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Instability and decay of solitary waves in the Davey-Stewartson 1 system

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Abstract

The exact solutions to the Davey-Stewartson 1 equations are constructed to describe the development of a transversal solitary wave instability. Two types of plane dark soliton decays into two-dimensional solitary waves are found and analysed. It is also shown that the transversal instability of one-dimensional waveguides leads only to a shift of their drift velocity under the action of nonlocalized, inhomogeneous perturbations.

1. Introduction

The discovery of the exact N-soliton solutions to several one-dimensional (1D) equations stimulated investigation of soliton dynamics in nonlinear systems (see, e.g., Ref. [1]). Using the N-soliton formulas Zakharov and Shabat succeeded in describing both the soliton scattering and the bound states of oscillating or stationary particles [2,3]. Later it was found that the range of applicability for this class of solutions exceeds the bounds of simple soliton dynamics. New resonant types of soliton interactions, accompanied by their decay and merging, were studied in the framework of the N-soliton solutions as well [4,5]. Moreover, the formal expansion of parameters of the solutions into the complex plane allowed one to analyse the development of instability both of the solitons and of the wave background of constant intensity [6-8].

In the two-dimensional (2D) case, the soliton dynamics is still more diverse and manifests simple and resonant interactions of plane and 2D solitary waves [9,10]. In order to describe the complete pattern of soliton dynamics in multidimensional integrable systems we need a more general representation of the explicit solutions than the N-soliton ones. Such a fundamental representation might be found by means of the dressing method [11]. For instance, the dressing method for a 1D stationary solution includes the construction of a complete set of eigenfunctions to a linearized problem, which was realized for the Kadomtsev-Petviashvili (KP) equation in Refs. [12,13]. Recently the nonlinear solutions to the KP1 equation were constructed in the framework of this approach to analyse the development of growing discrete-spectrum modes and the transformation of the original soliton into 2D solitary waves [14].

In the present paper we investigate the phenomenon of solitary wave instability in another 2D integrable model,

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$$2i\partial_t\Psi + \partial_x^2\Psi + \partial_y^2\Psi + 2\Psi(n + |\Psi|^2 - \rho^2) = 0, \qquad \partial_x^2 n - \partial_y^2 n + 2\partial_x^2|\Psi|^2 = 0. \tag{1}$$

This model is referred to as the Davey-Stewartson (DS1) equations and describes, for instance, surface gravity-capillary waves in a fluid of small depth [15]. The variable Ψ corresponds to the complex amplitude of a quasi-harmonic wave and the real variable n corresponds to the self-consistent mean flow.

The wave background (WB) $\Psi = \rho, n = 0$ is stable with respect to small perturbations depending on the coordinate x and is unstable with respect to the ones depending on the coordinate y. The development of the WB instability for plane quasi-periodic perturbations is described by the 1D nonlinear Schrödinger (NLS) equation and was studied earlier [7,8]. It was found that the growth of the perturbations alternated with their damping.

Here we shall consider the instability of other stationary solutions to Eqs. (1), namely, the 1D solitary waves. As is well-known [2,3], the soliton solutions of Eqs. (1) are presented by dark solitons, the amplitudes of which fall off along the x-axis to the value $|\Psi| = \rho$, and by waveguides, the amplitudes of which vanish to zero along the y-axis. We shall restrict our consideration of the dark soliton instability to only the localized discrete-spectrum modes evolving against the soliton background. The dark soliton instability with respect to the nonlocalized eigenfunctions of the continuum spectrum is known to originate from the WB instability and does not transform the original soliton into new solitary waves [6]. On the other hand, the waveguide instability discovered for Eqs. (1) by Ablowitz and Segur [15] is found to occur under the action of internal nonlocalized perturbations of the mean flow n and to have features of soliton instability with respect to the continuum-spectrum eigenfunctions.

2. Types of dark soliton instability

In order to investigate a dark soliton instability we consider the exact solution to Eqs. (1) at $\rho \neq 0$,

$$\Psi = G/F, \qquad |\Psi|^2 = \rho^2 + (\partial_y^2 - \partial_x^2)\ln F, \qquad n = 2\partial_y^2 \ln F, \tag{2}$$

$$G = \rho F + 2\phi\chi^*, \qquad F = \delta - \int_x^\infty (|\phi|^2 + |\chi|^2)\,\mathrm{d}x, \tag{3}$$

where δ is an arbitrary constant and the functions ϕ, χ are solutions to the linear system

$$\partial_x\phi + \partial_y\phi + \rho\chi = 0, \qquad \partial_x\chi - \partial_y\chi + \rho\phi = 0, \qquad i\partial_t\phi = \partial_x\partial_y\phi, \qquad i\partial_t\chi = \partial_x\partial_y\chi. \tag{4}$$

Formulas (2)-(4) follow from the general determinant of explicit solutions to the DS system which was recently found in the framework of the dressing method [16]. If we choose the functions ϕ, χ in the single-exponential form we get an ordinary one-soliton solution to Eqs. (1). Hence, the solution (2)-(4) is a direct functional generalization of the one-soliton solution.

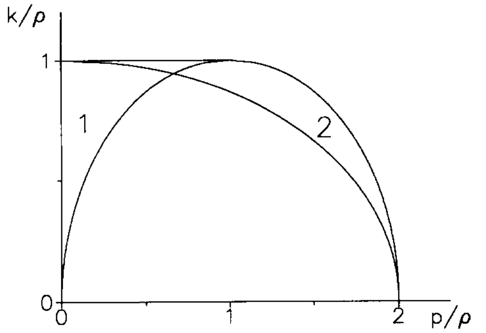
Let us choose $\delta = 1$ and the functions ϕ, χ in the form

$$\phi = \sqrt{k}\exp(kx - ivy - kvt) + \int \Phi(\kappa)\exp(\kappa x - i\nu y - \kappa\nu t)\,\mathrm{d}\kappa,$$

$$\chi = -\frac{k - iv}{\rho}\sqrt{k}\exp(kx - ivy - kvt) - \int \frac{\kappa - i\nu}{\rho}\Phi(\kappa)\exp(\kappa x - i\nu y - \kappa\nu t)\,\mathrm{d}\kappa, \tag{5}$$

where $v = \sqrt{\rho^2 - k^2}$, $\nu = \sigma\sqrt{\rho^2 - \kappa^2}$, and $\sigma = \pm 1$.

Obviously, if the spectral function Φ is infinitesimal we can expand the functions Ψ, n in the vicinity of the one-soliton solution with parameters (k, v) and find a solution to the linearized problem.



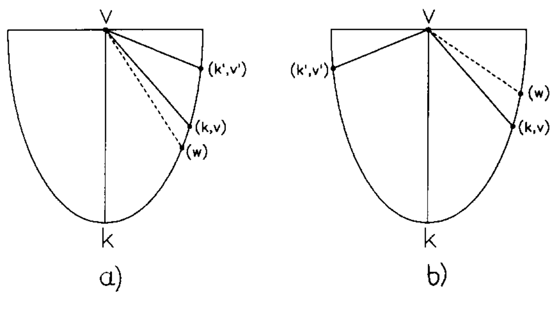


Fig. 1. Regions of dark-soliton instability in the parameter plane k, p.

Fig. 2. Scheme of dark-soliton decay for $\sigma = +1$ (a) and $\sigma = -1$ (b).

$$\Psi = \Psi_0(\xi) + \int \tilde{\Phi}(\kappa)[\Psi_+(\xi,\kappa)\exp(\mathrm{i}py) + \Psi_-(\xi,\kappa)\exp(-\mathrm{i}py)]\exp(\lambda t)\,\mathrm{d}\kappa + \mathrm{O}(\exp(2\lambda t)), \tag{6}$$

where

$$\Psi_0 = \frac{v - \mathrm{i}k}{\rho}[v - \mathrm{i}k\tanh(k\xi)], \qquad \xi = x - vt, \qquad \tilde{\Phi} = \frac{\sqrt{k}\Phi}{2\sqrt{k+\kappa}},$$

$$\Psi_+ = \frac{\exp(\kappa\xi)}{\cosh(k\xi)}\left[\rho\left(1 + \frac{(\nu - \mathrm{i}\kappa)(v - \mathrm{i}k - 2\mathrm{i}\kappa)}{\rho^2}\right) - \Psi_0\left(1 + \frac{(\nu - \mathrm{i}\kappa)(v + \mathrm{i}k)}{\rho^2}\right)\right],$$

$$\Psi_- = \frac{\exp(\kappa\xi)}{\cosh(k\xi)}\left[\rho\left(1 + \frac{(\nu - \mathrm{i}\kappa - 2\mathrm{i}k)(v - \mathrm{i}k)}{\rho^2}\right) - \Psi_0\left(1 + \frac{(\nu + \mathrm{i}\kappa)(v - \mathrm{i}k)}{\rho^2}\right)\right],$$

and the growth rate is related to the transversal wave number by the algebraic system

$$\lambda = \kappa p, \qquad p = \sqrt{\rho^2 - k^2} - \sigma\sqrt{\rho^2 - \kappa^2}. \tag{7}$$

Apparently, it is impossible to find a complete set of eigenfunctions to the linearized problem in the framework of the solution (2)-(4). Nevertheless, the unstable discrete-spectrum modes can be constructed by formulas (6), (7). Moreover, the non-reduced solution (2), (3) enables us to analyse simultaneously the linear growth and the nonlinear long-term evolution of the unstable soliton perturbations.

Restricting ourselves to a single-periodic perturbation, we choose the spectral function Φ to be

$$\Phi = \sqrt{\kappa}\,\delta(\kappa - k'). \tag{8}$$

The instability regions are bounded on the parameter plane k, p in the range $0 \leqslant |k'| \leqslant |k|$ because, otherwise, the linear corrections $\Psi_\pm$ grow exponentially along the coordinate ξ. The instability regions for both signs of σ are shown in Fig. 1.

The linear mode Ψ_+ at $\sigma = +1$ (region 1) is unstable only for long-wave transversal perturbations of the dark soliton Ψ_0. In the asymptotic limit $k \to 0$, the soliton transforms to the known soliton of the KP equation and the discrete-spectrum mode transforms to its instability mode [12-14]. Therefore, we shall call this type of dark soliton instability the KP-type.

Unlike this type, the linear mode Ψ_+ at $\sigma = -1$ (region 2) is unstable in a narrow range of the short-wave perturbations. At $k \to 0$ the instability region fuses at the point $p = 2\rho$ which is cut off for unstable transversal perturbations of the uniform wave background $|\Psi_0| = \rho$. Therefore, the second type of dark soliton instability will be referred to as the WB-type. It should be noted that this mode has an unusual feature: the growth rate at the long-wave boundary of the region 2 where $p_c = 2\sqrt{\rho^2 - k^2}$, does not vanish and coincides with the growth rate of the WB instability with respect to small y-dependent perturbations $\lambda \equiv \lambda_{WB} = p_c\sqrt{\rho^2 - p_c^2/4}$.

The conditional character of such a classification becomes obvious for the limiting case $k \to \rho$ when the original soliton is immobile. In this case, both modes are equivalent and represent two perturbations breaking the mirror symmetry of the original soliton $\Psi_0(x)$.

3. Types of dark soliton decay

At the nonlinear stage of a dark soliton instability of both types we observe a simple picture: the original plane soliton decays into two solitary waves. One of them is a 1D soliton which is described by the function Ψ_0 replacing $k \to k', v \to v'$, where $v' = \sigma\sqrt{\rho^2 - k'^2}$. The other, essentially 2D stationary wave, is described by the functions G, F given in (2), (3) for $\delta = 0$ and the functions ϕ, χ in the form (5), (8). This solution depends on two parameters k, k' and propagates with the velocity $w = (kv - k'v')/(k - k')$, the sign of σ determining two different branches. The scheme of the plane soliton decay is shown for each type of instability in Figs. 2a, 2b. In this section we discuss characteristic features of the processes outlined.

The development of the dark soliton instability of the KP-type generalizes the phenomenon of plane soliton decay in positive-dispersion media discovered recently in the KPI equation [10,14]. It is seen from Fig.2a that, for small values of the parameter k, both resulting solitary waves move in the same direction as an original unstable soliton, the plane wave having a smaller amplitude and greater velocity, while the 2D wave has a greater amplitude and smaller velocity.

The 2D stationary wave can be regarded as a periodic chain of 2D dark solitons, each of which is described by the following function,

$$\Psi = \rho\frac{(\xi - \mathrm{i}/v)^2 + k^2y^2/v^2 + \rho^2/4v^2k^2}{\xi^2 + k^2y^2/v^2 + \rho^2/4v^2k^2}, \tag{9}$$

where $\xi = x - (v^2 - k^2)t/v, v^2 + k^2 = \rho^2$.

For small k, 2D dark solitons look like inverted KP lumps, the region of lower intensity being concentrated near the global minimum at the point $\xi = y = 0$ and the regions of higher intensity near the points $\xi^2 = (3v^2 - k^2)/4v^2k^2, y = 0$.

For the WB-type dark soliton instability, the resulting solitary structures have slightly different characteristic features (see Fig. 2b). For $\sigma = -1$ and $|k'| < |k|$, a new 2D wave is also a stationary superposition of 2D perturbations, but it propagates faster than the 2D wave at $\sigma = +1$ and has much wider regions of higher intensity. Such a wave can also be regarded as a periodic chain of 2D dark solitons (9) but for large k when they have a pair of zero values at the points $\xi = 0, y^2 = (3k^2 - v^2)/4k^4$ and the global maximum in the origin $\xi = y = 0$.

Besides the 2D stationary wave, a new plane soliton appears and moves in the opposite direction with respect to the original soliton. We may call such a phenomenon the inversion of a dark soliton occurring as a result of its instability of the WB-type. In its pure form, the dark soliton inversion is observed for $k' = k$ at the boundary curve $p = p_c(k)$ where the growth rate $\lambda = \lambda_{WB}$. During this process, all energy of the plane soliton propagating with velocity $+v$ along the ξ-axis, transforms into energy of the plane soliton propagating with the velocity $-v$. Such a transformation is caused by growing and damping perturbations of the wave background which are uniform along the ξ-axis and are nothing but "the 2D wave" at $k' = k, v' = v, w = \infty$.

It should be emphasized that except for the limiting case $k' = k$ the dark soliton instability of the WB-type occurs under the action of localized discrete-spectrum modes (6). Nevertheless, its appearance is associated with the phenomenon of the WB transversal instability.

4. The waveguide instability

Here we consider solitary waves at the zero background, depending on the coordinate y. Since a quasi-harmonic wave propagates along the x-axis, such solitary waves are referred to as 1D waveguides [15]. As was found in Ref. [15], the 1D waveguides are also unstable with respect to transversal perturbations. In this section we construct the exact solutions describing the nonlinear stage of this instability.

The explicit solution to Eqs. (1) generalizing the one-soliton solution at $\rho = 0$ can be written in the form [17]

$$G = 2\phi\chi^*, \qquad F = 1 - \int_x^\infty |\phi|^2\, dx \cdot \int_x^\infty |\chi|^2\, dx, \tag{10}$$

where the functions ϕ, χ satisfy the linear system (4) at $\rho = 0$, and the functions G, F generate a solution to Eqs. (1) according to (2).

Applying the approach described above, we specify the functions ϕ, χ as follows,

$$\phi = c\exp[p(x-y) + ip^2 t] + \int \Phi(\alpha)\exp[\alpha(x-y) + i\alpha^2 t]\, d\alpha,$$

$$\chi = c\exp[q(x+y) - iq^2 t] + \int \Upsilon(\beta)\exp[\beta(x+y) - i\beta^2 t]\, d\beta. \tag{11}$$

Here the spectral functions Φ, Υ are independent and generate two partial sets of eigenfunctions to a linearized problem on the waveguide background. However, one set can be obtained from the other by the mirror transformation $x \to -x$. Therefore, we consider only the first one and restrict ourselves again to a single-periodic perturbation: $\Phi(\alpha) = c'\delta(\alpha - p'), \Upsilon(\beta) = 0$.

Substituting (11) into (2) and choosing a convenient parametrization, we obtain the following nonlinear solution,

$$\Psi = \frac{2v\exp(v\eta - i\theta)\{1 + [(v - i\kappa)/v]\exp[\lambda t + i\kappa(x - \kappa\eta)]\}}{1 + \exp(2v\eta)\{1 + [(v^2+\kappa^2)/v^2]\exp(2\lambda t) + 2\exp(\lambda t)\cos[\kappa(x - \kappa\eta)]\}}, \tag{12}$$

where

$$\eta = y + 2\,\mathrm{Im}\,q\,t, \qquad \theta = 2\,\mathrm{Im}\,q\,y + 2(\mathrm{Im}\,q^2 - \mathrm{Re}\,q^2)t, \qquad \kappa = \mathrm{Im}\,p' - \mathrm{Im}\,q,$$

$$\lambda = v\kappa, \qquad v = 2\,\mathrm{Re}\,q, \qquad p = -q^*, \qquad \mathrm{Re}\,p' = \mathrm{Re}\,p, \qquad c = \sqrt{v}, \qquad c' = (v - i\kappa)/\sqrt{v}.$$

As $t \to -\infty$, the solution (12) describes the 1D waveguide which drifts along the y-axis with velocity $-2\,\mathrm{Im}\,q$ and the growing perturbation which is periodic along the x-axis and oriented at an angle to the waveguide, depending on the parameter κ. At $\kappa \simeq 0$, the linear corrections $\Psi_\pm$ are asymmetrical, $\Psi_\pm \simeq \partial_\eta|\Psi_0|$ and represent the transversal modulation of the waveguide coordinate.

For the variable Ψ, the localized eigenfunction $\Psi_\pm$ might seem to be a discrete-spectrum mode which usually leads to instability of one-dimensional envelope solitons in equations of NLS-type [15]. Nevertheless, the other component of the wave field n is localized only along the direction $\eta \to -\infty$ for $v > 0$ and vice versa. Along the opposite direction, the field n is non-localized and looks like an oblique harmonic wave with a small but growing amplitude. Therefore, the waveguide instability is actually the instability with respect to the continuum eigenfunctions. It occurs only due to the action of internal inhomogeneous perturbations induced by the field n. This fact accounts for the unusual feature following from the linear dependence $\lambda = v\kappa$: the waveguide is unstable with respect to the oblique perturbations with an arbitrary period.

At the nonlinear stage, the growth of the perturbations is stabilized and is replaced by their damping. Only two effects appear as a result of the waveguide instability: the change of the drift velocity which becomes equal to $-(\operatorname{Im} q + \operatorname{Im} p')$ as $t \to +\infty$ and the bending of the phase fronts $\theta = \text{const}$ which become oblique in the x, y-plane. The energy of a waveguide is not changed by these effects.

Apparently, the damping of the 2D perturbations for the waveguide instability is associated with the absence of 2D solitary waves at the zero background in the model (1). The energy of the original waveguide cannot be distributed among other nonlinear structures of the wave field, and self-focusing of the 1D waveguides changes the defocusing stage.

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