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Traveling Monotonic Fronts in the Discrete Nagumo Equation

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Abstract We give an alternative proof of the theorem, which states that no propagation failure occurs for the discrete Nagumo equation with "translationally invariant" stationary monotonic fronts. The theorem was recently proved with the use of the invariant manifolds for lattice differential equations by Hupkes, Pelinovsky, and Sandstede. The alternative proof relies on the analysis of the advance-delay operator associated with the translationally invariant stationary front. This operator exhibits an infinite-dimensional kernel spanned by Fourier harmonics of front's translations, which are accounted when the stationary front is continued into the traveling one.

Keywords Lattice differential equations · Differential advance-delay operators · Propagation failure · Singular perturbation theory

1 Introduction

We consider the discrete Nagumo equation

$$\frac{du_n}{dt} = \frac{1}{h^2}\left(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)\right) + f(u_n(t)), \quad n \in \mathbb{Z}, \quad t \in \mathbb{R}, \tag{1.1}$$

where $f(u)$ is a function with three equilibrium states $\{-1, a, 1\}$ for a fixed $a \in (-1, 1)$ such that

$$f(\pm 1) = f(a) = 0, \quad f'(\pm 1) < 0, \text{ and } f'(a) > 0. \tag{1.2}$$

Under the condition (1.2), equilibrium states $\{-1, 1\}$ are stable with respect to the time evolution of the discrete Nagumo Eq. (1.1).

If $f(u)$ is cubic, e.g.

$$f(u_n) = 2(1 - u_n^2)(u_n - a), \quad a \in (-1, 1), \tag{1.3}$$

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it is known at least for large $h > 0$ [14] that a *propagation failure* occurs, that is, no traveling monotonic fronts between stable equilibrium states $\{-1, 1\}$ exist for small $a \neq 0$.

Propagation failure of monotonic fronts in discrete parabolic equations has been studied in many details in the recent years, in particular, by Erneux and Nicolis [9], Cahn et al. [4], Chow et al. [5], Fáth [10], Elmer and Van Vleck [8], Elmer [7], and Hoffman and Mallet-Paret [12].

We say that $[a_-, a_+]$ with $a_- \leq 0 \leq a_+$ is the propagation failure interval if no traveling front

$$u_n(t) = \phi(hn - ct) : \phi(z) \in C^1(\mathbb{R}), \quad \phi'(z) > 0, \quad \forall z \in \mathbb{R}, \quad \lim_{z \to \pm\infty} \phi(z) = \pm 1, \quad (1.4)$$

exists for $c \neq 0$ in the discrete Nagumo Eq. (1.1) for $a \in [a_-, a_+]$.

Note that the propagation failure is generic for other nonlinear functions $f(u)$ with two stable equilibrium states such as sinusoidal, piecewise linear (zigzag), and linear with a step (McKean) functions. On the other hand, traveling fronts in the discrete Nagumo equation generally exist for all values of $c > 0$ for $a \in (a_+, 1)$ and for all values of $c < 0$ for $a \in (-1, a_-)$. Theorem 2.1 and Corollary 2.5 in Mallet-Paret [17] state under some generic assumptions on $f(u)$ that for each value of $a \in (-1, a_-) \cup (a_+, 1)$, there is a unique speed $c = c_*(a)$ of the traveling monotonic front such that $\lim_{a \to a_\pm} c_*(a) = 0$. Numerical evidences of this result are given by Abell et al. [1].

If $a_- = 0 = a_+$, the propagation failure interval $[a_-, a_+]$ shrinks to the point $\{0\}$. It was found by Elmer [7] for zigzag nonlinearity that there may exist a countable set of points $\{h_k\}_{k \in \mathbb{N}}$ for parameter h of the discrete Nagumo Eq. (1.1), when no propagation failure occurs.

Traveling monotonic fronts (1.4) of the discrete Nagumo Eq. (1.1) are found from the differential advanced-delay equation

$$-c\frac{d\phi}{dz} = \frac{1}{h^2}\left(\phi(z+h) - 2\phi(z) + \phi(z-h)\right) + f(\phi(z)), \quad z \in \mathbb{R}. \quad (1.5)$$

In the anti-continuum limit $h \to \infty$, the differential advance-delay Eq. (1.5) reduces to the differential equation, which admits a simple solution for the cubic nonlinear function (1.3): traveling fronts exist for any $c > 0$ if $a = 1$ and for any $c < 0$ if $a = -1$. Therefore, the propagation failure for $h = \infty$ occurs for all values of a in $(-1, 1)$, which suggests that

$$a_\pm \to \pm 1 \text{ as } h \to \infty.$$

In what follows, we consider finite values of $h > 0$.

If $a_- < 0 < a_+$ (that is, the propagation failure occurs), Theorem 2.3 in Mallet-Paret [17] states that a sequence of traveling fronts $\{\phi_k(z)\}_{k \in \mathbb{N}}$ of the differential advance-delay Eq. (1.5) with speeds $\{c_k\}_{k \in \mathbb{N}}$ such that $\phi_k(0) = 0$ and $c_k \to 0$ as $k \to \infty$ converges to the stationary front at all but countable many points $z \in \mathbb{R}$. The limiting front

$$\phi_\infty(z) = \lim_{c_k \to 0} \phi_k(z), \quad z \in \mathbb{R} \quad (1.6)$$

is generally discontinuous on $\mathbb{R}$ and the limit $c \to 0$ corresponds to the singular perturbation theory.

Let us now discuss a different example of the nonlinear function $f(u)$ in the discrete Nagumo Eq. (1.1). We consider a class of gradient discrete Nagumo equation obtained from the variational principle

$$\frac{du_n}{dt} = \frac{\delta\Lambda}{\delta u_n}, \quad n \in \mathbb{Z}, \quad (1.7)$$

where the energy functional Λ is given by

$$\Lambda = \sum_{n \in \mathbb{Z}} \frac{u_n u_{n+1}}{h^2} + \frac{\log\left(1 + h^2\left(1 - u_n^2\right)\right)}{h^4} - \frac{2a\left(1 - \frac{1}{3}u_n^2\right)u_n}{1 + h^2(1 - a^2)}. \tag{1.8}$$

Variations of Λ shows that the nonlinear function $f(u_n)$ in the discrete Nagumo Eq. (1.1) takes the form

$$f(u_n) = \frac{2(1 - u_n^2)(u_n - a)\left(1 + h^2(1 + au_n)\right)}{\left(1 + h^2(1 - u_n^2)\right)\left(1 + h^2(1 - a^2)\right)}, \tag{1.9}$$

and it has only three equilibrium states $\{-1, a, 1\}$ on the interval $[-1, 1]$ if $a \in (-1, 1)$. For this example, the differential advance-delay Eq. (1.5) becomes

$$-c\frac{d\phi}{dz} = \frac{\phi(z+h) + \phi(z-h)}{h^2} - \frac{2\phi(z)}{h^2\left(1 + h^2(1 - \phi^2(z))\right)} - \frac{2a\left(1 - \phi^2(z)\right)}{1 + h^2(1 - a^2)}, \quad z \in \mathbb{R},$$

and there exists an exact traveling front solution for any $h > 0$ and $a \in (-1, 1)$

$$\phi(z) = \tanh(bz), \quad b = \frac{\operatorname{arcsinh}(h)}{h}, \quad c = \frac{2a}{b(1 + h^2(1 - a^2))}. \tag{1.10}$$

We can see from (1.10) that $c = 0$ at $a = 0$ and that the limiting solution (1.6) is smooth in z for all $h > 0$. These facts suggest that no propagation failure occurs in the discrete Nagumo Eq. (1.1) with the special nonlinear function (1.9).

We note that the nonlinearity similar to (1.9) is used in Section 3.3 of [1] to construct exact tanh-solutions for the traveling front with a particular speed c. Figure 5 in [1] shows that the propagation failure interval is nonempty for their model, whereas this interval is empty for the model with the nonlinearity (1.9).

The special nonlinear function (1.9) corresponds to the discrete Nagumo Eq. (1.1) with so-called "translationally invariant" stationary fronts, because the solution $u_n = \phi(hn - s)$ for $c = 0$ can be translated with the arbitrary parameter $s \in \mathbb{R}$. The construction of lattice differential equations with translationally invariant stationary fronts was considered in the past by Speight [18], Flach et al. [11] Kevrekidis [15], Dmitriev et al. [6], and Barashenkov et al. [2], in a more general context of nonlinear functions f that depend both on u_n and (u_{n-1}, u_{n+1}).

It is the goal of this work to confirm that the existence of translationally invariant stationary fronts in the discrete Nagumo Eq. (1.1) gives a sufficient condition for the absence of propagation failure for traveling fronts under some technical assumptions on the nonlinear function $f(u_n)$. The proof can be extended to the case of nonlinear functions $f(u_{n-1}, u_n, u_{n+1})$ for the price of more technical and lengthy computations.

The main theorem was proved recently by Hupkes et al. [13] using invariant manifolds for coupled differential equations on a lattice. Here we give an alternative proof of this theorem using spectral analysis of the linearized differential advance-delay operators. There exists no similarity between analysis of dynamics in coupled systems of differential equations (an initial-value problem) and analysis of existence of particular solutions in the differential advance-delay equation (a boundary-value problem). Differential advance-delay equations are considered by many researchers in the context of traveling waves in lattices, e.g. by Mallet-Paret [16,17]. Motivated by the alternative proof of Theorem 1, we shall clarify important properties associated to the differential advance-delay equations in the singular perturbation limit $c \to 0$, when these equations degenerate into advance-delay equations.

Assumption 1 Let f be a C^2-smooth function with respect to both u and a such that

$$f(u) = f_0(u) + af_1(u) + \mathcal{O}(a^2),$$

where f_0 is odd and f_1 is even. Assume that the stationary advance-delay equation

$$\frac{1}{h^2}\left(\varphi(z+h) - 2\varphi(z) + \varphi(z-h)\right) + f_0(\varphi(z)) = 0, \quad z \in \mathbb{R} \tag{1.11}$$

admits for any $h > 0$ a stationary front $\varphi(z)$ with the properties

$$\begin{aligned} &\varphi(z) \in C^1_{\text{odd}}(\mathbb{R}), \quad \varphi'(z) > 0, \quad \forall z \in \mathbb{R}, \\ &\varphi(z) \to \pm 1 \text{ as } z \to \pm\infty \text{ exponentially fast.} \end{aligned} \tag{1.12}$$

Also assume that

$$f_1(\varphi(z)) < 0, \quad z \in \mathbb{R}. \tag{1.13}$$

Remark 1 In the continuous limit $h \to 0$, the advance-delay Eq. (1.11) reduces formally to the second-order differential equation

$$\varphi''(x) + f_0(\varphi(x)) = 0, \quad x \in \mathbb{R}.$$

Under the constraint (1.2) on the odd nonlinear function $f_0(\varphi)$, there exists a unique odd stationary front $\varphi(x)$ such that $\lim_{x\to\pm\infty}\varphi(x) = \pm 1$.

The following main results state that, under Assumption 1, the translationally invariant stationary front (1.12) is uniquely continued as the traveling front (1.4) for $a \neq 0$ along a curve $c = c_*(a)$ with $c_*(0) = 0$ and $c_*'(0) \neq 0$. At the same time, the translationally invariant stationary front (1.12) can not be continued as a stationary front for any $a \neq 0$. Because of some technical limitations of the proof, we have to assume analyticity of the function $\varphi(z)$ and the smallness of the parameter $h > 0$.

Theorem 1 *Let Assumption 1 be met and assume that $\varphi(z)$ is real analytic for all $z \in \mathbb{R}$. There exists $h_0 > 0$ such that for any $h \in (0, h_0)$, no propagation failure occurs and there exists a unique family of traveling front solutions (1.4) of the differential advance-delay Eq. (1.5) for a unique $c = c_*(a)$ near $a = 0$. Moreover, $c_*(a)$ is C^1 near $a = 0$ and $0 < c_*'(0) < \infty$.*

Remark 2 The previous example with the nonlinear function (1.9) show that Assumption 1 and Theorem 1 hold with

$$c_*'(0) = \frac{2h}{(1+h^2)\operatorname{arcsinh}(h)}, \quad h \in \mathbb{R}.$$

In particular, $c_*'(0) \to 2$ as $h \to 0$ and $c_*'(0) \to 0$ as $h \to \infty$.

Corollary 1 *Let Assumption 1 be met. No stationary fronts of the difference equation*

$$\frac{1}{h^2}\left(\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}\right) + f(\varphi_n) = 0, \quad n \in \mathbb{Z}, \tag{1.14}$$

exist in a neighborhood of the translationally invariant stationary front (1.12) for a small $a \neq 0$.

We note that the zigzag nonlinearity considered by Elmer [7] provides a *different* example of the nonlinear function $f(u_n)$, where no propagation failure occurs for a countable set of values $\{h_k\}_{k\in\mathbb{N}}$ for the parameter h in the discrete Nagumo Eq. (1.1). While it is still true that a continuous set of stationary fronts exists for any $h = h_k$, $k \in \mathbb{N}$ and $a = 0$ (Lemma 21 in [7]) and no stationary fronts exist for $h = h_k$ and $a \neq 0$ (Fig. 7 in [7]), our translationally invariant family of stationary fronts exists for all sufficiently small $h > 0$. We do not claim that our results would imply existence of traveling fronts along the curve $c = c_*(a)$ with $\lim_{a\to 0} c_*(a) = 0$ in the case of zigzag nonlinearity for $h = h_k$ (no numerical data for $c \neq 0$ are included in [7]).

The article is organized as follows. Differential advance-delay operators in the singular limit $c \to 0$ are considered in Sect. 2. The proof of the main results is developed in Sect. 3. Section 4 discusses other examples.

2 Differential Advance-Delay Operators as $c \to 0$

We shall consider the differential advance-delay operator $L_c : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$\begin{aligned}(L_c\psi)(z) := -c\frac{d\psi}{dz} - \frac{1}{h^2}\left(\psi(z+h) - 2\psi(z) + \psi(z-h)\right)\\ - f_0'(\varphi(z))\psi(z), \quad z \in \mathbb{R}.\end{aligned} \tag{2.1}$$

We also define the adjoint operator $L_c^* : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\begin{aligned}(L_c^*\psi)(z) := c\frac{d\psi}{dz} - \frac{1}{h^2}\left(\psi(z+h) - 2\psi(z) + \psi(z-h)\right)\\ - f_0'(\varphi(z))\psi(z), \quad z \in \mathbb{R}.\end{aligned} \tag{2.2}$$

Sampling the advance-delay operator L_0 on $\{z_n(s) = hn - s\}_{n\in\mathbb{Z}}$ for a fixed $s \in \mathbb{R}$ gives the self-adjoint difference operator $M_s : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ defined by

$$(M_s\boldsymbol{\psi})_n := -\frac{1}{h^2}\left(\psi_{n+1} - 2\psi_n + \psi_{n-1}\right) - f_0'(\varphi(hn-s))\psi_n, \quad n \in \mathbb{Z}. \tag{2.3}$$

The following lemmas describe properties of M_s, L_0, and L_c for small $c \in \mathbb{R}$. In what follows, we use bold letters to denote vectors in $l^2(\mathbb{Z})$, plain letters to denote functions in $L^2(\mathbb{R})$, and subscripts to denote elements of sequences, i.e.,

$$\mathbf{a} \in l^2(\mathbb{Z}), \quad a(z) \in L^2(\mathbb{R}), \quad \text{and} \quad \{a_n\}_{n\in\mathbb{Z}}.$$

Lemma 1 *Let Assumption 1 be satisfied. For any $s \in \mathbb{R}$, there exists a one-dimensional kernel of M_s given by*

$$\text{Ker}(M_s) = \text{span}\{\boldsymbol{\chi}(s)\} \in l^2(\mathbb{Z}), \quad \boldsymbol{\chi}(s) = \{\chi(hn-s)\}_{n\in\mathbb{Z}},$$

where $\chi(z) = \varphi'(z) \in C_{\text{even}}(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\chi(z) > 0$ for all $z \in \mathbb{R}$. Moreover, 0 is the smallest isolated eigenvalue of M_s and the rest of the spectrum of M_s is real and strictly positive.

Proof The existence of $\boldsymbol{\chi}(s)$ is a consequence of Assumption 1. We note that M_s is a second-order self-adjoint difference operator in $l^2(\mathbb{Z})$ and the discrete Wronskian

$$W[\boldsymbol{\psi}, \boldsymbol{\theta}] = \psi_{n+1}\theta_n - \psi_n\theta_{n+1}$$

is constant in $n \in \mathbb{N}$ for any two eigenvectors $\boldsymbol{\psi}$ and $\boldsymbol{\theta}$. Therefore, there may exist at most one decaying eigenvector of M_s for the zero eigenvalue.

Since $\boldsymbol{\chi}(s) > 0$ for any $s \in \mathbb{R}$, Sturm's theory for difference equations implies that 0 is the smallest eigenvalue of M_s. Finally, as a consequence of the condition (1.2), operator

M_s is asymptotically hyperbolic in the sense that $(M_s\boldsymbol{\psi})_n$ is represented asymptotically as $n \to \pm\infty$ by

$$(M^{\pm}\boldsymbol{\psi})_n := -\frac{1}{h^2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) - f_0'(\pm 1)\psi_n, \quad n \in \mathbb{Z}. \tag{2.4}$$

Because $f_0'(\pm 1) < 0$, the purely continuous spectrum of $M^{\pm}$ in $l^2(\mathbb{Z})$ is strictly positive and bounded away from zero. The purely continuous spectrum of $M^{\pm}$ gives the continuous spectrum of M_s thanks to the fast (exponential) decay of $\{\chi(hn - s)\}_{n\in\mathbb{Z}}$ to 0 as $n \to \pm\infty$. □

Lemma 2 *Under the same conditions as in Lemma 1,* dimKer$(L_0) = \infty$ *and* Ker$(L_0) \subset L^2(\mathbb{R})$ *is spanned by the set of eigenvectors* $\{\chi^{(m)}\}_{m\in\mathbb{Z}}$*, where*

$$\chi^{(m)}(z) = \chi(z)e^{i\kappa m z}, \quad \kappa = \frac{2\pi}{h}, \tag{2.5}$$

and $\chi(z) = \varphi'(z)$.

Proof If $\chi \in \mathrm{Ker}(L_0)$, then $\chi^{(m)} \in \mathrm{Ker}(L_0)$ for any $m \in \mathbb{Z}$, thanks to the fact that

$$e^{i\kappa m h} = e^{2\pi i m} = 1, \quad m \in \mathbb{Z}.$$

If $\chi \in \mathrm{Ker}(L_0)$, then $f\chi \in \mathrm{Ker}(L_0)$ for any h-periodic function $f(z) \in L^2_{\mathrm{per}}(\mathbb{R})$. Completeness of the infinite set of eigenvectors $\{\chi(z)e^{i\kappa m z}\}_{m\in\mathbb{Z}}$ follows from the completeness of the orthogonal basis $\{e^{i\kappa m z}\}_{m\in\mathbb{Z}}$ in the space of h-periodic functions in $L^2_{\mathrm{per}}(\mathbb{R})$. □

Remark 3 Sampling at $\{z_n(s) = hn - s\}_{n\in\mathbb{Z}}$ for each $m \in \mathbb{Z}$ shows that

$$\chi_n^{(m)}(s) := \chi^{(m)}(hn - s) \equiv \chi_n^{(0)}(s)e^{-i\kappa m s}, \quad m \in \mathbb{Z}.$$

Therefore, $\boldsymbol{\chi}^{(m)}(s) = e^{-i\kappa m s}\boldsymbol{\chi}(s)$, that is, all eigenvectors (2.5) in $\mathrm{Ker}(L_0) \subset L^2(\mathbb{R})$ are parallel to the same eigenvector $\boldsymbol{\chi}(s) \in \mathrm{Ker}(M_s) \subset l^2(\mathbb{Z})$.

Remark 4 In the continuous limit $h \to 0$ (see Remark 1), the advance-delay operator L_0 converges formally to the continuous Schrödinger operator

$$L = -\partial_x^2 - f_0'(\varphi(x)).$$

The zero eigenvalue of L is isolated from the rest of the spectrum of L and it is simple with the even positive eigenfunction $\chi(x) = \varphi'(x)$.

Because the zero eigenvalue of L_0 has infinite multiplicity, the perturbation theory in the singular limit $c \to 0$ is a delicate analytical problem. It involves a variant of the Lyapunov–Schmidt reduction algorithm with an infinite number of projections. To get definite results in the infinite-dimensional projective algorithm, we will assume that the Fourier transform of $\chi(z)$ decays exponentially in the Fourier space [or, equivalently, that $\chi(z)$ is real analytic for all $z \in \mathbb{R}$ denoted by $\chi \in C^{\omega}(\mathbb{R})$] and that the lattice spacing $h > 0$ is sufficiently small.

Lemma 3 *Let Assumption 1 be satisfied and assume that* $\chi(z) \in C^{\omega}_{\mathrm{even}}(\mathbb{R})$*. There exist* $h_0 > 0$ *and* $c_0 > 0$ *such that for any* $h \in (0, h_0)$ *and* $c \in (0, c_0)$*, operator* L_c *has a unique eigenvector-eigenvalue pair*

$$(\lambda_c, \chi_c) \in \mathbb{C} \times L^2, \quad \lambda_c = \mathcal{O}(c^2) \quad as \quad c \to 0 \tag{2.6}$$

and an infinite set of simple eigenvalues

$$\lambda_c^{(m)} = \lambda_c - i\kappa m c, \quad \chi_c^{(m)}(z) = \chi_c(z) e^{i\kappa m z}, \quad m \in \mathbb{Z}. \tag{2.7}$$

All other eigenvalues of L_c are bounded away from zero as $c \to 0$.

Proof If $(\lambda_c, \chi_c) \in \mathbb{C} \times L^2$ is an eigenvector-eigenvalue pair of operator L_c for any $c \in \mathbb{R}$, then

$$(\lambda_c - i\kappa m c, \chi_c(z) e^{i\kappa m z}) \in \mathbb{C} \times L^2$$

is also an eigenvector-eigenvalue pair of operator L_c for any $m \in \mathbb{Z}$.

To study the pair $(\lambda_c, \chi_c) \in \mathbb{C} \times L^2$ for small $c \neq 0$, we write

$$\chi_c(z) = \chi_0(z) + c\chi_1(z) + c^2 \tilde{\chi}(z), \tag{2.8}$$

where $\chi_0(z) \in C^\omega_{\text{even}}(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\chi_1(z) \in C^\omega(\mathbb{R}) \cap L^2(\mathbb{R})$ will be specified below. The function $\tilde{\chi}(z) \in L^2(\mathbb{R})$ is the remainder term. Let us define the leading-order function

$$\chi_0(z) = \chi(z) \sum_{m \in \mathbb{Z}} a_m e^{i\kappa m z}, \tag{2.9}$$

where the set $\{a_m\}_{m\in\mathbb{Z}}$ is to be found from a projection algorithm. If $\chi_0(z)$ is even and real, then $\{a_m\}_{m\in\mathbb{Z}}$ satisfy the reduction

$$a_m = \bar{a}_m = a_{-m}, \quad m \in \mathbb{Z}. \tag{2.10}$$

We also add the normalization $a_0 = 1$. Consider the inhomogeneous equation for $\chi_1(z)$,

$$L_0 \chi_1 = F_1, \text{ where } F_1(z) := \chi_0'(z). \tag{2.11}$$

The solvability condition $F_1 \in \text{Ran}(L_0)$ for this equation is equivalent to the system

$$\langle \chi^{(n)}, F_1 \rangle_{L^2} = 0, \quad n \in \mathbb{Z}, \tag{2.12}$$

because L_0 is self-adjoint in $L^2(\mathbb{R})$.

Let $\hat{F}(k)$ be the Fourier transform of $F(z) \in L^2(\mathbb{R})$ given by

$$\hat{F}(k) = \int_{\mathbb{R}} F(z) e^{ikz} dz, \quad \text{so that } F(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{F}(k) e^{-ikz} dk. \tag{2.13}$$

Projection Eqs. (2.12) are equivalent to the following system on $\{a_m\}_{m\in\mathbb{Z}}$,

$$i\kappa \sum_{m\in\mathbb{Z}} m a_m \widehat{\chi^2}(\kappa(m-n)) + \sum_{m\in\mathbb{Z}} a_m \widehat{\chi'\chi}(\kappa(m-n)) = 0, \quad n \in \mathbb{Z}. \tag{2.14}$$

If $\chi(z) \in C^\omega(\mathbb{R})$, then $\hat{\chi}(k)$ decays to zero exponentially fast as $|k| \to \infty$. Since $\kappa = \frac{2\pi}{h}$ and $h > 0$ is small, the coefficients in the linear system (2.14) are strictly diagonally dominant and exponentially small in h for $|m - n| \geq 1$. We also note that $\widehat{\chi'\chi}(0) = 0$, whereas $\widehat{\chi^2}(0) > 0$. As a result, under the normalization $a_0 = 1$, there exists a unique solution of the system (2.14) for $\{a_m\}_{m\in\mathbb{Z}\backslash\{0\}}$ with the symmetry (2.10). Equation (2.14) at $n = 0$ is redundant under the symmetry (2.10).

Let $\{a_m\}_{m\in\mathbb{Z}}$ be uniquely defined by the system (2.14) under the symmetry (2.10). Let P_0 be the orthogonal projection operator from L^2 to $\text{Ran}(L_0)$. We write the solution of the inhomogeneous Eq. (2.11) in the form

$$\chi_1(z) = P_0 L_0^{-1} P_0 F_1(z) + \chi(z) \sum_{m\in\mathbb{Z}} b_m e^{i\kappa m z}, \tag{2.15}$$

where the set $\{b_m\}_{m\in\mathbb{Z}}$ is to be found from the projection algorithm under the normalization $b_0 = 0$. Note that $P_0 L_0^{-1} P_0 F_1(z) \in C^{\omega}_{\text{odd}}(\mathbb{R}) \cap L^2(\mathbb{R})$.

The remainder term $\tilde{\chi}$ satisfies the residual equation

$$(L_0 - c\partial_z - \lambda_c I)\tilde{\chi} = \tilde{F}, \quad \text{where } \tilde{F}(z) := \lambda_c(\chi_0(z) + c\chi_1(z)) + c^2\chi_1'(z). \tag{2.16}$$

For uniqueness, we require that $\tilde{\chi} \in \text{Ran}(L_0)$. The solvability condition for Eq. (2.16) is equivalent to the infinite system of equations,

$$\tilde{G}^{(n)} := \langle \chi^{(n)}, \tilde{F} + c\partial_z\tilde{\chi} + \lambda_c\tilde{\chi}\rangle_{L^2} = \langle \chi^{(n)}, \tilde{F}\rangle_{L^2} - c\langle \partial_z\chi^{(n)}, \tilde{\chi}\rangle_{L^2} = 0, \quad n \in \mathbb{Z}. \tag{2.17}$$

Recall that the spectrum of $P_0 L_0 P_0$ is bounded away from zero by a positive number and that $c\partial_z$ is a skew-adjoint perturbation to the self-adjoint operator L_0. As a result, the real part of the spectrum of $P_0(L_0 - c\partial_z)P_0$ is bounded from below by a positive number [3]. By the Inverse Function Theorem, for small c and λ_c there exists a unique solution of Eq. (2.16) for $\tilde{\chi} \in \text{Ran}(L_0)$ under the solvability condition (2.17) such that for some $D, D' > 0$,

$$\begin{aligned}(1 - \lambda_c)\|\tilde{\chi}\|_{L^2}^2 &\le D\langle (P_0 L_0 P_0 - \lambda_c I)\tilde{\chi}, \tilde{\chi}\rangle_{L^2} = D\langle (L_0 - c\partial_z - \lambda_c I)\tilde{\chi}, \tilde{\chi}\rangle_{L^2} \\ &= D\langle \tilde{F}, \tilde{\chi}\rangle_{L^2} \le D'\|\tilde{\chi}\|_{L^2}(c^2 + \lambda_c),\end{aligned}$$

that is,

$$\exists D > 0: \quad \|\tilde{\chi}\|_{L^2} \le D(c^2 + \lambda_c). \tag{2.18}$$

Thanks to the bound (2.18), the solvability system (2.17) can be written by

$$\tilde{G}^{(n)} = \langle \chi^{(n)}, \lambda_c\chi_0 + c^2\chi_1'\rangle_{L^2} + \mathcal{O}(c^3, c\lambda_c) = 0, \quad n \in \mathbb{Z}, \tag{2.19}$$

or explicitly,

$$\begin{aligned}\tilde{G}^{(n)} &= \lambda_c \sum_{m\in\mathbb{Z}} a_m \widehat{\chi^2}(\kappa(m-n)) + i\kappa c^2 \sum_{m\in\mathbb{Z}} m b_m \widehat{\chi^2}(\kappa(m-n)) + c^2 \sum_{m\in\mathbb{Z}} b_m \widehat{\chi'\chi}(\kappa(m-n)) \\ &\quad + c^2 \left\langle \chi^{(n)}, \left(P_0 L_0^{-1} P_0 F_1\right)' \right\rangle_{L^2} + \mathcal{O}(c^3, c\lambda_c) = 0, \quad n \in \mathbb{Z}.\end{aligned} \tag{2.20}$$

Again, because of the exponential decay of $\hat{\chi}(k)$ as $|k| \to \infty$, the coefficients in the linear system (2.20) are exponentially small in h for $|m - n| \ge 1$. For $n \ne 0$, the system (2.20) admits a unique solution for $\{b_m\}_{m\in\mathbb{Z}\setminus\{0\}}$ such that $\sup_{m\in\mathbb{Z}} |b_m| = \mathcal{O}(1)$ as $c \to 0$. For $n = 0$, the system (2.20) gives a uniquely solvable equation for λ_c, which shows that $\lambda_c = \mathcal{O}(c^2)$ as $c \to 0$. □

Remark 5 Expansion (2.8) with the leading-order term (2.9) shows that $\|\chi_c - \chi\|_{L^2}$ does not converge to zero as $c \to 0$ because of the projections to all other eigenvectors $\{\chi^{(m)}\}_{m\in\mathbb{Z}}$ in $\text{Ker}(L_0) \subset L^2(\mathbb{R})$.

Remark 6 For the Schrödinger operator $L = -\partial_x^2 - f_0'(\varphi(x))$ (see Remark 4), expansion (2.8) becomes

$$\chi_c = \chi + c\chi_1 + c^2\chi_2 + \mathcal{O}_{H^2}(c^3), \quad \lambda_c = c^2\lambda_2 + \mathcal{O}(c^3), \tag{2.21}$$

where

$$L\chi = 0, \quad L\chi_1 = \chi'(x), \quad L\chi_2 = \chi_1'(x) + \lambda_2\chi(x), \quad \text{etc.}$$

Since L is a self-adjoint operator and $\chi(x) \in C^{\omega}_{\text{even}}(\mathbb{R}) \cap H^2(\mathbb{R})$, we obtain uniquely

$$\chi_1(x) = L^{-1}\chi'(x) \in C^{\omega}_{\text{odd}}(\mathbb{R}) \cap H^2(\mathbb{R})$$

and

$$\lambda_2 = -\frac{\langle \chi, \chi_1' \rangle_{L^2}}{\|\chi\|^2_{L^2}} = \frac{\langle L^{-1}\chi', \chi' \rangle_{L^2}}{\|\chi\|^2_{L^2}} > 0,$$

after which a unique solution exists for $\chi_2(x) \in C^{\omega}_{\text{even}}(\mathbb{R}) \cap H^2(\mathbb{R}) \cap \text{Ran}(L)$. Since $c\partial_x$ is a regular perturbation to L, perturbation series expansion (2.21) is standard in the regular perturbation theory. In the case of the advance-delay operator L_0, $c\partial_x$ is a singular perturbation to L_0 and the perturbation series expansion can only be extended after infinitely many projections are accounted at each order in powers of c.

Lemma 4 *Let (λ_c, χ_c) denote the eigenvalue–eigenvector pair in Lemma 3 for $c \neq 0$. Then*

$$\dim\text{Ker}(L_c^* - \lambda_c I) = 1,$$

and there is a $\theta_c \in \text{Ker}(L_c^* - \lambda_c I)$. Similarly,

$$\text{Ker}(L_c^* - \lambda_c^{(m)} I) = \text{span}\left\{\theta_c^{(m)}\right\}, \tag{2.22}$$

where

$$\lambda_c^{(m)} = \lambda - i\kappa m c, \quad \theta_c^{(m)}(z) = \theta_c(z) e^{-i\kappa m z}, \quad m \in \mathbb{Z}. \tag{2.23}$$

Moreover, eigenvectors $\left\{\chi_c^{(m)}\right\}_{m\in\mathbb{Z}}$ and $\left\{\theta_c^{(n)}\right\}_{n\in\mathbb{Z}}$ are orthogonal in the sense

$$\langle \theta_c^{(n)}, \chi_c^{(m)} \rangle_{L^2} = \delta_{n,m} \langle \theta_c, \chi_c \rangle_{L^2}, \quad m, n \in \mathbb{Z}, \tag{2.24}$$

where $\langle \theta_c, \chi_c \rangle_{L^2} \neq 0$.

Proof The differential advance-delay operator L_c is asymptotically hyperbolic in the sense that $(L_c\psi)(z)$ is represented asymptotically as $z \to \pm\infty$ by

$$(L_c^{\infty}\psi)(z) := -c\frac{d\psi}{dz} - \frac{1}{h^2}\left(\psi(z+h) - 2\psi(z) + \psi(z-h)\right) - f_0'(\pm 1)\psi(z).$$

Using Fourier transform, we obtain for any $c \in \mathbb{R}$,

$$\sigma(L_c^{\infty}) = \left\{-ick + \frac{4}{h^2}\sin^2\left(\frac{k}{2}\right) - f_0'(\pm 1), \quad k \in \mathbb{R}\right\}.$$

Because $f_0'(\pm 1) < 0$, the spectrum of L_c^{∞} is bounded away from zero for any $c \in \mathbb{R}$. Assumptions of Theorem A of Mallet-Paret in [16] are satisfied for any $c \neq 0$ and the differential advance-delay operator L_c for $c \neq 0$ is a Fredholm operator of index zero. Existence of the one-dimensional $\dim\text{Ker}(L_c^* - \lambda_c I)$ spanned by $\theta_c \in L^2(\mathbb{R})$ follows from Fredholm's Alternative Theorem. Existence of the infinitely many eigenvalues (2.22) follows from the explicit form of L_c^*.

The orthogonality relations (2.24) follow from the fact that the isolated eigenvalues in the set $\{\lambda_c^{(m)}\}_{m\in\mathbb{Z}}$ are simple. □

Let us consider the inhomogeneous equation

$$(L_c - \lambda_c I)\,\psi = f, \quad c \neq 0, \tag{2.25}$$

for a given $f(z) \in L^2(\mathbb{R})$. The following lemma gives the main result needed for the proof of Theorem 1 in Sect. 3.

Lemma 5 *For any $f(z) \in L^2(\mathbb{R})$, there exists a unique solution $\psi(z) \in H^1(\mathbb{R}) \cap [\mathrm{Ker}(L_c - \lambda_c I)]^{\perp}$ of the inhomogeneous problem (2.25) if and only if $\langle \theta_c, f\rangle_{L^2} = 0$. Moreover, for any small $c > 0$, there is $D > 0$ such that*

$$\|\psi\|_{L^2} \leq \frac{D}{c}\|f\|_{L^2}. \tag{2.26}$$

If, in addition, for any $c > 0$, $f(z) \in L^2(\mathbb{R})$ satisfies

$$\langle \theta_c^{(n)}, f\rangle_{L^2} = 0, \quad \forall n \in \mathbb{Z}, \tag{2.27}$$

then there exists a unique solution

$$\psi(z) \in H^1(\mathbb{R}) \cap \left[\oplus_{m\in\mathbb{Z}} Ker\left(L_c - \lambda_c^{(m)} I\right)\right]^{\perp} \tag{2.28}$$

of the inhomogeneous problem (2.25) such that

$$\exists D > 0: \quad \|\psi\|_{L^2} \leq D\|f\|_{L^2}. \tag{2.29}$$

Proof Since L_c for $c \neq 0$ is a Fredholm operator of index zero, $\mathrm{Ker}(L_c - \lambda_c I) = \mathrm{span}\{\chi_c\}$, and $\mathrm{Ker}(L_c^* - \lambda_c I) = \mathrm{span}\{\theta_c\}$, Fredholm's Alternative Theorem gives existence of a unique solution $\psi \in H^1(\mathbb{R}) \cap [\mathrm{Ker}(L_c - \lambda_c I)]^{\perp}$ of the inhomogeneous problem (2.25) if and only if $\langle \theta_c, f\rangle_{L^2} = 0$. To prove bound (2.26), we use orthogonality of eigenvectors (2.24) and spectral decompositions. For any $f(z) \in L^2(\mathbb{R})$, we obtain

$$\psi(z) = \sum_{m\in\mathbb{Z}} \frac{i\langle \theta_c^{(m)}, f\rangle_{L^2}}{\kappa m c \langle \theta_c, \chi_c\rangle_{L^2}} \chi_c(z) + \tilde{\psi}(z),$$

where $\tilde{\psi} \in \left[\oplus_{m\in\mathbb{Z}} \mathrm{Ker}(L_c - \lambda_c^{(m)} I)\right]^{\perp}$ and the singular term $m = 0$ is missing thanks to the constraint $\langle \theta_c, f\rangle_{L^2} = 0$. The function $\tilde{\psi}(z)$ does not give a singular contribution to $\|\psi\|_{L^2}$ as $c \to 0$, whereas the summation term gives $\mathcal{O}(c^{-1})$ contribution to $\|\psi\|_{L^2}$. If $f(z)$ satisfies infinitely many constraints (2.27), the summation term is zero and $\psi(z) = \tilde{\psi}(z)$ satisfies (2.29). □

Corollary 2 *If $f(z) \in H^1(\mathbb{R})$ satisfies infinitely many constraints (2.27) for any small $c > 0$, then there exists a unique solution (2.28) of the inhomogeneous Eq. (2.25) such that*

$$\exists D > 0: \quad \|\psi\|_{H^1} \leq D\|f\|_{H^1}. \tag{2.30}$$

Proof To extend bound (2.29) to H^1, we note that the advance-delay operator L_0 is a bounded operator from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$ for any $s \geq 0$. Let P_c and P_c^* are the orthogonal projection operators to $\left[\oplus_{m\in\mathbb{Z}} \mathrm{Ker}(L_c^* - \lambda_c^{(m)} I)\right]^{\perp}$ and $\left[\oplus_{m\in\mathbb{Z}} \mathrm{Ker}(L_c - \lambda_c^{(m)} I)\right]^{\perp}$ respectively. Then, $P_c^*(L_c - \lambda_c I)^{-1} P_c$ is a bounded operator from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$ uniformly as $c \to 0$. □

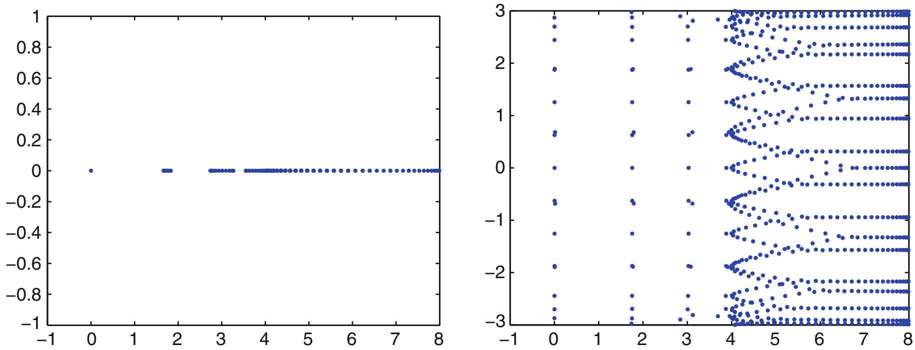

Fig. 1 Numerical approximation of spectrum of L_c for $c = 0$ (*left*) and $c = 0.1$ (*right*)

To illustrate Lemmas 3 and 5, we consider the example of the advance-delay operator L_0 associated with the special nonlinearity (1.9) and the exact solution (1.10). The corresponding operator L_0 is given by

$$(L_0\psi)(z) = -\frac{1}{h^2}(\psi(z+h) - 2\psi(z) + \psi(z-h)) + \frac{2(2 - 3\text{sech}^2(bz) - h^2\text{sech}^4(bz))\psi(z)}{(1 + h^2\text{sech}^2(bz))^2}. \tag{2.31}$$

Operator $L_0 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a self-adjoint operator and its kernel is generated by

$$\chi(z) = \text{sech}^2(bz).$$

Eigenvectors and eigenvalues of L_c and L_c^* are computed numerically using the six-order finite-difference approximation of the derivative operator with the step size of 0.05, the truncation of the computational domain on $[-20, 20]$, and the MATLAB eigenvalue solver.

The spectrum of L_c and L_0 is shown on Fig. 1. In agreement with Lemma 3, we observe many eigenvalues bifurcating from 0 for $c \neq 0$ along the imaginary axis. There is only one real eigenvalue λ_c in the neighborhood of 0 for small $c \neq 0$. Behavior of λ_c and $\|\chi_c - \chi\|_{L^2}$ as $c \to 0$ is illustrated on Fig. 2. We confirmed numerically the convergence rate $\lambda_c = \mathcal{O}(c^2)$ as $c \to 0$ (Lemma 3). On the other hand, the right panel of the figure shows that $\|\chi_c - \chi\|_{L^2}$ does not converge to zero as $c \to 0$ because of the projections to all other eigenvectors (Remark 5).

We compute eigenvector θ_c of the adjoint operator L_c^* numerically and use

$$f_c = f_0 - \frac{\langle\theta_c, f_0\rangle_{L^2}}{\|\theta_c\|_{L^2}^2}\chi, \quad \text{where} \quad f_0(z) = \text{sech}(bz).$$

The norm $\|\psi\|_{L^2}$ of the solution of the inhomogeneous Eq. (2.25) with $f = f_c$ is shown on the left panel of Fig. 3. The norm $\|\psi\|_{L^2}$ diverges as $c \to 0$ in agreement with Lemma 5. We confirm numerically the divergence rate $\|\psi\|_{L^2} = \mathcal{O}(c^{-1})$ as $c \to 0$. A typical solution $\psi(z)$ for $c = 0.1$ is shown on the right panel of Fig. 3.

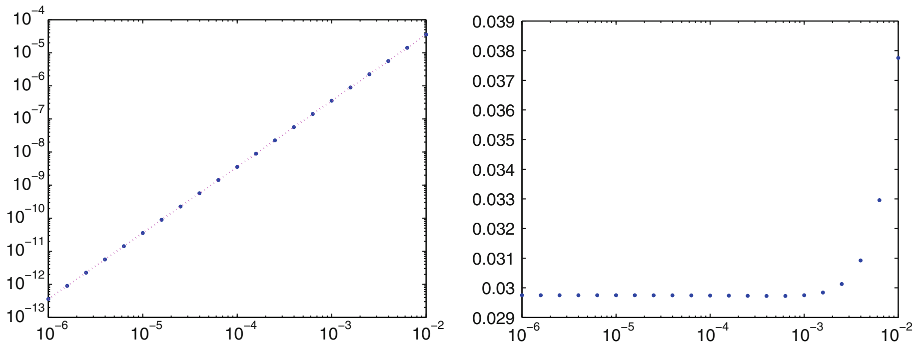

Fig. 2 *Left*: convergence of the smallest eigenvalue of L_c as $c \to 0$. The dotted curve shows the power fit with $c^{1.9997}$. *Right*: the norm $\|\chi_c - \chi\|_{L^2}$ versus c for the corresponding eigenvector

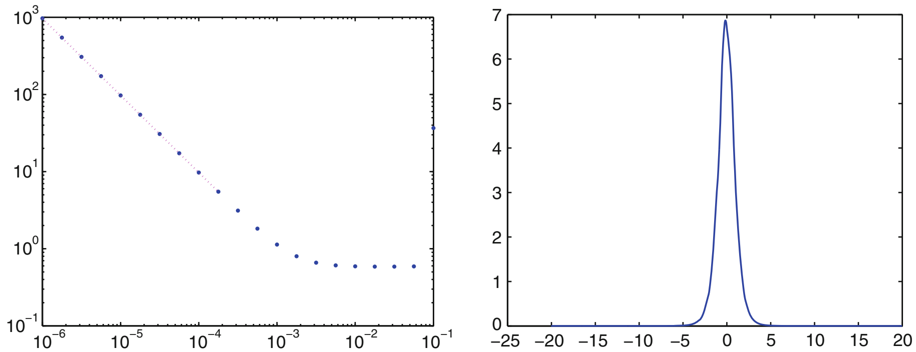

Fig. 3 *Left*: the norm $\|\psi\|_{L^2}$ versus c. The dotted curve shows the power fit with $c^{-0.9993}$. *Right*: the solution $\psi(z)$ for $c = 0.1$

3 Proof of the Main Results

Let us start with the proof of Corollary 1, which is shorter than the proof of Theorem 1.

Proof of Corollary 1 Let us consider the stationary front that solves the difference Eq. (1.14). Let $\{\varphi_n(s) = \varphi(hn - s)\}_{n \in \mathbb{Z}}$ be the translationally invariant front for $a = 0$ that exists by Assumption 1. Substituting

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}(s) + \boldsymbol{\psi}$$

to the difference Eq. (1.14), we obtain the persistence problem for $\boldsymbol{\psi}$,

$$M_s \boldsymbol{\psi} = a\mathbf{h} + \mathbf{N}(\boldsymbol{\psi}), \tag{3.1}$$

where the nth components of the vectors are given by $h_n = f_1(\varphi_n(s))$ and

$$\begin{aligned}(\mathbf{N}(\boldsymbol{\psi}))_n = & f(\varphi_n(s) + \psi_n) - f_0(\varphi_n(s) + \psi_n) - af_1(\varphi_n(s) + \psi_n) \\ & + f_0(\varphi_n(s) + \psi_n) - f_0(\varphi_n(s)) - f_0'(\varphi_n(s))\psi_n + a\left(f_1(\varphi_n(s) + \psi_n) - f_1(\varphi_n)\right).\end{aligned}$$

Since f is C^2 both in u_n and a, whereas $l^2(\mathbb{Z})$ is a Banach algebra with respect to pointwise multiplication, there is a positive constant C such that

$$\|\mathbf{N}_a(\boldsymbol{\psi})\|_{l^2} \leq C\left(a^2 + |a|\|\boldsymbol{\psi}\|_{l^2} + \|\boldsymbol{\psi}\|_{l^2}^2\right). \tag{3.2}$$

By Lemma 1, we have

$$\mathrm{Ker}(M_s) = \mathrm{span}\{\boldsymbol{\chi}(s)\} \subset l^2(\mathbb{Z}).$$

Since M_s is a Fredholm operator of index zero, Fredholm's Alternative Theorem tells us that no solution $\boldsymbol{\psi} \in l^2(\mathbb{Z})$ of Eq. (3.1) exists in a neighborhood of $\mathbf{0} \in l^2(\mathbb{Z})$ if

$$\langle \boldsymbol{\chi}(s), a\mathbf{h} + \mathbf{N}(\boldsymbol{\psi})\rangle_{l^2} \neq 0 \tag{3.3}$$

for a fixed $s \in \mathbb{R}$. By Assumption 1, $\boldsymbol{\chi}(s)$ is positive and $\mathbf{h}$ is negative, so that $\langle \boldsymbol{\chi}(s), \mathbf{h}\rangle_{l^2} < 0$ for all $s \in \mathbb{R}$. In view of the bound (3.2) and the smallness of a, we infer that (3.3) holds for small $a \neq 0$. Therefore, no stationary fronts of the difference Eq. (1.14) exist in a neighborhood of $\boldsymbol{\varphi}(s)$ for any $s \in \mathbb{R}$ and small $a \neq 0$. □

Let us now consider the traveling front that solves the differential advance-delay Eq. (1.5). Let $\varphi(z)$ be a solution of the advance-delay Eq. (1.11) for $a = 0$. The following result shows that the solution is not unique and it is in fact a member of an infinite-dimensional family of solutions.

Lemma 6 *Let $\varphi(z)$ be a solution of the advance-delay Eq.* (1.11) *with properties* (1.12). *Let $g \in C^1_{\mathrm{per}}(0, h)$ satisfy $\|g'\|_{L^\infty} < 1$. Then, $\tilde{\varphi}(\tilde{z})$ is also a solution of the same Eq.* (1.11) *with the same properties* (1.12) *if*

$$z = \tilde{z} - g(\tilde{z}), \quad \varphi(z) = \varphi(\tilde{z} - g(\tilde{z})) = \tilde{\varphi}(\tilde{z}). \tag{3.4}$$

Proof If $g(\tilde{z} \pm h) = g(\tilde{z})$, then

$$\varphi(z \pm h) = \varphi(\tilde{z} \pm h - g(\tilde{z} \pm h)) = \tilde{\varphi}(\tilde{z} \pm h), \tag{3.5}$$

so that $\tilde{\varphi}(\tilde{z})$ satisfies the same equation as $\varphi(z)$. To ensure that $\tilde{\varphi}(\tilde{z})$ is a monotonically increasing front, we need invertibility of the mapping $z \mapsto \bar{z}$, that is, $1 - g'(\tilde{z}) > 0$ for all $\tilde{z} \in \mathbb{R}$, which is guaranteed by the assumption that $\|g'\|_{L^\infty} < 1$. □

We need the periodic function $g(\tilde{z})$ in order to cancel the terms of the order $\mathcal{O}(c, a)$ in solutions of the differential advance-delay Eq. (1.5) in a neighborhood of $\varphi(z)$. Using the Fourier series, we represent

$$\frac{d\tilde{z}}{dz} = \frac{1}{1 - g'(\tilde{z})}\Big|_{z=\tilde{z}-g(\tilde{z})} = 1 + \sum_{m\in\mathbb{Z}} b_m e^{i\kappa m z}, \tag{3.6}$$

where coefficients $\{b_m\}_{m\in\mathbb{Z}}$ are to be found from a projection algorithm. To this end, we substitute

$$\phi(z) = \tilde{\varphi}(\tilde{z}) + \psi(z)$$

to the differential advance-delay Eq. (1.5) and obtain the persistence problem for $\psi(z)$,

$$L_c\psi = h + N(\psi), \tag{3.7}$$

where

$$h(z) = c\tilde{\chi}(z)\left(1 + \sum_{m\in\mathbb{Z}} b_m e^{i\kappa m z}\right) + aF_1(z), \quad \tilde{\chi}(z) = \tilde{\varphi}'(\tilde{z}), \quad F_1(z) = f_1(\tilde{\varphi}(\tilde{z})),$$

and

$$\begin{aligned} N(\psi) = & f(\varphi+\psi) - f_0(\varphi+\psi) - af_1(\varphi+\psi) \\ & + f_0(\varphi+\psi) - f_0(\varphi) - f_0'(\varphi)\psi + a\left(f_1(\varphi+\psi) - f_1(\varphi)\right), \end{aligned}$$

where we have used identities (3.4) and (3.5). The following result describes properties of the source term and the nonlinear vector field in Eq. (3.7).

Lemma 7 *Let Assumption* 1 be satisfied. Then, $h \in H^1(\mathbb{R})$ and $N(\psi)$ maps $H^1(\mathbb{R})$ to $H^1(\mathbb{R})$. Moreover, for small a, c, and $\psi, \tilde{\psi} \in H^1(\mathbb{R})$, there exists $C > 0$ such that

$$\begin{aligned} \|h\|_{L^2} &\leq C(|c|+|a|), \\ \|N(\psi)\|_{H^1} &\leq C\left(a^2 + |a|\|\psi\|_{H^1} + \|\psi\|^2_{H^1}\right), \\ \|N(\psi) - N(\tilde{\psi})\|_{H^1} &\leq C\left(|a| + \|\psi\|_{H^1} + \|\tilde{\psi}\|_{H^1}\right)\|\psi - \tilde{\psi}\|_{H^1}. \end{aligned}$$

Proof The proof follows from the properties of f in Assumption 1 and the fact that $H^1(\mathbb{R})$ is a Banach algebra with respect to pointwise multiplication. □

Remark 7 If f is the special nonlinear function (1.9) and $b_m = 0$ for all $m \in \mathbb{Z}$, then

$$h(z) = c\varphi'(z) - \frac{2a(1-\varphi^2(z))}{1+h^2(1-a^2)}.$$

If $\varphi(z) = \tanh(bz)$ and $c = \frac{2a}{1+h^2(1-a^2)}$, then $h(z) \equiv 0$. In this case, $\psi(z) \equiv 0$ satisfies (3.7). Hence expression (1.10) is an exact solution of Eq. (1.5).

Proof of Theorem 1 Let P_c and P_c^* be the projection operator to subspaces

$$\left[\oplus_{m\in\mathbb{Z}}\mathrm{Ker}(L_c^* - \lambda_c^{(m)}I)\right]^\perp \quad \text{and} \quad \left[\oplus_{m\in\mathbb{Z}}\mathrm{Ker}(L_c - \lambda_c^{(m)}I)\right]^\perp.$$

Using an infinite-dimensional analogue of the method of Lyapunov–Schmidt reductions, we are looking for a solution of the persistence problem (3.7) in function space

$$\psi \in H^1(\mathbb{R}) \cap \left[\oplus_{m\in\mathbb{Z}}\mathrm{Ker}(L_c - \lambda_c^{(m)}I)\right]^\perp. \tag{3.8}$$

The solvability condition for the persistence problem (3.7) is given by

$$\mathbf{G}^{(n)}(\psi) := \langle\theta_c^{(n)}, h + N(\psi)\rangle_{L^2} + \lambda_c^{(n)}\langle\theta^{(n)}, \psi\rangle_{L^2} = 0, \quad n \in \mathbb{Z}. \tag{3.9}$$

Because of the system (3.9), we can write

$$\psi = H(\psi) := P_c^* L_c^{-1} P_c(h + N(\psi)). \tag{3.10}$$

Let $\bar{B}_\delta(H^1)$ denotes a closed ball of radius $\delta > 0$ in $H^1(\mathbb{R})$. Thanks to Corollary 2 and Lemma 7, $H(\psi) : \bar{B}_\delta(H^1) \to \bar{B}_\delta(H^1)$ is a Lipschitz continuous map for sufficiently small $|c|$, $|a|$, and $\delta > 0$ such that

$$\exists C > 0 : \quad \|H(0)\|_{H^1} \leq C(|c|+|a|).$$

Moreover, the map is a contraction for sufficiently small $|c|$, $|a|$, and $\delta > 0$. By Banach's Fixed-Point Theorem, for sufficiently small a and c, there exists a unique map $\mathbb{R}^2 \ni (c, a) \mapsto \psi \in H^1(\mathbb{R})$ such that ψ satisfies Eq. (3.10) and

$$\exists C > 0 : \quad \|\psi\|_{H^1} \leq C(|c|+|a|). \tag{3.11}$$

Therefore, for small a and c, we have

$$\mathbf{G}^{(n)}(\psi) := \langle \theta_c^{(n)}, h \rangle_{L^2} + \mathcal{O}(a^2 + |a||c| + c^2) \quad n \in \mathbb{Z}. \tag{3.12}$$

In the explicit form, we can write

$$\langle \theta_c^{(n)}, h \rangle_{L^2} = c \left(\widehat{\theta_c \tilde{\chi}}(-\kappa n) + \sum_{m \in \mathbb{Z}} b_m \widehat{\theta_c \tilde{\chi}}(\kappa(m-n)) \right) + a \widehat{\theta_c F_1}(\kappa n),$$

where $\hat{F}$ denotes again the Fourier transform (2.13). Without the loss of generality, we set $b_0 = 0$.

By Lemmas 3 and 4, there exists a limiting function

$$\theta_0(z) = \lim_{c \to 0} \theta_c(z), \quad z \in \mathbb{R},$$

and the distance $\|\theta_0 - \chi\|_{L^\infty}$ is exponentially small in h as $h \to 0$. Since $\chi(z) > 0$ for all $z \in \mathbb{R}$, this implies that $\theta_0(z) > 0$ for all $z \in \mathbb{R}$ for sufficiently small $h > 0$. As a result, thanks to the assumption $f_1'(\varphi(z)) < 0$ for all $z \in \mathbb{R}$, we have

$$\widehat{\theta_0 \tilde{\chi}}(0) > 0, \quad \widehat{\theta_0 F_1}(0) < 0. \tag{3.13}$$

Thanks to the analyticity of $\chi(z)$ for $z \in \mathbb{R}$ and smallness of $h > 0$, the Fourier transform of $\theta_c(z)$ and $\tilde{\chi}(z)$ decays to zero exponentially fast as $|k| \to \infty$. Since $\kappa = \frac{2\pi}{h}$, this means that $\widehat{\theta_c \tilde{\chi}}(\kappa(m-n))$ is exponentially small in h for $|m-n| \geq 1$. Solving the system (3.12) separately for the component $n = 0$ and for the components $n \in \mathbb{Z}\backslash\{0\}$, we have for $n = 0$,

$$\mathbf{G}^{(0)}(\psi) = c \left(\widehat{\theta_c \tilde{\chi}}(0) + \sum_{m \in \mathbb{Z}} b_m \widehat{\theta_c \tilde{\chi}}(\kappa m) \right) + a \widehat{\theta_c F_1}(0) + \mathcal{O}\left(a^2 + |a||c| + c^2\right) = 0. \tag{3.14}$$

Under the condition (3.13), the scalar Eq. (3.14) is solved using the Implicit Function Theorem in a local neighborhood of $c = 0$ and $a = 0$ for sufficiently small $h > 0$. For a given $\{b_n\}_{n \in \mathbb{Z}}$ with $b_0 = 0$, there exists a unique solution

$$c = c_*(a) = c_1 a + \mathcal{O}(a^2), \quad c_1 = -\frac{\widehat{\theta_0 F_1}(0)}{\widehat{\theta_0 \tilde{\chi}}(0) + \sum_{m \in \mathbb{Z}} b_m \widehat{\theta_0 \tilde{\chi}}(\kappa m)} > 0. \tag{3.15}$$

For the components $n \in \mathbb{Z}\backslash\{0\}$, the system (3.12) under the relation (3.15) is rewritten in the form

$$\frac{b_n \widehat{\theta_c \tilde{\chi}}(0) + \widehat{\theta_c \tilde{\chi}}(-\kappa n) + \sum_{m \in \mathbb{Z}\backslash\{n\}} b_m \widehat{\theta_c \tilde{\chi}}(\kappa(m-n))}{\widehat{\theta_0 \tilde{\chi}}(0) + \sum_{m \in \mathbb{Z}} b_m \widehat{\theta_0 \tilde{\chi}}(\kappa m)} = \frac{\widehat{\theta_c F_1}(\kappa n)}{\widehat{\theta_0 F_1}(0)} + \mathcal{O}(a), \quad n \in \mathbb{Z}\backslash\{0\}.$$

Again, thanks to fact that $\widehat{\theta_c \tilde{\chi}}(\kappa(m-n))$ is exponentially small in $h > 0$ for $|m-n| \geq 1$, there is a unique solution for $\{b_m\}_{m \in \mathbb{Z}\backslash\{0\}}$ such that $\sup_{m \in \mathbb{Z}} |b_m|$ is exponentially small in $h > 0$. This construction completes the proof of Theorem 1. □

Remark 8 It is an essential ingredient of the proof of Theorem 1 that we invert the differential advance-delay operator $(L_c - \lambda_c I)$ for $c \neq 0$ instead of the limiting advance-delay operator L_0. The Lyapunov–Schmidt reductions can not be constructed from the bifurcation point $c = 0$ directly because $c\partial_z$ is a singular perturbation operator to L_0.

4 Discussions

Theorem 1 and Corollary 1 were also proven recently in [13], where nontrivial examples are constructed numerically. If the condition $f_1'(\varphi(z)) < 0$ for all $z \in \mathbb{R}$ is violated, the numerical examples show that propagation failure may occur in the discrete Nagumo equation in spite of the existence of the translationally invariant stationary fronts for $c = 0$. Here we will discuss another example of the same kind.

Let us consider a hybrid nonlinearity between the cubic nonlinearity (1.3) and the variational nonlinearity (1.9) in the form

$$f(u_n) = \frac{2(1-u_n^2)(u_n - a)(1 + h^2(1 + au_n))}{(1 + h^2(1-u_n^2))(1 + h^2(1-a^2))} + 2\mu(1-u_n^2)(u_n - a), \tag{4.1}$$

where (a, μ) are real parameters. For any small nonzero μ, we claim that the translationally invariant stationary front of the advance-delay Eq. (1.11) with properties (1.12) is destroyed. This fact suggests that the propagation failure interval is non-empty for any $\mu \neq 0$.

Let us look for solutions of the advance-delay Eq. (1.11) in the form $\varphi(z) = \varphi_0(z) + \psi(z)$, where $\varphi_0(z)$ is the translationally invariant stationary front for $\mu = 0$ and $\psi(z)$ is found from the persistence problem in the form

$$L_0 \psi = h + N(\psi),$$

where

$$h(z) = 2\mu(1 - \varphi_0^2(z))\varphi_0(z) \in H^1_{\text{odd}}(\mathbb{R})$$

and $N(\psi)$ is the nonlinear vector field such that $N(\psi) : H^1_{\text{odd}}(\mathbb{R}) \to H^1_{\text{odd}}(\mathbb{R})$ and

$$\exists C > 0 : \quad \|N(\psi)\|_{H^1} \leq C\left(|\mu| \|\psi\|_{H^1} + \|\psi\|^2_{H^1}\right).$$

If we only consider the eigenvector $\chi_0 \in \text{Ker}(L_0)$, where $\chi_0(z) = \varphi_0'(z)$, the Fredholm condition is satisfied trivially because h and $N(\psi)$ are defined in $L^2_{\text{odd}}(\mathbb{R})$ whereas χ_0 is defined in $L^2_{\text{even}}(\mathbb{R})$. In this case, we could (errously) conclude that L_0 is invertible on the space $H^1_{\text{odd}}(\mathbb{R})$ with a bounded inverse in $H^1_{\text{odd}}(\mathbb{R})$ and that the translationally invariant stationary front with properties (1.12) persists with respect to parameter μ near $\mu = 0$.

However, this result is false and the reason for this failure is the fact that $\text{Ker}(L_0)$ is spanned by an infinite set of eigenvectors $\{\chi_0^{(m)}\}_{m\in\mathbb{Z}}$. We will show that h is not orthogonal to $\text{Ker}(L_0)$ spanned by $\{\chi_0^{(m)}\}_{m\in\mathbb{Z}}$ by showing that the translationally invariant stationary front with properties (1.12) does not persist in the advance-delay Eq. (1.11) for the hybrid nonlinear function (4.1) with small $\mu \neq 0$.

By a contradiction, we shall assume that there exists a smooth family of the translationally invariant stationary front $\varphi(z)$ with properties (1.12) for small $\mu \neq 0$ and prove that $\text{Ker}(M_s)$ disappears for any $\mu \neq 0$, where M_s is the difference operator associated with the stationary front $\{\varphi(hn - s)\}_{n\in\mathbb{Z}}$.

Expanding the family of the stationary front $\varphi(z)$ in the perturbation series,

$$\varphi(z) = \varphi_0(z) + \mu\varphi_1(z) + \mathcal{O}(\mu^2) \in H^1_{\text{odd}}(\mathbb{R}),$$

we obtain a similar expansion of the difference operator for any $s \in \mathbb{R}$,

$$M_s = M_s^{(0)} + \mu M_s^{(1)} + \mathcal{O}(\mu^2),$$

where $\mathrm{Ker}(M_s^{(0)}) = \mathrm{span}(\boldsymbol{\chi}_0(s)) \subset l^2(\mathbb{Z})$ and $(\boldsymbol{\chi}_0(s))_n = \varphi_0'(hn - s)$, $n \in \mathbb{Z}$. Perturbation theory for zero eigenvalue

$$\boldsymbol{\chi}(s) = \boldsymbol{\chi}_0(s) + \mu \boldsymbol{\chi}_1(s) + \mathcal{O}(\mu^2), \quad \lambda = \mu\lambda_1 + \mathcal{O}(\mu^2),$$

leads to the perturbation equation

$$M_s^{(0)} \boldsymbol{\chi}_1(s) + M_s^{(1)} \boldsymbol{\chi}_0(s) = \lambda_1 \boldsymbol{\chi}_0(s),$$

where $\boldsymbol{\chi}_0(s), M_s^{(1)} \boldsymbol{\chi}_0(s) \in l^2(\mathbb{Z})$.

For $s = 0$, $\boldsymbol{\chi}_0(0)$ and $M_0^{(1)} \boldsymbol{\chi}_0(0)$ are even on $n \in \mathbb{Z}$ so that $\lambda_1 \neq 0$ (generally) and the smallest eigenvalue of M_0 moves from zero if $\mu \neq 0$. This gives the contradiction with the existence of translationally invariant stationary front with properties (1.12) for small $\mu \neq 0$.

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