\|D_\alpha \psi\|_{L^2_{\rm per}} = \|D_\alpha (c - D_\alpha)^{-1}\psi^2\|_{L^2_{\rm per}} \le \|\psi^2\|_{L^2_{\rm per}} = \|\psi\|^2_{L^4_{\rm per}}.$$

In order to show that $\psi \in L^4_{\rm per}(-\pi,\pi)$, we use the generalized Young and Hölder inequalities:

$$\|\psi\|_{L^r_{\rm per}} \le \|G\|_{L^p_{\rm per}} \|\psi^2\|_{L^q_{\rm per}}, \qquad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad p,q,r \ge 1, \tag{2.34}$$

$$\le \|G\|_{L^p_{\rm per}} \|\psi\|_{L^{sq}_{\rm per}} \|\psi\|_{L^{sq/(s-1)}_{\rm per}}, \quad s \ge 1. \tag{2.35}$$

By using the Hausdorff–Young inequality

$$\|G\|_{L^p_{\rm per}} \le C_p \|(c + |n|^\alpha)^{-1}\|_{\ell^{p/(p-1)}}, \quad p \ge 2,$$

we can see that $\|G\|_{L^p_{\rm per}} < \infty$ if $\alpha p/(p-1) > 1$. If $\alpha \ge 1$, then $G \in L^p_{\rm per}(-\pi,\pi)$ for every $p \in [2,\infty)$. Applying (2.34) with $r = p$ and $q = 1$, we have $\psi \in L^p_{\rm per}(-\pi,\pi)$ for every $p \in [2,\infty)$. If $\alpha \in (\alpha_0, 1)$, we set $p_0 = 1/(1-\alpha_0) > 2$ and obtain with the same argument that $G, \psi \in L^{p_0}_{\rm per}(-\pi,\pi)$. Then, using bound (2.35) with $sq = 2$ and $sq/(s-1) = p_0$, that is, with $s = 1 + 2/p_0$ and $q = 2p_0/(2+p_0)$, we obtain $\psi \in L^r_{\rm per}(-\pi,\pi)$ with $r = 2p_0/(4-p_0) > p_0$ (because $p_0 > 2$). Iterating bound (2.35) with $sq = 2$ and $sq/(s-1) = r$, we obtain a bigger value for $r = p_0/(3-p_0) > 2p_0/(4-p_0)$; hence by further iterations, we get $\psi \in L^p_{\rm per}(-\pi,\pi)$ for every $p \in [2,\infty)$ including $p = 4$.

Step 5. Leray–Schauder index along distinct branches of fixed points. The fixed point $\psi \in P_{c,\alpha} \cap (\bar{B}_{r_+} \backslash B_{r_-})$ for $r_- < r_+$ satisfying (2.33) exists for every $c > 0$. However, the constant periodic solution

$$\psi_c(x) = c, \quad x \in [-\pi,\pi], \tag{2.36}$$

is a fixed point of $A_{c,\alpha}$ in $P_{c,\alpha} \cap (\bar{B}_{r_+} \backslash B_{r_-})$ for every $c > 0$ and $\alpha > 0$. Indeed, $A_{c,\alpha}(\psi_c) = \psi_c$ for every $\alpha > 0$ and $\psi_c \in P_{c,\alpha} \cap (\bar{B}_{r_+} \backslash B_{r_-})$ for every $c > 0$ thanks to the condition $\sqrt{2\pi} m_{c,\alpha} \le M_{c,\alpha}$. In order to be able to claim that there exists a nontrivial fixed point $\psi \in P_{c,\alpha} \cap (\bar{B}_{r_+} \backslash B_{r_-})$ for $c > 1$ in addition to the constant fixed point ψ_c, we look at the Leray–Schauder index of the fixed point in the subspace of even functions in $L^2_{\rm per}(-\pi,\pi)$, defined as $(-1)^N$, where N is the number of unstable eigenvalues of $A'_{c,\alpha}(\psi)$ outside the unit disk.

For the fixed point ψ_c in (2.36), we have $A'_{c,\alpha}(\psi_c) = 2c(c - D_\alpha)^{-1}$; hence there exist $N = K + 1$ unstable eigenvalues of $A'_{c,\alpha}(\psi_c)$ outside the unit disk for every $c \in (K^\alpha, (K+1)^\alpha)$, where $K \in \mathbb{N}$. Therefore, the index of ψ_c changes sign every time c crosses values in the set $\{K^\alpha\}_{K\in\mathbb{N}}$, as shown in Figure 1. On the other hand, for $K = 1$, $c = 1$ is a bifurcation value by Theorem 2.1, and two nontrivial fixed points $\psi \in P_{c,\alpha} \cap (\bar{B}_{r_+} \backslash B_{r_-})$ bifurcate for $c > 1$ if $\alpha > \alpha_0$, one being single-lobe with maximum at $x = 0$ and the other with minimum at $x = 0$. For the nontrivial fixed points ψ, we have

$$A'_{c,\alpha}(\psi) = 2(c - D_\alpha)^{-1}\psi = \mathrm{Id} - (c - D_\alpha)^{-1}\tilde{\mathcal{H}}_{c,\alpha},$$

where it follows from positivity of ψ that $A'_{c,\alpha}(\psi) \geq 0$. By Lemma 2.4 for $c > 1$ and $\alpha \in (\alpha_0, 2]$, $\tilde{\mathcal{H}}_{c,\alpha} = \mathcal{H}_{c,\alpha}$ has only one simple negative eigenvalue, and hence there exists $N = 1$ unstable eigenvalue of $A'_{c,\alpha}(\psi)$. Therefore, the pair of nontrivial fixed points $\psi \in P_{c,\alpha} \cap (\bar{B}_{r_+} \backslash B_{r_-})$ is distinct from the constant fixed point ψ_c for every $c > 1$, as shown in Figure 1.

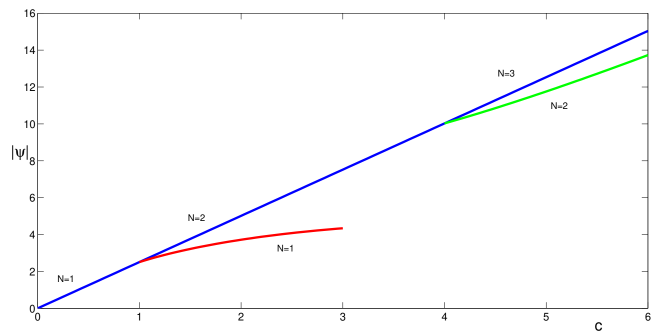


FIG. 1. *Schematic representation of the constant fixed point ψ_c and pairs of nontrivial fixed points on the $(c, \|\psi\|_{L^2_{\rm per}})$ plane for $\alpha = 2$.*

The pair of nontrivial fixed points for the single-lobe solution remains inside $P_{c,\alpha} \cap (\bar{B}_{r_+} \backslash B_{r_-})$ in continuation of the solution family in c for a fixed $\alpha \in (\alpha_0, 2]$ thanks to the conditions (2.31), (2.32), and (2.33). Their indices also remain invariant with respect to c thanks to Lemma 2.4. Therefore, these fixed points cannot coalesce with any other fixed points of $A_{c,\alpha}$ in $P_{c,\alpha} \cap (\bar{B}_{r_+} \backslash B_{r_-})$. By continuity, these fixed points coincide with the single-lobe solutions, existence of which is proven for every $c > 1$ in Proposition 2.1 in [25]. $\square$

Remark 2.7. At every bifurcation point $c = K^\alpha$ with $K \geq 2$, a pair of additional fixed points of $A_{c,\alpha}$ bifurcates in $P_{c,\alpha} \cap (\bar{B}_{r_+} \backslash B_{r_-})$, as shown in Figure 1 for $K = 2$ and $\alpha = 2$. These fixed points are not single-lobe solutions for $K \geq 2$, but instead these are concatenations of the single-lobe solutions with K periods on $[-\pi, \pi]$.

Remark 2.8. Theorem 4.1 in [6] states that $\tilde{\mathcal{H}}_{c,\alpha} = \mathcal{H}_{c,\alpha}$ in (1.13) has only one simple negative eigenvalue and a simple zero eigenvalue if ψ and its Fourier transform are strictly positive. These properties have been verified in [6] for the integrable cases $\alpha = 2$ and $\alpha = 1$, for which the exact solutions (2.37) and (2.38) are available. With Theorem 3.5 in [10] and Theorem 2.2 above, Theorem 4.1 in [6] can be applied to the periodic waves for every $c > 1$ and $\alpha \in (\alpha_0, 2]$. This argument gives an alternative proof of Lemma 2.4.

Let us illustrate positivity of ψ for the classical cases $\alpha = 2$ and $\alpha = 1$. For the KdV equation with the solution (2.18) and (2.19), we use $\psi(x) = c + \phi(x)$ and obtain

$$\psi(x) = \frac{2K(k)^2}{\pi^2}\left[1 - 2k^2 + \sqrt{1 - k^2 + k^4} + 3k^2 \mathrm{cn}^2\left(\frac{K(k)}{\pi}x; k\right)\right], \tag{2.37}$$

from which $\psi(x) \geq \psi(\pm\pi) > 0$ holds for every $x \in [-\pi, \pi]$ and every $k \in (0, 1)$. Indeed, if $\alpha = 2$, the boundary-value problem (1.2) can be formulated as a planar Hamiltonian system on the phase plane (ψ, ψ'), and a set of closed orbits for periodic solutions is located on the phase plane inside a positive homoclinic orbit to the saddle point $(0,0)$; hence $\psi(x) > 0$ for every $x \in [-\pi,\pi]$.

For the BO equation with the solution (2.21), we use $\psi(x) = c + \phi(x)$ and obtain

$$\psi(x) = \frac{\sinh\gamma}{\cosh\gamma - \cos x}, \tag{2.38}$$

from which $\psi(x) \geq \psi(\pm\pi) = \tanh(\gamma/2) > 0$ holds for every $x \in [-\pi,\pi]$ and every $\gamma \in (0,\infty)$.

3. Proof of Theorem 1.1. In what follows, we always use ϕ to denote the single-lobe periodic wave, which is even with a maximum at $x = 0$ and minimum at $x = \pm\pi$. We always assume that

$$\int_{-\pi}^{\pi} \phi^3 dx \neq 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \phi(\phi')^2 dx \neq 0. \tag{3.1}$$

Recall that although $\phi \in H^{\alpha}_{\rm per}(-\pi,\pi)$, it is extended to $\phi \in H^{\infty}_{\rm per}(-\pi,\pi)$ by bootstrapping arguments similar to those used in Step 4 in the proof of Theorem 2.2.

Linearizing $T_{c,\alpha}$ at ϕ with $w_n = \phi + \omega_n$, where $\omega_n \in H^{\alpha}_{\rm per}(-\pi,\pi)$, yields the linearized iterative rule

$$\omega_{n+1} = -\frac{2\langle \mathcal{L}_{c,\alpha}\phi, \omega_n\rangle}{\langle \mathcal{L}_{c,\alpha}\phi, \phi\rangle}\phi + \mathcal{L}_{c,\alpha}^{-1}(2\phi\omega_n), \quad n \in \mathbb{N}. \tag{3.2}$$

Since $\mathcal{L}_{c,\alpha}^{-1}(\phi^2) = \phi$ and $\mathcal{L}_{c,\alpha}^{-1}(2\phi\phi') = \phi'$, the linearized iterative rule (3.2) is invariant in the constrained space

$$L^2_c := \left\{\omega \in L^2_{\rm per}(-\pi,\pi) : \quad \langle \phi^2, \omega\rangle = \langle \phi\phi', \omega\rangle = 0\right\}. \tag{3.3}$$

To satisfy the two constraints, one can expand $\omega_n = a_n\phi + b_n\phi' + \beta_n$ with $\beta_n \in H^{\alpha}_{\rm per}(-\pi,\pi) \cap L^2_c$ and derive the following from (3.2):

$$a_{n+1} = 0, \quad b_{n+1} = b_n, \quad \beta_{n+1} = \mathcal{L}_T\beta_n, \tag{3.4}$$

where

$$\mathcal{L}_T := \mathcal{L}_{c,\alpha}^{-1}(2\phi\cdot) = \mathrm{Id} - \mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha} : \quad H^{\alpha}_{\rm per}(-\pi,\pi) \cap L^2_c \mapsto H^{\alpha}_{\rm per}(-\pi,\pi) \cap L^2_c \tag{3.5}$$

is the linearized iterative operator with $\mathcal{H}_{c,\alpha}$ given by (1.8). The following two results provide sufficient conditions for divergence or convergence of the iterative method (1.6).

THEOREM 3.1. *Assume $\int_{-\pi}^{\pi}\phi^3 dx \neq 0$. There exists $w_0 \in H^{\alpha}_{\rm per}(-\pi,\pi)$ near $\phi \in H^{\alpha}_{\rm per}(-\pi,\pi)$ such that the iterative method* (1.6) *diverges from ϕ if $\sigma(\mathcal{L}_T)$ in L^2_c includes at least one eigenvalue outside the unit disk.*

Proof. If $\sigma(\mathcal{L}_T)$ in L^2_c admits at least one eigenvalue outside the unit disk, the corresponding eigenfunction of $\mathcal{L}_T$ defines a direction in $H^{\alpha}_{\rm per}(-\pi,\pi)$ along which the sequence $\{w_n\}_{n\in\mathbb{N}}$ diverges from the fixed point ϕ, as follows from the unstable manifold theorem. ∎

THEOREM 3.2. *Assume $\int_{-\pi}^{\pi}\phi^3 dx \neq 0$ and $\int_{-\pi}^{\pi}\phi(\phi')^2 dx \neq 0$. There exists a small $\epsilon_0 > 0$ such that for every $w_0 \in H^{\alpha}_{\rm per}(-\pi,\pi)$ satisfying*

$$\epsilon := \|w_0 - \phi\|_{H^{\alpha}_{\rm per}} \leq \epsilon_0, \tag{3.6}$$

there exist b_* satisfying $|b_*| \leq C\epsilon$ for some ϵ-independent $C > 0$ such that the iterative method (1.6) *converges to $\phi(\cdot - b_*)$ if $\sigma(\mathcal{L}_T)$ in L^2_c is located inside the unit disk.*

Proof. Let us first assume that $w_0 \in H^\alpha_{\rm per}(-\pi,\pi)$ is even, in which case the assertion is true with $b_* = 0$. Since $\mathcal{L}_{c,\alpha}$ maps even functions to even functions, the sequence of functions $\{w_n\}_{n\in\mathbb{N}}$ in $H^\alpha_{\rm per}(-\pi,\pi)$ generated by the iterative rule (1.6) is even. Therefore, the linearization $w_n = \phi + \omega_n$ and the decomposition $\omega_n = a_n\phi + b_n\phi' + \beta_n$ yield $b_n = 0$ for every $n \geq 0$. The linear iterative rule (3.4) yields $a_n = 0$ for every $n \geq 1$ even if $a_0 \neq 0$. The linearized operator $\mathcal{L}_T$ given by (3.5) is a strict contraction if $\sigma(\mathcal{L}_T)$ in L^2_c is located inside the unit disk. Convergence of the sequence to ϕ follows by Banach's fixed-point theorem (Theorem 1.A in [41]).

Let us now relax the condition that the initial guess $w_0 \in H^\alpha_{\rm per}(-\pi,\pi)$ is even. In order to control the projection b_n in the decomposition $\omega_n = a_n\phi + b_n\phi' + \beta_n$, we need to use tools of the modulation theory for periodic waves; see, e.g., section 6 in [21]. Instead of defining b_n by $\omega_n = a_n\phi + b_n\phi' + \beta_n$, we define $b_n \in \mathbb{R}$ by using the decomposition

$$w_n(x) = \phi(x-b_n) + \omega_n(x-b_n) \tag{3.7}$$

and the orthogonality condition

$$\langle \phi\phi', \omega_n\rangle = 0. \tag{3.8}$$

By a standard application of the implicit function theorem (see, e.g., Lemma 6.1 in [21]), for every $w_n \in H^\alpha_{\rm per}(-\pi,\pi)$ satisfying

$$\epsilon_n := \inf_{b\in[-\pi,\pi]} \|w_n - \phi(\cdot - b)\|_{H^\alpha_{\rm per}} \leq \epsilon_0, \tag{3.9}$$

the decomposition (3.7)–(3.8) is unique under the assumption $\int_{-\pi}^{\pi} \phi(\phi')^2 dx \neq 0$ with uniquely defined b_n near the argument of the infimum in (3.9) and uniquely defined ω_n satisfying

$$\|\omega_n\|_{H^\alpha_{\rm per}} \leq C_0\epsilon_n \tag{3.10}$$

for some ϵ_n-independent constant $C_0 > 0$.

Substituting the decomposition (3.7) into the iterative method (1.6) and using the translational invariance in x, we obtain the equivalent iterative scheme:

$$\omega_{n+1} = \phi(\cdot + \Delta b_n) - \phi + T'(\phi(\cdot+\Delta b_n))\omega_n(\cdot+\Delta b_n) + N(\omega_n(\cdot+\Delta b_n)), \tag{3.11}$$

where $\Delta b_n := b_{n+1} - b_n$, $T'(\phi)\omega_n$ denotes the linearized iterative operator given by the right-hand side of (3.2), and $N(\omega_n)$ is the nonlinear terms satisfying

$$\|N(\omega_n)\|_{H^\alpha_{\rm per}} \leq C\|\omega_n\|^2_{H^\alpha_{\rm per}} \tag{3.12}$$

for every $\omega_n \in B_\rho(0) := \left\{\omega \in H^\alpha_{\rm per}(-\pi,\pi) : \quad \|\omega\|_{H^\alpha_{\rm per}} \leq \rho\right\}$, where the constant $C > 0$ does not depend on ρ, provided the radius ρ of the ball $B_\rho(0)$ is small. Thanks to (3.6) and (3.10), we work with $\rho = C\epsilon$ for some positive ϵ-independent constant C.

By using the constraint (3.8) both for ω_n and ω_{n+1}, we derive the following equation for Δb_n:

$$\begin{aligned} 0 = {} & \langle \phi\phi', \phi(\cdot+\Delta b_n) - \phi\rangle + \langle \phi\phi', T'(\phi(\cdot+\Delta b_n))\omega_n(\cdot+\Delta b_n)\rangle \\ & + \langle \phi\phi', N(\omega_n(\cdot+\Delta b_n))\rangle. \end{aligned} \tag{3.13}$$

This equation can be treated as the root-finding problem $F(\Delta b_n, \omega_n) = 0$, where

$$F : \mathbb{R} \times H^\alpha_{\rm per}(-\pi,\pi) \mapsto \mathbb{R}$$

is a smooth function in its variables satisfying $F(0,0) = 0$ and $\partial_{\Delta b_n} F(0,0) \neq 0$ thanks to the smoothness of $\phi \in H^\infty_{\rm per}(-\pi,\pi)$ and $N(\omega_n)$ as well as the assumption $\int_{-\pi}^{\pi} \phi(\phi')^2 dx \neq 0$. By the implicit function theorem, the root-finding problem (3.13) is uniquely solvable in Δb_n for every $\omega_n \in B_\rho(0)$ with small $\rho > 0$. Moreover, thanks to $\langle \phi\phi', T'(\phi)\omega_n\rangle = \langle \phi\phi', \omega_n\rangle = 0$ and (3.12), the uniquely found Δb_n satisfies the bound

$$|\Delta b_n| \leq C \|\omega_n\|^2_{H^\alpha_{\rm per}} \tag{3.14}$$

for some constant $C > 0$ that does not depend on the small radius ρ.

Substituting Δb_n satisfying (3.14) into (3.11) and decomposing $\omega_n = a_n\phi + \beta_n$ with $a_n \in \mathbb{R}$ and $\beta_n \in H^\alpha_{\rm per}(-\pi,\pi) \cap L^2_c$, we obtain the linearized problem

$$a_{n+1} = 0, \quad \beta_{n+1} = \mathcal{L}_T \beta_n. \tag{3.15}$$

Since $\mathcal{L}_T$ is a strict contraction in L^2_c, convergence $a_n \to 0$, $\Delta b_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$ follows by Banach's fixed-point theorem (Theorem 1.A in [41]). Moreover, these sequences converge exponentially fast so that the sequence $\{b_n\}_{n\in\mathbb{N}}$ converges to a limit denoted by b_*. Since $|b_* - b_0| \leq C\epsilon^2$ thanks to (3.10) and (3.14), whereas $|b_0| \leq C\epsilon$ thanks to (3.6), (3.9), and triangle inequality, we also have $|b_*| \leq C\epsilon$ for some ϵ-independent $C > 0$. The assertion is proven thanks to the decomposition (3.7) with $\omega_n = a_n\phi + \beta_n$. ∎

Remark 3.1. Compared to section 6 in [21], where standard orthogonality condition $\langle \phi', \omega\rangle = 0$ was used together with the energy conservation, we have to use the modified orthogonality condition $\langle \phi\phi', \omega\rangle = 0$ in order to comply with the iterative scheme (3.11), which results in the non-self-adjoint linearized operator $T'(\phi)\omega_n$ given by the right-hand side of (3.2).

In order to compute $\sigma(\mathcal{L}_T)$ in L^2_c used in Theorems 3.1 and 3.2, we study the spectrum of $\mathcal{L}^{-1}_{c,\alpha}\mathcal{H}_{c,\alpha}$ in $L^2_{\rm per}(-\pi,\pi)$. Analytical results on convergence of the method for $c \in (1, c_0)$ for some $c_0 \in (1, 2^\alpha)$ and divergence for $c > 2^\alpha$ are obtained in sections 3.1 and 3.2, respectively. These results give the proof of Theorem 1.1. Numerical results showing convergence or divergence of the method for c in $(1, 2^\alpha)$ are obtained in section 3.3 for $\alpha = 2$ and $\alpha = 1$.

3.1. Convergence results for $c \in (1, c_0)$. Here we prove that the iterative method converges near the single-lobe periodic wave ϕ in the small-amplitude limit if $\alpha > \alpha_1$ and diverges if $\alpha \in (\alpha_0, \alpha_1)$, where α_0 and α_1 are given by (1.9). Note that $\alpha_0 \approx 0.585$ and $\alpha_1 \approx 1.322$. The following lemma characterizes the spectrum of $\mathcal{L}^{-1}_{c,\alpha}\mathcal{H}_{c,\alpha}$ in $L^2_{\rm per}(-\pi,\pi)$.

LEMMA 3.1. *For every $\alpha > \alpha_0$ there exists $c_0 \in (1, 2^\alpha)$ such that for every $c \in (1, c_0)$, $\sigma(\mathcal{L}^{-1}_{c,\alpha}\mathcal{H}_{c,\alpha})$ in $L^2_{\rm per}(-\pi,\pi)$ consists of a countable sequence of eigenvalues in a neighborhood of* 1 *and simple eigenvalues $\{-1, 0, \lambda_1, \lambda_2\}$ with*

$$\lambda_1 \to \frac{2^{\alpha+1} - 5}{2^{\alpha+1} - 3} \quad \text{and} \quad \lambda_2 \to 2 \quad \text{as} \quad c \to 1.$$

Moreover, $\lambda_2 < 2$, whereas $\lambda_1 < 0$ if $\alpha \in (\alpha_0, \alpha_1)$ and $\lambda_1 \in (0, 1)$ if $\alpha > \alpha_1$.

Proof. It follows from (2.10) that for every $c \in (1, 2^\alpha)$, the operator $\mathcal{L}_{c,\alpha}$ in $L^2_{\rm per}(-\pi,\pi)$ is invertible and

$$\sigma(\mathcal{L}^{-1}_{c,\alpha}) = \{(-c + |n|^\alpha)^{-1}, \ \ n \in \mathbb{Z}\}.$$

Since the sequence of eigenvalues is squared summable if $\alpha > 1/2$, the linear bounded operator $\mathcal{L}_{c,\alpha}^{-1}$ is of the Hilbert–Schmidt class (see Example 2 in section 5.16 of [41]), and thus it is compact. The linear operator $\mathcal{L}_T$ in $L^2_{\rm per}(-\pi,\pi)$ is a composition of a bounded operator $2\phi\cdot$ and a compact (Hilbert–Schmidt) operator $\mathcal{L}_{c,\alpha}^{-1}$; hence $\mathcal{L}_T$ is a compact operator and $\sigma(\mathcal{L}_T)$ in $L^2_{\rm per}(-\pi,\pi)$ consists of a sequence of eigenvalues converging to 0. Thanks to the representation (3.5), $\sigma(\mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha})$ in $L^2_{\rm per}(-\pi,\pi)$ consists of a sequence of eigenvalues converging to 1.

Eigenvalues $\{-1,0\}$ of $\mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha}$ in $L^2_{\rm per}(-\pi,\pi)$ follow from exact computations for every $c>1$:

$$\mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha}\phi = -\phi \quad \text{and} \quad \mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha}\phi' = 0. \tag{3.16}$$

In order to identify other eigenvalues of $\mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha}$ in $L^2_{\rm per}(-\pi,\pi)$, we consider the generalized eigenvalue problem (1.7) defined by linear operators $\mathcal{L}_{c,\alpha}$ and $\mathcal{H}_{c,\alpha}$ in $L^2_{\rm per}(-\pi,\pi)$ with the domains in $H^\alpha_{\rm per}(-\pi,\pi)$.

Since $\mathcal{H}_{c=1,\alpha}$ coincides with $\mathcal{L}_{c=1,\alpha}$, the generalized eigenvalue problem (1.7) for $c=1$ admits only one solution $\lambda=1$ for every $v\in H^\alpha_{\rm per}(-\pi,\pi)\backslash\{e^{ix},e^{-ix}\}$. Since (ϕ,c) depend analytically on a in Theorem 2.1, by the analytic perturbation theory (Theorem VII.1.7 in [28]), the eigenvalues of $\mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha}$ in $L^2_{\rm per}(-\pi,\pi)$ for every $c>1$ are divided into two sets: a countable sequence of eigenvalues near 1 and converging to 1 related to the subspace $L^2_{\rm per}(-\pi,\pi)\backslash\{e^{ix},e^{-ix}\}$ and a finite number of eigenvalues related to the subspace $\{e^{ix},e^{-ix}\}$. The second set includes eigenvalues $\{-1,0\}$ due to the exact solutions (3.16) for every $c>1$. The subspace $\{e^{ix},e^{-ix}\}$ may be related to more than two simple eigenvalues in the generalized eigenvalue problem (1.7) because both $\mathcal{H}_{c=1,\alpha}$ and $\mathcal{L}_{c=1,\alpha}$ vanish on the subspace.

In order to study all possible eigenvalues of $\mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha}$ in $L^2_{\rm per}(-\pi,\pi)$ related to the subspace $\{e^{ix},e^{-ix}\}$, we perform perturbation expansions. Since $\mathcal{L}_{c,\alpha}$ and $\mathcal{H}_{c,\alpha}$ are closed in the subspaces of even and odd functions in $L^2_{\rm per}(-\pi,\pi)$, the generalized eigenvalue problem (1.7) can be uncoupled in these subspaces. By using (2.2) and (2.11), we rewrite the generalized eigenvalue problem (1.7) in the perturbed form

$$\begin{aligned}&(\lambda-1)\left[1+D_\alpha+c_2a^2+c_4a^4+\mathcal{O}(a^6)\right]v\\ &-2\left[a\cos(x)+a^2\phi_2(x)+a^3\phi_3(x)+a^4\phi_4(x)+\mathcal{O}(a^5)\right]v=0.\end{aligned} \tag{3.17}$$

Assuming $\lambda\neq 1$, we are looking for perturbative expansions of the eigenvalues related to the even and odd subspaces of $\{e^{ix},e^{-ix}\}$ separately from each other. For the even subspace, we set

$$v(x)=\cos(x)+av_1(x)+a^2v_2(x)+\mathcal{O}(a^3) \tag{3.18}$$

and obtain recursively

$$\left\{\begin{array}{ll}\mathcal{O}(a): & (\lambda-1)\,(1+D_\alpha)\,v_1 = 1+\cos(2x),\\ \mathcal{O}(a^2): & (\lambda-1)\,(1+D_\alpha)\,v_2+(\lambda-1)c_2\cos(x)=2\cos(x)(v_1+\phi_2).\end{array}\right.$$

At $\mathcal{O}(a)$, we obtain the exact solution in $H^\alpha_{\rm per}(-\pi,\pi)$:

$$v_1(x)=\frac{1}{\lambda-1}\left[1-\frac{\cos(2x)}{2^\alpha-1}\right]. \tag{3.19}$$

The linear inhomogeneous equation at $\mathcal{O}(a^2)$ admits a solution $v_2\in H^\alpha_{\rm per}(-\pi,\pi)$ if and only if λ satisfies

$$\left[\lambda-\frac{2}{\lambda-1}\right]c_2=0.$$

If $\alpha > \alpha_0$, then $c_2 \neq 0$ and λ satisfies the quadratic equation $\lambda(\lambda - 1) = 2$ with two roots $\{-1, 2\}$. For each of the two roots, we obtain the exact solution in $H^\alpha_{\rm per}(-\pi, \pi)$:

$$v_2(x) = \frac{(3 - \lambda)\cos(3x)}{2(\lambda - 1)^2(2^\alpha - 1)(3^\alpha - 1)}. \tag{3.20}$$

For the odd subspace, we set

$$v(x) = \sin(x) + av_1(x) + a^2 v_2(x) + \mathcal{O}(a^3) \tag{3.21}$$

and obtain recursively

$$\left\{ \begin{array}{ll} \mathcal{O}(a): & (\lambda - 1)(1 + D_\alpha) v_1 = \sin(2x), \\ \mathcal{O}(a^2): & (\lambda - 1)(1 + D_\alpha) v_2 + (\lambda - 1) c_2 \sin(x) = 2(\cos(x) v_1 + \sin(x)\phi_2). \end{array} \right.$$

At $\mathcal{O}(a)$, we obtain the exact solution in $H^\alpha_{\rm per}(-\pi, \pi)$:

$$v_1(x) = -\frac{\sin(2x)}{(\lambda - 1)(2^\alpha - 1)}. \tag{3.22}$$

The linear inhomogeneous equation at $\mathcal{O}(a^2)$ admits a solution $v_2 \in H^\alpha_{\rm per}(-\pi, \pi)$ if and only if λ satisfies

$$\lambda c_2 + \frac{\lambda}{(\lambda - 1)(2^\alpha - 1)} = 0.$$

If $\alpha > \alpha_0$, then $c_2 \neq 0$ and λ satisfies the quadratic equation

$$\lambda\left[(2^{\alpha+1} - 3)\lambda - (2^{\alpha+1} - 5)\right] = 0$$

with two roots $\{0, \frac{2^{\alpha+1}-5}{2^{\alpha+1}-3}\}$. For each of the two roots, we obtain the exact solution in $H^\alpha_{\rm per}(-\pi, \pi)$:

$$v_2(x) = \frac{(3 - \lambda)\sin(3x)}{2(\lambda - 1)^2(2^\alpha - 1)(3^\alpha - 1)}. \tag{3.23}$$

Summarizing, we have obtained four eigenvalues related to the subspace $\{e^{ix}, e^{-ix}\}$, which are located as $c \to 1$ at the points $\{-1, 0, \frac{2^{\alpha+1}-5}{2^{\alpha+1}-3}, 2\}$.

The eigenvalues $\{-1, 0\}$ are preserved for every $c > 1$ thanks to the exact solution (3.16). However, the eigenvalues $\{\lambda_1, \lambda_2\}$ near $\{\frac{2^{\alpha+1}-5}{2^{\alpha+1}-3}, 2\}$ depend generally on c. It follows by the perturbation theory that $\lambda_1 < 0$ if $\alpha \in (\alpha_0, \alpha_1)$ and $\lambda_1 \in (0, 1)$ if $\alpha > \alpha_1$. We now claim that $\lambda_2 < 2$ if $\alpha > \alpha_0$ and $c > 1$. To prove this claim, we use the extended spectral problem (3.17) up to the order $\mathcal{O}(a^4)$. Hence, instead of the expansion (3.18) with (3.19) and (3.20), we use the expansions

$$\left\{ \begin{array}{l} v(x) = \cos(x) + av_1(x) + a^2 v_2(x) + a^3 v_3(x) + a^4 v_4(x) + \mathcal{O}(a^5), \\ \lambda = 2 + \Lambda_2 a^2 + \mathcal{O}(a^4), \end{array} \right. \tag{3.24}$$

where

$$v_1(x) = 1 - \frac{\cos(2x)}{2^\alpha - 1}, \quad v_2(x) = \frac{\cos(3x)}{2(2^\alpha - 1)(3^\alpha - 1)}.$$

We obtain from the extended spectral problem (3.17) the linear inhomogeneous equations

$$\left\{ \begin{array}{ll} \mathcal{O}(a^3): & (1 + D_\alpha)v_3 + \Lambda_2(1 + D_\alpha)v_1 + c_2 v_1 = 2\left[\cos(x)(v_2 + \phi_3) + \phi_2 v_1\right], \\ \mathcal{O}(a^4): & (1 + D_\alpha)v_4 + \Lambda_2(1 + D_\alpha)v_2 + c_2 v_2 + (c_4 + c_2\Lambda_2)\cos(x) \\ & \qquad = 2\left[\cos(x)(v_3 + \phi_4) + \phi_2 v_2 + \phi_3 v_1\right]. \end{array} \right.$$

The linear inhomogeneous equation at $\mathcal{O}(a^3)$ admits the explicit solution:

$$v_3(x) = \frac{3^\alpha - 2^{\alpha+1} + 1}{2(2^\alpha - 1)^2(3^\alpha - 1)(4^\alpha - 1)} \cos(4x) + \left[\frac{\Lambda_2}{2^\alpha - 1} - \frac{1 + (2 + c_2)(3^\alpha - 1)}{(2^\alpha - 1)^2(3^\alpha - 1)}\right] \cos(2x) - \left(\Lambda_2 + c_2 + 1 + \frac{1}{2(2^\alpha - 1)^2}\right).$$

The linear inhomogeneous equation at $\mathcal{O}(a^4)$ admits a solution $v_4 \in H^\alpha_{\text{per}}(-\pi, \pi)$ if and only if Λ_2 is given by

$$\Lambda_2 = -1 + \frac{3}{2^\alpha - 1} - \frac{7}{2^{\alpha+1} - 3}. \tag{3.25}$$

It is easy to see that Λ_2 has a vertical asymptote at $\alpha = \alpha_0$. By plotting Λ_2 versus α in Figure 2, we verify that $\Lambda_2 < 0$ for every $\alpha > \alpha_0$. Hence $\lambda_2 = 2 + \Lambda_2 a^2 + \mathcal{O}(a^4)$ satisfies $\lambda_2 < 2$. $\square$

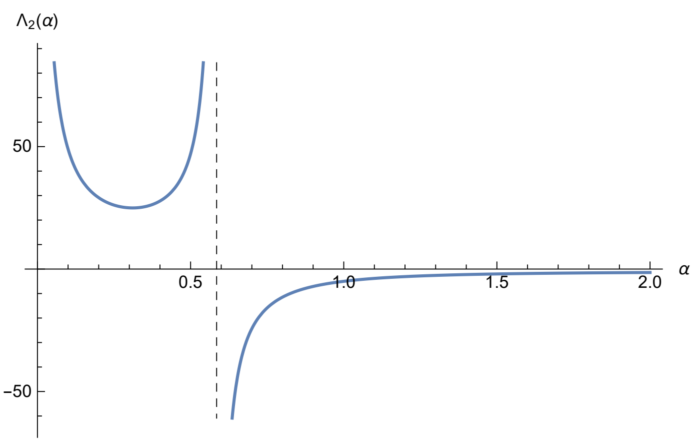


FIG. 2. *Plot of Λ_2 versus α.*

COROLLARY 3.1. *For every $c \in (1, c_0)$ in Lemma* 3.1, *the iterative method* (1.6) *converges to ϕ in $H^\alpha_{\text{per}}(-\pi, \pi)$ if $\alpha > \alpha_1$ and diverges from ϕ if $\alpha \in (\alpha_0, \alpha_1)$.*

Proof. Assumptions (3.1) used in Theorems 3.1 and 3.2 have been verified in Lemma 2.1.

If $\alpha > \alpha_1$, then $\lambda_1 \in (0, 1)$ by Lemma 3.1. By using the representation (3.5) and the count of eigenvalues of the generalized eigenvalue problem (1.7) in Lemma 3.1, we can see that $\sigma(\mathcal{L}_T)$ in $L^2_{\text{per}}(-\pi, \pi)$ consists of a countable sequence of eigenvalues in a neighborhood of 0 and converging to 0, two simple eigenvalues inside the interval $(-1, 1)$, and two additional simple eigenvalues: 1 related to the eigenfunction ϕ' and 2 related to the eigenfunction ϕ. The two constraints in (3.3) remove the latter two eigenvalues so that the operator $\mathcal{L}_T$ is a strict contraction in L^2_c for every $c \in (1, c_0)$ if $\alpha > \alpha_1$. Convergence of the iterative method (1.6) for $\alpha > \alpha_1$ follows by Theorem 3.2.

If $\alpha \in (\alpha_0, \alpha_1)$, then $\lambda_1 < 0$ by Lemma 3.1. Then $\sigma(\mathcal{L}_T)$ in $L^2_{\text{per}}(-\pi, \pi)$ consists of a countable sequence of eigenvalues in a neighborhood of 0 and converging to 0, one simple eigenvalue inside the interval $(-1, 1)$, simple eigenvalue 1 related to the eigenfunction ϕ', simple eigenvalue 2 related to the eigenfunction ϕ, and an additional simple eigenvalue bigger than 1 with an odd eigenfunction denoted by v_*. Because of the orthogonality conditions

$$\langle \mathcal{L}_{c,\alpha} v_j, v_k \rangle = 0, \quad j \neq k,$$

between eigenfunctions v_j and v_k of the generalized eigenvalue problem (1.7) for distinct eigenvalues, we verify that $\langle \phi^2, v_* \rangle = \langle \phi\phi', v_* \rangle = 0$, which implies that $v_* \in L^2_c$. Therefore, $\sigma(\mathcal{L}_T)$ in L^2_c contains exactly one eigenvalue outside the unit disk for every $c \in (1, c_0)$ if $\alpha \in (\alpha_0, \alpha_1)$. Divergence of the iterative method (1.6) for $\alpha \in (\alpha_0, \alpha_1)$ follows by Theorem 3.1. ∎

Remark 3.2. Since the unstable eigenfunction v_* is odd, divergence of the iterative method (1.6) for $\alpha \in (\alpha_0, \alpha_1)$ is only observed if the initial guess $w_0 \in H^\alpha_{\rm per}(-\pi, \pi)$ is not even but of a general form.

Although Theorem 1.1 follows already from Corollary 3.1, we would like to add more details to the eigenfunctions of the generalized eigenvalue problem (1.7).

The numbers of negative eigenvalues of operators $\mathcal{L}_{c,\alpha}$ and $\mathcal{H}_{c,\alpha}$ are affected by the constraint $\langle \phi^2, \alpha \rangle = 0$ in (3.3). As is well known (see, e.g., Theorem 4.1 in [35]), if n is the number of negative eigenvalues of a self-adjoint invertible operator $\mathcal{L}$ in a Hilbert space and if $\langle \mathcal{L}^{-1}\phi^2, \phi^2 \rangle < 0$, then the restriction of the self-adjoint operator to the constraint $\langle \phi^2, \alpha \rangle = 0$ has one less negative eigenvalue $n - 1$. We compute

$$\langle \mathcal{L}^{-1}_{c,\alpha}\phi^2, \phi^2 \rangle = \langle \phi, \phi^2 \rangle, \quad \langle \mathcal{H}^{-1}_{c,\alpha}\phi^2, \phi^2 \rangle = -\langle \phi, \phi^2 \rangle. \tag{3.26}$$

By Lemma 2.1, we have $\langle \phi, \phi^2 \rangle = \int_{-\pi}^{\pi} \phi(x)^3 dx < 0$ for every $c \in (1, c_0)$ if $\alpha > \alpha_0$. Therefore, $\mathcal{L}_{c,\alpha}$ restricted to L^2_c has one less negative eigenvalue compared to $\mathcal{L}_{c,\alpha}$ in $L^2_{\rm per}(-\pi, \pi)$, whereas $\mathcal{H}_{c,\alpha}$ restricted to L^2_c has still one simple negative eigenvalue. In the space of even functions, $\mathcal{L}_{c,\alpha}$ and $\mathcal{H}_{c,\alpha}$ restricted to L^2_c have only one simple negative eigenvalue. By Theorem 1 in [13], the generalized eigenvalue problem (1.7) admits one of the following in the subspace of even functions in $L^2_{\rm per}(-\pi, \pi)$:

- two simple negative eigenvalues λ with the two eigenfunctions v for which the sign of $\langle \mathcal{L}_{c,\alpha} v, v \rangle$ is opposite the sign of $\langle \mathcal{H}_{c,\alpha} v, v \rangle$;
- a simple positive eigenvalue λ with the eigenfunction v for which the signs of $\langle \mathcal{L}_{c,\alpha} v, v \rangle$ and $\langle \mathcal{H}_{c,\alpha} v, v \rangle$ are negative;
- a double defective real eigenvalue λ with only one eigenfunction v for which $\langle \mathcal{L}_{c,\alpha} v, v \rangle = \langle \mathcal{H}_{c,\alpha} v, v \rangle = 0$;
- a complex-conjugate pair of eigenvalues λ.

The following result shows that the second option from the list above is true if $\alpha > \alpha_1$.

LEMMA 3.2. *For every $\alpha > \alpha_1$ there exists $c_0 \in (1, 2^\alpha)$ such that for every $c \in (1, c_0)$, the generalized eigenvalue problem* (1.7) *admits*

- *a simple positive eigenvalue λ with the eigenfunction v for which the signs of $\langle \mathcal{L}_{c,\alpha} v, v \rangle$ and $\langle \mathcal{H}_{c,\alpha} v, v \rangle$ are negative;*
- *a simple negative eigenvalue λ with the eigenfunction v for which the sign of $\langle \mathcal{L}_{c,\alpha} v, v \rangle$ is negative;*
- *a simple zero eigenvalue with the eigenfunction v for which the sign of $\langle \mathcal{L}_{c,\alpha} v, v \rangle$ is negative.*

The eigenfunction v for all other eigenvalues corresponds to positive values of $\langle \mathcal{L}_{c,\alpha} v, v \rangle$ and $\langle \mathcal{H}_{c,\alpha} v, v \rangle$.

Proof. We utilize the perturbative expansions in the proof of Lemma 3.1. For the even expansion (3.18), we obtain

$$\mathcal{L}_{c,\alpha} v = -\frac{a}{\lambda-1}\left[1+\cos(2x)\right] + \frac{(3-\lambda)a^2}{2(2^\alpha-1)(\lambda-1)^2}\cos(3x) - \left(1-\frac{1}{2(2^\alpha-1)}\right) a^2 \cos(x) + \mathcal{O}(a^3).$$

By evaluating elementary integrals, we obtain

$$\langle \mathcal{L}_{c,\alpha} v, v\rangle = -\frac{\pi a^2 (2^{\alpha+1}-3)}{2(\lambda-1)^2(2^\alpha-1)}\left[2+(\lambda-1)^2\right] + \mathcal{O}(a^3).$$

If $\alpha > \alpha_0$, then $\langle \mathcal{L}_{c,\alpha} v, v\rangle$ is negative for both roots $\{-1, 2\}$ of the quadratic equation $\lambda(\lambda-1) = 2$. Since $\langle \mathcal{H}_{c,\alpha} v, v\rangle = \lambda \langle \mathcal{L}_{c,\alpha} v, v\rangle$, the eigenvalue at $\lambda = 2 + \mathcal{O}(a^2)$ corresponds to the simple positive eigenvalue, for which both quadratic forms are negative, whereas the eigenvalue $\lambda = -1$ corresponds to the negative eigenvalue, for which only $\langle \mathcal{L}_{c,\alpha} v, v\rangle$ is negative and $\langle \mathcal{H}_{c,\alpha} v, v\rangle$ is positive.

For the odd expansion (3.21), we obtain

$$\mathcal{L}_{c,\alpha} v = -\frac{a}{\lambda-1}\sin(2x) + \frac{(3-\lambda)a^2}{2(2^\alpha-1)(\lambda-1)^2}\sin(3x) - \left(1-\frac{1}{2(2^\alpha-1)}\right) a^2 \sin(x) + \mathcal{O}(a^3).$$

By evaluating elementary integrals, we obtain

$$\langle \mathcal{L}_{c,\alpha} v, v\rangle = \frac{\pi a^2}{2(2^\alpha-1)(\lambda-1)^2}\left[2-(2^{\alpha+1}-3)(\lambda-1)^2\right] + \mathcal{O}(a^3).$$

The sign of $\langle \mathcal{L}_{c,\alpha} v, v\rangle$ depends on the value of α for the two roots $\{0, \frac{2^{\alpha+1}-5}{2^{\alpha+1}-3}\}$ of the quadratic equation $\lambda[(2^{\alpha+1}-3)\lambda + (5-2^{\alpha+1})] = 0$. If $\alpha > \alpha_1$, then $\langle \mathcal{L}_{c,\alpha} v, v\rangle$ is negative for the eigenvalue $\lambda = 0$, for which $\langle \mathcal{H}_{c,\alpha} v, v\rangle$ is zero, and positive for the eigenvalue $\lambda = \frac{2^{\alpha+1}-5}{2^{\alpha+1}-3} + \mathcal{O}(a^2)$, for which $\langle \mathcal{H}_{c,\alpha} v, v\rangle$ is also positive.

Every other eigenvalue bifurcating from $\lambda = 1$ corresponds to the positive eigenvalues, for which both quadratic forms $\langle \mathcal{L}_{c,\alpha} v, v\rangle$ and $\langle \mathcal{H}_{c,\alpha} v, v\rangle$ are positive. □

Remark 3.3. For $\alpha \in (\alpha_0, \alpha_1)$ the eigenvalue $\lambda = \frac{2^{\alpha+1}-5}{2^{\alpha+1}-3} + \mathcal{O}(a^2)$ is negative and the third item of Lemma 3.2 changes as follows: The sign of $\langle \mathcal{L}_{c,\alpha} v, v\rangle$ is now positive for the eigenvalue $\lambda = 0$ and negative for the eigenvalue $\lambda = \frac{2^{\alpha+1}-5}{2^{\alpha+1}-3} + \mathcal{O}(a^2)$. Nevertheless, we still count three eigenfunctions v of the generalized eigenvalue problem (1.7) with negative values of $\langle \mathcal{L}_{c,\alpha} v, v\rangle$ and one eigenfunction v with positive values of $\langle \mathcal{H}_{c,\alpha} v, v\rangle$, in agreement with Theorem 1 in [13].

3.2. Divergence results for $c > 2^\alpha$. The first resonance occurs at $c = 2^\alpha$, when a double eigenvalue of the operator $\mathcal{L}_{c,\alpha}$ crosses zero and becomes a negative eigenvalue for $c > 2^\alpha$. Some eigenvalues of the operator $\mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha}$ may diverge as $c \to 2^\alpha$ and the conclusion on convergence of the iterative method (1.6) may change after the resonance. Here we prove that the iterative method (1.6) diverges for every $c > 2^\alpha$ and $\alpha \in (\alpha_0, 2]$, for which $\mathcal{L}_{c,\alpha}^{-1}$ exists. Compared to the perturbative results in section 3.1, the restriction $\alpha \le 2$ is necessary to apply the results of [25] in the proof of Lemma 2.4. The following lemma specifies the number of negative eigenvalues of $\mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha}$ in $L^2_{\rm per}(-\pi,\pi)$.

LEMMA 3.3. *For every $c > 2^\alpha$ and $\alpha \in (\alpha_0, 2]$, for which $\mathcal{L}_{c,\alpha}^{-1}$ exists, $\sigma(\mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha})$ in $L^2_{\rm per}(-\pi,\pi)$ includes N negative eigenvalues (counting with their algebraic multiplicities) with $N \ge 1$ in addition to the simple negative eigenvalue -1.*

Proof. It follows from (2.10) that for $c > 2^\alpha$, $\sigma(\mathcal{L}_{c,\alpha})$ in $L^2_{\rm per}(-\pi,\pi)$ admits n negative eigenvalues (counting with their algebraic multiplicities) with $n \geq 5$. By Lemma 2.4, $\sigma(\mathcal{H}_{c,\alpha})$ in $L^2_{\rm per}(-\pi,\pi)$ admits only one simple negative eigenvalue and the simple zero eigenvalue with an odd eigenfunction ϕ'. In the space of even functions, $\mathcal{H}_{c,\alpha}$ has only one simple negative eigenvalue and is invertible, whereas $\mathcal{L}_{c,\alpha}$ has n_{ev} negative eigenvalues with $n_{ev} \geq 3$. Both $\mathcal{H}_{c,\alpha}$ and $\mathcal{L}_{c,\alpha}$ are self-adjoint in $L^2_{\rm per}(-\pi,\pi)$ with the domain in $H^\alpha_{\rm per}(-\pi,\pi)$, as well as in the corresponding subspaces of even functions. By Theorem 4.1 in [35], the constraints in L^2_c may only reduce one negative eigenvalue in either $\mathcal{L}_{c,\alpha}$ or $\mathcal{H}_{c,\alpha}$ (the choice between the two operators depends on the sign of $\int_{-\pi}^{\pi} \phi^3(x)dx$). In either case, by Theorem 1 in [13], there exist at least $N \geq 1$ negative eigenvalues λ of the generalized eigenvalue problem (1.7) in L^2_c. ∎

COROLLARY 3.2. *Assume* $\int_{-\pi}^{\pi} \phi^3 dx \neq 0$. *The iterative method* (1.6) *diverges from* ϕ *for every* $c > 2^\alpha$ *and* $\alpha \in (\alpha_0, 2]$.

Proof. It follows from Lemma 3.3 and the representation (3.5) that $\sigma(\mathcal{L}_T)$ in L^2_c includes $N \geq 1$ positive eigenvalues larger than 1. These eigenvalues of $\mathcal{L}_T$ outside the unit disk correspond to the eigenfunctions in the constrained subspace (3.3), which satisfy the orthogonality conditions

$$\langle \mathcal{L}_{c,\alpha}\phi, \alpha\rangle = \langle \mathcal{L}_{c,\alpha}\phi', \alpha\rangle = 0.$$

Divergence of the iterative method (1.6) follows by Theorem 3.1. ∎

Remark 3.4. It follows from the proof of Lemma 3.3 that the divergence of the iterative method (1.6) for $c > 2^\alpha$ and $\alpha \in (\alpha_0, 2]$ is observed if the initial guess $w_0 \in H^\alpha_{\rm per}(-\pi,\pi)$ is even.

3.3. Numerical results for $c \in (1, 2^\alpha)$. Here we address numerically convergence of the iterative method (1.6) near the single-lobe periodic wave for $c \in (1, 2^\alpha)$. For simplicity of computations, we only consider the classical KdV and BO equations.

In the case of the KdV equation ($\alpha = 2$), the following numerical results illustrate the convergence of the method for $c \in (1, c_0)$ with $c_0 \approx 2.3$ in agreement with Corollary 3.1, the transition to instability at $c = c_0$, and the divergence for $c \in (c_0, 4)$.

Figure 3 shows eigenvalues of the generalized eigenvalue problem (1.7) computed numerically with the Fourier method for $c \in (1,4)$. The five largest and five smallest eigenvalues of the operator $\mathcal{L}^{-1}_{c,\alpha}\mathcal{H}_{c,\alpha}$ are shown in the left panel. In agreement with the result of Lemma 3.1, we observe eigenvalues λ near points $\{-1, 0, \frac{3}{5}, 2\}$ in addition to a countable sequence of eigenvalues near 1. The right panel depicts a closeup of the eigenvalues near $c = 1$ and shows the asymptotic approximation of the eigenvalue near 2 given by (3.24) and (3.25) with $\alpha = 2$.

For $c_* \approx 1.2$, two real eigenvalues coalesce to create a pair of complex eigenvalues that exist for every $c > c_*$. This transformation of eigenvalues compared to the result of Lemma 3.2 does not contradict the count of eigenvalues in Theorem 1 of [13]. Figure 4 shows that $|1-\lambda|$ for the eigenvalues of $\mathcal{L}_T$ remains inside the unit disk for $c \in (c_*, 4)$. Therefore, the complex eigenvalue pair does not introduce instability to the iterative method. For $c \in (1, c_0)$ with $c_0 \approx 2.3$, the spectrum of $\mathcal{L}_T$ in L^2_c remains inside the unit disk for $c \in (1, c_0)$. However, the largest eigenvalue of $\mathcal{L}^{-1}_{c,\alpha}\mathcal{H}_{c,\alpha}$ crosses the level 2 for $c = c_0$ and the corresponding eigenvalue of $\mathcal{L}_T$ is smaller than -1 for $c \in (c_0, 4)$. This numerical result suggests that the iterative method (1.6) converges for $c \in (1, c_0)$ and diverges for $c \in (c_0, 4)$. Moreover, for $c_1 \approx 2.7$, the second largest eigenvalue of $\mathcal{L}^{-1}_{c,\alpha}\mathcal{H}_{c,\alpha}$ crosses the level 2, and hence the iterative method (1.6) diverges with two unstable eigenvalues for $c \in (c_1, 4)$.

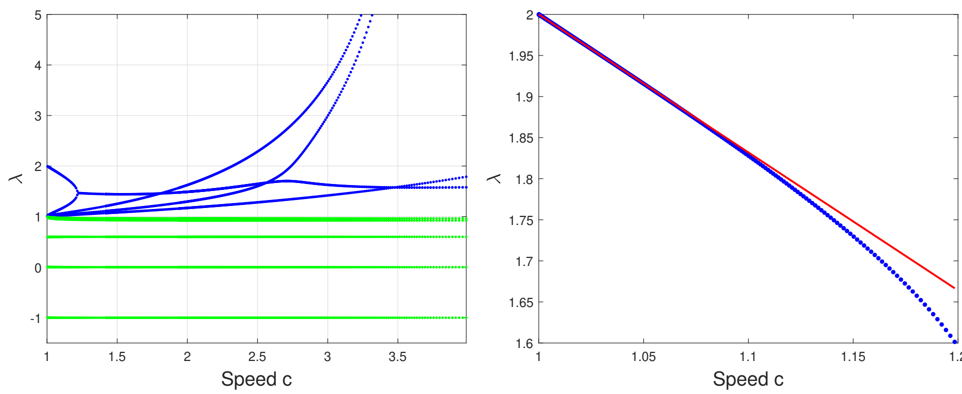


FIG. 3. *Left: Eigenvalues of the operator* $\mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha}$ *for* $\alpha = 2$. *Right: Zoom-in, with the asymptotic dependence given by* (3.24) *and* (3.25).

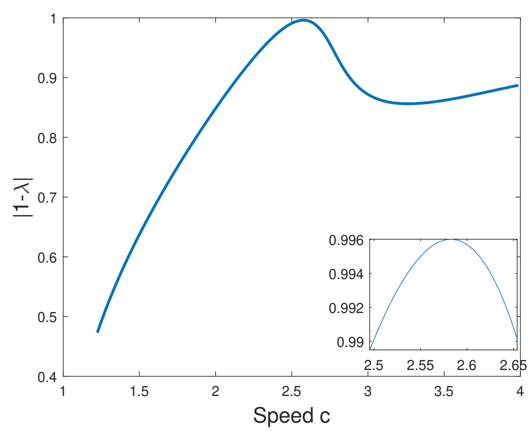


FIG. 4. *The plot of* $|1-\lambda|$ *for the complex eigenvalues* λ. *The inset shows that the complex eigenvalues do not reach the boundary of the unit disk.*

To illustrate convergence of the iterative method (1.6) for $\alpha = 2$, we use the initial function

$$u_0(x) = a\cos(x) + \frac{1}{2}a^2\left(\cos(2x) - 3\right) + \varepsilon \sin(x), \tag{3.27}$$

where $a > 0$ and $\varepsilon \in \mathbb{R}$ are small parameters at our disposal. We have included the $\mathcal{O}(a^2)$ correction term of the Stokes expansion (2.1) in the initial function (3.27) in order to avoid vanishing denominators in the Petviashvili quotient M defined by (1.5). Indeed, $\int_{\pi}^{\pi} \cos(x)^3 dx = 0$, whereas $\int_{-\pi}^{\pi} \phi^3 dx < 0$ for every $c > 1$ and $\alpha = 2$ by Lemma 2.5. Computations reported below correspond to $a = 0.4$ and $\varepsilon = 0$; we have checked that computations for other small values of a and ε return similar results.

We measure the computational errors by using the quantity $|1 - M_n|$, where $M_n = M(u_n)$, and the residual error $\|cu_n + u_n'' + u_n^2\|_{L^\infty}$. If iterations do not converge, we stop the algorithm after 500 iterations.

Figure 5 shows the profile of the last iteration and the two computational errors versus the number of iterations in the case $c = 2$. It is seen that the iterative method

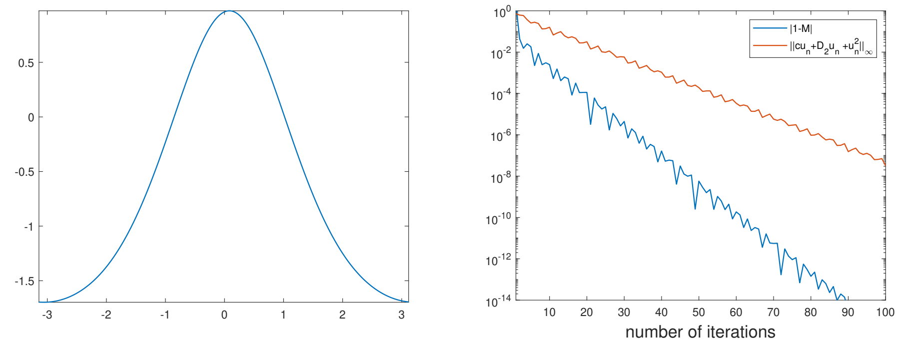


FIG. 5. *Iterations for* $c = 2$ *and* $\alpha = 2$. (a) *The last iteration versus* x. (b) *Computational errors versus* n.

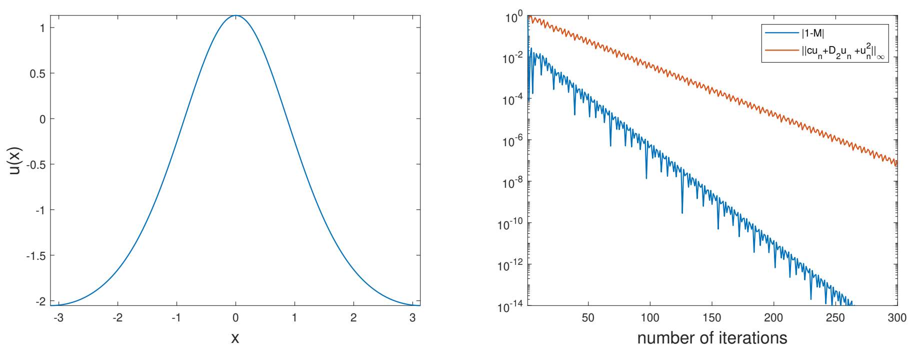


FIG. 6. *Iterations for* $c = 2.3$ *and* $\alpha = 2$. (a) *The last iteration versus* x. (b) *Computational errors versus* n.

(1.6) converges to the single-lobe periodic wave, in agreement with Corollary 3.1. Since the exact periodic wave is given by (2.18)–(2.19), we can also compute the distance between the last iteration and the exact solution, in which case we find $\|u - \phi\|_{L^\infty} \approx 2 \cdot 10^{-11}$. If $\varepsilon \neq 0$ in the initial function (3.27), the convergence to the periodic wave is still observed but the last iteration is shifted from $x = 0$, in agreement with Theorem 3.2.

Figure 6 illustrates the case $c = 2.3$. Since the largest eigenvalue of $\mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha}$ crosses level 2 at this value of c (see Figure 3), this case is marginal for convergence of iterations. As we can see from Figure 6, iterations still converge to a single-lobe periodic wave, but the convergence is slow.

Figure 7 illustrates the case $c = 3$. The iterative method (1.6) diverges from the single-lobe periodic wave. The instability is related to the eigenvalue of $\mathcal{L}_T$ which is smaller than -1; hence the period-doubling instability leads to an alternating sequence which oscillates between two double-lobe profiles shown in the left panel. The right panel shows that the factor M no longer converges to 1 but to -4.3737 and the residual error does not converge to 0 but remains strictly positive with the number of iterations. Therefore, the two limiting states of the iterative method (1.6) in the 2-periodic orbit are not periodic solutions of the boundary-value problem (1.3).

In the case of the BO equation ($\alpha = 1$), the following numerical results illustrate the divergence of the method for $c \in (1, 2)$ in agreement with Corollary 3.1. Figure 8

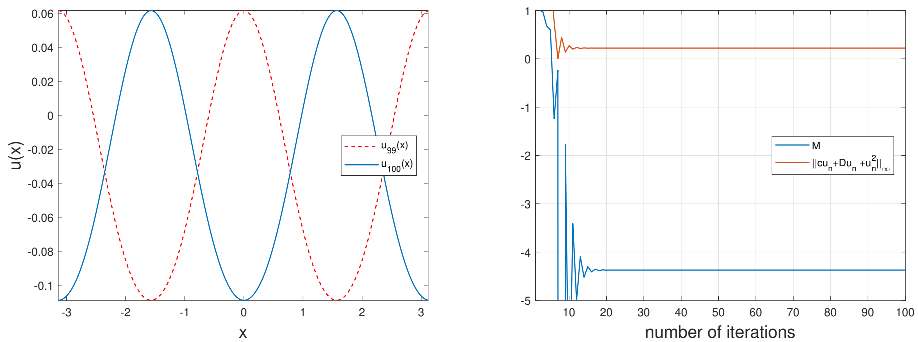


FIG. 7. *Iterations for $c = 3$ and $\alpha = 2$.* (a) *The last two iterations versus x.* (b) *Computational errors versus n.*

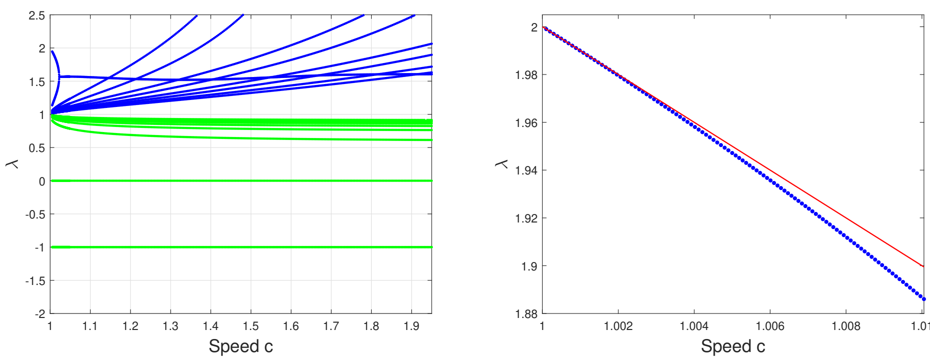


FIG. 8. *Left: Eigenvalues of the operator $\mathcal{L}_{c,\alpha}^{-1}\mathcal{H}_{c,\alpha}$ for $\alpha = 1$. Right: Zoom-in, with the asymptotic dependence given by* (3.24) *and* (3.25).

shows the eigenvalues of the generalized eigenvalue problem (1.7) for $\alpha = 1$. The eigenvalue $\lambda_1 = \frac{2^{\alpha+1}-5}{2^{\alpha+1}-3}$ in Lemma 3.1 yields $\lambda_1 = -1$ for $\alpha = 1$ in addition to the other eigenvalue -1 in $\{-1, 0, \lambda_1, \lambda_2\}$. Hence, $\lambda = -1$ is a double eigenvalue and the left panel shows that this double eigenvalue is preserved in c. The right panel depicts a closeup of the eigenvalues near $c = 1$ and shows the asymptotic approximation of the eigenvalue near 2 given by (3.24) and (3.25) with $\alpha = 1$.

To illustrate divergence of the iterative method (1.6) for $\alpha = 1$, we use the initial function

$$u_0(x) = a\cos(x) + \frac{1}{2}a^2(\cos(2x) - 1) + \varepsilon\sin(x), \tag{3.28}$$

where $a > 0$ and $\varepsilon \in \mathbb{R}$. Again, we have verified that $\int_{-\pi}^{\pi}\phi^3 dx < 0$ for every $c > 1$ and $\alpha = 1$ by the explicit computations in (2.22); therefore, we included the second term of the Stokes expansion (2.1) in the initial function (3.28) in order to ensure that $\int_{-\pi}^{\pi} u_0^3 dx < 0$. In the computations below, we take $a = 0.4$.

As predicted by Corollary 3.1 for $\alpha = 1$, the iterative method (1.6) diverges for the BO equation, and this divergence is due to an odd eigenfunction of the generalized eigenvalue problem (1.7) for the eigenvalue $\lambda_1 = -1$.

Figure 9 illustrates the case $c = 1.1$, showing the last four iterations in the left

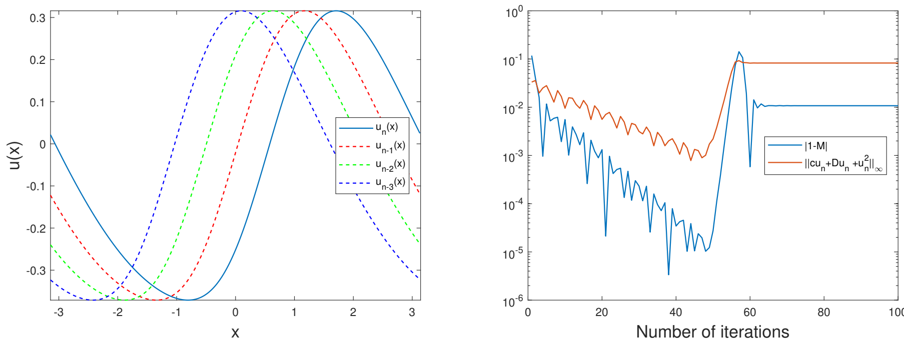


FIG. 9. *Iterations for $c = 1.1$ and $\alpha = 1$.* (a) *The last four iterations versus x.* (b) *Computational errors versus n.*

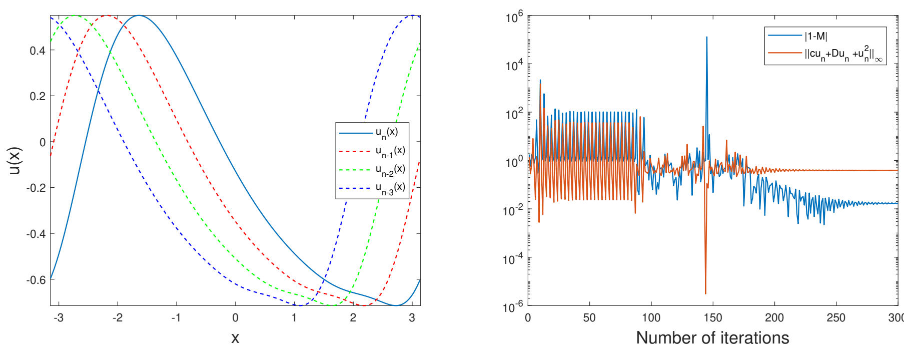
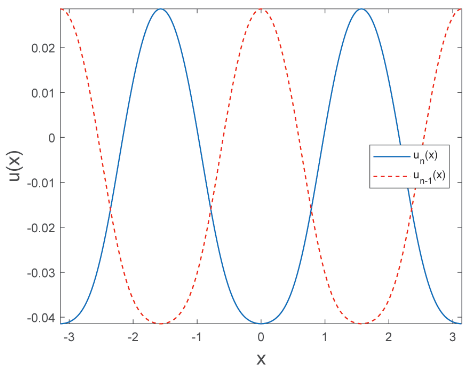


FIG. 10. *Iterations for $c = 1.3$ and $\alpha = 1$.* (a) *The last four iterations versus x.* (b) *Computational errors versus n.*

panel, and the factor M converging to 1.0107 and the residual error converges to 0.0826 in the right panel. In this computation, we take $\varepsilon = 0$. Although the residual error starts to decrease initially due to contracting properties of $\mathcal{L}_T$ on the even subspace of $L^2_{\text{per}}(-\pi, \pi)$, round-off errors induce odd perturbations, resulting in slow instability. Consequently, the periodic wave of amplitude 0.458 is not captured by the iterative method (1.6), and instead iterations converge to the periodic profile of amplitude 0.344, which drifts by every iteration to the right. This drifted periodic profile of the iterative method (1.6) is not a periodic solution to the boundary-value problem (1.3). If $\varepsilon \neq 0$, the instability develops much faster and the drifted periodic profile drifts to the right if $\varepsilon > 0$ and to the left if $\varepsilon < 0$.

Figure 10 shows the marginal case $c = 1.3$, where another unstable eigenvalue of $\mathcal{L}_T$ related to the even eigenfunction crosses level -1. Although the instability pattern of Figure 9 is repeated in Figure 10, the periodic profile becomes more complicated and the instability process is accompanied by many intermediate oscillations. Here again we set $\varepsilon = 0$. If $\varepsilon \neq 0$, the drifted periodic profile is formed much faster and intermediate oscillations are reduced.

Figure 11 illustrates the case $c = 1.6$ when several eigenvalues of $\mathcal{L}_T$ are located below -1. After short intermediate iterations, the iterative method starts to oscillate between two iterations, similarly to the pattern of Figure 7. The right panel of Figure 11 shows that the factor M converges to -5.1447 and the residual error remains



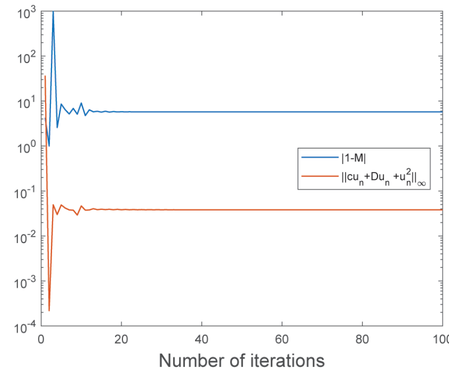


FIG. 11. *Iterations for $c = 1.6$ and $\alpha = 1$.* (a) *The last two iterations versus x.* (b) *Computational errors versus n.*

strictly positive. The two limiting states of the iterative method (1.6) are not the periodic solutions of the boundary-value problem (1.3).

4. Proof of Theorem 1.2. By linearizing $\tilde{T}_{c,\alpha}$ at ψ with $w_n = \psi + a_n\psi + b_n\psi' + \beta_n$, where $\beta_n \in H^\alpha_{\rm per}(-\pi,\pi) \cap L^2_c$ satisfies the two constraints in

$$L^2_c := \left\{\omega \in L^2_{\rm per}(-\pi,\pi) : \quad \langle \psi^2, \omega\rangle = \langle \psi\psi', \omega\rangle = 0\right\}, \tag{4.1}$$

we obtain the linearized iterative rule

$$a_{n+1} = 0, \quad b_{n+1} = b_n, \quad \beta_{n+1} = \tilde{\mathcal{L}}_T \beta_n, \tag{4.2}$$

where

$$\tilde{\mathcal{L}}_T := \tilde{\mathcal{L}}^{-1}_{c,\alpha}(2\psi\cdot) = {\rm Id} - \tilde{\mathcal{L}}^{-1}_{c,\alpha}\tilde{\mathcal{H}}_{c,\alpha} : \quad H^\alpha_{\rm per}(-\pi,\pi)\cap L^2_c \mapsto H^\alpha_{\rm per}(-\pi,\pi)\cap L^2_c, \tag{4.3}$$

with $\tilde{\mathcal{L}}_{c,\alpha} = c - D_\alpha$ and $\tilde{\mathcal{H}}_{c,\alpha} = \mathcal{H}_{c,\alpha}$ given by (1.13). Hence Lemmas 2.2 and 2.4 apply to $\tilde{\mathcal{H}}_{c,\alpha} = \mathcal{H}_{c,\alpha}$. In addition, Theorem 2.2 ensures positivity of $\psi(x) > 0$ for every $x \in [-\pi,\pi]$.

The following lemma characterizes the spectrum of $\tilde{\mathcal{L}}^{-1}_{c,\alpha}\tilde{\mathcal{H}}_{c,\alpha}$ in $L^2_{\rm per}(-\pi,\pi)$ for every $c > 1$ and $\alpha \in (\alpha_0, 2]$. Convergence of the iterative method (1.11) to the positive periodic wave $\psi \in H^\alpha_{\rm per}(-\pi,\pi)$ of the boundary-value problem (1.2) follows from this lemma. This construction yields the proof of Theorem 1.2.

LEMMA 4.1. *For every $c > 1$ and $\alpha \in (\alpha_0, 2]$, $\sigma(\tilde{\mathcal{L}}^{-1}_{c,\alpha}\tilde{\mathcal{H}}_{c,\alpha}) \in (0,1)$ in L^2_c.*

Proof. We note that $\tilde{\mathcal{L}}_{c,\alpha}$ is positive for every $c > 1$ and $\alpha > 0$, whereas $\tilde{\mathcal{H}}_{c,\alpha}$ has one simple negative eigenvalue and a simple zero eigenvalue for every $c > 1$ and $\alpha \in (\alpha_0, 2]$ by Lemma 2.4.

By Theorem 1 in [13], $\sigma(\tilde{\mathcal{L}}^{-1}_{c,\alpha}\tilde{\mathcal{H}}_{c,\alpha})$ in $L^2_{\rm per}(-\pi,\pi)$ is real and contains one simple negative eigenvalue and a simple zero eigenvalue; the rest of the spectrum is positive and bounded away from zero. The negative and zero eigenvalues correspond to the exact solutions

$$\tilde{\mathcal{L}}^{-1}_{c,\alpha}\tilde{\mathcal{H}}_{c,\alpha}\psi = -\psi \quad \text{and} \quad \tilde{\mathcal{L}}^{-1}_{c,\alpha}\tilde{\mathcal{H}}_{c,\alpha}\psi' = 0. \tag{4.4}$$

These eigenvalues are removed by adding two constraints in the definition of L^2_c in (4.1). The positive eigenvalues are bounded from above by 1 because the operator

$$\tilde{\mathcal{L}}_{\tilde{T}} = \tilde{\mathcal{L}}^{-1}_{c,\alpha}(2\psi\cdot) = {\rm Id} - \tilde{\mathcal{L}}^{-1}_{c,\alpha}\tilde{\mathcal{H}}_{c,\alpha}$$

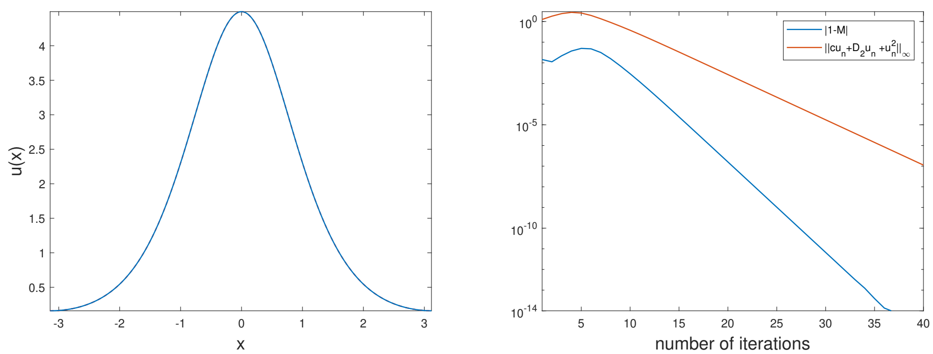


FIG. 12. *Iterations for* $c = 3$ *and* $\alpha = 2$. (a) *The last iteration versus* x. (b) *Computational errors versus* n.

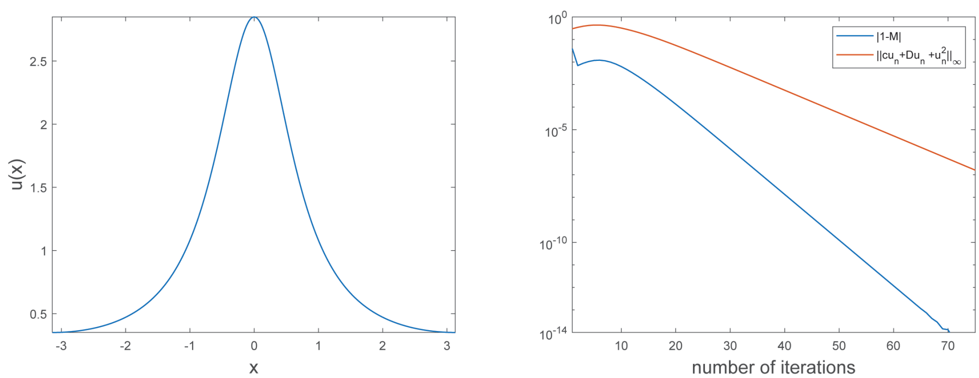


FIG. 13. *Iterations for* $c = 1.6$ *and* $\alpha = 1$. (a) *The last iteration versus* x. (b) *Computational errors versus* n.

is strictly positive due to positivity of $\tilde{\mathcal{L}}_{c,\alpha}$ and ψ. Hence, $\sigma(\tilde{\mathcal{L}}_{c,\alpha}^{-1}\tilde{\mathcal{H}}_{c,\alpha}) \in (0,1)$ in L^2_c. $\square$

COROLLARY 4.1. *For every* $c > 1$ *and* $\alpha \in (\alpha_0, 2]$, *the iterative method* (1.11) *converges to* ψ *in* $H^\alpha_{\rm per}(-\pi,\pi)$.

Proof. Conditions $\int_{-\pi}^{\pi} \psi^3 dx > 0$ and $\int_{-\pi}^{\pi} \psi(\psi')^2 dx > 0$ follow by positivity of ψ in Theorem 2.2. By Lemma 4.1, the operator $\tilde{\mathcal{L}}_{\tilde{T}}$ is a strict contraction in L^2_c for every $c > 1$ and $\alpha \in (\alpha_0, 2]$. Convergence of the iterative method (1.11) follows by Theorem 3.2. $\square$

To demonstrate the convergence of the iterative method (1.11), we use the initial condition

$$u_0(x) = c + a\cos(x)$$

with $a = 0.4$. This initial guess corresponds to the first two terms of the Stokes expansion (2.1) for $\psi(x) = c + \phi(x)$. We do not need to include the $\mathcal{O}(a^2)$ in the initial guess because $\int_{-\pi}^{\pi} u_0^3 dx > 0$ and the denominator of the Petviashvili quotient (1.10) does not vanish at u_0.

Figure 12 shows the result of iterations for $c = 3$ and $\alpha = 2$. It is seen that iterations converge quickly to a positive, single-lobe periodic wave ψ in agreement with Corollary 4.1. Note that the iterative method (1.6) diverges for $c = 3$ and $\alpha = 2$, as seen in Figure 7. We can also compute the distance between the last iteration and the exact solution, in which case we find $\|u - \phi\|_{L^\infty} \approx 1.3 \cdot 10^{-11}$.

Figure 13 reports similar results for $c = 1.6$ and $\alpha = 1$. Again, the iterative method (1.6) diverges for these values of c and α, as seen in Figure 11. We can also compute the distance between the last iteration and the exact solution, in which case we find $\|u - \phi\|_{L^\infty} \approx 5.9 \cdot 10^{-11}$.

Acknowledgments. DEP thanks D. Clamond for posing a problem on the lack of convergence of Petviashvili's method for periodic waves in the KdV equation. The authors also thank H. Chen, A. Durán, V. Hur, M. Johnson, F. Natali, and P. Torres for discussion of various technical aspects of this manuscript.

REFERENCES

[1] J. Álvarez and A. Durán, *An extended Petviashvili method for the numerical generation of traveling and localized waves*, Commun. Nonlinear Sci. Numer. Simul., 19 (2014), pp. 2272–2283.

[2] J. Álvarez and A. Durán, *Petviashvili type methods for traveling wave computations:* I. *Analysis of convergence*, J. Comput. Appl. Math., 266 (2014), pp. 39–51.

[3] J. Álvarez and A. Durán, *Petviashvili type methods for traveling wave computations:* II. *Acceleration with vector extrapolation methods*, Math. Comput. Simulation, 123 (2016), pp. 19–36.

[4] J. Álvarez and A. Durán, *Numerical generation of periodic traveling wave solutions of some nonlinear dispersive wave systems*, J. Comput. Appl. Math., 316 (2017), pp. 29–39.

[5] C. J. Amick and J. F. Toland, *Uniqueness and related analytic properties for the Benjamin–Ono equation: A nonlinear Neumann problem in the plane*, Acta Math., 167 (1991), pp. 107–126.

[6] J. Angulo Pava and F. M. A. Natali, *Positivity properties of the Fourier transform and the stability of periodic travelling-wave solutions*, SIAM J. Math. Anal., 40 (2008), pp. 1123–1151, https://doi.org/10.1137/080718450.

[7] Z. Bai and H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl., 311 (2005), pp. 495–505.

[8] T. B. Benjamin, J. L. Bona, and J. J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Phil. Trans. Royal Soc. London, Ser. A, 272 (1972), pp. 47–78.

[9] J. C. Bronski and V. M. Hur, *Modulational instability and variational structure*, Stud. Appl. Math., 132 (2014), pp. 285–331.

[10] H. Chen, *Existence of periodic traveling-wave solutions of nonlinear, dispersive wave equations*, Nonlinearity, 17 (2004), pp. 2041–2056.

[11] H. Chen and J. Bona, *Periodic traveling wave solutions of nonlinear dispersive evolution equations*, Discrete Contin. Dyn. Syst., 33 (2013), pp. 4841–4873.

[12] M. Chugunova and D. Pelinovsky, *Two-pulse solutions in the fifth-order KdV equation: Rigorous theory and numerical approximations*, Discrete Contin. Dyn. Syst. Ser. B, 8 (2007), pp. 773–800.

[13] M. Chugunova and D. Pelinovsky, *Count of eigenvalues in the generalized eigenvalue problem*, J. Math. Phys., 51 (2010), 052901.

[14] D. Clamond and D. Dutykh, *Fast accurate computation of the fully nonlinear solitary surface gravity waves*, Comput. & Fluids, 84 (2013), pp. 35–38.

[15] D. Clamond and D. Dutykh, *Accurate fast computation of steady two-dimensional surface gravity waves in arbitrary depth*, J. Fluid Mech., 844 (2018), pp. 491–518.

[16] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Heidelberg, 1985.

[17] L. Demanet and W. Schlag, *Numerical verification of a gap condition for a linearized NLS equation*, Nonlinearity, 19 (2006), pp. 829–852.

[18] A. Durán, *An efficient method to compute solitary wave solutions of fractional Korteweg–de Vries equations*, Int. J. Comput. Math., 95 (2018), pp. 1362–1374.

[19] D. Dutykh and D. Clamond, *Efficient computation of steady solitary gravity waves*, Wave Motion, 51 (2014), pp. 86–99.

[20] S. A. Dyachenko, P. M. Lushnikov, and A. O. Korotkevich, *Complex singularity of a*

Stokes wave, JETP Lett., 98 (2014), pp. 675–679.

[21] T. Gallay and D. E. Pelinovsky, *Orbital stability in the cubic defocusing NLS equation. Part* I: *Cnoidal periodic waves*, J. Differential Equations, 258 (2015), pp. 3607–3638.

[22] A. Garijo and J. Villadelprat, *Algebraic and analytical tools for the study of the period function*, J. Differential Equations, 257 (2014), pp. 2464–2484.

[23] M. Haragus, J. Li, and D. E. Pelinovsky, *Counting unstable eigenvalues in Hamiltonian spectral problems via commuting operators*, Comm. Math. Phys., 354 (2017), pp. 247–268.

[24] V. Hur and M. A. Johnson, *Modulational instability in the Whitham equation for water waves*, Stud. Appl. Math., 134 (2014), pp. 120–143.

[25] V. M. Hur and M. A. Johnson, *Stability of periodic traveling waves for nonlinear dispersive equations*, SIAM J. Math. Anal., 47 (2015), pp. 3528–3554, https://doi.org/10.1137/12090215X.

[26] M. A. Johnson, *Nonlinear stability of periodic traveling wave solutions of the generalized Korteweg–de Vries equation*, SIAM J. Math. Anal., 41 (2009), pp. 1921–1947, https://doi.org/10.1137/090752249.

[27] M. A. Johnson, *Stability of small periodic waves in fractional KdV-type equations*, SIAM J. Math. Anal., 45 (2013), pp. 3168–3193, https://doi.org/10.1137/120894397.

[28] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, Heidelberg, Germany, 1995.

[29] T. I. Lakoba and J. Yang, *A generalized Petviashvili iteration method for scalar and vector Hamiltonian equations with arbitrary form of nonlinearity*, J. Comput. Phys., 226 (2007), pp. 1668–1692.

[30] T. I. Lakoba and J. Yang, *A mode elimination technique to improve convergence of iteration methods for finding solitary waves*, J. Comput. Phys., 226 (2007), pp. 1693–1709.

[31] F. Linares, D. Pilod, and J.-C. Saut, *Dispersive perturbations of Burgers and hyperbolic equations* I: *Local theory*, SIAM J. Math. Anal., 46 (2014), pp. 1505–1537, https://doi.org/10.1137/130912001.

[32] Y. Matsuno, *Bilinear Transformation Method*, Math. Sci. Engrg. 174, Academic Press, Orlando, FL, 1984.

[33] L. Molinet, D. Pilod, and S. Vento, *On well-posedness for some dispersive perturbations of Burgers' equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire 35 (2018), pp. 1719–1756.

[34] J. J. Nieto, *Maximum principles for fractional differential equations derived from Mittag–Leffler functions*, Appl. Math. Lett., 23 (2010), pp. 1248–1251.

[35] D. E. Pelinovsky, *Localization in Periodic Potentials: From Schrödinger Operators to the Gross-Pitaevskii Equation*, London Math. Soc. Lecture Note Ser. 390, Cambridge University Press, Cambridge, UK, 2011.

[36] D. E. Pelinovsky and Yu. A. Stepanyants, *Convergence of Petviashvili's iteration method for numerical approximation of stationary solutions of nonlinear wave equations*, SIAM J. Numer. Anal., 42 (2004), pp. 1110–1127, https://doi.org/10.1137/S0036142902414232.

[37] V. I. Petviashvili, *Equation of an extraordinary soliton*, Plasma Phys., 2 (1976), pp. 469–472.

[38] R. Schaaf, *A class of Hamiltonian systems with increasing periods*, J. Reine Angew. Math., 363 (1985), pp. 96–109.

[39] P. J. Torres, *Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem*, J. Differential Equations, 190 (2003), pp. 643–662.

[40] J. Yang, *Nonlinear Waves in Integrable and Nonintegrable Systems*, SIAM, Philadelphia, 2010, https://doi.org/10.1137/1.9780898719680.

[41] E. Zeidler, *Applied Functional Analysis: Applications to Mathematical Physics*, Appl. Math. Sci. 108, Springer-Verlag, New York, 1995.