KP-II limit of 2D FPU

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Overview of Thesis

- Chapter 2: Linear Dispersion Relationships and Formal Expansions
- Chapter 3: Well-Posedness for the Kadomtsev-Petviashvili Equation
- Chapter 4: Small-amplitude, long-wavelength limit for a 2D "α-Model"
- Chapter 5: Propagation Along a Diagonal for a 2D "α-Model"
- Chapter 6: Small-amplitude, long-wavelength limit for a 2D "β-Model"
- Chapter 7: Line solitary waves in linearized 2D FPU

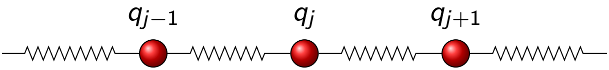
Well-Posedness for the Kadomtsev-Petviashvili Equation

- $\partial_\xi^{-1} A = \int_{-\infty}^{\xi} A(\xi') d\xi'$
- We need a bound in Sobolev norm for terms of the form $\partial_\xi^{-1}\partial_\tau^2 A$, which is equivalent to a bound for $\partial_\tau^3 A$
- A solves a KP-II equation

$$2c_1\partial_\xi\partial_\tau A + \frac{c_1^2}{12}\partial_\xi^4 A + 2\alpha\partial_\xi\left(A\partial_\xi A\right) + c_2^2\partial_\eta^2 A = 0,$$

- Regularity in time results are extended so that solutions to the KP-II equation are in $C^3\left(\left[-\tau_0, \tau_0\right], H^s\left(\mathbb{R}^2\right)\right)$.

The Fermi-Pasta-Ulam problem



- System of particles of a line
- Nearest neighbour interactions with Hamiltonian given by
  $$H = \sum_j \frac{1}{2} p_j^2 + V(q_{j+1} - q_j)$$
- Potential a cubic or quartic function in displacements
- Numerical experiment showed the system was nearly periodic for long time scales

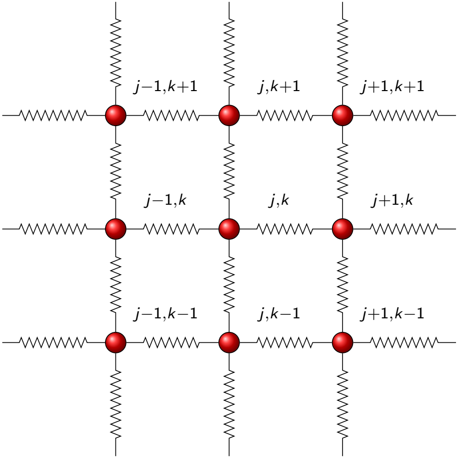
Small-amplitude, long-wavelength limit - 1D

- Ansatz: $r_j(t) = q_{j+1}(t) - q_j(t) = \varepsilon^2 R\left(\varepsilon\left(j - c_s t\right), \varepsilon^3 t\right) + \mathrm{error}$
- Satisfies FPU system (with cubic potential) to $O(\varepsilon^6)$ if R satisfies the KdV equation:

$$\partial_\tau R + \frac{\alpha}{c_s} R \partial_\xi R + \frac{c_s}{24} \partial_\xi^3 R = 0$$

- Rigorous justification for this limit given by Schneider and Wayne in 1999
- This limit has been extensively studied with more complicated potentials (u^p by Khan and Pelinovsky, Hertzian potential by Dumas and Pelinovsky), and in the polyatomic case (e.g. by Gaison, Moskow, Wright, and Zhang)

2D Square Lattice



Small-amplitude, long-wavelength limit - 2D

- Two-dimensional lattice, with nearest neighbour interactions, in the thesis we look at a square lattice
- Results for this limit regarding KdV-like solitary wave (e.g. Chen and Herrmann) in a lattice with diagonal interactions
- Allowing for motion in the transverse direction Duncan, Eilbeck, and Zakharov formally derives a Kadomtsev-Petviashvili (KP-II) in this limit
- Thesis gives rigorous justification of this limit for several models

2D FPU Models studied

- $\alpha-$model is an analogue of the one-dimensional FPU with a cubic potential
- $\beta-$model is an analogue of the one-dimensional FPU with a quartic potential
- The case for propagation along a diagonal in the α-model is studied

2D α-Model equations of motion

$$
\begin{aligned}
\dot{u}^{(1)}_{j,k} =& w_{j+1,k} - w_{j,k}, \, \dot{u}^{(2)}_{j,k} = w_{j,k+1} - w_{j,k}, \\
\dot{v}^{(1)}_{j,k} =& z_{j+1,k} - z_{j,k}, \, \dot{v}^{(2)}_{j,k} = z_{j,k+1} - z_{j,k}, \\
\dot{w}_{j,k} =& c_1^2 \left(u^{(1)}_{j,k} - u^{(1)}_{j-1,k} \right) + c_2^2 \left(u^{(2)}_{j,k} - u^{(2)}_{j,k-1} \right) + \alpha_1 \left[\left(u^{(1)}_{j,k} \right)^2 - \left(u^{(1)}_{j-1,k} \right)^2 \right] \\
&+ \alpha_2 \left[u^{(2)}_{j,k} v^{(2)}_{j,k} - u^{(2)}_{j,k-1} v^{(2)}_{j,k-1} + \frac{1}{2} \left(v^{(1)}_{j,k} \right)^2 - \frac{1}{2} \left(v^{(1)}_{j-1,k} \right)^2 \right] \\
\dot{z}_{j,k} =& c_1^2 \left(v^{(2)}_{j,k} - v^{(2)}_{j,k-1} \right) + c_2^2 \left(v^{(1)}_{j,k} - v^{(1)}_{j-1,k} \right) + \alpha_1 \left[\left(v^{(2)}_{j,k} \right)^2 - \left(v^{(2)}_{j,k-1} \right)^2 \right] \\
&+ \alpha_2 \left[u^{(1)}_{j,k} v^{(1)}_{j,k} - u^{(1)}_{j-1,k} v^{(1)}_{j-1,k} + \frac{1}{2} \left(u^{(2)}_{j,k} \right)^2 - \frac{1}{2} \left(u^{(2)}_{j,k-1} \right)^2 \right]
\end{aligned}
$$

2D α-Model equations of motion

$$
\begin{aligned}
u_{j,k}^{(1)} &= \varepsilon^2 A(\xi,\eta,\tau) + \varepsilon^2 U_{j,k}^{(1)} \\
u_{j,k}^{(2)} &= \varepsilon^2 U_\varepsilon(\xi,\eta,\tau) + \varepsilon^2 U_{j,k}^{(2)} \\
v_{j,k}^{(1)} &= \varepsilon^2 V_{j,k}^{(1)} \\
v_{j,k}^{(2)} &= \varepsilon^2 V_{j,k}^{(2)} \\
w_{j,k} &= \varepsilon^2 W_\varepsilon(\xi,\eta,\tau) + \varepsilon^2 W_{j,k} \\
z_{j,k} &= \varepsilon^2 Z_{j,k},
\end{aligned}
$$

where $\xi = \varepsilon j, \eta = \varepsilon^2 k, \tau = \varepsilon^3 t$

Small-amplitude, long-wavelength limit for a 2D α-Model

- Chapter 4 shows that solutions to the 2D FPU system remain $\varepsilon^{\frac{5}{2}}$-close to a approximating function of the form

$$u_{j,k}^{(1)} = x_{j+1,k} - x_{j,k} = \varepsilon^2 A(\varepsilon\,(j - c_1 t)\,, \varepsilon^2 k, \varepsilon^3 t) + error,$$

  where A solves a KP-II equation, for time scales of $O\left(\frac{1}{\varepsilon^3}\right)$

- Remaining variables found through asymptotic expansions of the equations of motion

$$W_\varepsilon = -c_1 A + \varepsilon \frac{c_1}{2} \partial_\xi A + \varepsilon^2 \left(\partial_\xi^{-1} \partial_\tau A - \frac{c_1}{12} \partial_\xi^2 A \right) - \varepsilon^3 \frac{1}{2} \partial_\tau A$$
$$U_\varepsilon = \varepsilon \partial_\xi^{-1} \partial_\eta A - \varepsilon^2 \frac{1}{2} \partial_\eta A + \varepsilon^3 \left(\frac{1}{2} \partial_\xi^{-1} \partial_\eta^2 A + \frac{1}{12} \partial_\eta \partial_\xi A \right)$$

- Equations of motion used to give evolution equations of the error

Small-amplitude, long-wavelength limit for a 2D α-Model

Lemma

Let $u_{j,k} = U(\varepsilon j, \varepsilon^2 k)$, *with* $U \in H^s(\mathbb{R}^2)$, $s > 1$. *Then, there is a constant* $C_s > 0$, *such that for every* $\varepsilon \in (0,1)$ *we have*

$$\|u\|_{\ell^2(\mathbb{Z}^2)} \leq C_s \varepsilon^{-\frac{3}{2}} \|U\|_{H^s(\mathbb{R}^2)}, \qquad \forall U \in H^s\left(\mathbb{R}^2\right).$$

- Used to find bounds on the residual
- One-dimensional result loses only $\varepsilon^{-\frac{1}{2}}$, extra power of epsilon due to the ε^2 scaling on the second variable

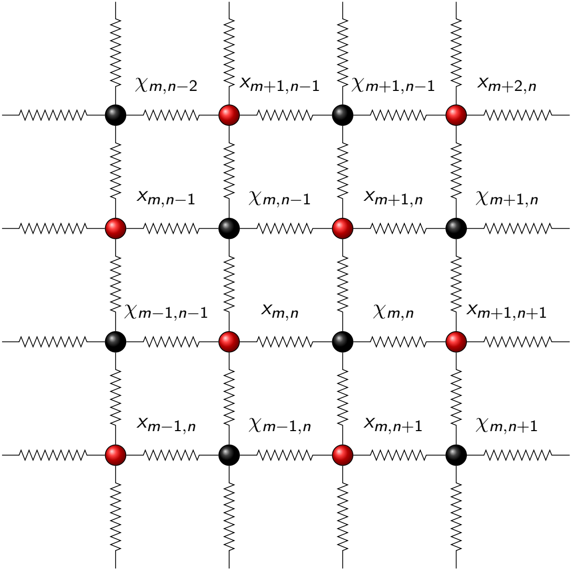
Small-amplitude, long-wavelength limit for a 2D α-Model

- Energy type quantity is introduced to control the growth of the error
- Energy is shown to be coercive
- Bounds are given by a Gronwall lemma
- Gronwall argument loses ε^{-3}, this with previous lemma requires asymptotic expansion accurate to $O(\varepsilon^5)$
- Improving beyond this difficult – need to solve linearized nonhomogeneous KP-II equation, where the nonhomogeneous term contains higher order nonlocal terms

Propagation Along a Diagonal for a 2D α-Model

- Change of coordinates on the lattice is introduced: $m = \frac{j+k}{2}, n = \frac{j-k}{2}$
- Parameters of the model carefully chosen in chapter 1.
- Other parameters lead to propagation in the transverse direction or appearance of nonlocal terms, for which we may not have bounds
- Other parameters also introduce a perturbation to the PDE which our approximating function must satisfy

Propagation Along a Diagonal for a 2D α-Model



Propagation Along a Diagonal for a 2D α-Model

- Chapter 5 shows that solutions to the 2D FPU system remain $\varepsilon^{\frac{5}{2}}$-close to a approximating function of the form

  $$x_{m+1,n} - x_{m,n} = \varepsilon^2 A(\varepsilon\,(m - c_1 t)\,, \varepsilon^2 n, \varepsilon^3 t) + error,$$

  for time scales of $O\left(\frac{1}{\varepsilon^3}\right)$
- This is not one of the strain variables, but a linear combination of them, choosing a strain variable and performing asymptotic expansions introduces terms which may not be bounded
- Once the expansions are performed, and an appropriate energy type quantity is chosen, remainder of the chapter is similar to before

Small-amplitude, long-wavelength limit for a 2D β-Model

- Chapter 6 shows that solutions to the 2D FPU system with a cubic nonlinearity remain $\varepsilon^{\frac{5}{2}}$-close to a approximating function of the form

$$u_{j,k}^{(1)} = x_{j+1,k} - x_{j,k} = \varepsilon A(\varepsilon (j - c_1 t), \varepsilon^2 k, \varepsilon^3 t) + error,$$

  for time scales of $O\left(\frac{1}{\varepsilon^3}\right)$
- A solves the following cubic KP-II equation

$$2\partial_\xi \partial_\tau A + \frac{1}{12}\partial_\xi^4 A + \beta \partial_\xi^2 \left(A^3\right) + \partial_\eta^2 A = 0,$$

Small-amplitude, long-wavelength limit for a 2D β-Model

- Scaling of the amplitude differs due to the different nonlinearity.
- A different energy type quantity is chosen to accomodate the nonlinearity
- Argument for the Gronwall lemma modified due to growth of the energy function

FPU Solitary Waves - 1D

- So far - approximations of FPU through KdV (1D) and KP-II (2D)
- Seeking a solitary wave solution, of the form $R(\xi,\tau) = \phi_\gamma(\xi - \gamma\tau)$, for KdV yields the exact solution

$$\phi_\gamma(\xi - \gamma\tau) = \frac{3c_s\gamma}{\alpha}\,\mathrm{sech}^2\left(\sqrt{\frac{6\gamma}{c_s}}(\xi - \gamma\tau)\right).$$

- Q: Does the FPU system admit solitary wave solutions? Can the KdV equation give us information about them?

FPU Solitary Waves - 1D

- Solitary waves of the one-dimensional FPU system can be approximated by solitary waves of the KdV equations
- Exponentially weighted space: $\ell^2_a = \left\{ u : \mathbb{Z} \to \mathbb{R}^2 | e^{aj} u(j) \in \ell^2 \right\}$
- In a series of papers Friesecke and Pego proved that solitary waves of FPU converge to KdV
- Low energy solitary waves of the FPU system are stable – solutions close to a solitary wave of FPU converge to a solitary wave of FPU in ℓ^2_a

FPU Solitary Waves - 1D

- Result on the essential spectrum of FPU
- Proof that the linearized FPU is asymptotically stable in the sense that
$$\|w(t)\|_{\ell^2_a} \leq Ke^{-\beta t}\|w(0)\|_{\ell^2_a}$$
- Previous paper proved that asymptotic stability in the sense above implies the stability of the solitary waves
- Convergence of a certain operator in the small amplitude long wavelength limit
- Stability results for the KdV equation

FPU Solitary Waves - 2D

- Conjecture about asymptotic stability of line solitary waves for linearized FPU system in 2D
- Equations of motion: 2D α-model linearized around a horizontally propagating solitary wave
- Limiting problem is a linearized KP-II equation, stability for line solitary waves established by Mizumachi in 2015
- Rigorous results obtained for the essential spectrum
- Partial result obtained for convergence of the linearized FPU operator in the small amplitude long wavelength limit

FPU Solitary Waves - 2D

Seeking a solution of the form $r(x,t) = e^{\lambda t} R(x,t)$

Lemma

Suppose that $\hat{k} \in [-\pi, \pi] \setminus \{0\}$. Then the essential spectrum is given by the following eigenvalue problem

$$\left(\lambda - c\frac{d}{dx}\right)^2 R_1 = \left(c_1^2 \Delta^+ \Delta^- + 2\alpha \Delta^+ \Delta^- u_c(x)\right) R_1 - 4c_2^2 \sin^2\left(\hat{k}\right) R_1 + (1 - e^{-i\hat{k}}) r_2, \tag{1}$$

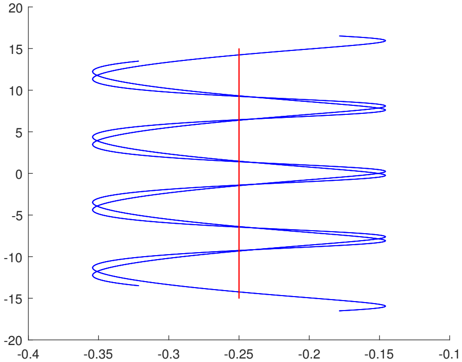
where r_2 satisfies

$$\left(\lambda - c\frac{d}{dx}\right) r_2 = 0. \tag{2}$$

FPU Solitary Waves - 2D

Lemma

Suppose that $\hat{k} \in [-\pi, \pi] \setminus \{0\}$. If $c > c_1$ and $0 < a < a_c$ where $a_c > 0$ is the solution of $\sinh(\frac{1}{2}a_c)\left(\frac{1}{2}a_c\right)^{-1} = \frac{c}{c_1}$ then the essential spectrum of the operator $c\partial_x + L$ in L^2_a does not intersect the closed right half plane.



Overview of Results

- Small-amplitude, long-wavelength limit for a 2D α-Model
- Propagation Along a Diagonal for a 2D α-Model
- Small-amplitude, long-wavelength limit for a 2D β-Model
- Line solitary waves in linearized 2D FPU (conjecture with some rigorous results)

Further work

- The KP-II approximation could be considered on other lattices
- Arbitrary angle of propagation can be considered
- Stability of line solitary waves for the nonlinear FPU system, only linearized FPU was considered in the 2D case