ELSEVIER

Physica D 80 (1995) 246–255

PHYSICA D

The structure of the rational solutions to the Boussinesq equation

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Received 13 January 1994; accepted 30 July 1994
Communicated by H. Flaschka

Abstract

Rational solutions of the one-dimensional Boussinesq equation both with zero and nonzero boundary conditions at the infinity are obtained by reducing the known solutions of the Kadomtsev–Petviashvili equation. The structure of the found solutions generalizes a family of rational solutions to the Korteweg–de Vries equation to the case of two-wave processes.

1. Rational solutions, similarly to finite - zone and multi-soliton ones, have an important role in the study of some nonlinear equations in modern theory of nonlinear waves and structures [1 − 8]. Sometimes solitons are expressed by rational functions [1,2]. In other cases, they are used to describe the explode-decay waves [3,4], the motion of the vortexes [5] as well as the particle dynamics in some finite-dimensional Hamiltonian systems [6–8].

Although the existence of some families of rational solutions has been proved for many equations of mathematical physics, there are still no regular methods for finding a complete set of meromorphic solutions of these equations, and even explicit formulas of the found families of solutions cannot always be written. A search for rational solutions to the one-dimensional Boussinesq equation (the equation of a nonlinear string) is one of such problems which have not been solved until now.

The Boussinesq equation describes waves in weakly nonlinear and weakly dispersive media [9]:

$$3\sigma^2 u_{t_2 t_2} - 12p^2 u_{xx} + 3u_{xx}^2 + u_{xxxx} = 0\,, \tag{1}$$

where $\sigma^2 = \pm 1$, and p is an arbitrary parameter.

The simplest rational solutions of this equation in the hyperbolic case for negative-dispersion media ($\sigma^2 = -1$ and p is an imaginary value) were found long ago by the pole expansion of solutions to equation (1) [7] and by finding a long-wave limit of the known N-soliton solution [10]. Besides, two partial families of solutions to (1) for $p = 0$ were constructed in [11] by expand-

SSDI 0167-2789(94)00178-2

ing the wave function of the inverse scattering problem in power series of spectral parameter.

A new interest in the search for all rational solutions to this equation was stimulated by the problem of a description of two-dimensional stationary multisoliton structures in the framework of the Kadomtsev–Petviashvili (KP) equation for positive-dispersion media [8,12–14]. Such structures are expressed by rational solutions of the equation (1) in the elliptic case ($\sigma^2 = -1$ and p is a real value). It was wrongly proved in [12] that the real rational solutions do not exist in the equation (1) for $\sigma^2 = -1$, except the simplest one which corresponds to a single soliton and was found earlier [7,10]. But shortly after, it becomes clear that the calculation of coefficients of the characteristic polynomial was incorrect, and the real rational solutions describing two-dimensional multisoliton stationary structures do exist; some similar solutions were found in an explicit form in [13,14]. It was also proved in [13] by means of the degeneration of the N-soliton solution in a long-wave limit, that the multisoliton solutions are expressed by the polynomials with the degrees $N(N+1)$, where N is an arbitrary natural number. However, this method did not allow the authors to determine the structure of all rational solutions of equation (1) and to express them in an explicit form.

In this paper we construct a broad class of rational solutions to equation (1) in the form of a determinant with polynomial terms. This class includes all the partial solutions which have been found earlier [7,10–14]. The various forms of the solutions to the Boussinesq equation are discussed in section 2; precisely, the wronskian form, the determinant with integral coefficients and the intermediate determinant which have common properties with both the forms. The solutions to equation (1) for $p = 0$ and one family of the wronskians to equation (1) for arbitrary p are constructed in section 3. The structure of the general rational solutions to equation (1) for arbitrary p is considered in section 4. The concluding section 5 is devoted to the discussion of rational solutions in an alternative modification of the model under consideration, which is the system of classical Boussinesq equations.

2. In this paper we proceed from the fact that all the solutions of equation (1) can be obtained from the KP equation in the form:

$$(-4u_{t_3} + 6uu_{t_1} + u_{t_1t_1t_1})_{t_1} + 3\sigma^2 u_{t_2t_2} = 0\,,$$
$$\sigma^2 = \pm 1 \tag{2}$$

by reduction to the stationary coordinate $x = t_1 + 3p^2 t_3$. As a result, the solution of equation (2) depends on two variables: $u(t_1, t_2, t_3) \equiv u(x, t_2)$. So, the task is to find from the general set of rational solutions of equation (2) the ones that satisfy the reduction.

As was shown by direct [15,16] and inverse [17,18] methods, equation (2) has a broad class of partial solutions which are expressed by the τ-function in the form of the determinant with integral coefficients:

$$u(t_1, t_2, t_3) = 2\frac{\partial^2}{\partial t_1^2}\ln \tau(t_1, t_2, t_3)\,,$$
$$\tau(t_1, t_2, t_3) = \det|c_{nk} + I_{nk}|\,, \tag{3}$$

where $I_{nk} = \int_{-\infty}^{t_1} \Psi_n^+(s, t_2, t_3)\cdot\Psi_k^-(s, t_2, t_3)\,\mathrm{d}s$, c_{nk} are arbitrary constants, $1 \le n,\ k \le N$, and the functions $\Psi_n^{\pm}(t_1, t_2, t_3)$ satisfy the system of the linear partial differential equations:

$$\pm\sigma\frac{\partial\Psi_n^{\pm}}{\partial t_2} = \frac{\partial^2\Psi_n^{\pm}}{\partial t_1^2}\,,$$
$$\frac{\partial\Psi_n^{\pm}}{\partial t_3} = \frac{\partial^3\Psi_n^{\pm}}{\partial t_1^3}\,. \tag{$4_\pm$}$$

Further on we shall consider the case $\sigma = +1$. Besides, in order to find meromorphic solutions one needs to set all $c_{nk} = 0$.

As is well known [19,20], besides this form of the τ-function, there exists another form where the τ-function is expressed by the wronskian of the functions Ψ_n^+ (or Ψ_n^-) satisfying the system (4_+) (or (4_-), respectively):

$$\tau(t_1, t_2, t_3) = \det|J_{nk}^{\pm}| \equiv W[\Psi_1^{\pm}, \Psi_2^{\pm}, \ldots, \Psi_N^{\pm}], \tag{$5_\pm$}$$

where $J_{nk}^{\pm} = \partial^{k-1}\Psi_n^{\pm}/\partial t_1^{k-1}$ and $1 \le n,\ k \le N$.

The construction of any solution to equation (1) which is expressed by the τ-function in the form of the determinant with the integral coefficients (3) can be made in the wronkian form too. But a twice higher order of the wronskian is necessary in the latter case (see, for example, [15,16]).

On the other hand, some solutions can be found from the determinant (3) only by special degeneration which changes their functional structure. Therefore, in order to write the solutions of equation (2) in a convenient form we need to find the form that does not demand additional degeneration and that agrees with the known forms (3) and (5). For this purpose one needs to determine the correspondence between both the forms.

The solution of the form (5_+) can be obtained from (3) by choosing the function Ψ_k^- to be:

$$\Psi_k^- = \exp(p_k t_1 - p_k^2 t_2 + p_k^3 t_3) \equiv \exp(\Phi_k^-), \quad 1 \le k \le N \tag{6}$$

and by the limit transition: $p_k \to +\infty$ for all $k = 1, \ldots, N$. Indeed, integrating I_{nk} by parts, given that $(\partial^m \Psi_n^+/\partial t_1^m)\cdot e^{p_k t_1}|_{t_1=-\infty} = 0$ for any m, we obtain:

$$I_{nk} = \left(\frac{1}{p_k}\cdot \Psi_n^+ - \frac{1}{p_k^2}\cdot\frac{\partial \Psi_n^+}{\partial t_1} + \frac{1}{p_k^3}\cdot\frac{\partial^2\Psi_n^+}{\partial t_1^2} - \cdots\right) \times \exp(\Phi_k^-). \tag{7}$$

Substituting the expression (7) into (3) and taking into account that the determinant with the same columns is equal to zero, we have an expansion for the τ-function when $p_k \gg 1$, and all p_k are supposed to have the same order of magnitude but not to be equal to each other:

$$\tau = \frac{\prod\prod_{N\ge k>m\ge 1} (p_k - p_m)}{\prod_{k=1}^{N} p_k^N} \cdot W[\Psi_1^+, \Psi_2^+, \ldots, \Psi_N^+] \times \exp\left(\sum_{k=1}^{N} \Phi_k^-\right) + O(p^{-L}), \tag{8}$$

where $L = N(N+1)/2 + 1$.

Since the τ-function in the solution (3) can be multiplied by an arbitrary exponential factor which depends linearly on t_1 and by any constant factor, the solution of equation (2) is expressed by the wronskian (5_+) on the limit transition $\{p_k \to +\infty\}_{k=1}^N$. Below we shall neglect the constant and exponential factors in the τ-function without special mention.

The limit transition described above can be made not for all but only for part of the functions Ψ_k^- at $K+1 \le k \le N$. Then we obtain the intermediate form of a solution to (2):

$$\tau(t_1, t_2, t_3) = \det|S_{nk}|, \tag{9}$$

where $S_{nk} = I_{n,k}$ at $1 \le k \le K$; $S_{nk} = J_{n,k-K}^+$ at $K+1 \le k \le N$, and $1 \le n \le N$. We shall use this intermediate formula in the construction of general rational solutions to the Boussinesq equation.

3. We consider the solution of the set of equations (4_+) in the following form:

$$\varphi_m^+(t_1, t_2, t_3; p) = \frac{\partial^m}{\partial p^m}\exp[\Phi^+(t_1, t_2, t_3; p)] \equiv P_m^+(\theta_1^+, \ldots, \theta_m^+)\cdot\exp[\Phi^+(t_1, t_2, t_3; p)], \tag{10}$$

where $\Phi^+(t_1, t_2, t_3; p) = \Sigma_{j=1}^{\infty} p^j t_j$ and

$$\theta_m^+(t_m, t_{m+1}, \ldots) = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\Phi^+(t_1, t_2, t_3; p).$$

Here the variables t_j for $j \ge 4$, which are essential for the construction of solutions to the infinite hierarchy of equations related to equation (2) [21], are arbitrary parameters. Besides, the phases of the variables t_1, t_2, t_3 are obviously arbitrary parameters too, but we do not write them in the formula for Φ^+.

The polynomial $P_m^+(\theta_1^+,\dots,\theta_m^+)$ can be expressed explicitly [20]:

$$P_m^+(\theta_1^+,\dots,\theta_m^+) = m!\cdot\sum_{(i_1,i_2,\dots,i_m)}\prod_{j=1}^{m}\frac{(\theta_j^+)^{i_j}}{i_j!}, \tag{11}$$

where the sum consists of all possible combinations of integer non-negative numbers $(i_1,i_2,\dots,i_m)$ which satisfy the condition $i_1+2i_2+\dots+mi_m=m$. Using the formula (11) one can easily prove the properties of the polynomials P_m^+ that are important for our analysis:

$$P_m^+(\lambda\theta_1^+,\dots,\lambda^m\theta_m^+) = \lambda^m P_m^+(\theta_1^+,\dots,\theta_m^+), \tag{12}$$

$$\frac{\partial P_m^+}{\partial\theta_j^+} = \frac{\partial^j P_m^+}{\partial t_i^j} = H_m^j\cdot P_{m-j}^+, \quad H_m^1 = \frac{m!}{(m-j)!} \tag{13}$$

If we take the functions Ψ_n^+ in the wronskian $\tau = W[\Psi_1^+,\Psi_2^+,\dots,\Psi_N^+]$ in the form $\Psi_n^+ = \varphi_{m_n}^+(t_1,t_2,t_3;p_n)$ we obtain a general rational solution of equation (2). Restricting the consideration to the solutions of the Boussinesq equation, we must find from all the wronskians obtained in this fashion, only the ones which have a stationary dependence on the variable t_3 (i.e., on the variable $x=t_1+3p^2t_3$). As follows from (12), the variables θ_j^+, $1\le j\le m$ have different degrees of homogeneity in the polynomials P_m^+ and hence in the τ-function. Therefore, the necessary condition for finding the solutions in the form $\tau(x,t_2)$ is the equality $p_n=p$ for any n. Then the solution (5_+) can be written in the form of the wronskian with polynomial terms:

$$\tau(t_1,t_2,t_3;p) = W[P_{m_1}^+,P_{m_2}^+,\dots,P_{m_N}^+] \times\exp[N\cdot\Phi^+(t_1,t_2,t_3;p)]\,. \tag{14}$$

It generates a non-zero solution for the function u only when there are no equal polynomials in the set $\{P_{m_n}^+\}_{n=1}^N$.

First, we construct the rational solutions to equation (1) at $p=0$. The solutions of equation (2) in the form $u(t_1,t_2)$ correspond to them. It is clear from the definition of the variables θ_m^+ that all the solutions which are independent of θ_3^+ do not depend on t_3 at $p=0$. We shall prove that the following N-order wronskian is such a solution:

$$\tau_{m,k}(t_1,t_2;p=0) = W[P_1^+,P_4^+,\dots,P_{3\mu-2}^+,\dots; P_2^+,P_5^+,\dots,P_{3\kappa-1}^+,\dots], \tag{15}$$

where $1\le\mu\le m$, $1\le\kappa\le k$, and $m+k=N$.

Indeed, using the determinant properties and the formula (13) one can easily establish that

$$\begin{aligned}\frac{\partial\tau_{m,k}}{\partial\theta_3} = {}&\sum_{\mu=2}^{m} H_{3\mu-2}^3 \times W[P_1^+,P_4^+,\dots,P_{3(\mu-1)-2}^+,\dots; P_2^+,P_5^+,\dots,P_{3\kappa-1}^+,\dots]\\ &+\sum_{\kappa=2}^{k} H_{3\kappa-1}^3\cdot W[P_1^+,P_4^+,\dots,P_{3\mu-2}^+,\dots; P_2^+,P_5^+,\dots,P_{3(\kappa-1)-1}^+,\dots]=0\,.\end{aligned}$$

So, $N+1$ possible wronskian of N-order in the form (15) represent a general solution of equation (1) for $p=0$. These solutions form a characteristic tree-like structure (Fig. 1) which indicates the existence of the Backlund-transform between the neighboring wronskians of N and $N+1$ orders. Indeed, it was shown in [19,20] that the arbitrary solutions of equation (2) in the form $\tau=W[\Psi_1^+,\Psi_2^+,\dots,\Psi_N^+]$ and $\tau'=W[\Psi_1^+,\Psi_2^+,\dots,\Psi_N^+,\Psi_{N+1}^+]$ satisfy the equations of the Backlund-transform for the variable $\tau(t_1,t_2,t_3)$.

The degree of the polynomial $\tau_{m,k}$ in terms of the variable $\theta_1^+=t_1$ can be calculated from the formula (15). We designate this quantity by $R(m,k)$:

$$R(m,k) = m^2+k(k+1)-mk \tag{16a}$$

Each polynomial is parametrized by the phases of the variables t_j on which it depends. Using our proof, one can reveal that the polynomials $\tau_{m,k}$ depend neither on the variable t_3 nor on the variables t_{3n} for any n. Besides, detailed analysis

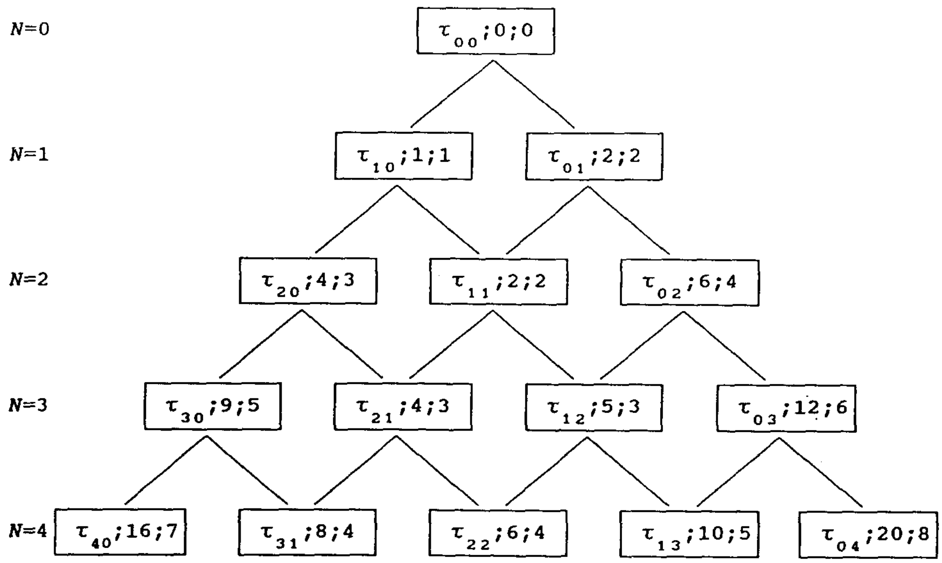

Fig. 1. The structure of the rational solutions of the Boussinesq equation for $p = 0$; the τ-function expressed by formula (15), and the parameters R and G of the solutions.

shows that the solutions $\tau_{m,k}$ for $m \neq 0$, $k \neq 0$, $m \neq k$ and $m \neq k+1$ do not depend on the additional set of the variables:

$$t_{(3k-1)-(3\mu-2)}, \quad \mu = 1, 2, \ldots, m$$
for $N > k \geq 2m$;

$$t_{(3k-1)-(3\mu-2)}, \quad \mu = 1, 2, \ldots, k-m$$
for $2m > k > m$;

$$t_{(3m-2)-(3\kappa-1)}, \quad \kappa = 1, 2, \ldots, m-k-1$$
for $m-1 > k \geq m/2$;

$$t_{(3m-2)-(3\kappa-1)}, \quad \kappa = 1, 2, \ldots, k$$
for $m/2 > k > 0$.

As a result, the number of the independent parameters of the solution $\tau_{m,k}$ which is designated by $G(m,k)$, is determined by the formula:

$$G(m,k) = \begin{cases} 2k - m\,, & N \geq k \geq 2m \\ k + m\,, & 2m > k \geq m/2 \\ 2m - k - 1\,, & m/2 > k \geq 0 \end{cases} \tag{16b}$$

Some first polynomials with various $R(m,k)$ and $G(m,k)$ are presented schematically in Fig. 1.

Note that the extreme families of the solutions to equation (1) for $p = 0$ (when $m = N$, $k = 0$ or $m = 0$, $k = N$) were found in [11]. It is also worthy of notice that for $m, k \neq 0$ there exist the polynomials with the same values of R and G at higher levels N. These new solutions can be obtained from the known ones by replacing $t_{2n} \to -t_{2n}$ for any n admissible by the equation (2). These solutions can be constructed more easily in the framework of the wronskians (5_-) which are determined by the function $\Psi_n^- = \varphi_{m_n}^-$ with the same set $\{m_n\}_{n=1}^N$ but satisfying the adjoint system (4_-). However, among the solutions with $m, k \neq 0$ there exist essentially new solutions which cannot be reduced to the extreme families.

Obviously, the solutions of the Boussinesq equation at $p = 0$ are also the solutions of this equation for arbitrary $p \neq 0$ which exist on the background of the magnitude $2p^2$. But all rational solutions of equation (1) do not reduce only to such solutions.

Considering the solutions with zero boundary conditions at the infinity ($u \to 0$ for x, $t_2 \to \pm\infty$), we can prove that the family of the solutions in the form (5_+) to the equation (1) for arbitrary $p \neq 0$ may be represented by the N-order wronskian:

$$\tau_{N,0}(x, t_2; p) = W[S^{(N-1)}(\nu)P_1^+, \\ S^{(N-2)}(\nu)P_3^+, \dots, P_{2N-1}^+] \tag{17}$$

The displacement of the variables of the polynomials P_m^+ in each line of the wronskian is made by a vertex operator in the form [21, 22]:

$$S(\nu) = \exp\left(-\sum_{m=1}^{\infty} \frac{\nu^m}{m} \cdot \frac{\partial}{\partial \theta_m^+}\right), \tag{18}$$

where $\nu = -1/3p$.

Using the expression (18) one can show that the action of the operator $S(\nu)$ on the polynomials $P_n^\pm$ has a simple form:

$$S(\nu)P_n^\pm = \left(1 - \nu \cdot \frac{\partial}{\partial t_1}\right)P_n^\pm .$$

As a result we can obtain a number of formulas which are necessary for the further analysis:

$$S^{(m)}(\nu)P_n^\pm = \sum_{k=0}^{m} (-\nu)^k \cdot C_m^k \cdot \frac{\partial^k P_n^\pm}{\partial t_1^k},$$

$$C_m^k = \frac{m!}{k!(m-k)!} \tag{19}$$

$$S^{-(k+1)}(\nu)P_n^\pm = \sum_{m=0}^{n} \nu^m \cdot C_{m+k}^k \cdot \frac{\partial^m P_n^\pm}{\partial t_1^m}. \tag{20}$$

In order to prove that the wronskian $\tau_{N,0}$ in (17) is a solution to (1) at arbitrary p, we note that the variable $\theta_1^+ = t_1 + 2pt_2 + 3p^2 t_3$ always satisfies the reduction to the stationary coordinate $x = t_1 + 3p^2 t_3$. Therefore, the solution (17) as a whole satisfies this reduction if the variables θ_2^+ and θ_3^+ enter $\tau_{N,0}$ in the form: $\theta_{23}^+ = \theta_3^+ - \theta_2^+/3p = -t_2/3p$ which does not depend on t_1 and t_3. For this purpose it is sufficient to prove that the function $\tau_{N,0}$ satisfies the linear partial differential equation:

$$\frac{\partial \tau_{N,0}}{\partial \theta_2^+} = -\frac{1}{3p}\frac{\partial \tau_{N,0}}{\partial \theta_3^+} \tag{21}$$

To prove equation (21) we differentiate the n-th line $(n > 1)$ of the determinant (17) with respect to θ_2^+, and then subtract the $(n-1)$-th line of the determinant from the nth line. If we take into account the formulas (13) and (19), the result of these calculations will be presented by the following expression:

$$\begin{aligned}\left(\frac{\partial \tau_{N,0}}{\partial \theta_2^+}\right)^n &= H_{2n-1}^2 \cdot W[\dots, S^{(N-n)}(\nu)P_{2n-3}^+ \\ &\quad - S^{(N-n+1)}(\nu)P_{2n-3}^+, \dots] \\ &= \nu \cdot H_{2n-1}^3 \cdot W[\dots, S^{(N-n)}(\nu)P_{2n-4}^+, \dots] \\ &= \nu \cdot \left(\frac{\partial \tau_{N,0}}{\partial \theta_3^+}\right)^n .\end{aligned} \tag{22}$$

Because $\nu = -1/3p$ and n is an arbitrary number, it follows from the formula (22) that the function $\tau_{N,0}$ satisfies equation (21), i.e. it has a form $\tau_{N,0}(\theta_1^+, \theta_{23}^+) = \tau_{N,0}(x, t_2)$.

The family of the rational solutions (17) which are represented by the unique N-order wronskian, consists of the polynomials of the degree $R(N) = N(N+1)/2$ with $G(N) = N$ arbitrary parameters. Since the even variables θ_{2n}, $n \geq 1$, are not independent quantities, the phases of the odd variables θ_{2n-1}, $1 \leq n \leq N$ can be chosen as the parameters of the solution $\tau_{N,0}$. The family of the solutions (17) transforms in the limit $p \to 0$ to the solutions (15) of the Boussinesq equation for $p = 0$. As a result of this limit transition the structure of the rational solution and the sets of the parameters R and G change completely Another limit ($p \gg 1$) leads to the well-known rational solutions of the Korteweg–de Vries (KdV) equation [7,10,11,22,23]. This equation can be derived from equation (2), provided that the function u does not depend on the variable t_2:

$$-4u_{t_3} + 6uu_{t_1} + u_{t_1t_1t_1} = 0 . \tag{23}$$

The rational solutions of the KdV equation (23) can also be written in the wronskian form [11,23]:

$$\tau_N(t_1, t_3; p = 0) = W[P_1^+, P_3^+, \dots, P_{2N-1}^+] \tag{24}$$

They are the polynomials of the same degree $R(N) = N(N+1)/2$ and with the same number of parameters $G(N) = N$ as the solutions (17) of equation (1) for $p \neq 0$. Moreover, both the solutions (17) and (24) coincide in the limit $p \gg 1$, and the variable $\theta_1^+ = x + 2pt_2$ which is determined in the reference frame moving with the velocity of infinitely long linear perturbations in equation (1), has a role of "spatial" coordinate t_1 in equation (23), and the variable $\theta_{23}^+ = -t_2/3p$ which has a sense of "slow" time, corresponds to the "time' coordinate t_3.

This asymptotic correspondence of the rational solutions in both the equations indicates the well-known one-wave approximation which is used for reduction of the Boussinesq equation to the KdV equation when only waves propagating in one direction are considered (see, for example, [22]). Hence, the family of the solutions to equation (1) expressed by the formula (17) can be regarded as the generalization of the known rational solutions of the KdV equation to the case of two-wave processes which are described by the Boussinesq equation for $p \neq 0$.

4. In section 3 we constructed the solutions of equation (1) in the wronskian form. It means that we made the reduction to the stationary coordinate x for the case $K = 0$ in the general solution of equation (2), which is expressed by the intermediate determinant (9). It is natural to suppose that, at $K > 0$, the determinant (9) is also a solution of equation (1) if the functions Ψ_n^+ and Ψ_k^- are chosen as

$$\begin{aligned}\Psi_n^+ &= S^{(N-n)}(\nu)P_{2n-1}^+(\theta_1^+, \dots, \theta_{2n-1}^+) \\ &\quad \times \exp(\Phi^+(t_1, t_2, t_3; p)) , \quad 1 \le n \le N , \\ \Psi_K^- &= S^{(K-k)}(\nu)P_{2k-1}^-(\theta_1^-, \dots, \theta_{2k-1}^-) \\ &\quad \times \exp(\Phi^-(t_1, t_2, t_3; p)) , \quad 1 \le k \le K , \end{aligned} \tag{25}$$

where the variables θ_m^- and Φ^- can be obtained from θ_m^+ and Φ^+ by replacing $t_{2k} \to -t_{2k}$ for any k, and again $\nu = -1/3p$.

To prove this statement we transform the τ-function of the form (9) to the factorized form. First, we calculate the integrals I_{nk}, $1 \le n \le N$, $1 \le k \le K$:

$$\begin{aligned} I_{nk} &= \frac{i_{nk}}{2p} \cdot \exp(\Phi^+(t_1, t_2, t_3; p) \\ &\quad + \Phi^-(t_1, t_2, t_3; p)) , \end{aligned}$$

where

$$\begin{aligned} i_{nk} &= \sum_{m=0}^{2(n+k-1)} \mu^m \cdot \frac{\partial^m}{\partial t_1^m} (S^{(N-n)}(\nu)P_{2n-1}^+ \\ &\quad \times S^{(K-k)}(\nu)P_{2k-1}^-) , \quad \mu = -1/2p . \end{aligned}$$

Using the formula for the combinatorial analysis:

$$C_{n+k}^n = \sum_{m=0}^{n} C_n^m \cdot C_k^m ,$$

and the property of the vertex operator (20), it is convenient to rewrite i_{nk} in the form of the formal expansion in terms of μ^2:

$$\begin{aligned} i_{nk} &= \sum_{r=0}^{2n-1} \sum_{s=0}^{2k-1} C_{r+s}^{\min(r,s)} \\ &\quad \times \mu^{r+s} \cdot \frac{\partial^r}{\partial t_1^r} (S^{(N-n)}(\nu)P_{2n-1}^+) \\ &\quad \times \frac{\partial^s}{\partial t_1^s} (S^{(K-k)}(\nu)P_{2k-1}^-) \\ &= \sum_{m=0}^{M} \mu^{2m} \\ &\quad \times \left[\frac{\partial^m}{\partial t_1^m} (S^{-(m+1)}(\mu)S^{(N-n)}(\nu)P_{2n-1}^+) \right] \\ &\quad \times \left[\frac{\partial^m}{\partial t_1^m} (S^{-(m+1)}(\mu)S^{(K-k)}(\nu)P_{2k-1}^-) \right] , \end{aligned} \tag{26}$$

where $M = \min(2n-1, 2k-1)$.

On the other hand, one can establish by an inductive procedure that:

$$\begin{aligned} \frac{\partial^k}{\partial t_1^k} (S^{(N-n)}(\nu)P_{2n-1}^+) &= \sum_{m=k}^{2n-1} (-\mu)^{m-k} \\ &\quad \times C_m^k \cdot \left[\frac{\partial^m}{\partial t_1^m} (S^{-(m+1)}(\mu)S^{(N-n)}(\nu)P_{2n-1}^+) \right] \end{aligned} \tag{27}$$

Using the formulas (26) and (27) we express the solution (9) with the functions Ψ_n^+ and Ψ_k^-

specified by (25), in the form of the determinant with polynomial terms of the product of two matrices with the sizes $N \times 2N-1$:

$$\tau_{N,K}(x, t_2; p) = \det|w^+ \cdot w^{-T}| , \tag{28}$$

where

$$(w^+)_{nk} = (-\mu)^{k-1} \cdot \frac{\partial^{k-1}}{\partial t_1^{k-1}} \times (S^{-(k)}(\mu) S^{(N-n)}(\nu) P^+_{2n-1}) ,$$
$$1 \le n \le N , \quad 1 \le k \le 2N-1 ;$$

$$(w^-)_{nk} = (-\mu)^{k-1} \cdot \frac{\partial^{k-1}}{\partial t_1^{k-1}} (S^{-(k)}(\mu) S^{(K-n)}(\nu) P^-_{2n-1}) ,$$
$$1 \le n \le K , \ 1 \le k \le 2N-1 ;$$
$$= C_{k-1}^{n-K-1} , \quad K+1 \le n \le N , \ n-K \le k \le 2N-1 ;$$
$$= 0 \quad , \quad K+1 \le n \le N , \ 1 \le k \le n-K-1$$

According to the known Binet–Cauchy formula [24], the determinant of the product of two matrices with the size $N \times M$, $M > N$, is equal to the sum of products of the determinants of all possible N-order minors of one matrix and the corresponding minors of the same order of magnitude of the other matrix. Since the vertex operators $S^{-(k)}(\mu)$ act on the columns of the matrices $w^{\pm}$, and the vertex operators $S^{(N-n)}(\nu)$ act on the lines of these matrices, each factor in the sum which is formed from the columns of the matrices $w^{\pm}$, depends only on the variables $\theta_1^{\pm}$ and $\theta_{2,3}^{\pm}$, as was proved in section 3. Consequently, the complete solution expressed by $\tau_{N,K}(x, t_2; p)$ satisfies equation (1) for arbitrary p and represents a general rational solution of this equation.

Since the degrees of the polynomials P^+_{2n-1}, P^-_{2k-1}, $1 \le n \le N$, and $1 \le k \le K$ decrease in each column of the matrices w^+ and w^-, the leading term of the τ-function (28) is presented by the product of the determinants obtained from the matrices $w^{\pm}$ by neglecting the columns with $k > N$. Using the formula (19), one can show that these determinants correspond exactly to the solutions $\tau_{N,0}$ and $\tau_{0,K}$ which are written in the wronskian form (17) with the sets of the polynomials $\{S^{(N-n)}(\nu) P^+_{2n-1}\}_{n=1}^N$ and $\{S^{(K-k)}(\nu) P^-_{2k-1}\}_{k=1}^K$ depending on the variables θ_m^+ and θ_m^- that are displaced by the same magnitude $N\mu^m/m$:

$$\tau_{N,K}(x, t_2) \sim S^{-(N)}(\mu) \tau_{N,0} \cdot S^{-(N)}(\mu) \tau_{0,K} \tag{29}$$

as $t_n \to \pm\infty$ for any n.

As a consequence, the degree of the polynomial $R(N, K)$ and the number of the parameters $G(N, K)$ for the general solution (28) of the Boussinesq equation at $p \ne 0$ are the sums of the values of R and G for extreme solutions: $R(N, K) = N(N+1)/2 + K(K+1)/2$, $G(N, K) = N + K$.

Thus, the structure of the general solutions which is shown in Fig. 2 differs from the structure of the rational solutions of this equation at $p = 0$.

All the rational solutions of equation (1) in the hyperbolic case $\sigma^2 = +1$ have pole singularities and, hence, are physically meaningless. On the other hand, in the elliptic case $\sigma^2 = -1$, the solutions $\tau_{N,K}$ with $K \ne N$ are complex valued, which is clear from the formulas $(4_{\pm})$ and (28). But at $K = N$ and by choosing $P^-_{2k-1} = \bar{P}^+_{2k-1}$, $1 \le k \le N$ the family of the $2N$-parametric rational solutions with the degree $R = N(N+1)$ is real valued and nonsingular that follows directly from the factorized form (28) as a sum of positively determined terms. These solutions correspond to the bound states of a certain number $R/2$ of two-dimensional solitons of the KP equation with positive dispersion. The existence of this family of solutions was discovered in [13] and physically accounted for in [8] by the features of the potential of soliton interaction.

Note that nonsingular rational solutions exist neither in the hyperbolic nor in the elliptic equations (1) at imaginary p.

5. The structure of the rational solutions of the Boussinesq equation generalizes the structure of the rational solutions of the KdV equation which is a one-wave reduction of the Boussinesq equation in an asymptotic limit $p \to \infty$. It is interesting to point out that other modifica-

	$K=0$	$K=1$	$K=2$	$K=3$	$K=4$
$N=0$	τ_{00};0;0	τ_{01};1;1	τ_{02};3;2	τ_{03};6;3	τ_{04};10;4
$N=1$	τ_{10};1;1	τ_{11};2;2	τ_{12};4;3	τ_{13};7;4	τ_{14};11;5
$N=2$	τ_{20};3;2	τ_{21};4;3	τ_{22};6;4	τ_{23};9;5	τ_{24};13;6
$N=3$	τ_{30};6;3	τ_{31};7;4	τ_{32};9;5	τ_{33};12;6	τ_{34};16;7
$N=4$	τ_{40};10;4	τ_{41};11;5	τ_{42};13;6	τ_{43};16;7	τ_{44};20;8

Fig. 2. The structure of the rational solutions of the Boussinesq equation for arbitrary p: the τ-function expressed by the formula (28) and the parameters R and G of the solutions. The determinants of the same order of magnitude are outlined.

tions of the equation (the so-called classical Boussinesq systems) are also known, for example, in the surface water wave theory [25]. These equations have different forms of nonlinear and dispersive terms as compared to equation (1). Since the KdV-reduction is general for all the considered models, it is natural to suppose that a similar structure of rational solutions exist also in the framework of the Boussinesq system. However, only the families of meromorphic solutions of these equations [3,4] which correspond to the waves localized near the bottom of a reservoir (the case $p = 0$ in the equation (1)) were found by now. We suppose that a still further interesting problem in the study of the classical Boussinesq system is to reveal the rational solutions corresponding to the waves localized near the surface of a reservoir.

This work was carried out under the financial support of the Russian Foundation of Fundamental Research (grant Nr. 93-05-8073 and of the International Science Foundation (grant Nr. RST 000).

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