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An asymptotic approach to solitary wave instability and critical collapse in long-wave KdV-type evolution equations

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Abstract

Instability development and critical collapse of solitary waves are considered in the framework of generalized Korteweg–de Vries (KdV) equations in one and two dimensions. An analytical theory of the solitary wave dynamics and generation of radiation is constructed for the critical case when the solitary waves are weakly unstable. Characteristic types of the global, essentially nonlinear evolution of the unstable solitary waves are analyzed for some typical generalized KdV equations. The scaling laws of the self-similar wave field transformation are found analytically for the power-like KdV equation in the critical case $p = 4$. The asymptotic approach is also developed for the modified Zakharov–Kuznetsov equation in two dimensions and the rate of the singularity formation is found to be smaller than in one dimension due to diffractive wave effects.

1. Introduction

Collapse, that is, the formation of singularities in finite time, is a feature of some nonlinear dispersive wave equations when a small-amplitude solitary wave, realized as a balance between nonlinearity and dispersion, becomes unstable (for a review, see [1,2]). Strictly speaking, these weakly nonlinear equations displaying solitary wave instability and subsequent collapse are not really valid for the description of the wave evolution towards a singularity. Instead the original set of governing equations should be investigated and possibly, in place of an actual collapse, there could be formation of strongly nonlinear coherent structures. However, basic information about such large-amplitude structures can be extracted from the analysis of the self-similar collapsing waves in the weakly nonlinear model equations. Thus, such simple models displaying collapse phenomena have been intensively studied for several years.

In contemporary nonlinear physics, two types of nonlinear evolution equations have been found especially important for the description of wave propagation (see [1]). They are the so-called *NLS-type* equations for the envelopes of quasiharmonic waves, and the *KdV-type* equations for weakly nonlinear long waves.

For the NLS-type model equations, solitary wave stability and collapse formation have been investigated in much detail. In many cases, these models can be reduced to the *generalized NLS equation*,

$$i\Psi_t + \Delta_D\Psi + f(|\Psi|)\Psi = 0, \qquad (1.1)$$

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where Δ_D stands for the Laplacian operator in D dimensions and $f(|\Psi|)$ is proportional to the nonlinear correction to the frequency of linear waves. It has been proved (see [2a] and reference therein) that the solitary wave solutions of the generalized NLS equation are stable if and only if the power (number of particles) invariant grows with increase of the solitary wave amplitude. For the *power-like* NLS equation, when $f(|\Psi|) \sim |\Psi|^p$, this condition is satisfied for the so-called *subcritical* case when $pD < 4$. In the *supercritical* region, i.e. $pD > 4$, the solitary waves are exponentially unstable and collapse can be observed from a large class of initial conditions.

The *critical* case, $pD = 4$, reveals a number of remarkable properties which has attracted much interest (see, e.g., [2b]). In this case, the power invariant has a unique (threshold) value for all solitary wave solutions and this value separates two different global scenarios of the evolution of a localized initial perturbation. If the power of the localized perturbation exceeds the threshold value, the initial pulse collapses while, in the opposite case, it spreads out and decays into linear dispersive waves. During the collapse the localized spike remains self-similar to the underlying solitary wave solution with varying parameters [3]. The scaling laws for the variation of the solitary wave parameters were found to be different at different stages of the critical collapse [2b]. At an early stage, the scaling laws are described by *exact self-similar solutions* to the power-like NLS equation. However, at a later stage, the collapse is accompanied by the generation of a radiation field and this effect modifies the scaling laws of the solitary wave evolution by a logarithmic factor. Finally, for the threshold value of the power of a localized initial pulse, the collapse is completely free of radiation and it is governed by an exact solution describing the nonlinear stage of instability of a stationary solitary wave.

The development of solitary wave instability and collapse has also been investigated in the framework of the KdV-type evolution equations. In the one-dimensional case, one such model is the *generalized KdV equation*,

$$u_t + \left(c + f'(u)\right) u_x + u_{xxx} = 0, \tag{1.2}$$

where $f'(u) = \mathrm{d}f/\mathrm{d}u$. For instance, this equation describes the propagation of nonlinear long acoustic-type waves. The function $f'(u)$ is regarded as a nonlinear correction to the limiting long-wave phase speed c. In the weakly nonlinear approximation, this function is linear or quadratic in u and, in this case, the model (1.2) reduces to the integrable KdV, or MKdV equation, respectively [4]. However, if wave amplitude is not supposed to be small, the generalized KdV equation serves as an approximate model for the description of weak dispersive effects on the propagation of nonlinear waves along a characteristic direction [5].

Solitary waves and their linear stability in the framework of (1.2) have been studied by many authors [6–9] in analogy with the generalized NLS equation (1.1). The criterion of the solitary wave stability was found to be expressed through the *momentum* $P_s(v)$ evaluated at the stationary solution,

$$\frac{\mathrm{d}P_s}{\mathrm{d}v} > 0, \tag{1.3a}$$

$$P_s(v) = \frac{1}{2} \int_{-\infty}^{+\infty} u_0^2(x; v)\,\mathrm{d}x, \tag{1.3b}$$

where the solitary wave solution has the form $u = u_0(x - vt; v)$ and v is the wave velocity.

However, a more detailed study by Pego and Weinstein [10] revealed that the birth of the solitary wave instability in the generalized KdV equation (1.2) is completely different from that in the generalized NLS model (1.1). In the latter case, a pair of positive and negative real eigenvalues (which give the instability growth rate) arise as a result of the merging of two imaginary eigenvalues. In the former case, the real eigenvalues emerge from the origin when the solitary wave velocity v, or the model parameters, cross the bifurcation (critical) point. Inside the stability region, the eigenfunctions corresponding to the real eigenvalues are nonlocalized and exponentially growing along a spatial coordinate [10].

We would like to point out that these results are similar to the linear analysis of the transverse stability of the plane KdV solitary waves in two dimensions described by the Kadomtsev– Petviashvili equation [11]. In the latter case, Burtsev [12] found that the existence of nonlocalized eigenfunctions inside the stability region is associated with the generation of strong radiation escaping from the front of the stable solitary waves and resulting in effective (Landau-type) damping of the front oscillations. Note that the perturbation theory is quite effective in this case for the description of the solitary wave dynamics and the generation of the radiation field [13,14].

Just as in the power-like NLS equation, the generalized KdV model (1.2) with power-like nonlinearity,

$$f(u) = \tfrac{1}{2}(p+2)u^{p+1}, \tag{1.4}$$

displays instability of solitary wave solutions for $p \geq 4$. Originally, Kodama and Ablowitz [15] revealed that the perturbation theory for adiabatic solitary wave dynamics in the presence of external perturbations breaks down at $p = 4$ and they suggested that the power-like KdV model (1.2) and (1.4) admits the formation of singularities at $p \geq 4$. Then, Blaha et al. [16] proved that the Hamiltonian of the system is unbounded from below for $p > 4$ and the solitary wave solutions realize global maximum of the Hamiltonian. These results indicate that the collapse formation is a global scenario for solitary wave dynamics. Furthermore, Blaha et al. [16a] used the method of variation of action in order to get not only qualitative but also quantitative features of the focusing singularities for $p \geq 4$. However, we will show that the strong radiation excited due to the collapse development was omitted in this approach and that radiation-induced damping of the momentum of the perturbed solitary wave changes drastically the scaling laws of the collapse formation. Recently the structure of the singularities was observed by Bona et al. [17] in numerical simulation of the power-like KdV model. Their results (see [17, Fig. 8]) are in obvious disagreement with the results of [16a, Fig. 6]. Thus, the detailed analysis of the critical collapse in the power-like KdV equation awaits investigation.

There exist different two-dimensional generalizations of the power-like KdV model which also display critical solitary wave dynamics [16,18]. Here, we will consider the following anisotropic generalization referred to as the modified Zakharov–Kuznetsov (mZK) equation [19]:

$$u_t + 6u^2 u_x + u_{xxx} + u_{xyy} = 0. \tag{1.5}$$

This equation can be derived for Alfven waves in magnetized plasma at a special, critical angle to the magnetic field by means of an asymptotic multi-scale technique [20]. It was pointed out [3,16] that this equation describes critical collapse but a detailed investigation of its development has not yet been carried out.

In this paper we present a universal analytical theory for the description of the development of solitary wave instability and the related critical collapse in long-wave evolution equations. This theory is based on a standard perturbation approach for solitary wave dynamics (see [21,22]). However, in contrast to previous versions of this technique, the evolution of the solitary waves occurs under the action of *their own perturbations* which develop in the vicinity of the instability threshold slowly in time. In this case, the original KdV-type equation can be reduced through this asymptotic technique to an ordinary differential equation for the slowly varying velocity of the perturbed solitary wave. This result is very useful for the prediction of the long-term (global) dynamics of unstable solitary waves. Moreover, it enables us to construct approximate solutions for self-similar critical collapse in the KdV-type evolution equations and, in this case, the validity of our approach is guaranteed by a collapse theorem proved by Laedke et al. [3]. According to this theorem, for critical collapse, the field near the singularity spike remains self-similar to the shape of the solitary wave solutions but with varying parameters. We note that a similar technique was recently developed for the generalized NLS equation (1.1) and, in the critical collapse case, it led to the same results as the exact self-similar solutions [23].

Our paper is organized as follows. In Section 2 we reduce the generalized KdV model (1.2) to the governing equation for the velocity of the perturbed solitary wave. Using this equation we discuss in Section 3 the general features of the evolution of the unstable solitary waves for the generalized KdV equation. Then, in Section 4 we analyze the critical solitary wave collapse described by the model (1.2) with a power-like nonlinearity (1.4) at $p = 4$. We compare the results of our theory with those of the variational approach [16a] and of exact self-similar reductions [16a,17] and discuss the difference between these approaches. Next, in Section 5 we consider the mZK equation (1.5) and show that the excitation of the radiation field in two dimensions is accompanied by diffractive effects which slows down the collapse development. Finally, our results are summarized in Section 6. In Appendix A we show that the asymptotic equations for the KdV-type solitary wave dynamics can be alternatively derived from analysis of conservation laws.

2. Asymptotic model for solitary wave dynamics

Here we consider the generalized KdV equation (1.2) for an arbitrary nonlinear function $f(u)$. We assume that this model exibits solitary wave solutions in the form $u = u_0(x - vt; v)$ and that these solutions are *weakly unstable*, i.e. the inequality (1.3a) is violated for some values of the velocity v and the model parameters. In the vicinity of the instability threshold, the KdV-type solitary waves evolve slowly in time and, therefore, we can describe their long-term dynamics by the formal asymptotic series,

$$u = u_0(\xi; v) + u_1(\xi; v, v_t) + u_2(\xi; v, v_t, v_{tt}) + \cdots \tag{2.1}$$

Here ξ is the coordinate of the wave center, $\xi = x - \int_0^t \left(c + v(t')\right) \mathrm{d}t'$, and v is a (positive) correction to the limiting long-wave phase speed c induced due to dispersive and nonlinear effects. Although we did not introduce a small parameter in the asymptotic expansion (2.1), we have supposed that the solitary wave changes slowly (*adiabatically*) in time so that the time-derivative of each term u_{nt} is much less than $vu_{n\xi}$. In the next sections we will show that the small parameter, say ϵ, should be introduced in the asymptotic series (2.1) in a different manner for some particular problems related to the generalized KdV equation (1.2).

The leading order of the asymptotic series (2.1) coincides with the profile of the solitary wave solution $u_0(\xi; v)$. This solution can be found from the equation,

$$u_{0\xi\xi} + f(u_0) - vu_0 = 0, \tag{2.2}$$

with vanishing boundary conditions at infinity, $u_0 \to 0$ as $\xi \to \pm\infty$. We assume that the function $f(u)$ vanishes faster than u for small u and admits the existence of exponentially localized solutions of (2.2) [see conditions (C.1),(C.2) in [6] which are necessary for existence of such solutions].

Substitution of (2.1) into (1.2) reduces the generalized KdV equation to a set of linear inhomogeneous equations for the higher-order corrections u_1, u_2 and so on,

$$\begin{aligned}(\mathbf{L}u_n)_\xi &= \mathbf{F}_n(u_0, u_1, \ldots, u_{n-1}) \\ &= F_n(\xi; v, v_t, \ldots, \underbrace{v_{tt\ldots t}}_{n}), \quad n \geq 1.\end{aligned} \tag{2.3}$$

Here $\mathbf{L}$ is the linear operator associated with the nonlinear equation (2.2),

$$\mathbf{L} = \partial_\xi^2 + f'(u_0) - v, \tag{2.4}$$

and the right-hand side operators $\mathbf{F}_n$ can be expressed through terms of lower order. For our purposes, we write down the explicit form only for first two operators $\mathbf{F}_n$,

$$\mathbf{F}_1 = -u_{0t}, \tag{2.5a}$$

$$\mathbf{F}_2 = -u_{1t} - \tfrac{1}{2}\left(f''(u_0)u_1^2\right)_\xi. \tag{2.5b}$$

Being general solutions of the linear inhomogeneous equations (2.3), the functions u_n, $n \geq 1$, contain terms which diverge exponentially as $\xi \to \pm\infty$. To avoid this, it is necessary to impose a compatability condition on F_n resulting in an equation for the variation of $v(t)$. This is most easily obtained by multiplying (2.3) by u_0, and integrating with respect to ξ over the whole ξ-axis. Thus we get the *compatability conditions* [15],

$$\int_{-\infty}^{+\infty} u_0(\xi; v) F_n(\xi; v, v_t, \ldots, \underbrace{v_{tt\ldots t}}_{n}) \,\mathrm{d}\xi = 0, \quad n \geq 1, \tag{2.6}$$

which gives an n-order differential equation for the velocity $v(t)$. So, substituting F_1 from (2.5a) into (2.6) we immediately obtain a first-order equation,

$$\frac{\mathrm{d}P_s}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}t} = 0, \tag{2.7}$$

where the momentum of the solitary wave $P_s(v)$ is given by (1.3b). It follows from (2.7) that nontrivial solitary wave dynamics ($v \neq$ const.) can be described by the asymptotic theory only if $\mathrm{d}P_s/\mathrm{d}v = 0$, i.e. in the vicinity of the instability threshold. Of course, if the momentum P_s depends on v, condition (2.7) cannot be *exactly* fulfilled. However, near the critical value of the velocity the left-hand side of (2.7) remains *small* and we can remove this condition to the next, second-order approximation. This procedure leads to a nontrivial, second-order differential equation for $v(t)$. Note that for the critical collapse models the momentum P_s does not depend on v and, in this case, equality (2.7) is satisfied *identically*.

Thus, the first-order term u_1 does not contain exponentially diverging terms near the instability threshold. However, it still possesses a *shelf*, i.e. a nonlocalized part tending to a constant at infinity [15]. Integrating (2.3) at $n = 1$ we find that the first-order term tends to the following values at infinity:

$$u_1 \to u^{\pm} = C \pm \frac{v_t}{2v}\frac{\mathrm{d}M_s}{\mathrm{d}v} \quad \text{as } \xi \to \pm\infty. \tag{2.8}$$

Here $M_s(v)$ is the integral of the *solitary wave mass* given by

$$M_s(v) = \int_{-\infty}^{+\infty} u_0(\xi; v)\,\mathrm{d}\xi \tag{2.9}$$

and C is an arbitrary integration constant. The appearance of shelfs in asymptotic expansions for the KdV-type solitary waves has been discussed by many authors (see, e.g., [14,15,21]). As is well known, this effect is associated with generation of radiative waves escaping from the solitary wave. From a formal point of view, the nonlocalized part of the asymptotic series (2.1) in the *inner* region ($\xi \sim O(1)$) should be matched with a corresponding series in the *outer* region ($\xi \to \pm\infty$). In this paper we avoid the further details of this technique but reproduce only the principal results.

Here, the radiation field remains behind the core of the KdV-type solitary wave and, therefore, we should choose the constant C in the form

$$C = -\frac{v_t}{2v}\frac{\mathrm{d}M_s}{\mathrm{d}v}. \tag{2.10}$$

In this case, there is no wave field in front of the solitary wave. Furthermore, the radiation field behind propagates with the constant velocity c in the time scale of the solitary wave dynamics. The profile of such a stationary radiation field $u_r(x)$ can be found from (2.8),

$$u_r = -\frac{v_t}{v}\frac{\mathrm{d}M_s}{\mathrm{d}v}\bigg|_{t=T_s(x)} = -\frac{\partial M_s[v(t = T_s(x))]}{\partial x}, \tag{2.11}$$

where $T_s(x)$ is the inverse function to the solitary wave coordinate $X_s(t) = \int_0^t v(t')\,\mathrm{d}t'$ in the reference frame moving with the limiting phase speed c.

In order to find the governing equation for $v(t)$ we need to evaluate the first-order term u_1. Let us present it in the implicit form of a sum of even and odd components,

$$u_1 = C u_{1\mathrm{ev}}(\xi; v) + v_t u_{1\mathrm{od}}(\xi; v), \tag{2.12}$$

where C is given by (2.10) and $u_{1\mathrm{ev}}$, $u_{1\mathrm{od}}$ satisfy the equations,

$$\mathbf{L}u_{1\mathrm{ev}} = -v, \qquad \mathbf{L}u_{1\mathrm{od}} = -\int_0^{\xi} \frac{\partial u_0}{\partial v}\,\mathrm{d}\xi. \tag{2.13a,b}$$

Now we turn to the second-order approximation and combine the compatability conditions (2.6) for $n = 1$ and $n = 2$ to get

$$\int_{-\infty}^{+\infty} \left(u_0 u_{0t} + u_0 u_{1t} + \frac{1}{2} u_0 \left[f''(u_0) u_1^2 \right]_{\xi} \right) \mathrm{d}\xi = 0. \tag{2.14}$$

Next, we integrate the last term in this equation by parts and use Eq. (2.3) for $n = 1$ to reduce (2.14) to the form,

$$\frac{\mathrm{d}P_0}{\mathrm{d}t} = \frac{v}{2}\left(u^{+2} - u^{-2}\right), \tag{2.15}$$

where $P_0(t) = P_s(v) + C\int_{-\infty}^{+\infty} u_0 u_{1\mathrm{ev}}\,\mathrm{d}\xi$. In Appendix A we show that P_0 is a part of the *momentum* localized at the perturbed solitary wave and, generally, the governing equation (2.15) can also be obtained from the momentum conservation law of the generalized KdV equation (see also [22]).

Eq. (2.15) can be rewritten in an explicit form as a second-order differential equation for $v(t)$. To do this, we first note that

$$\mathbf{L}\frac{\partial u_0}{\partial v} = u_0. \tag{2.16}$$

Then multiplication of (2.13a) with $\partial u_0/\partial v$, followed by integration with respect to ξ and substitution of (2.16) leads to the expression,

$$\int_{-\infty}^{+\infty} u_0 u_{1\mathrm{ev}}\,\mathrm{d}\xi = -v\frac{\mathrm{d}M_\mathrm{s}}{\mathrm{d}v}. \tag{2.17}$$

Combining formulas (2.8),(2.10),(2.15), and (2.17) we finally obtain the equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[P_\mathrm{s}(v) + \frac{1}{2}\left(\frac{\mathrm{d}M_\mathrm{s}}{\mathrm{d}v}\right)^2\frac{\mathrm{d}v}{\mathrm{d}t}\right] = -\frac{1}{2v}\left(\frac{\mathrm{d}M_\mathrm{s}}{\mathrm{d}v}\right)^2\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2. \tag{2.18}$$

A nonlinear second-order differential equation cannot generally be integrated. However, (2.18) has the first integral,

$$\frac{v}{2}\left(\frac{\mathrm{d}M_\mathrm{s}}{\mathrm{d}v}\right)^2\frac{\mathrm{d}v}{\mathrm{d}t} = H_\mathrm{s}(v) - H_0, \tag{2.19}$$

where H_0 is an integration constant and $H_\mathrm{s}(v)$ is given by

$$\mathrm{d}H_\mathrm{s}(v) = -v\,\mathrm{d}P_\mathrm{s}(v). \tag{2.20}$$

It is well-known [10] that this equation defines the *Hamiltonian* $H_\mathrm{s}(v)$ calculated at the solitary wave solution $u_0(\xi; v)$,

$$H_\mathrm{s}(v) = \int_{-\infty}^{+\infty}\left[\frac{1}{2}(u_{0\xi})^2 - \int_0^{u_0} f(u)\,\mathrm{d}u\right]\mathrm{d}\xi. \tag{2.21}$$

Thus, our asymptotic theory for the dynamics of the KdV-type solitary waves reduces in the vicinity of the instability threshold to the first-order differential equation (2.19) with only one parameter H_0. In Appendix A we show that the conserved quantity H_0 represents the Hamiltonian of the perturbed solitary wave while (2.19) is equivalent to the energy conservation law for the generalized KdV equation (1.2). Furthermore, we note the remarkable relation between the momentum and energy of the perturbed solitary wave which follows from (2.18) and (2.19),

$$\Delta H + v\Delta P = 0, \tag{2.22}$$

where $\Delta H = H_0 - H_\mathrm{s}(v)$ and $\Delta P = P_0(t) - P_\mathrm{s}(v)$.

The system of asymptotic equations (2.11),(2.18), and (2.19) allows us to consider two *different* problems. First, if the dependence $H_\mathrm{s}(v)$ has an extremal point for a special, critical value of the velocity v we can simplify (2.19) in a weakly nonlinear (quadratic) approximation and analyze the local and global features of the solitary wave instability in the generalized KdV equation (1.2). This is discussed in Section 3. On the other hand, when $H_\mathrm{s}(v)$ is identically equal to zero, the system (2.11) and (2.19) represents a governing model for the critical collapse of the KdV-type solitary waves [16,17] and leads to approximate solutions describing the collapse development. These solutions are presented in Section 4.

3. Linear and weakly nonlinear analysis of the KdV-type solitary wave instability

Using the asymptotic equation (2.18) we can reproduce the results of the linear theory for solitary wave instability of the generalized KdV equation [6–10]. Let the stationary solitary wave solution have the constant velocity v_0. Then, in a linear approximation, when $v(t) = v_0 + v_1\exp(\lambda t)$, (2.18) transforms to the following linear algebraic equation for the eigenvalue λ:

$$D(\lambda; v_0) \equiv \lambda\left[\left.\frac{\mathrm{d}P_\mathrm{s}}{\mathrm{d}v}\right|_{v=v_0} + \frac{1}{2}\left(\left.\frac{\mathrm{d}M_\mathrm{s}}{\mathrm{d}v}\right|_{v=v_0}\right)^2\lambda\right] = 0. \tag{3.1}$$

There are two roots for λ. The first is always zero and it is associated with a spatial translation of the solitary wave with respect to the ξ-axis. However, the second root is generally different from zero. It indicates an appearance of solitary wave instability in the parameter region where the slope $\mathrm{d}P_s/\mathrm{d}v$ becomes negative. In this case, there exists a real positive eigenvalue λ.

In the critical case, when $\mathrm{d}P_s/\mathrm{d}v|_{v=v_0} = 0$, the eigenvalue λ has a double zero and there is instead a weak, power-like instability of the solitary wave (see [8b]) so that $v(t) = v_0 + v_1 t$. Note that in this case the asymptotic series (2.1) is truncated at the first two terms within the linear approximation and our approach allows us to obtain an exact but implicit solution to the linearized equation for the solitary wave background $\tilde{u}(\xi, t; v_0) = u(\xi, t) - u_0(\xi; v_0)$,

$$\tilde{u} = v_1 \left[u_1(\xi; v_0) + \left.\frac{\partial u_0}{\partial v}\right|_{v=v_0} t - \frac{1}{2} \left.\frac{\partial u_0}{\partial \xi}\right|_{v=v_0} t^2 \right]. \tag{3.2}$$

Here the function u_1 can be found by means of inversion of the linear equation,

$$(\mathbf{L}u_1)_\xi = -\frac{\partial u_0}{\partial v}. \tag{3.3}$$

When $\mathrm{d}P_s/\mathrm{d}v|_{v=v_0}$ is nonzero but small, the asymptotic series (2.1) is not truncated even if we neglect the nonlinear terms. However, in this case all the higher-order correction terms u_n in (2.1) are proportional to powers of λ. Therefore, in the linear approximation the series (2.1) represents an asymptotic solution of the linear problem while the expansion of the exact algebraic equation $D(\lambda; v_0) = 0$ given by (3.1) determines approximately the eigenvalue of the unstable linear mode.

These results completely coincide with the general criterion for solitary wave instability of the generalized KdV equation (1.2) [6–9] and with the linear bifurcation analysis of Pego and Weinstein [10]. However, our asymptotic theory enables us to investigate not only the linear properties of this bifurcation but also the long term, essentially nonlinear dynamics of the unstable solitary waves. Moreover, if the linear theory does not possess any localized eigenfunction for positive $\mathrm{d}P_s/\mathrm{d}v$ (see [10]) the nonlinear asymptotic analysis of the solitary wave dynamics works equally well for both the branches of the function $P_s(v)$. This fact can be explained by a slow divergence of the linear eigenfunction near the bifurcation point. At the critical point, this divergence degenerates into a shelf produced by the linear equation (3.3) [see (2.8)]. Such secularly divergent terms of the asymptotic expansion (2.1) for the linear eigenfunction can be removed by means of matching with the corresponding expansion for the radiation field outside the solitary wave core [21].

Next we go beyond the linear theory and consider nonlinear effects on the development of solitary wave instability. Although the extremal points defining the instability threshold for the curves $P_s(v)$ and $H_s(v)$ coincide, the role of these integral invariants is different for the dynamics of the KdV-type solitary waves. Because of the radiation, the momentum invariant P_0 permanently decreases in time and the final stage for long-term solitary wave dynamics cannot be predicted from the curve $P_s(v)$. On the other hand, the energy H_0 is constant for the time scale of the wave evolution and therefore the curve $H_s(v)$ is a sufficient tool to analyze different scenarios of wave dynamics in the framework of the asymptotic equation (2.19).

Since our asymptotic approach is valid only near the instability threshold, we can approximate the function $H_s(v)$ in the general case by a Taylor-series expansion. To do this, we now introduce explicitly a small parameter ϵ which describes the slow time scale of the instability growth rate near the bifurcation point,

$$T = \epsilon t, \qquad v(t) = v_0 + \epsilon v_1(T). \tag{3.4}$$

Here T is the slow time for the solitary wave evolution, v_0 is constant which corresponds to the initial value of the energy H_0 so that $H_s(v_0) = H_0$, and v_1 is a slowly varying correction to the solitary wave velocity. Using (3.4) we reduce the governing equation (2.19) to a weakly nonlinear (quadratic) model,

$$\frac{v_0}{2} \left(\left.\frac{\mathrm{d}M_s}{\mathrm{d}v}\right|_{v=v_0} \right)^2 \frac{\mathrm{d}v_1}{\mathrm{d}T} = \frac{1}{\epsilon} \left.\frac{\mathrm{d}H_s}{\mathrm{d}v}\right|_{v=v_0} v_1 + \frac{1}{2} \left.\frac{\mathrm{d}^2 H_s}{\mathrm{d}v^2}\right|_{v=v_0} v_1^2. \tag{3.5}$$

Here we have supposed that $\mathrm{d}H_s/\mathrm{d}v|_{v=v_0}$ is positive and small (of the order of $\mathrm{O}(\epsilon)$). It follows from (3.5) that the evolution of the unstable solitary waves essentially depends on the sign of the perturbation. For one case, this evolution is always bounded and it results in a monotonic transition from an unstable state to a stable one realized at the same value of the initial energy H_0. We can find the general solution to (3.5) describing this transition,

$$v_1 = \frac{\Delta V}{1 + c\exp(-\lambda T)}, \tag{3.6}$$

where c is an arbitrary constant determined by the initial condition for v_1, and

$$\lambda = -\left.\frac{2\,\mathrm{d}P_s/\mathrm{d}v}{\epsilon(\mathrm{d}M_s/\mathrm{d}v)^2}\right|_{v=v_0},$$

$$\Delta V = -\left.\frac{2\,\mathrm{d}H_s/\mathrm{d}v}{\epsilon\,\mathrm{d}^2H_s/\mathrm{d}v^2}\right|_{v=v_0}.$$

If $c > 0$, that is the sign of v_1 coincides initially with that of ΔV, then $v_1 \to \Delta V$ as $T \to +\infty$. However, if $c < 0$ and the perturbation v_1 has initially the opposite sign to ΔV, then the evolution of the solitary wave is unbounded and depends on the global behavior of $H_s(v)$ far from the bifurcation point. In the case, when there exists another stable solitary wave solution realized at the same value of H_0, the nonlinear dynamics of the unstable solitary wave will result in a monotonic transition to the stable state. In the opposite case, when the other stable solitary wave solutions do not exist, the perturbed KdV-type solitary wave transforms either to a collapsing ($v \to \infty$) or to a spreading ($v \to 0$) structure.

All these scenarios of the wave dynamics are determined by the dependence of the energy H_s on the velocity v. In Fig. 1(a) and (b) we present typical forms of this curve using, as a particular example, the nonlinear function $f(u)$ in the generalized KdV equation (1.2) in the form, $f = au^{p+1} + bu^{2p+1}$, where b, $p > 0$ (see [23]). For $a > 0$ and $p = 3$ the curve $H_s(v)$ is shown in Fig. 1(a) where different scenarios of solitary wave dynamics are depicted, namely the transition to a stable solitary wave with smaller velocity described by (3.6) for $\Delta V < 0$ [curve 1 in Fig. 1(a)] and the transformation into a collapsing state [curve 2]. The dependence shown in Fig. 1(a) is typical for finite amplitude free surface solitary waves in shallow water [24]. Therefore, although the final equation (2.19) has been derived here only for the generalized KdV equation (1.2), we conjecture that evolution of the solitary waves in shallow water essentially depends only on the dependence of $H_s(v)$ and displays basically the same characteristic features. Indeed, the singularities are well known in water-wave theory (see [24b]).

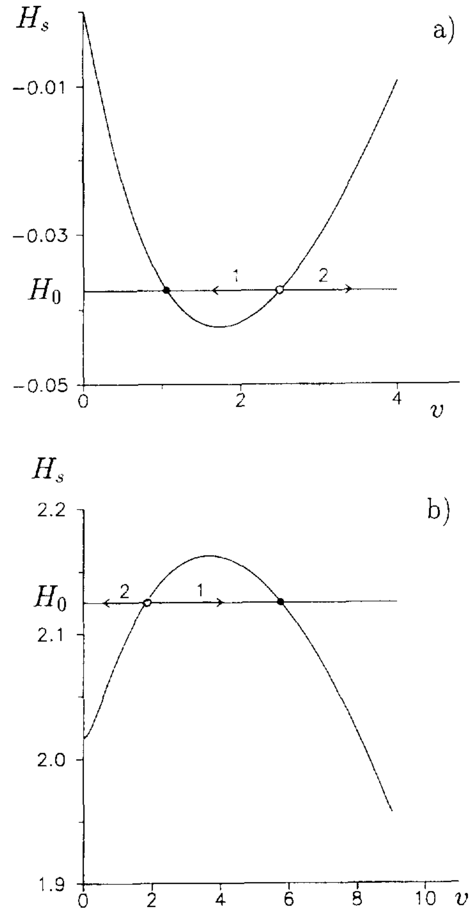

Fig. 1. The dependence of the solitary wave energy H_s on the velocity v for the generalized KdV equation (1.2) with $f = au^{p+1} + bu^{2p+1}$ for (a) $a = 5$, $b = 4$, and $p = 3$; (b) $a = -3.5$, $b = 2.5$, and $p = 1.5$. H_0 stands for an initial value of the energy of the perturbed solitary wave. The dot points designate the unstable solitary wave solution while the black points designate the stable solutions. The curve 1 represents the transition from unstable to stable solitary waves described by the asymptotic formula (3.6). The curve 2 represents the unbounded scenario of the solitary wave dynamics, either tranformation into collapsing states (a) or decay into quasilinear dispersive wave packets (b).

For $a < 0$ and $p = \frac{3}{2}$ the curve $H_s(v)$ is shown in Fig. 1(b). In this case, we also find two branches of the solitary wave solutions with the same value of the energy but the stable branch now has greater velocities than the unstable one. As a result, unstable small-amplitude solitary waves transform either into stable large-amplitude solitary waves according to (3.6) with $\Delta V > 0$ [curve 1 in Fig. 1(b)] or into packets of quasi-linear dispersive waves [curve 2].

4. Critical collapse in one dimension

4.1. Approximate solutions of the asymptotic theory

Here we apply the general theory developed in Section 2 to the power-like KdV model (1.2) and (1.4). In the reference frame moving with the limiting phase speed c this model takes the form

$$u_t + \tfrac{1}{2}(p+1)(p+2)u^p u_x + u_{xxx} = 0. \tag{4.1}$$

Solitary wave solutions of this equation can be found explicitly,

$$u_0(\xi; v) = \left(\sqrt{v}\,\mathrm{sech}\left[\tfrac{1}{2}p\sqrt{v}\xi\right]\right)^{2/p}. \tag{4.2}$$

They are known to be unstable for $p \geq 4$ [6–10]. Our theory is valid in the critical case $p = 4$ where the compatability condition reduced to (2.7) is identically satisfied. In this case, the asymptotic series (2.1) can be rewritten in the form of an expansion in powers of ϵ, where the small parameter ϵ defines the slow time scale of the solitary wave evolution,

$$T = \epsilon t, \qquad v = v(T).$$

Furthermore, we can easily find from (1.3b), (2.9) and (2.21) the dependence of the invariants on the velocity,

$$M_s(v) = \frac{m}{v^{1/4}}, \quad P_s(v) = P_{cr}, \quad H_s(v) = 0, \tag{4.3}$$

where $m = \Gamma^2(1/4)/(2\sqrt{2\pi})$, $P_{cr} = \frac{1}{4}\pi$, and $\Gamma(z)$ is the Gamma function. The governing equation (2.19) reduces with the help of (4.3) to the following explicit form:

$$\frac{m^2}{32v^{3/2}}\frac{dv}{dT} = -H_0, \tag{4.4}$$

where H_0 in (2.19) is now replaced by ϵH_0. It is obvious that the initial value of the energy H_0 which depends on the first-order perturbation u_1 to the solitary wave (4.2) determines completely the resulting dynamics. If $H_0 < 0$, the velocity increases indefinitely and it indicates critical collapse of the solitary wave. Otherwise, for $H_0 > 0$, the velocity monotonically decreases. In the case $H_0 = 0$, the localized initial perturbation is self-similar to the solitary wave solution and it does not result in nontrivial dynamics since $v(T) = v(0)$.

It is important to note that the criterion separating two cardinally different scenarios of the wave evolution is directly related to the value of the momentum of the perturbed solitary wave P_0. It follows from (2.22) that the initial values of the momentum $P_0(0)$, velocity $v(0)$, and energy H_0 are related by

$$P_0(0) = P_{cr} - \frac{H_0}{v(0)}. \tag{4.5}$$

Therefore, the collapse develops for $P_0(0) > P_{cr}$ while the decay occurs for $P_0(0) < P_{cr}$. This fact is well known for all models displaying critical collapse phenomena [2b,3].

Next, we integrate (4.4) and obtain the scaling law of the self-similar solitary wave dynamics,

$$v = v_0\left(\frac{\tau}{\tau - T}\right)^2. \tag{4.6}$$

Here $v_0 = v(0)$ and the parameter τ is given by $\tau = -m^2/(16H_0\sqrt{v_0})$. If $\tau > 0$ ($H_0 < 0$) singularities occur at $t = \tau$, while for $\tau < 0$ ($H_0 > 0$) the velocity $v(t)$ decreases monotonically from the initial value v_0. According to the general linear theory in the critical case (see [8b] and formulas (3.2) and (3.3)), the explicit solution (4.6) describes the nonlinear stage of the weak, power-like instability of the solitary wave with respect to small but finite perturbations,

$$v = v_0\left[1 + \frac{2T}{\tau} + \mathrm{O}\left(\frac{T}{\tau}\right)^2\right]. \tag{4.7}$$

Implicit solutions to the linearized problem indicating the existence of such a weak instability of the KdV-type solitary waves were first found by Laedke and Spatschek [8b]. We would like to mention that, in contrast to NLS-type solitary waves [1], this instability is very special because the nonadiabatic perturbation u_1

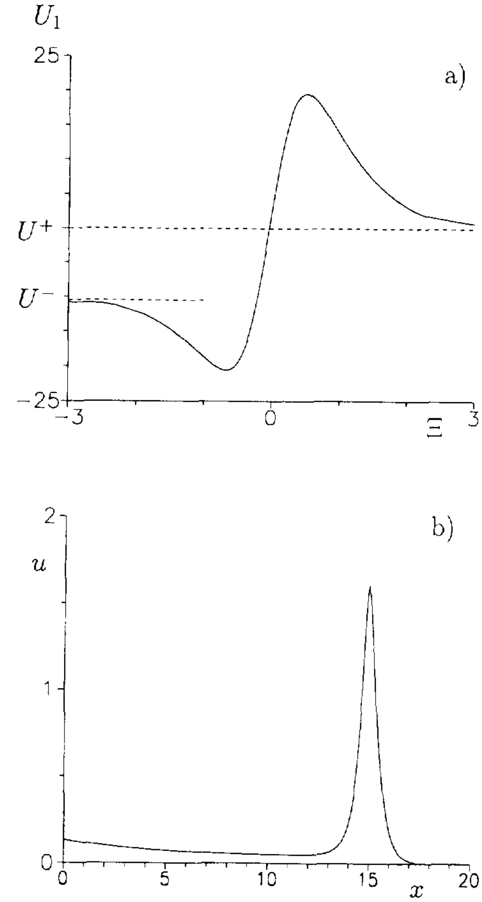

Fig. 2. (a) The profile of the linear eigenfunction $U_1(\Xi)$ found from (3.3) in a parameterless form. The dashed lines represent the asymptotic values $U^{\pm}$ of the eigenfunction in the limit $\Xi \to \pm\infty$, $U^+ = 0$, $U^- \approx -10.48$. (b) The instantaneous profile $u(x)$ of the collapsing solitary wave in the power-like KdV equation (4.1) at $p = 4$ reconstructed from the asymptotic approach at $v_0 = 1$, $\tau = 10$, and $T = 6$.

to the solitary wave u_0 is nonlocalized in the vicinity of the solitary wave location. This is obvious from Fig. 2(a) where we present the solution $u_1(\xi; v)$ to (3.3) for the power-like nonlinear function (1.4) at $p = 4$ which can be expressed in a parameterless form, $u_1 = -v_T/(4v^{9/4})U_1(\Xi)$, $\Xi = \sqrt{v}\xi$. Thus, the instability and collapse of the solitary waves (4.2) in the critical case $p = 4$ can be observed only under the action of initial perturbations of a special type. This has been confirmed in numerical experiments reported in [17] (see formula (5.8) and the discussion there).

Now we consider the radiation escaping from the solitary wave during its self-similar dynamics. Integrating (4.6) we find the coordinate of the moving solitary wave,

$$X_s = \frac{v_0 \tau T}{\tau - T}. \tag{4.8}$$

Then, substitution of (4.6) and (4.8) into (2.11) gives the profile of the radiation field ϵu_r, where

$$u_r = \frac{m}{2}\sqrt{\frac{\tau\sqrt{v_0}}{(X + X_\infty)^3}}, \tag{4.9}$$

$X = \epsilon x$ and $X_\infty = v_0 \tau$.

If the solitary wave is collapsing ($\tau > 0$), the radiation field u_r is distributed between the initial position of the solitary wave (which is specified to be zero) and infinity (where the collapse actually occurs, see (4.8)). Further, the radiation remains positive and small throughout this interval, and the larger solitary waves have the smaller radiation. We reconstruct the evolution of the collapsing solitary wave and the generated radiation field using the leading order of our asymptotic series (2.1) and present it in Fig. 2(b). This picture seems to be in full agreement with direct numerical simulations of the critical power-like KdV model (see [16b, Fig. 7;17, Fig. 8]). Moreover, we can show that the higher-order terms u_n of the asymptotic series (2.1) are vanishing in the vicinity of the solitary wave core at $T \to \tau$ so that the leading order of (2.1), i.e. the solitary wave solution u_0 with the varying parameter $v(T)$, represents the asymptotic profile of the collapsing state. This fact completely agrees with the general theorem for critical collapse [3].

Note that outside of the perturbed solitary wave the emitted wave field does not vanish as $T \to \tau$. Indeed, since for u_r nonlinear effects are negligible, the evolution of the radiation field $u_r(X, \sigma)$ obeys the linearized KdV equation,

$$u_{r\sigma} + u_{rXXX} = 0, \tag{4.10}$$

where $\sigma = \epsilon^3 t$ represent a slow time scale of the radiation evolution and the initial profile $u_r(X, 0)$ is given by (4.9). Thus, the radiation field generated by the collapsing solitary wave [see Fig. 2(b)] transforms to linear dispersive wave packets which completely decay due to dispersive effects.

In the other case, when the solitary wave is decaying ($\tau < 0$), the velocity decreases and, as a result, the solitary wave stops at the point $X = -X_\infty = v_0|\tau|$. Then, expression (4.9) indicates that the radiation field diverges at this point as $u_r \sim (X + X_\infty)^{-3/2}$. Of course, this divergence of the profile of the radiation field is not a real singularity of Eq. (4.1). For small values of the velocity our asymptotic theory becomes invalid because the solitary wave width grows indefinitely. In this case, it is impossible to distinguish the weakly localized solitary wave and the extended radiation field. As a result, the first-order correction term u_1 (and terms of higher order) grow secularly like $\sim T^{3/2}$. Therefore, the final stage of the self-similar spreading of the perturbed solitary wave remains unclear in the framework of our approach.

We conclude that the structure of the collapsing localized perturbations is self-similar to the solitary wave solutions with varying parameters according to the theorem for critical collapse [3]. However, this theorem cannot be directly used for predicting the scaling laws of the self-similar solitary wave dynamics while our approach enables us to construct approximate analytical solutions for the KdV-type solitary wave collapse *and* the generation of the radiative waves. In Section 4.2 we compare our results with other approaches used for the derivation of the scaling laws of the critical collapse in the power-like KdV equation (4.1).

4.2. Discussion

The scaling laws for the solitary wave collapse in the power-like KdV equation (4.1) were first constructed by Blaha et al. [16a] by means of a *variational principle*. This approach leads to a reduction of the power-like KdV equation (4.1) to a set of ordinary differential equations for the parameters of the trial functions used for minimization of the Lagrangian function. The trial functions have to be localized in order to make all integrals converge. Then, using conservation laws of the momentum and energy evaluated for these localized trial functions Blaha et al. [16a] integrated their 'variational' equations for $p \geq 4$ and predicted the scaling laws for singularity formation. In the critical case $p = 4$, their results can be expressed through our parameter v determining the amplitude, width and velocity of the solitary wave solution (4.2) as follows:

$$v = v_0 \left(\frac{\tau}{\tau - t} \right)^{2/3}. \tag{4.11}$$

We see that this result differs from our formula (4.6). The reason for this disagreement is explained by the *essential necessity* to include the radiation field when one considers solitary wave collapse in these KdV-type models. The radiation field cannot be expressed by the localized trial functions used in the variational approach. Moreover, the radiation leads to a decrease in the momentum of the perturbed solitary wave. As a result, the correct scaling laws of the KdV-type solitary wave collapse (4.6) are three times as large as the result obtained from the variational approach.

The other technique which is often used for the analysis of collapse formation is to find *exact self-similar reductions* of the original equations. For the generalized NLS model (1.1) this technique has been shown to be a good approximation to the early stage of critical collapse [2b]. However, for the KdV-type models this technique also fails to describe the correct scaling laws for the solitary wave collapse. Indeed, the power-like KdV model (4.1) for $p = 4$ can be reduced to an ordinary differential equation by means of the following self-similar substitution,

$$u(x,t) = [v(t)]^{1/4} U(z), \quad z = [v(t)]^{1/2} (x - X_s(t)). \tag{4.12}$$

Here $v(t) = dX_s/dt$ is given by formula (4.11), and the function $U(z)$ satisfies the equation,

$$U_{zzz} + 15U^4 U_z - U_z = -\mu (U + 2zU_z), \tag{4.13}$$

where $\mu = 1/(6\tau v_0^{3/2})$. We note that the left-hand side of (4.13) admits a parameterless solution in the form $U_0 = u_0(z; 1)$ while the right-hand side can be very small in the *inner region* where $|z| \ll 1/\mu$. If $\mu \ll 1$ solutions to (4.13) in this inner region can be analyzed by means of the same asymptotic technique used in Section 2 but with a fixed law for the velocity $v(t)$ given by (4.11),

$$U = U_0(z) + \mu U_1(z) + O(\mu^2). \quad (4.14)$$

Note that the right-hand side of (4.13) is just $\partial u/\partial v$ expressed in a parameterless form so that the second term of expansion (4.14) possesses a nonlocalized secular term [see formula (2.8)] which should be matched with the corresponding asymptotic expansion for the *outer region* where $|z| \gg O(1/\mu)$. However, we have found in Section 2 that the asymptotic series (4.14) also generates exponentially diverging terms unless the compatability conditions (2.6) are met. Since the scaling law (4.11) does not satisfy the compatability condition (2.15), we have to make another choice for the integration constant C than that given by (2.10). Suppose we specify $C = 0$ so that $u^+ = -u^-$. Then, we could proceed to the third-order approximation where the scaling law (4.11) actually appears. Therefore, this asymptotic expansion implies that the scaling law (4.11) can only be realized if the extended radiation wave field is also given initially *in front* of the solitary wave. It is obvious that such critical collapse is induced by a very special initial condition, rather than through its own dynamics. We believe that this difficulty explains the poor agreement between the integral characteristics of the critical collapse calculated numerically and analytically from the self-similar reductions of the power-like KdV equation (4.1) for $p = 4$ (cf. [17, Tables 17 and 20]).

Thus, both the variational approach and the method of self-similar reductions fail to describe the correct scaling law (4.6) of the critical KdV-type solitary wave collapse. The first approach neglects the nonlocalized component of radiation field while the second implies the existence of incoming radiation in front of the collapsing solitary wave. Only the direct asymptotic theory developed in Section 2 enables us to find the correct scaling laws for the self-similar solitary wave dynamics.

5. Critical collapse in two dimensions

It is well known (see, e.g., [13,14]) that the dynamics of KdV-type solitary waves in two or higher dimensions is also accompanied by strong radiation. Thus, the asymptotic approach developed for the generalized KdV equation (1.2) should be modified when we analyze solitary wave dynamics in two dimensions. In this section we consider critical solitary wave collapse in a special, anisotropic generalization of (1.2) which was introduced by Zakharov and Kuznetsov [19] for the description of plasma waves in a strong magnetic field. If the plasma waves propagate at a special, critical angle to the magnetic field, the usual quadratic nonlinear term in the long-wave evolution equation vanishes [20], and the governing model equation can be rewritten in the modified form (1.5).

The solitary wave solutions of (1.5) are expressed by function $u_0(\xi, y; v)$, where $\xi = x - vt$, which satisfies the equation

$$u_{0\xi\xi} + u_{0yy} + 2u_0^3 - vu_0 = 0. \quad (5.1)$$

Note that this equation coincides with that for stationary optical (bright) solitary waves in the two-dimensional NLS equation [1,2]. We consider only the *ground state* solutions which are radially symmetrical and nodeless, $u_0(\xi, y; v) = u_0(r; v)$, where $r = \sqrt{\xi^2 + y^2}$ and the function $u_0(r; v)$ is positive everywhere. Fig. 3(a) presents the ground state solution at $v = 1$ found numerically by a standard 'shooting' scheme.

Our aim is to describe the self-similar dynamics of localized wave perturbations which evolve close to the stationary wave solution $u_0(r; v)$, with a slowly varying parameter v. Thus, we introduce a slow time scale $T = \epsilon t$, where $\epsilon \ll 1$, so that

$$v = v(T), \qquad \xi = x - \frac{1}{\epsilon}\int_0^T v(T')\,dT'. \quad (5.2)$$

Then, we seek solutions to the mZK equation (1.5) in the form of an asymptotic series

$$u = u_0(\xi, y; v) + \epsilon u_1(\xi, y; v, v_T) + \epsilon^2 u_2(\xi, y; v, v_T, v_{TT}) + O(\epsilon^3). \quad (5.3)$$

In the first-order approximation, we obtain the following linear equation for function u_1:

$$\left(u_{1\xi\xi} + u_{1yy} + 6u_0^2 u_1 - vu_1\right)_\xi = -v_T \frac{\partial u_0}{\partial v}. \quad (5.4)$$

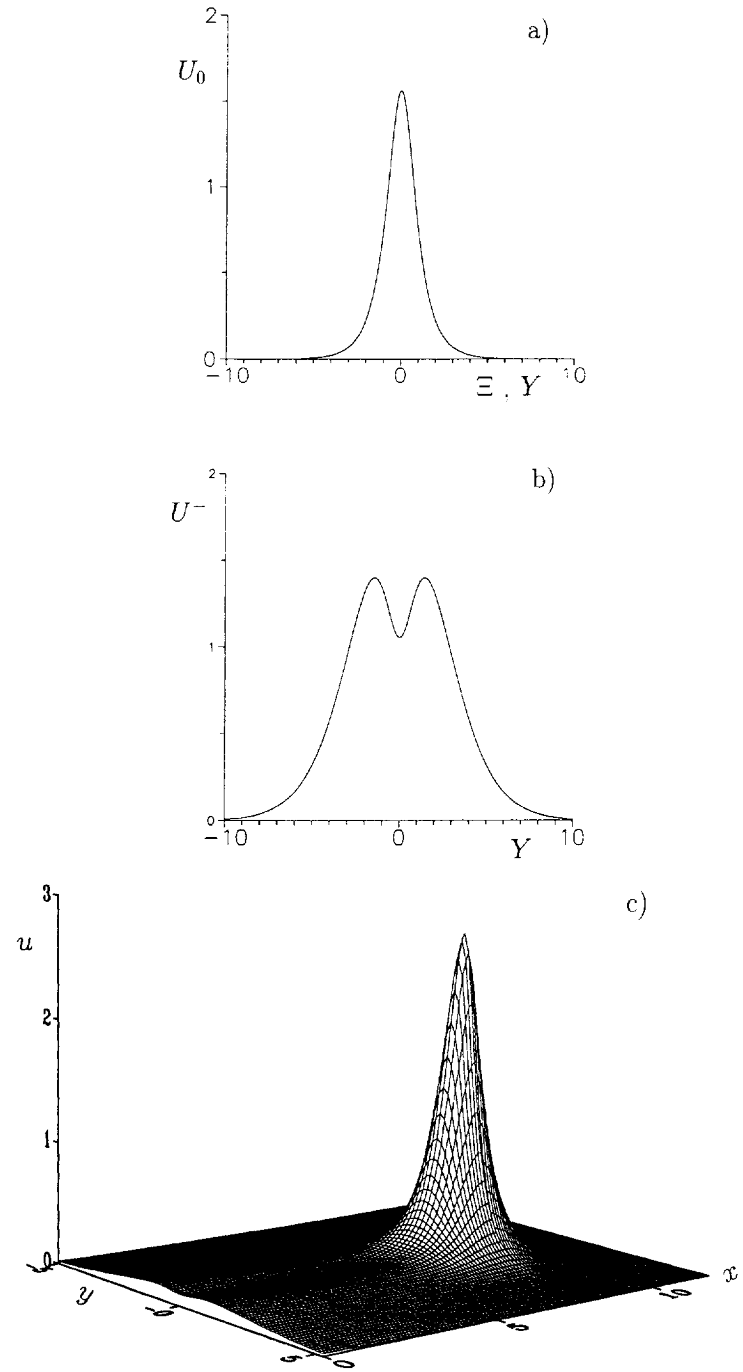

Fig. 3. (a) The projection of the ground state solution $U_0(\Xi, Y)$ to (5.1) at Ξ and Y-axes in the parameterless form (5.15). (b) The transverse profile of the radiation field $U^-(Y)$ generated behind the solitary wave core according to (5.8) in the parameterless form (5.15). (c) The development of two-dimensional solitary wave collapse in the mZK equation (1.5) in the framework of the asymptotic approach at $v_0 = 1$, $\tau = 10$, and $T = 5$.

It can be readily checked from the scaling properties of u_0 [see (5.15)] that the linear inhomogeneous equation (5.4) can be solved for u_1 without the necessity of a compatability condition to exclude exponentially divergent terms. However, the solution u_1 still contains a nonlocalized part (shelf) which now depends on the coordinate y which is transverse to the direction of the solitary wave propagation. In order to find the shelf we integrate (5.4) with respect to ξ,

$$u_{1\xi\xi} + u_{1yy} + 6u_0^2 u_1 - vu_1 = -vC(y,T) - v_T \int_0^{\xi} \frac{\partial u_0}{\partial v}\,\mathrm{d}\xi, \tag{5.5}$$

where C is an integration constant which depends on y and T. Using arguments similar to those discussed in Section 2 we choose this constant in the form

$$C = -\frac{v_T}{2v}\frac{\mathrm{d}m_{xs}}{\mathrm{d}v}, \tag{5.6}$$

where $m_{xs}(y; v)$ is a density of the solitary wave mass defined by

$$m_{xs} = \int_{-\infty}^{+\infty} u_0(\xi, y; v)\,\mathrm{d}\xi. \tag{5.7}$$

Then, considering solutions of (5.5) as $\xi \to \pm\infty$ we find that the radiation field in front of the solitary wave is absent, while behind it obeys the differential equation for $u^-(y; v, v_T) = \lim_{\xi\to-\infty} u_1(\xi, y; v, v_t)$,

$$u^-_{yy} - vu^- = v_T \frac{\mathrm{d}m_{xs}}{\mathrm{d}v}, \tag{5.8}$$

with the boundary conditions $u^- \to 0$ as $y \to \pm\infty$. We have found numerically the profile of the radiation field u^- in a parameterless form [see formula (5.15)] and present it in Fig. 3(b). Note that in the two-dimensional mZK equation the radiation field is effectively excited only in a direction parallel to the solitary wave propagation direction, but the maximum of the radiation field is shifted at some distance from the ξ-axis. The other novel feature of the solitary wave dynamics in two dimensions is the existence of a nonlocal equation governing the radiation field generation.

Next, we consider the second-order approximation and obtain the following linear equation for u_2:

$$\left(u_{2\xi\xi} + u_{2yy} + 6u_0^2 u_2 - vu_2\right)_\xi = -u_{1T} - 6\left(u_0 u_1^2\right)_\xi. \tag{5.9}$$

The compatability condition for this equation can be transformed to the form of a balance equation for the x-projection of the momentum of the perturbed solitary wave, $P = \frac{1}{2}\iint_{-\infty}^{+\infty} u^2\,\mathrm{d}x\,\mathrm{d}y$, that is,

$$\frac{\mathrm{d}\Delta P}{\mathrm{d}T} = -\frac{1}{2}\int_{-\infty}^{+\infty}\left(vu^{-2} + u_y^{-2}\right)\mathrm{d}y, \tag{5.10}$$

where

$$\Delta P = \iint_{-\infty}^{+\infty} u_0 u_1\,\mathrm{d}\xi\,\mathrm{d}y = \frac{1}{2}v_T \int_{-\infty}^{+\infty}\left(\frac{\mathrm{d}m_{xs}}{\mathrm{d}v}\right)^2 \mathrm{d}y. \tag{5.11}$$

The last equality can be directly established from (5.1) and (5.5). The quantity ΔP stands for the deviation of the x-projection of the momentum of the perturbed solitary wave P from the critical value P_{cr} realized for a stationary solution. Using (5.8) and (5.11) we rewrite the governing equation for the solitary wave velocity in the form

$$\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\mathrm{d}v}{\mathrm{d}T}\int_{-\infty}^{+\infty}\left(\frac{\mathrm{d}m_{xs}}{\mathrm{d}v}\right)^2\mathrm{d}y\right) = \frac{\mathrm{d}v}{\mathrm{d}T}\int_{-\infty}^{+\infty} u^- \frac{\mathrm{d}m_{xs}}{\mathrm{d}v}\,\mathrm{d}y. \tag{5.12}$$

Besides the momentum balance equation, we can also find the balance equation for the energy, $H = \frac{1}{2}\iint_{-\infty}^{+\infty}\left(u_x^2 + u_y^2 - u^4\right)\mathrm{d}x\,\mathrm{d}y$. This equation has the form

$$\frac{\mathrm{d}H_0}{\mathrm{d}T} = -\frac{1}{2}\int_{-\infty}^{+\infty}\left(vu_y^{-2} + u_{yy}^{-2}\right)\mathrm{d}y = -\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}T}\int_{-\infty}^{+\infty} u^-_{yy}\frac{\mathrm{d}m_{xs}}{\mathrm{d}v}\,\mathrm{d}y, \tag{5.13}$$

where $H_0(T)$ is the energy of the perturbed solitary wave which is related to the momentum $\Delta P(T)$ according to the formula,

$$H_0 = -v\Delta P. \tag{5.14}$$

Of course, (5.13) also follows directly from (5.8), (5.10), and (5.14). However, in contrast to the one-dimensional case, the energy of the perturbed solitary wave $H_0(T)$ is no longer constant during the solitary wave evolution for the time scales considered. Consequently, the asymptotic equation (5.12) has no first integral which can simplify its analysis. The reason for this is caused by a stronger radiation from the KdV-type solitary waves in two dimensions.

Nevertheless, for the mZK equation there exists a special scaling transformation which allows us to rewrite (5.12) in an explicit form with respect to $v(T)$. Indeed, we note that the solutions to (5.1) and (5.8) and the density (5.7) can be expressed in a parameterless form:

$$\begin{aligned} u_0(\xi, y; v) &= \sqrt{v}U_0(\Xi, Y), \\ u^-(y; v, v_T) &= \frac{v_T}{2v^2}U^-(Y), \\ m_{xs}(y; v) &= M_{xs}(Y), \end{aligned} \tag{5.15}$$

where $\Xi = \sqrt{v}\xi$, $Y = \sqrt{v}y$. Using this transformation we reduce (5.12) to the form

$$\frac{\mathrm{d}}{\mathrm{d}T}\left[\frac{1}{v^{5/2}}\frac{\mathrm{d}v}{\mathrm{d}T}\right] = \alpha\frac{1}{v^{7/2}}\left(\frac{\mathrm{d}v}{\mathrm{d}T}\right)^2, \tag{5.16}$$

where

$$\alpha = \left(\int_{-\infty}^{+\infty} Y^2\left(\frac{\mathrm{d}M_{xs}}{\mathrm{d}Y}\right)^2 \mathrm{d}Y\right)^{-1} \times \left(\int_{-\infty}^{+\infty} U^- Y\frac{\mathrm{d}M_{xs}}{\mathrm{d}Y}\,\mathrm{d}Y\right).$$

The coefficient α has been calculated numerically, $\alpha \approx -0.83$. Then, the general solution to (5.16) is

$$v(T) = v_0\left(\frac{\tau}{\tau - T}\right)^{\gamma}, \tag{5.17}$$

with $\gamma = 2/(3 + 2\alpha) \approx 1.50$. Similarly to the one-dimensional case, there is a self-similar collapse of the solitary waves for $\tau > 0$ and self-similar decay for $\tau < 0$. It follows from (5.11) and (5.14) that critical collapse in two dimensions is possible only if $\Delta P(0) > 0$ or $H_0(0) < 0$ in agreement with the general theory [3].

Finally, let us consider the mZK equation (1.5) outside the solitary wave core, $u \to \epsilon u_r(X, y, T)$, where $X = \epsilon x$. Then, we find that the radiation field evolves according to the evolution equation,

$$u_{rT} + u_{rXyy} = 0. \tag{5.18}$$

The radiation field is determined in the domain $0 < X < X_s(T)$ and the profile u_r is generated by the perturbed solitary wave at the location $X = X_s(T)$. Using (5.8),(5.15), and (5.17) we obtain this profile of the radiation field

$$u_r = \frac{\eta}{X + X_\infty}U^-\left(\sqrt{v_0}\left[\frac{X + X_\infty}{X_\infty}\right]^{\eta} y\right)$$
$$\text{at } X = X_s(T) = X_\infty\left[\left(\frac{\tau}{\tau - T}\right)^{\gamma - 1} - 1\right], \tag{5.19}$$

where $\eta = \gamma/[2(\gamma - 1)]$ and $X_\infty = v_0\tau/(\gamma - 1)$. Thus, diffractive effects make the evolution of the radiation field escaping from the collapsing solitary wave more complicated in two dimensions so that u_r evolves on the time scale of the solitary wave dynamics. It results in the nonlocal equation (5.8) for the generation of the radiation field and in the slowing down of the rate of the singularity formation compared to the one-dimensional case [see (4.6)].

We can solve (5.18) and (5.19) by a Fourier transform and reconstruct the radiation field at any point X, y for a fixed time T. The instantaneous view of the collapse development found thus from the asymptotic equations (5.15)–(5.19) is presented in Fig. 3(c). Note that the two-dimensional radiation field forms two beams diverging from the solitary wave.

Thus, the rate of critical collapse for these KdV-type solitary waves in two dimensions is slower than in one dimension. This is caused by diffractive effects resulting in the damping not only of the momentum $\Delta P(T)$ of the perturbed solitary wave but also of its energy $H_0(T)$ [see (5.10) and (5.13)]. If the energy were constant we would find the same scaling laws as in (4.6) because in this case, $\alpha = -1$ and $\gamma = 2$. Note that if momentum were conserved, then $\alpha = 0$, and the scaling laws (5.17) of the solitary wave collapse in the mZK equation would reproduce the incorrect result (4.11), $\gamma = \frac{2}{3}$.

6. Summary

We have presented a general approach to investigating solitary wave instability and related critical collapse in KdV-type equations. This approach leads to an asymptotic reduction of the original equation to an evolution equation for the solitary wave parameters. This method can be regarded as an alternative to the variational technique but importantly it can be applied even if the critical collapse generates strong radiation. Note that the asymptotic technique enables us to investigate the *self-consistent* dynamics of the solitary wave and its radiation field.

We have shown that the KdV-type solitary waves are described by an equation for motion of a *dissipative particle* [see (2.19)]. This equation describes monotonic transitions from unstable to stable states if the latter exist. If the stable stationary structures do not exist the evolution equation describes the formation of singularities, or the decay of localized perturbations. In particular, the scaling laws for critical collapse can always be found by means of this technique.

We have found that radiation makes the critical solitary wave collapse faster than the predictions of other approaches which do not take into account the generation of the radiative waves. On the other hand, critical collapse in this case becomes less generic and can be observed only from special *nonlocalized* and *asymmetrical* initial conditions. Moreover, we have found for the mZK equation that critical collapse in two dimensions is accompanied by damping both the momentum and energy of the perturbed solitary waves and this slows down the rate of the singularity formation.

Finally, we note that there are also noncritical types of singularity formation when the collapsing structures are not self-similar to the stationary solitons. Our asymptotic theory is invalid for such noncritical situations. However, even in this case analysis of the scaling properties of the conserved energy and momentum remains very important for description of the collapse formation (see [1,2]) and, therefore, we believe that our technique can be developed for this noncritical case as well. Nevertheless, this problem needs further investigation.

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Appendix A. Derivation of the asymptotic equations from conservation laws

In the reference frame of the solitary wave moving with velocity $c + v$, the conservation laws for mass, momentum and energy of the generalized KdV equation (1.2) can be written in the local form,

$$(u)_t = \left(vu - f(u) - u_{\xi\xi}\right)_\xi , \tag{A.1a}$$

$$\left(\tfrac{1}{2}u^2\right)_t = \left(\tfrac{1}{2}vu^2 + \int_0^u f(u)\,du - uf(u) + \tfrac{1}{2}u_\xi^2 - uu_{\xi\xi}\right)_\xi , \tag{A.1b}$$

$$\left(\tfrac{1}{2}u_\xi^2 - \int_0^u f(u)\,du\right)_t = \left(\tfrac{1}{2}vu_\xi^2 - v\int_0^u f(u)\,du + f(u)u_{\xi\xi} - u_\xi^2 f(u) + \tfrac{1}{2}f^2(u) + \tfrac{1}{2}u_{\xi\xi}^2 - u_\xi u_{\xi\xi\xi}\right)_\xi , \tag{A.1c}$$

Let us integrate these equations with respect to ξ and then expand the function u in the asymptotic series

(2.1). As a result, the mass conservation law (A.1a) leads to the jump of the radiation wave field lying outside the solitary wave at the solitary wave position $X_s(t)$,

$$\left(u^+ - u^-\right)\Big|_{x=X_s(t)} = \frac{v_t}{v}\frac{\mathrm{d}M_s}{\mathrm{d}v}, \tag{A.2}$$

which is equivalent to (2.8). If there is no radiation field in front of the solitary wave, then $u^+ = 0$, and the mass conservation law (A.2) reduces to (2.11).

Next, integration of (A.1b) gives (2.15) and, therefore, the momentum $P_0(t)$ of the localized part of the nonlinear wave field is not conserved in time due to the radiation. Note that the decrease of the momentum P_0 of the perturbed solitary wave is completely compensated by an increase in the momentum of the extended wave field so that *the solitary wave and the radiating waves together form a closed system* which conserves the total momentum (see [14b] for details).

Finally, we analyze the energy conservation law (A.1c). It is obvious that the energy of the radiating waves is of the fourth-order in terms of v_t and, therefore, the energy of the perturbed solitary wave H_0 does not change on the time scales where our asymptotic theory works so that $H_0 = \text{const}$. This leads to the existence of the first integral of (2.18) which is just the energy conservation law for the localized part of the nonlinear wave field. Indeed, analysis of (A.1c) shows that

$$\begin{aligned} H_0 &= H_s(v) + \int_{-\infty}^{+\infty} \left(u_{0\xi}u_{1\xi} - f(u_0)u_1\right) \mathrm{d}\xi \\ &= H_s(v) - vC \int_{-\infty}^{+\infty} u_0 u_{1\mathrm{ev}} \,\mathrm{d}\xi \\ &= H_s(v) - \frac{v}{2}\left(\frac{\mathrm{d}M_s}{\mathrm{d}v}\right)^2 v_t, \end{aligned} \tag{A.3}$$

which coincides with (2.19).

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