Existence and Stability of Periodic Waves in KdV type equations

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Introduction

1. Fractional Korteweg De-Vries equation (fKdV)

$$u_t + 2uu_x = (D^\alpha u)_x,$$

2. Fractional modified Korteweg De-Vries equation (fmKdV)

$$u_t + 6u^2 u_x = (D^\alpha u)_x,$$

We consider 2π periodic solutions on $\mathbb{T} := (-\pi, \pi)$ so the operator D^α is defined via Fourier series

$$f(x) = \sum_{n\in\mathbb{Z}} f_n e^{inx}, \quad (D^\alpha f)(x) = \sum_{n\in\mathbb{Z}} |n|^\alpha f_n e^{inx}.$$

Background on the fKdV equation

- Well–posedness in Sobolev spaces
  - F. Linares, D. Pilod, J.C. Saut (2014)
  - L. Molinet, D. Pilod, S. Vento (2018)
- Existence and stability of periodic waves
  - M. Johnson: perturbation method. (2013)
  - V. Hur, M. Johnson: variational method. (2015)
- Convergence of Petviashvili method in periodic waves
  - J. Alvarez, A. Duran (2017)
  - D. Clamond, D. Dutykh (2018)

Main Results

Key contributions of the thesis are

- New representation and positivity of Green's function of $(D^\alpha + c)$.
- Existence of positive, single–lobe solution of the fKdV equation.
- Convergence of Petviashvili method in fKdV equation.
- New variational method for fKdV
  - Justify C^1 dependence between the solution and the wave speed parameter.
  - Fully characterize the kernel of linearized operator
  - Obtain spectral stability criteria.
- Extending the new variational method to fmKdV

Fractional Korteweg De-Vries equation

The fractional KdV equation :

$$u_t + 2uu_x = (D^\alpha u)_x.$$

The periodic travelling wave solution takes the form

$$u(x,t) = \varphi(x - \omega t).$$

Substituting the travelling wave anzat in the fKdV equation and integrating once, we to obtain the boundary value problem

$$(D^\alpha + \omega)\,\varphi = \varphi^2, \quad \varphi \in H^\alpha_{\mathrm{per}}.$$

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Definition: The periodic wave φ is said to have single–lobe profile if there is only one maximum and one minimum of φ on the period.

In general, when we use the travelling wave anzat $u(x,t) = \psi(x - ct)$ and integrate the fKdV equation once, there is a constant of integration

$$(D^{\alpha} + c)\,\psi + b = \psi^2.$$

In general, when we use the travelling wave anzat $u(x,t) = \psi(x - ct)$ and integrate the fKdV equation once, there is a constant of integration

$$(D^\alpha + c)\,\psi + b = \psi^2.$$

Thanks to Galilean transformation $\psi = \varphi + \frac{1}{2}\left(c - \sqrt{c^2 + 4b}\right)$,

$$(D^\alpha + c)\,\psi + b = \psi^2 \Rightarrow (D^\alpha + \omega)\,\varphi = \varphi^2.$$

where $\omega = \sqrt{c^2 + 4b}$.

Positive solution of fKdV equation

Theorem (L, Pelinovsky - 2019)

For every $\omega > 1$ *and* $\alpha \in (\alpha_0, 2]$ *with* $\alpha_0 \approx 0.585$, *there exists an even, single–lobe solution* $\varphi \in H^{\alpha}_{\mathrm{per}}$ *to the BVP*

$$(D^{\alpha} + \omega)\varphi = \varphi^2,$$

such that $\varphi(x) > 0$ *on* $[-\pi, \pi]$

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Elements of analysis:

1) Green's function of $D^\alpha + \omega$.
2) Krasnoselskii's fixed point theorem in a positive cone.
3) Leray–Schauder index.

Green's function of $D^\alpha + \omega$

The Green's function of $D^\alpha + \omega$ is obtained from the solution of the inhomogeneous equation

$$(D^\alpha + \omega)\phi(x) = h, \quad h \in L^2_{\mathrm{per}}$$

with

$$\phi(x) = \int_{-\pi}^{\pi} G(x - s)h(s)ds.$$

Equivalently, in Fourier series form

$$G(x) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \frac{\cos(nx)}{\omega + |n|^\alpha} \tag{1}$$

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Lemma: $G(x)$ as defined in (1) is positive on $[-\pi, \pi]$ for $\omega > 0$ and $\alpha \in (0, 2]$.

Krasnoselskii's fixed point in positive cone

We define the solution operator

$$A(\varphi)(x) := (D^\alpha + \omega)^{-1}\varphi^2 = \int_{-\pi}^{\pi} G(x-s)\varphi(s)^2 ds,$$

and the positive cone

$$P := \left\{\varphi \in L^2_{\mathrm{per}} : \varphi(x) \geq \frac{m}{M}\|\varphi\|_{L^2_{\mathrm{per}}}\right\}, \quad \text{where} \left\{ \begin{array}{l} G(x) \geq m \\ \|G(x)\|_{L^2_{\mathrm{per}}} \leq M. \end{array} \right.$$

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Krasnoselskii's fixed point theorem provides the existence of a fixed point the cone. However, from the BVP

$$(D^\alpha + \omega)\varphi = \varphi^2,$$

the fixed point could be a constant!

Leray–Schauder index

The Leray–Schauder index of φ is given by $(-1)^N$ where N is the number of unstable eigenvalues of $A'(\varphi)$ outside of the unit disk, counting multiplicities.

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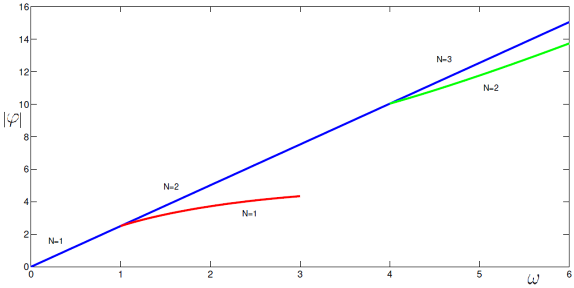
If $\varphi = \omega$, then in the space of even function, $A'(\omega) = 2\omega(D^\alpha + \omega)^{-1}$ has $k+1$ unstable eigenvalues lying outside the unit disk for $\omega \in (k^\alpha, (k+1)^\alpha)$ with $k \in \mathbb{N}$

$\Rightarrow$ the index of the constant solution changes sign when ω crosses k^α



fKdV: Variational characterization

The stationary equation

$$(D^\alpha + c)\,\psi - \psi^2 + b = 0.$$

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Standard variational approach: minimize the energy

$$E(u) = \frac{1}{2}\int_{-\pi}^{\pi}\left(D^{\frac{\alpha}{2}}u\right)^2 dx - \frac{1}{3}\int_{-\pi}^{\pi} u^3 dx,$$

subject to fixed mass M and momentum F

$$M(u) = \int_{-\pi}^{\pi} u dx, \quad F(u) = \frac{1}{2}\int_{-\pi}^{\pi} u^2 dx.$$

The stationary equation is the Euler–Lagrange equation for the action functional

$$\Lambda(u) = E(u) + cF(u) + bM(u).$$

The stationary equation

$$(D^\alpha + c)\,\psi - \psi^2 + b = 0.$$

New variational approach: minimize the quadratic part of the action functional

$$\mathcal{B}_c(u) = \int_{-\pi}^{\pi} \left[\left(D^{\frac{\alpha}{2}} u \right)^2 + cu^2 \right] dx,$$

subject to fixed cubic part of the energy and zero mean constraints

$$Y_0 := \left\{ u \in H^{\frac{\alpha}{2}}_{\mathrm{per}} : \int_{-\pi}^{\pi} u^3 dx = 1, \quad \int_{-\pi}^{\pi} u dx = 0 \right\}.$$

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Let $\psi(x)$ has zero mean, then

$$b = b(c) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi^2 dx.$$

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Theorem (Natali, L, Pelinovsky-2020)

For every $\alpha > 1/3$ and $c > -1$, there exists a constrained minimizer $u_* \in Y_0$.

Continuation of solution

Consider the Hessian operator:

$$\mathcal{H} = D^\alpha + c - 2\psi : H^\alpha_{\rm per} \subset L^2_{\rm per} \mapsto L^2_{\rm per}$$

We can verify that for $b(c) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi^2 dx$,

$$\mathcal{H}\psi = -\psi^2 - b(c), \qquad \mathcal{H}1 = -2\psi + c.$$

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Lemma (Natali,L, Pelinovsky - 2019)

If $\mathrm{Ker}(\mathcal{H}|_{1^\perp}) = \mathrm{span}(\partial_x\psi)$ *at* $c = c_0$, *then* $c \mapsto \psi(\cdot, c)$ *is* C^1 *at* $c = c_0$.

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We can now differentiate in c and get

$$\mathcal{H}\partial_c\psi = -\psi - b'(c).$$

From the 3 equations

$$\mathcal{H}\psi = -\psi^2 - b(c),$$
$$\mathcal{H}1 = -2\psi + c,$$
$$\mathcal{H}\partial_c\psi = -\psi - b'(c),$$

we have a condition to fully characterize the kernel of $\mathcal{H}$

Corollary

If $c + 2b'(c) \neq 0$, *then* $\mathrm{Ker}(\mathcal{H}) = \mathrm{span}(\partial_x\psi)$. *Otherwise,* $\mathrm{Ker}(\mathcal{H}) = \mathrm{span}\left(\partial_x\psi, 1 - 2\partial_c\psi\right)$.

Spectral stability result

We have $\mathcal{H}|_{\{1,\psi^2\}^\perp} \geq 0$ since ψ is the constrained minimizer.

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Theorem (Natali,L, Pelinovsky, 2020)

Let $\alpha \in (1/3, 2]$ and $c \in (-1, \infty)$ and ψ be the minimizer of new variational problem. Assume Ker$(\mathcal{H}|_{\{1\}^\perp}) =$ span$(\partial_x \psi)$. Then, the periodic wave ψ is spectrally stable if $b'(c) \geq 0$ and is spectrally unstable if $b'(c) < 0$.

Numerical method

Fix α. Given $c > -1$, solve for ψ

$$(D^\alpha + c)\,\psi + b = \psi^2, \quad b(c) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi^2 dx.$$

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Recall the Galilean transformation, $\psi = \varphi + \frac{1}{2}\left(c - \sqrt{c^2 + 4b}\right)$, we numerically solve instead

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Theorem (L, Pelinovsky - 2019)

For $\omega > 1$ and $\alpha \in (\alpha_0, 2]$, the single–lobe solution φ is a stable fixed point of the Petviashvili iteration

$$w_{n+1} = T(w_n) = \left(\frac{\langle (D^\alpha + c)\, w_n, w_n\rangle}{\langle w_n^2, w_n\rangle}\right)^2 (D^\alpha + \omega)^{-1} w_n^2$$

Numerical illustration: fKdV

Numerical scheme:

- Fix α. Given $\omega > 1$, solve for φ using fixed point iterations.
- $c = \omega - \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi dx$, and $b = \frac{1}{4}(\omega^2 - c^2)$.

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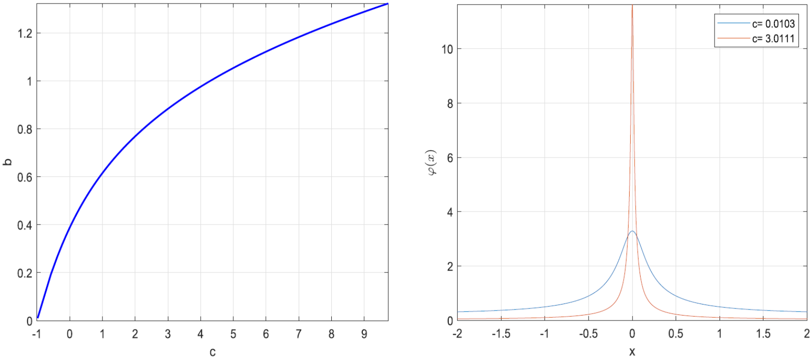
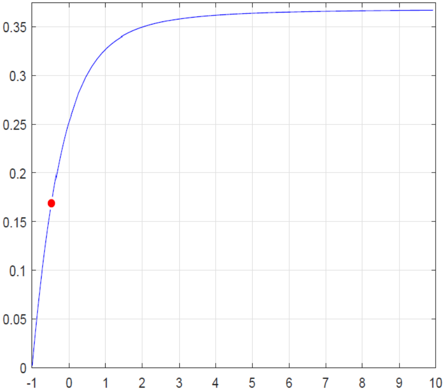


Figure: Left: b vs c. Right: Profiles of φ at two values of c, $\alpha = 0.6$

Numerical illustration: fKdV



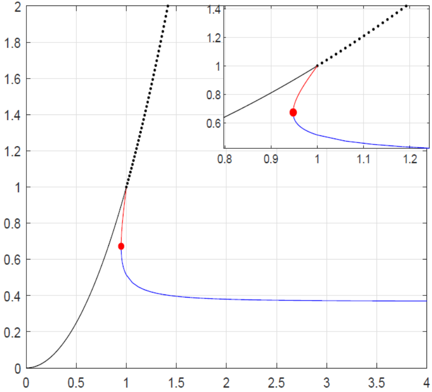


Figure: Left: b vs c, $\alpha = 0.5$. Right: μ vs ω, with $\mu = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi^2 dx$ and $\alpha = 0.5$

Fractional modified KdV equation

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By Galilean transformation, let $\psi = \beta + \varphi$

$$\Rightarrow D^\alpha \varphi + (c - 6\beta^2)\varphi - 6\beta\varphi^2 - 2\varphi^3 + (b + c\beta - 2\beta^3) = 0.$$

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$\Rightarrow$ Partial solution: set $b = 0$ and obtain 2 families: odd and even periodic waves.

Variational characterization

We want to minimize

$$\mathcal{B}_c(u) = \int_{-\pi}^{\pi} \left[(D^{\frac{\alpha}{2}} u)^2 + cu^2 \right] dx,$$

subject to fixed quartic part of the energy

$$Y_{\mathrm{odd(even)}} := \left\{ u \in H^{\frac{\alpha}{2}}_{\mathrm{per,odd(even)}} : \int_{-\pi}^{\pi} u^4 dx = 1 \right\}.$$

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Theorem (Natali, L, Pelinovsky-2021)

Odd waves:
For $\alpha > 1/2$ and $c > -1$ there exists a constrained minimizer u_* in Y_{odd}. Moreover, u_* has single–lobe profile and zero mean.

Even waves:
For $\alpha > 1/2$ and $c > 0$ there exists a constrained minimizer $\tilde{u}$ in Y_{even}. For $c \in (0, 1/2)$, $\tilde{u}$ is constant. For $c > 1/2$, $\tilde{u}$ has single–lobe profile but might have non zero mean.

Numerical illustrations

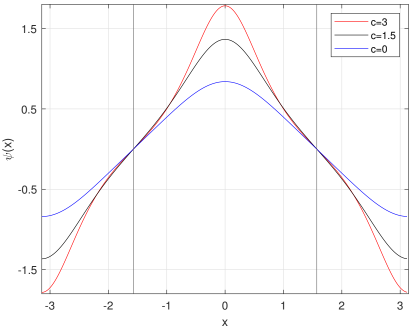

Figure: Odd waves: $\alpha = 2$

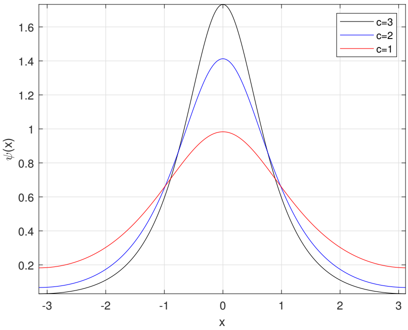

Figure: Even waves: $\alpha = 2$

Summary

For the periodic waves in the fractional KdV equation satisfying

$$(D^\alpha + c)\,\psi + b = \psi^2$$

we showed:

- $\psi > 0$ for $c > 1$, $b = 0$ and $\alpha > \alpha_0 \approx 0.585$.
- Convergence of Petviashvili fixed point method for positive wave.
- Periodic waves with zero mean with $b \neq 0$ for both $\alpha > \alpha_0$ and $\alpha < \alpha_0$ via new variational method.
- Spectral stability of ψ

For the periodic waves in the fractional modified KdV equation satisfying

$$(D^\alpha + c)\,\psi + b = \psi^3$$

we showed:

- Odd and even periodic, single–lobe waves for $b = 0$ via new variational method.

Further direction: Characterize all periodic, single–lobe waves for arbitrary b.