



Periodic Travelling Waves of the Modified KdV Equation and Rogue Waves on the Periodic Background

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Abstract

We address the most general periodic travelling wave of the modified Korteweg–de Vries (mKdV) equation written as a rational function of Jacobian elliptic functions. By applying an algebraic method which relates the periodic travelling waves and the squared periodic eigenfunctions of the Lax operators, we characterize *explicitly* the location of eigenvalues in the periodic spectral problem away from the imaginary axis. We show that Darboux transformations with the periodic eigenfunctions remain in the class of the same periodic travelling waves of the mKdV equation. In a general setting, there exist three symmetric pairs of simple eigenvalues away from the imaginary axis, and we give a new representation of the second non-periodic solution to the Lax equations for the same eigenvalues. We show that Darboux transformations with the non-periodic solutions to the Lax equations produce rogue waves on the periodic background, which are either brought from infinity by propagating algebraic solitons or formed in a finite region of the time-space plane.

Keywords Modified Korteweg-de Vries equation · Periodic travelling waves · Rogue waves · Lax operators · Darboux transformations

Mathematics Subject Classification 35Q53 · 37K10 · 37K20

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1 Introduction

We address periodic travelling waves of the modified Korteweg–de Vries (mKdV) equation, which we take in the normalized form:

$$u_t + 6u^2u_x + u_{xxx} = 0. \quad (1.1)$$

As is well-known since the pioneer paper (Ablowitz et al. 1974), the mKdV equation (1.1) is a compatibility condition of the following pair of two linear equations written for the vector $\varphi = (\varphi_1, \varphi_2)^t$:

$$\varphi_x = U(\lambda, u)\varphi, \quad U(\lambda, u) = \begin{pmatrix} \lambda & u \\ -u & -\lambda \end{pmatrix}, \quad (1.2)$$

and

$$\varphi_t = V(\lambda, u)\varphi, \quad V(\lambda, u) = \begin{pmatrix} -4\lambda^3 - 2\lambda u^2 & -4\lambda^2 u - 2\lambda u_x - 2u^3 - u_{xx} \\ 4\lambda^2 u - 2\lambda u_x + 2u^3 + u_{xx} & 4\lambda^3 + 2\lambda u^2 \end{pmatrix}. \quad (1.3)$$

Assuming $\varphi(x, t) \in C^{2,2}(\mathbb{R} \times \mathbb{R})$ and $u(x, t) \in C^{3,1}(\mathbb{R} \times \mathbb{R})$, the compatibility condition $\varphi_{xt} = \varphi_{tx}$ is equivalent to the mKdV equation (1.1) satisfied in the classical sense.

Among the periodic travelling wave solutions, the mKdV equation (1.1) admits the normalized constant wave $u(x, t) = 1$ and two families of the normalized periodic waves given by

$$u(x, t) = \operatorname{dn}(x - ct; k), \quad c = 2 - k^2 \quad (1.4)$$

and

$$u(x, t) = \operatorname{cn}(x - ct; k), \quad c = 2k^2 - 1, \quad (1.5)$$

where dn and cn are Jacobian elliptic functions and $k \in (0, 1)$ is the elliptic modulus (see Chapter 8.1 in Gradshteyn and Ryzhik 2005 for review of elliptic functions and integrals).

The normalized constant wave $u(x, t) = 1$ is linearly and nonlinearly stable in the time evolution of the mKdV equation (1.1) in the sense that any small perturbation to the constant wave in the energy space $H^1(\mathbb{R})$ remains small in the $H^1(\mathbb{R})$ norm globally in time; see, e.g., (Kenig et al. 1993; Koch and Tataru 2018). Among the exact solutions to the mKdV equation (1.1) on the normalized constant wave, we note the following *algebraic soliton*

$$u(x, t) = 1 - \frac{4}{1 + 4(x - 6t - x_0)^2}, \quad (1.6)$$

where $x_0 \in \mathbb{R}$ is arbitrary. The algebraic soliton propagates on the normalized constant background with the speed $c_0 = 6$.

In our previous work (Chen and Pelinovsky 2018), we have constructed new solutions on the periodic background given by the normalized periodic waves (1.4) and (1.5). In doing so, we have adopted the formal algebraic method from Cao and Geng (1990), Cao et al. (1999), Geng and Cao (2001) and elaborated the following algorithm for constructing new solutions to the mKdV equation (1.1):

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1. Impose a constraint between a solution u to the mKdV equation (1.1) and a solution $\varphi = (p_1, q_1)^t$ to the Lax system (1.2)–(1.3) with $\lambda = \lambda_1$ and deduce closed differential equations on u . These equations are satisfied if u is a periodic travelling wave
 2. Characterize the set of admissible values for λ_1 and the relations between u and the squared components $p_1^2 + q_1^2$, $p_1^2 - q_1^2$, and $p_1 q_1$. The solution $\varphi = (p_1, q_1)^t$ is periodic in x and is travelling in t
 3. Obtain the second solution $\varphi = (\hat{p}_1, \hat{q}_1)^t$ to the Lax system (1.2)–(1.3) for the same values of λ_1 . The second solution is non-periodic; it grows linearly in x and t almost everywhere as $|x| + |t| \rightarrow \infty$
 4. Apply Darboux transformation with the second solution $\varphi = (\hat{p}_1, \hat{q}_1)^t$ and obtain new solutions to the mKdV equation (1.1) on the periodic background u
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As the main outcome of step 2 in Chen and Pelinovsky (2018), we obtained two pairs of admissible values for λ_1 with $\text{Re}(\lambda_1) \neq 0$. For the dn-periodic wave (1.4), the two pairs are real $\pm\lambda_+$ and $\pm\lambda_-$ with

$$\lambda_{\pm} = \frac{1}{2}(1 \pm \sqrt{1 - k^2}). \tag{1.7}$$

For the cn-periodic wave (1.5), the two pairs are complex-conjugate $\pm\lambda_+$ and $\pm\lambda_-$ with

$$\lambda_{\pm} = \frac{1}{2}(k \pm i\sqrt{1 - k^2}). \tag{1.8}$$

As the main outcome of step 4 in Chen and Pelinovsky (2018), we constructed an algebraic soliton propagating on the background of the dn-periodic wave (1.4) and a fully localized rogue wave on the (x, t) plane growing and decaying on the background of the cn-periodic wave (1.5). Since the dn-periodic wave (1.4) converges to the constant wave $u(x, t) = 1$ as $k \rightarrow 0$, the algebraic soliton on the dn-periodic wave background generalizes the exact solution (1.6). The rogue wave on the cn-periodic wave background satisfies the following mathematical definition of a rogue wave.

Definition 1 Let u be a periodic travelling wave of the mKdV equation (1.1) with the period L and \tilde{u} be another solution to the mKdV equation (1.1). We say that \tilde{u} is a rogue wave on the background u if \tilde{u} is different from the orbit $\{u(x - x_0)\}_{x_0 \in [0, L]}$ for $t \in \mathbb{R}$ and it satisfies

$$\inf_{x_0 \in [0, L]} \sup_{x \in \mathbb{R}} |\tilde{u}(x, t) - u(x - x_0)| \rightarrow 0 \text{ as } t \rightarrow \pm\infty. \tag{1.9}$$

Definition 1 corresponds to the physical interpretation of a rogue wave as *the wave that comes from nowhere and disappears without any trace*. Rogue waves in physics are associated with the gigantic waves on the ocean's surface and in optical fibers which arise due to the modulation instability of the background wave (Kharif et al. 2009; Kibler et al. 2018). Several recent publications were devoted to numerical and analytical studies of rogue waves on the background of periodic (Agafontsev and Zakharov 2016; Grinevich and Santini 2018), quasi-periodic (Bertola et al. 2016; Bertola and Tovbis 2017; Calini and Schober 2017), and multi-soliton (Bilman and Buckingham 2019; Bilman and Miller 2019) wave patterns in the framework of the nonlinear Schrödinger (NLS) equation. Since internal waves are modeled by the mKdV equation (Grimshaw et al. 2010), formation of rogue internal waves was also studied in the framework of the mKdV equation (1.1) as a result of multi-soliton interactions (Pelinovsky and Shurgalina 2016; Shurgalina and Pelinovsky 2016; Shurgalina 2018; Slunyaev and Pelinovsky 2016).

The difference between the two outcomes of the algorithm applied in Chen and Pelinovsky (2018) to the normalized periodic waves (1.4) and (1.5) in the mKdV equation is related to the fact that the dn-periodic waves are modulationally stable with respect to perturbations of long periods, whereas the cn-periodic waves are modulationally unstable (Bronski et al. 2011, 2016).

The purpose of this work is to consider the most general periodic travelling wave of the mKdV equation (1.1) and to characterize *explicitly* location of eigenvalues λ with $\text{Re}(\lambda) \neq 0$ in the periodic spectral problem (1.2). Although it may seem to be an incremental goal, advancement from the normalized periodic waves (1.4) and (1.5) to the most general periodic wave of the mKdV equation (1.1) require us to consider Riemann Theta functions of genus $g = 2$, which are expressed as rational functions of Jacobian elliptic functions. As is well known (Belokolos et al. 1994; Gesztesy and Holden 2003), Riemann Theta functions of genus g represent quasi-periodic solutions to many integrable evolution equations including the mKdV equation (1.1). Hence, having successfully solved the problem for $0 \leq g \leq 2$, we can move to the next goal of solving this problem for general g .

The algebraic method developed here is different from construction of multi-soliton solutions on the background of quasi-periodic solutions developed in Gesztesy and Svirsky (1995) by using commutation methods. It is also different from other analytical techniques for explicit characterization of eigenvalues related to the periodic solutions of the mKdV equation in the Whitham modulation theory (Kamchatnov et al. 2012, 2013) (see also Kamchatnov 1990; Pavlov 1994 for earlier works).

Let us now present the main results of our work. The travelling wave to the mKdV equation (1.1) has the form $u(x, t) = u(x - ct)$, where c is wave speed. The wave profile u satisfies the third-order differential equation:

$$\frac{d^3 u}{dx^3} + 6u^2 \frac{du}{dx} - c \frac{du}{dx} = 0. \quad (1.10)$$

Integrating it once yields the second-order differential equation:

$$\frac{d^2 u}{dx^2} + 2u^3 - cu = e, \quad (1.11)$$

where e is the integration constant. Integrating it once again yields the first-order invariant:

$$\left(\frac{du}{dx}\right)^2 + u^4 - cu^2 + d = 2eu, \tag{1.12}$$

where d is another integration constant. Thus, the most general periodic travelling wave in the mKdV equation (1.1) is characterized by the parameters (c, d, e) . The previous case considered in Chen and Pelinovsky (2018) corresponds to $e = 0$.

Our first result is about classification of the most general periodic travelling wave solution to the mKdV equation (1.1). As is well-known (see, e.g., Vassilev et al. 2008), there exists two explicit families of the periodic solutions to Eqs. (1.11) and (1.12) depending on parameters (c, d, e) . When the polynomial

$$P(u) := u^4 - cu^2 + d - 2eu, \tag{1.13}$$

admits four real roots ordered as $u_4 \leq u_3 \leq u_2 \leq u_1$, where (u_1, u_2, u_3, u_4) are related to the parameters (c, d, e) , the exact periodic solution to the system (1.11) and (1.12) is given by

$$u(x) = u_4 + \frac{(u_1 - u_4)(u_2 - u_4)}{(u_2 - u_4) + (u_1 - u_2)\text{sn}^2(\nu x; \kappa)}, \tag{1.14}$$

where $\nu > 0$ and $\kappa \in (0, 1)$ are parameters given by

$$\begin{cases} 4\nu^2 = (u_1 - u_3)(u_2 - u_4), \\ 4\nu^2\kappa^2 = (u_1 - u_2)(u_3 - u_4). \end{cases} \tag{1.15}$$

When the polynomial $P(u)$ in (1.13) admits two real roots $b \leq a$ and two complex-conjugate roots $\alpha \pm i\beta$, where (a, b, α, β) are related to parameters (c, d, e) , the exact periodic solution to the system (1.11) and (1.12) is given by

$$u(x) = a + \frac{(b - a)(1 - \text{cn}(\nu x; \kappa))}{1 + \delta + (\delta - 1)\text{cn}(\nu x; \kappa)}, \tag{1.16}$$

where $\delta > 0$, $\nu > 0$, and $\kappa \in (0, 1)$ are parameters given by

$$\begin{cases} \delta^2 = \frac{(b-\alpha)^2 + \beta^2}{(a-\alpha)^2 + \beta^2}, \\ \nu^2 = \sqrt{[(a - \alpha)^2 + \beta^2][(b - \alpha)^2 + \beta^2]}, \\ 2\kappa^2 = 1 - \frac{(a-\alpha)(b-\alpha) + \beta^2}{\sqrt{[(a-\alpha)^2 + \beta^2][(b-\alpha)^2 + \beta^2]}}. \end{cases} \tag{1.17}$$

The trivial case when the polynomial $P(u)$ in (1.13) admits no real roots does not produce any real solution to the system (1.11) and (1.12). The following theorem characterizes the periodic travelling waves to the mKdV equation.

Theorem 1 Fix $c > 0$ and $e \in (-e_0, e_0)$ with $e_0 := 2\sqrt{c^3}/(3\sqrt{6})$. There exist $-\infty < d_1 < d_2 < \infty$ such that for every $d \in (d_1, d_2)$, the system (1.11) and (1.12) admits the exact periodic solution in the form (1.14) with (1.15) and three other periodic solutions of the same period obtained with the following three symmetry transformations

$$\begin{aligned} \text{(S1)} \quad & u_1 \leftrightarrow u_2, \quad u_3 \leftrightarrow u_4, \quad \text{(S2)} \quad u_1 \leftrightarrow u_3, \quad u_2 \leftrightarrow u_4, \\ \text{(S3)} \quad & u_1 \leftrightarrow u_4, \quad u_2 \leftrightarrow u_3. \end{aligned} \quad (1.18)$$

In addition, if $e \neq 0$, there exists $d_3 > d_2$ such that for every $d \in (-\infty, d_1) \cup (d_2, d_3)$ the system (1.11) and (1.12) admits the exact periodic solution in the form (1.16) with (1.17) and another periodic solution of the same period obtained with the symmetry transformation

$$\text{(S0)} \quad a \leftrightarrow b. \quad (1.19)$$

For every other value of (c, e) , there exists $d_1 > 0$ such that for every $d \in (-\infty, d_1)$ only the periodic solution in the form (1.16)–(1.17) exists together with another solution obtained by the symmetry transformation (1.19). All the solutions are unique up to the translational symmetry $u(x) \mapsto u(x + x_0)$, $x_0 \in \mathbb{R}$.

Remark 1 Periodic solutions of the third-order equation (1.10) are invariant with respect to the reflection $u \mapsto -u$. The reflection corresponds to the transformation $e \mapsto -e$ in the second-order equation (1.11).

Remark 2 The proof of Theorem 1 is elementary. It is based on the phase-plane analysis and properties of the Jacobian elliptic functions. We included Theorem 1 for clarity of our presentation.

Remark 3 The periodic travelling wave of the mKdV equation (1.1) in Theorem 1 is also the periodic travelling wave of the following Gardner equation:

$$v_t + 12avv_x + 6v^2v_x + v_{xxx} = 0, \quad (1.20)$$

where $a \in \mathbb{R}$ is arbitrary. Indeed, if $u(x, t) \in C^{3,1}(\mathbb{R} \times \mathbb{R})$ satisfies the mKdV equation (1.1) and is represented by $u(x, t) = a + v(x - 6a^2t, t)$ with $a \in \mathbb{R}$, then $v(x, t) \in C^{3,1}(\mathbb{R} \times \mathbb{R})$ satisfies the Gardner equation (1.20). The Gardner equation is commonly used in modeling of internal waves (Grimshaw et al. 2010).

Next, we use the algebraic method from Cao and Geng (1990), Cao et al. (1999), Geng and Cao (2001) and relate the solutions u in Theorem 1 with squared eigenfunctions of the periodic spectral problem (1.2). Compared to our previous work in Chen and Pelinovsky (2018), we have to use two squared eigenfunctions for two different eigenvalues λ in order to obtain periodic solutions of the third-order differential equation (1.10). The following theorem represents the outcome of the algebraic method.

Theorem 2 *The spectral problem (1.2) with u given by the periodic waves (1.14) and (1.16) admits three pairs $\pm\lambda_1, \pm\lambda_2, \pm\lambda_3$ of eigenvalues with $\text{Re}(\lambda) \neq 0$ that corresponds to the periodic eigenfunctions φ . For the periodic wave (1.14), the eigenvalues are located at*

$$\lambda_1 = \frac{1}{2}(u_1 + u_2), \quad \lambda_2 = \frac{1}{2}(u_1 + u_3), \quad \lambda_3 = \frac{1}{2}(u_2 + u_3). \tag{1.21}$$

For the periodic wave (1.16), the eigenvalues are located at

$$\lambda_1 = \frac{1}{4}(a - b) + \frac{i}{2}\beta, \quad \lambda_2 = \frac{1}{4}(a - b) - \frac{i}{2}\beta, \quad \lambda_3 = \frac{1}{2}(a + b). \tag{1.22}$$

Remark 4 The algebraic method in the proof of Theorem 2 gives also explicit characterization of the periodic eigenfunctions φ of the spectral problem (1.2) at the three pairs of eigenvalues λ . If $u(x, t) = u(x - ct)$ is a travelling wave solution to the mKdV equation (1.1), the time evolution problem (1.3) is also satisfied with the solution $\varphi(x, t) = \varphi(x - ct)$.

Remark 5 The algebraic method does not allow us to conclude that no other eigenvalues λ with $\text{Re}(\lambda) \neq 0$ exist in the periodic spectral problem (1.2).

Next, we proceed with the multi-fold Darboux transformations (Gu et al. 2005; Matveev and Salle 1991) by using the general form proven with explicit computations in Appendix A of Chen and Pelinovsky (2018). Since we only have up to three pairs of simple eigenvalues in Theorem 2 in a generic case, we should only use one-fold, twofold, and threefold Darboux transformations. The corresponding Darboux transformations are given by the explicit expressions:

$$\tilde{u} = u + \frac{4\lambda_1 p_1 q_1}{p_1^2 + q_1^2}, \tag{1.23}$$

$$\tilde{u} = u + \frac{4(\lambda_1^2 - \lambda_2^2) [\lambda_1 p_1 q_1 (p_2^2 + q_2^2) - \lambda_2 p_2 q_2 (p_1^2 + q_1^2)]}{(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 2\lambda_1 \lambda_2 [4p_1 q_1 p_2 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)]}, \tag{1.24}$$

and

$$\tilde{u} = u + \frac{4N}{D}, \tag{1.25}$$

with

$$\begin{aligned} N := & (\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)\lambda_3 p_3 q_3 \left[(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) \right. \\ & \left. - 2\lambda_1 \lambda_2 (p_1^2 - q_1^2)(p_2^2 - q_2^2) \right] \\ & + (\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_3^2)\lambda_2 p_2 q_2 \left[(\lambda_1^2 + \lambda_3^2)(p_1^2 + q_1^2)(p_3^2 + q_3^2) \right. \\ & \left. - 2\lambda_1 \lambda_3 (p_1^2 - q_1^2)(p_3^2 - q_3^2) \right] \end{aligned}$$

$$\begin{aligned}
& +(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)\lambda_1 p_1 q_1 \left[(\lambda_2^2 + \lambda_3^2)(p_2^2 + q_2^2)(p_3^2 + q_3^2) \right. \\
& \left. - 2\lambda_2 \lambda_3 (p_2^2 - q_2^2)(p_3^2 - q_3^2) \right] \\
& - 8(\lambda_1^4 + \lambda_2^4 + \lambda_3^4 - \lambda_1^2 \lambda_2^2 - \lambda_1^2 \lambda_3^2 - \lambda_2^2 \lambda_3^2)\lambda_1 \lambda_2 \lambda_3 p_1 p_2 p_3 q_1 q_2 q_3
\end{aligned}$$

and

$$\begin{aligned}
D := & (\lambda_1 + \lambda_2)^2(\lambda_1 + \lambda_3)^2(\lambda_2 - \lambda_3)^2(p_1^2 q_2^2 q_3^2 + q_1^2 p_2^2 p_3^2) \\
& + (\lambda_1 + \lambda_2)^2(\lambda_2 + \lambda_3)^2(\lambda_1 - \lambda_3)^2(p_2^2 q_1^2 q_3^2 + q_2^2 p_1^2 p_3^2) \\
& + (\lambda_1 + \lambda_3)^2(\lambda_2 + \lambda_3)^2(\lambda_1 - \lambda_2)^2(p_3^2 q_1^2 q_2^2 + q_3^2 p_1^2 p_2^2) \\
& + (\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2(p_1^2 p_2^2 p_3^2 + q_1^2 q_2^2 q_3^2) \\
& - 8(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)\lambda_1 \lambda_2 p_1 p_2 q_1 q_2 (p_3^2 + q_3^2) \\
& - 8(\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_3^2)\lambda_1 \lambda_3 p_1 p_3 q_1 q_3 (p_2^2 + q_2^2) \\
& - 8(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)\lambda_2 \lambda_3 p_2 p_3 q_2 q_3 (p_1^2 + q_1^2),
\end{aligned}$$

where \tilde{u} stands for a new solution to the mKdV equation (1.1) and $\varphi_j = (p_j, q_j)^t$, $j = 1, 2, 3$ stands for a nonzero solution to the Lax system (1.2)–(1.3) with potential u and eigenvalue λ_j assuming $\lambda_i \neq \pm \lambda_j$ for $i \neq j$.

The following theorems represent the outcomes of the Darboux transformations (1.23), (1.24), and (1.25) with the periodic solutions to the Lax system (1.2)–(1.3).

Theorem 3 Assume $u_4 < u_3 < u_2 < u_1$ such that $u_1 + u_2 + u_3 + u_4 = 0$, $u_1 + u_2 \neq 0$, $u_1 + u_3 \neq 0$, and $u_2 + u_3 \neq 0$. The onefold Darboux transformation with the periodic eigenfunctions for each eigenvalue in (1.21) transforms the periodic wave (1.14) to the periodic wave of the same period obtained after the corresponding symmetry transformation in (1.18) and the reflection $u \mapsto -u$. The twofold Darboux transformation with the periodic eigenfunctions for any two eigenvalues from (1.21) transforms the periodic wave (1.14) to the periodic wave of the same period obtained after the complementary third symmetry transformation in (1.18). The threefold Darboux transformation with the periodic eigenfunctions for all three eigenvalues in (1.21) maps the periodic wave (1.14) to itself reflected with $u \mapsto -u$.

Remark 6 Under the term “complementary third transformation” in Theorem 3, we mean that if (λ_1, λ_2) is selected in (1.21), then (S3) is selected in (1.18), and so on.

Remark 7 The three eigenvalues in (1.21) with the three periodic eigenfunctions used in the onefold transformation (1.23) recover the three symmetry transformations in (1.18). It is interesting to note that due to the constraint $u_1 + u_2 + u_3 + u_4 = 0$, the choice of each eigenvalue indicate explicitly the choice of the transformation between u_1, u_2, u_3 , and u_4 , e.g., $\lambda_1 = (u_1 + u_2)/2$ corresponds to (S1) with $u_1 \leftrightarrow u_2$ and $u_3 \leftrightarrow u_4$, and so on.

Theorem 4 Assume $a \neq \pm b$. The onefold Darboux transformation with the periodic eigenfunction for eigenvalue λ_3 in (1.22) transforms the periodic wave (1.16) to the periodic wave of the same period obtained after the symmetry transformation in (1.19)

and the reflection $u \mapsto -u$. The twofold Darboux transformation with the periodic eigenfunctions for two eigenvalues λ_1 and λ_2 in (1.22) transforms the periodic wave (1.16) to the periodic wave of the same period obtained after the symmetry transformation in (1.19). The threefold Darboux transformation with the periodic eigenfunctions for all three eigenvalues in (1.22) maps the periodic wave (1.16) to itself reflected with $u \mapsto -u$.

Remark 8 Other possible onefold and twofold Darboux transformations missing in the formulation of Theorem 4 return complex-valued solutions u to the mKdV equation (1.1).

Finally, we use the algorithm above and construct new solutions to the mKdV equation (1.1). To do so, we obtain the closed form expression for the second solution $\varphi = (\hat{p}_1, \hat{q}_1)^t$ of the Lax system (1.2)–(1.3) with $\lambda = \lambda_1$ given by an eigenvalue in Theorem 2. The explicit representation of the second solution has been already obtained in our previous work (Chen and Pelinovsky 2018); however, the expression obtained there is singular at any point of (x, t) where one of the two components of the periodic solution $\varphi = (p_1, q_1)^t$ vanishes. As a result, the corresponding expression was only used for the eigenvalue λ_+ in (1.7) but not for the eigenvalue λ_- . Here we obtain a different explicit representation of the second solution φ which is free of singularities for every eigenvalue λ in Theorem 2. As a result, we are able to construct new solutions to the mKdV equation by using the Darboux transformations with the second solution to the Lax system (1.2)–(1.3) for every eigenvalue in Theorem 2. The following two theorems describe the new solutions as a function of (x, t) .

Theorem 5 Assume $u_4 < u_3 < u_2 < u_1$ such that $u_1 + u_2 + u_3 + u_4 = 0$, $u_1 + u_2 \neq 0$, $u_1 + u_3 \neq 0$, and $u_2 + u_3 \neq 0$. Under three non-degeneracy conditions (7.5), (7.6), and (7.7) below, there exist $c_1, c_2, c_3 \neq c$ such that the second solutions to the Lax system (1.2)–(1.3) for the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ in (1.21) are linearly growing in x and t everywhere on the (x, t) plane except for the straight lines $x - c_{1,2,3}t = \xi_{1,2,3}$, where ξ_1, ξ_2, ξ_3 are phase parameters which are not uniquely defined. The onefold Darboux transformation with the second solution for each eigenvalue $\lambda_{1,2,3}$ in (1.21) adds an algebraic soliton with the corresponding speed $c_{1,2,3}$ on the background of the periodic wave (1.14) transformed by the corresponding symmetry in (1.18) and reflection $u \mapsto -u$. The twofold Darboux transformation with the second solutions for any two eigenvalues from (1.21) adds two algebraic solitons with the corresponding two wave speeds on the background of the periodic wave (1.14) transformed by the complementary symmetry in (1.18). The threefold Darboux transformation with the second solutions for all three eigenvalues in (1.21) adds all three algebraic solitons with the three wave speeds on the background of the periodic wave (1.14) reflected with $u \mapsto -u$.

Theorem 6 Assume $a \neq \pm b$. Under two non-degeneracy conditions (7.11) and (7.12) below, the second solutions to the Lax system (1.2)–(1.3) for the eigenvalues λ_1 and λ_2 in (1.22) are linearly growing in x and t everywhere, whereas there exists $c_0 \neq c$ such that the second solution to the Lax system (1.2)–(1.3) for the eigenvalue λ_3 in (1.22) is linearly growing in x and t everywhere except for the straight line $x -$

$c_0 t = \xi_0$, where ξ_0 is the phase parameter which is not uniquely defined. The onefold Darboux transformation with the second solution for eigenvalue λ_3 in (1.22) adds an algebraic soliton with the wave speed c_0 on the background of the periodic wave (1.16) transformed by the symmetry in (1.19) and reflection $u \mapsto -u$. The twofold Darboux transformation with the second solutions for two eigenvalues λ_1 and λ_2 in (1.22) adds a rogue wave on the background of the periodic wave (1.16) transformed by the symmetry in (1.19). The threefold Darboux transformation with the second solutions for all three eigenvalues in (1.22) adds both the algebraic soliton with the wave speed c_0 and the rogue wave on the background of the periodic wave (1.16) reflected with $u \mapsto -u$.

Remark 9 The only rogue wave in Theorem 6 satisfies Definition 1, whereas the other two solutions do not satisfy the limit (1.9) due to the algebraic solitons propagating on the periodic background along the straight line $x - c_0 t = \xi_0$. None of new solutions in Theorem 5 satisfy Definition 1 due to the same reason.

The paper is organized as follows. Section 2 describes the algebraic method which allows us to complete step 1 in the algorithm above. Section 3 characterizes the most general periodic wave of the mKdV equation (1.1) and gives the proof of Theorem 1. Step 2 of the algorithm and the proof of Theorem 2 are given in Sect. 4. The proof of Theorems 3 and 4 can be found in Sect. 5. Step 3 of the algorithm is developed in Sect. 6, where the second non-periodic solution to the Lax system (1.2)–(1.3) is obtained. Step 4 of the algorithm is completed in Sect. 7, where the new solutions to the mKdV equation (1.1) are constructed, Theorems 5 and 6 are proven, and the surface plots are included for graphical illustrations. Finally, Sect. 8 contains a conjecture on the possible generalization of our results to the case of quasi-periodic solutions to the mKdV equation (1.1).

2 The Algebraic Method

We showed in Chen and Pelinovsky (2018) that imposing a constraint $u = p_1^2 + q_1^2$ between a solution u to the mKdV equation (1.1) and components of the solution $\varphi = (p_1, q_1)^t$ to the Lax system (1.2)–(1.3) with fixed parameter $\lambda = \lambda_1$ results in the second-order differential equation on u . This equation is given by (1.11) for $e = 0$ and it does not recover the most general periodic wave of the mKdV equation (1.1). Here we extend the algebraic method of Cao and Geng (1990), Cao et al. (1999), Geng and Cao (2001) by imposing the constraint

$$u = p_1^2 + q_1^2 + p_2^2 + q_2^2 \quad (2.1)$$

between a solution u to the mKdV equation (1.1) and components of two solutions $\varphi = (p_1, q_1)^t$ and $\varphi = (p_2, q_2)^t$ to the Lax system (1.2)–(1.3) with two fixed parameters $\lambda = \lambda_1$ and $\lambda = \lambda_2$. Assume $\lambda_1 \neq \pm\lambda_2$.

The spectral problem (1.2) for two vectors $\varphi = (p_1, q_1)^t$ and $\varphi = (p_2, q_2)^t$ at λ_1 and λ_2 can be written as the Hamiltonian system of degree two

$$\frac{dp_j}{dx} = \frac{\partial H_0}{\partial q_j}, \quad \frac{dq_j}{dx} = -\frac{\partial H_0}{\partial p_j}, \quad j = 1, 2, \quad (2.2)$$

generated from the Hamiltonian function

$$H_0(p_1, q_1, p_2, q_2) = \frac{1}{4}(p_1^2 + q_1^2 + p_2^2 + q_2^2)^2 + \lambda_1 p_1 q_1 + \lambda_2 p_2 q_2. \quad (2.3)$$

Since H_0 is x -independent, we introduce the constant $E_0 := 4H_0(p_1, q_1, p_2, q_2)$ in addition to parameters (λ_1, λ_2) . As is shown in Cao et al. (1999), the Hamiltonian system (2.2)–(2.3) of degree two is integrable in the sense of Liouville and there exists another conserved quantity

$$\begin{aligned} H_1(p_1, q_1, p_2, q_2) &= 4(\lambda_1^3 p_1 q_1 + \lambda_2^3 p_2 q_2) - 4(\lambda_1 p_1 q_1 + \lambda_2 p_2 q_2)^2 \\ &\quad - (\lambda_1(p_1^2 - q_1^2) + \lambda_2(p_2^2 - q_2^2))^2 \\ &\quad + 2(p_1^2 + q_1^2 + p_2^2 + q_2^2) \\ &\quad (\lambda_1^2(p_1^2 + q_1^2) + \lambda_2^2(p_2^2 + q_2^2)). \end{aligned} \quad (2.4)$$

Since H_1 is x -independent, we introduce another constant $E_1 := 4H_1(p_1, q_1, p_2, q_2)$. Thus, the algebraic method includes four parameters $(\lambda_1, \lambda_2, E_0, E_1)$. It remains to establish differential equations on the class of admissible solutions u .

2.1 Lax–Novikov Equations

By differentiating (2.1) in x and using the Hamiltonian system (2.2)–(2.3), we obtain the following first-order differential equation:

$$\frac{du}{dx} = 2\lambda_1(p_1^2 - q_1^2) + 2\lambda_2(p_2^2 - q_2^2). \quad (2.5)$$

It follows from (2.1) and (2.3) that

$$E_0 - u^2 = 4\lambda_1 p_1 q_1 + 4\lambda_2 p_2 q_2. \quad (2.6)$$

By differentiating (2.5) in x and using the Hamiltonian system (2.2)–(2.3) and the relation (2.6), we obtain the following second-order differential equation:

$$\begin{aligned} \frac{d^2u}{dx^2} + 2u^3 &= 2E_0u + 4\lambda_1^2(p_1^2 + q_1^2) + 4\lambda_2^2(p_2^2 + q_2^2) \\ &= cu - 4\lambda_2^2(p_1^2 + q_1^2) - 4\lambda_1^2(p_2^2 + q_2^2), \end{aligned} \quad (2.7)$$

where we have introduced a parameter

$$c := 2E_0 + 4\lambda_1^2 + 4\lambda_2^2. \quad (2.8)$$

Taking yet another derivative of (2.7) in x and using the Hamiltonian system (2.2)–(2.3) again, we obtain the following third-order differential equation:

$$\frac{d^3u}{dx^3} + 6u^2 \frac{du}{dx} = c \frac{du}{dx} - 8\lambda_1\lambda_2 \left[\lambda_2(p_1^2 - q_1^2) + \lambda_1(p_2^2 - q_2^2) \right]. \quad (2.9)$$

Differential equations (2.5), (2.7), and (2.9) are not closed for u . However, we show that one more differentiation gives a closed fourth-order differential equation on u . To do so, we first note that it follows from (2.1), (2.3), (2.4), (2.5), and (2.7) that

$$E_1 = 16(\lambda_1^3 p_1 q_1 + \lambda_2^3 p_2 q_2) - (E_0 - u^2)^2 + 2u \left(\frac{d^2u}{dx^2} + 2u^3 - 2E_0u \right) - \left(\frac{du}{dx} \right)^2. \quad (2.10)$$

By using (2.6), this relation can be rewritten in the equivalent form

$$E_1 + E_0^2 - 4E_0(\lambda_1^2 + \lambda_2^2) + \left(\frac{du}{dx} \right)^2 - 2u \frac{d^2u}{dx^2} - 3u^4 + cu^2 = -16\lambda_1\lambda_2(\lambda_2 p_1 q_1 + \lambda_1 p_2 q_2). \quad (2.11)$$

By differentiating (2.9) in x and using the Hamiltonian system (2.2)–(2.3) and the relation (2.11), we obtain a closed fourth-order differential equation on u :

$$\frac{d^4u}{dx^4} + 10u^2 \frac{d^2u}{dx^2} + 10u \left(\frac{du}{dx} \right)^2 + 6u^5 = c \left(\frac{d^2u}{dx^2} + 2u^3 \right) + 2du, \quad (2.12)$$

where we have introduced another parameter

$$d := E_1 + E_0^2 - 4E_0(\lambda_1^2 + \lambda_2^2) - 8\lambda_1^2\lambda_2^2. \quad (2.13)$$

The fourth-order differential equation (2.12) belongs to the class of Lax–Novikov equations (see Varley and Seymour 1998 and references therein), which combine the stationary flows of the mKdV equation and higher-order members of its integrable hierarchy.

2.2 Conserved Quantities for the Stationary Fourth-Order Equation

Let us introduce

$$W(\lambda) = \begin{pmatrix} W_{11}(\lambda) & W_{12}(\lambda) \\ W_{12}(-\lambda) & -W_{11}(-\lambda) \end{pmatrix}, \quad (2.14)$$

where the components of $W(\lambda)$ are given explicitly by

$$W_{11}(\lambda) = 1 - \frac{2\lambda_1 p_1 q_1}{\lambda^2 - \lambda_1^2} - \frac{2\lambda_2 p_2 q_2}{\lambda^2 - \lambda_2^2}, \quad (2.15)$$

$$W_{12}(\lambda) = \frac{p_1^2}{\lambda - \lambda_1} + \frac{q_1^2}{\lambda + \lambda_1} + \frac{p_2^2}{\lambda - \lambda_2} + \frac{q_2^2}{\lambda + \lambda_2}. \tag{2.16}$$

In order to simplify the presentation, we denote derivatives of u in x by subscripts. By using (2.1), (2.5), (2.7), and (2.9), the expression for W_{12} can be rewritten in the equivalent form:

$$W_{12}(\lambda) = \frac{\lambda^3 u + \frac{1}{2} \lambda^2 u_x + \frac{1}{4} \lambda (u_{xx} + 2u^3 - cu) + \frac{1}{8} (u_{xxx} + 6u^2 u_x - cu_x)}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)}. \tag{2.17}$$

By using (2.6), (2.11), and (2.13), the expression for W_{11} can also be rewritten in the equivalent form:

$$W_{11}(\lambda) = 1 - \frac{\lambda^2 (E_0 - u^2)}{2(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)} - \frac{d + 8\lambda_1^2 \lambda_2^2 + (u_x)^2 - 2uu_{xx} - 3u^4 + cu^2}{8(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)}. \tag{2.18}$$

The Lax equation

$$\frac{d}{dx} W(\lambda) = U(\lambda, u)W(\lambda) - W(\lambda)U(\lambda, u), \tag{2.19}$$

is satisfied for every $\lambda \in \mathbb{C}$ if and only if (p_1, q_1, p_2, q_2) satisfy the Hamiltonian system (2.2) with (2.3), where u is represented by (2.1). In particular, the (1, 2)-entry in the above relations yields the equation

$$\frac{d}{dx} W_{12}(\lambda) = 2\lambda W_{12}(\lambda) - 2u W_{11}(\lambda). \tag{2.20}$$

Substituting (2.17) and (2.18) into (2.20) yields the same fourth-order differential equation (2.12).

It was shown in a similar context in Tu (1989) that $\det[W(\lambda)]$ has only simple poles at $\lambda = \pm\lambda_1$ and $\lambda = \pm\lambda_2$. This can be confirmed from (2.14), (2.15), and (2.16) with straightforward computations. Substituting the representations (2.17) and (2.18) into $\det[W(\lambda)]$ and removing the double poles at $\lambda = \pm\lambda_1$ and $\lambda = \pm\lambda_2$ yield the following two constraints:

$$4\lambda_j^2 \left[u_{xx} + 2u^3 - cu + 4\lambda_j^2 u \right]^2 - \left[u_{xxx} + 6u^2 u_x - cu_x + 4\lambda_j^2 u_x \right]^2 - \left[d + 8\lambda_1^2 \lambda_2^2 + (u_x)^2 - 2uu_{xx} - 3u^4 + cu^2 + 4\lambda_j^2 (E_0 - u^2) \right]^2 = 0 \tag{2.21}$$

These two equations (2.21) for $j = 1$ and $j = 2$ represent lower-order invariants for the fourth-order differential equation (2.12). By performing elementary operations, the system of two constraints (2.21) can be rewritten in the equivalent form:

$$\begin{aligned}
& \left(u_{xx} + 2u^3 - 2E_0u\right)^2 - 16\lambda_1^2\lambda_2^2u^2 - 2u_x \left(u_{xxx} + 6u^2u_x - cu_x\right) \\
& - 2(E_0 - u^2) \left[d + 8\lambda_1^2\lambda_2^2 + (u_x)^2 - 2uu_{xx} - 3u^4 + cu^2\right] \\
& - 4(\lambda_1^2 + \lambda_2^2) \left[(u_x)^2 + (E_0 - u^2)^2\right] = 0
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
& \left(u_{xxx} + 6u^2u_x - 2E_0u_x\right)^2 + \left(E_1 + E_0^2 + (u_x)^2 - 2uu_{xx} - 3u^4 + 2E_0u^2\right)^2 \\
& - 4(\lambda_1^2 + \lambda_2^2)(u_{xx} + 2u^3 - 2E_0u)^2 - 16\lambda_1^2\lambda_2^2 \left[(u_x)^2 + (E_0 - u^2)^2\right] \\
& + 32\lambda_1^2\lambda_2^2u(u_{xx} + 2u^3 - 2E_0u) = 0.
\end{aligned} \tag{2.23}$$

In order to verify conservation of (2.22) and (2.23), we have checked that differentiating these constraints in x recovers the fourth-order differential equation (2.12).

2.3 Dubrovin Equations

One can characterize solutions u to the fourth-order differential equation (2.12) from the algebro-geometric point of view, which is now commonly accepted in the context of quasi-periodic solutions of integrable equations (Belokolos et al. 1994; Gesztesy and Holden 2003). Since the numerator of $W_{12}(\lambda)$ is a polynomial of degree three, it admits three roots denoted as μ_1 , μ_2 , and μ_3 . Writing the representation (2.17) in the form

$$W_{12}(\lambda) = \frac{u(\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)}, \tag{2.24}$$

yields the following differential expressions for μ_1 , μ_2 , and μ_3 :

$$\mu_1 + \mu_2 + \mu_3 = -\frac{1}{2u} \frac{du}{dx}, \tag{2.25}$$

$$\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 = \frac{1}{4u} \left[\frac{d^2u}{dx^2} + 2u^3 - cu \right], \tag{2.26}$$

$$\mu_1\mu_2\mu_3 = -\frac{1}{8u} \left[\frac{d^3u}{dx^3} + 6u^2 \frac{du}{dx} - c \frac{du}{dx} \right]. \tag{2.27}$$

By substituting the representations (2.17) and (2.18) into $\det[W(\lambda)]$, removing the double poles with the constraints (2.21), and simplifying the remaining expressions, we obtain the following representation:

$$\det[W(\lambda)] = -1 + \frac{4E_0(\lambda^2 - \lambda_1^2 - \lambda_2^2) + E_1}{4(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)} = -\frac{b(\lambda)}{a(\lambda)}, \tag{2.28}$$

where

$$\begin{cases} a(\lambda) := (\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2), \\ b(\lambda) := (\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2) - E_0(\lambda^2 - \lambda_1^2 - \lambda_2^2) - \frac{1}{4}E_1. \end{cases}$$

By substituting (2.24) and (2.28) into (2.20) and evaluating (2.20) at $\lambda = \mu_{1,2,3}$, we recover the Dubrovin equations (Dubrovin 1981) in the form:

$$\frac{d\mu_j}{dx} = \frac{2\sqrt{a(\mu_j)b(\mu_j)}}{\prod_{i \neq j} (\mu_j - \mu_i)}. \quad (2.29)$$

These equations characterize the algebro-geometric structure of the quasi-periodic solutions. The variables $\{\mu_1, \mu_2, \mu_3\}$ are called Dubrovin's variables for the quasi-periodic solutions. The presence of three Dubrovin variables indicates that the general solution to the fourth-order equation (2.12) is expressed by the Riemann Theta function of *genus three* (Belokolos et al. 1994; Gesztesy and Holden 2003).

2.4 Degeneration Procedure

For the scopes of our work, we reduce the algebraic method in order to recover the third-order differential equation (1.10) instead of the fourth-order equation (2.12). This scope is achieved by imposing the constraint $\mu_3 = 0$ on Dubrovin variables satisfying the system of equations (2.25), (2.26), and (2.27). Indeed, if $\mu_3 = 0$, Eq. (2.27) is equivalent to the third-order differential equation (1.10) rewritten again as

$$\frac{d^3u}{dx^3} + 6u^2 \frac{du}{dx} - c \frac{du}{dx} = 0, \quad (2.30)$$

whereas the other two Eqs. (2.25) and (2.26) yield relations

$$\mu_1 + \mu_2 = -\frac{1}{2u} \frac{du}{dx}, \quad (2.31)$$

$$\mu_1\mu_2 = \frac{1}{4u} \left[\frac{d^2u}{dx^2} + 2u^3 - cu \right]. \quad (2.32)$$

The presence of two Dubrovin variables indicates that the general solution to the third-order equation (2.30) is expressed by the Riemann Theta function of *genus two* (Belokolos et al. 1994; Gesztesy and Holden 2003).

Substituting (2.30) into (2.12) yields the following invariant for the third-order equation (2.30):

$$\left(\frac{du}{dx} \right)^2 - 2u \frac{d^2u}{dx^2} - 3u^4 + cu^2 + d = 0. \quad (2.33)$$

Indeed, differentiating (2.33) in x recovers the third-order equation (2.30).

Note in passing that the representations (2.17) and (2.18) are factorized by one power of λ , after equations (2.30) and (2.33) are used, namely:

$$\lambda^{-1}W_{12}(\lambda) = \frac{\lambda^2 u + \frac{1}{2}\lambda u_x + \frac{1}{4}(u_{xx} + 2u^3 - cu)}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)}$$

and

$$\lambda^{-1}W_{11}(\lambda) = \frac{\lambda^3 - \frac{1}{2}\lambda(E_0 + 2\lambda_1^2 + 2\lambda_2^2 - u^2)}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)}.$$

These representations coincide with those used in Wright (2016) for integration of ultra-elliptic solutions to the cubic NLS equation.

Remark 10 It may be possible to modify the algebraic method by starting with polynomials of even degree for $W_{12}(\lambda)$ and odd degree for $W_{11}(\lambda)$ and to derive the same solutions as those obtained by the degeneration procedure here. This possible modification remains to be developed.

By substituting differential equations (2.30) and (2.33) into the constraints (2.22) and (2.23), we rewrite the constraints in the equivalent form:

$$\begin{aligned} & \left(\frac{d^2 u}{dx^2} + 2u^3 - 2E_0 u \right)^2 - 4(\lambda_1^2 + \lambda_2^2) \left[\left(\frac{du}{dx} \right)^2 + (E_0 - u^2)^2 \right] \\ & - 16\lambda_1^2 \lambda_2^2 E_0 = 0 \end{aligned} \quad (2.34)$$

and

$$\begin{aligned} & 4(\lambda_1^4 + \lambda_1^2 \lambda_2^2 + \lambda_2^4) \left[\left(\frac{du}{dx} \right)^2 + (E_0 - u^2)^2 \right] \\ & + 16\lambda_1^2 \lambda_2^2 (\lambda_1^2 + \lambda_2^2) (E_0 - u^2) + 16\lambda_1^4 \lambda_2^4 \\ & - (\lambda_1^2 + \lambda_2^2) \left(\frac{d^2 u}{dx^2} + 2u^3 - 2E_0 u \right)^2 \\ & + 8\lambda_1^2 \lambda_2^2 u \left(\frac{d^2 u}{dx^2} + 2u^3 - 2E_0 u \right) = 0. \end{aligned} \quad (2.35)$$

In order to simplify characterization of the general solution of the third-order equation (2.30), we integrate it once and obtain the differential equation (1.11) rewritten again as

$$\frac{d^2 u}{dx^2} + 2u^3 - cu = e, \quad (2.36)$$

where e is the integration constant. Substituting (2.36) into (2.33) yields the first-order invariant (1.12) for the second-order equation (2.36) rewritten again as

$$\left(\frac{du}{dx}\right)^2 + u^4 - cu^2 + d = 2eu. \tag{2.37}$$

Indeed, differentiating (2.37) in x recovers the second-order equation (2.36).

Substituting (2.36) and (2.37) into (2.34) and (2.35) yields the following two constraints on the parameters of the algebraic method:

$$e^2 = 4(\lambda_1^2 + \lambda_2^2)(E_0^2 - d) + 16\lambda_1^2\lambda_2^2E_0 \tag{2.38}$$

and

$$4(\lambda_1^4 + \lambda_1^2\lambda_2^2 + \lambda_2^4)(E_0^2 - d) + 16\lambda_1^2\lambda_2^2(\lambda_1^2 + \lambda_2^2)E_0 + 16\lambda_1^4\lambda_2^4 - (\lambda_1^2 + \lambda_2^2)e^2 = 0. \tag{2.39}$$

The system (2.38) and (2.39) is solved in the explicit form:

$$d = E_0^2 - 4\lambda_1^2\lambda_2^2 \tag{2.40}$$

and

$$e^2 = 16\lambda_1^2\lambda_2^2(E_0 + \lambda_1^2 + \lambda_2^2). \tag{2.41}$$

The relation (2.40) and the definition (2.13) imply that the parameter E_1 is uniquely expressed by

$$E_1 = 4E_0(\lambda_1^2 + \lambda_2^2) + 4\lambda_1^2\lambda_2^2. \tag{2.42}$$

Summarizing, the three parameters (c, d, e) for the differential equations (2.30), (2.36), and (2.37) are related to the parameters $(\lambda_1, \lambda_2, E_0)$ of the algebraic method with the help of relation (2.8), (2.40), and (2.41), whereas parameter E_1 is uniquely expressed by (2.42).

Remark 11 Compared to the algebraic method with only one eigenfunction $\varphi = (p_1, q_1)^t$ in Chen and Pelinovsky (2018), where solutions to the second-order equation (2.36) with $e = 0$ are obtained, we have flexibility to recover the same solutions with the two eigenfunctions $\varphi = (p_1, q_1)^t$ and $\varphi = (p_2, q_2)^t$ as in (2.1), as long as there are two nonzero eigenvalues for λ_1 and λ_2 such that $\lambda_1 \neq \pm\lambda_2$. Thus, solutions to the second-order equation (2.36) with $e = 0$ can be recovered by two equivalent algebraic methods.

3 Proof of Theorem 1

Here we develop the phase-plane analysis of the second-order differential equation (1.11). Therefore, we treat it as a dynamical system on the phase plane (u, u') , where $u' := \frac{du}{dx}$. If $c > 0$ and $e \in (-e_0, e_0)$, where $e_0 := 2\sqrt{c^3}/(3\sqrt{6})$, the cubic polynomial

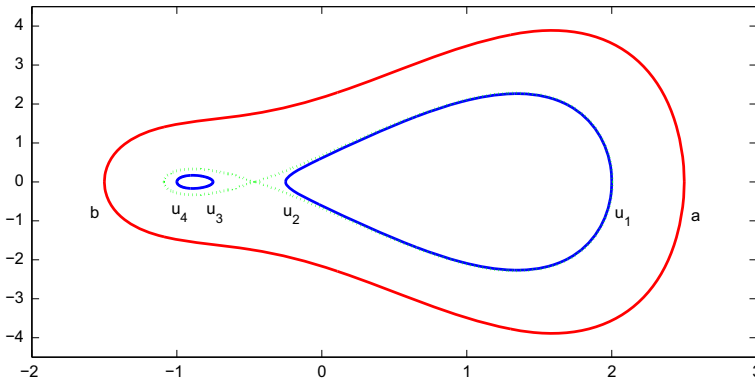


Fig. 1 Phase plane (u, u') for the second-order equation (1.11)

$$Q(u) := 2u^3 - cu - e, \tag{3.1}$$

has three real roots ordered as $u_* < u_{**} < u_{***}$. Since $Q'(u) = 6u^2 - c$, we have $Q'(u_*) > 0$, $Q'(u_{**}) < 0$, and $Q'(u_{***}) > 0$, hence $(u_*, 0)$ and $(u_{***}, 0)$ are center points on the phase plane, whereas $(u_{**}, 0)$ is a saddle point. Trajectories on the phase plane coincide with the level curves of the energy function

$$\mathcal{H}(u, u') = (u')^2 + u^4 - cu^2 - 2eu,$$

which are given by the constant value $-d$ in the first-order invariant (1.12). The level curves of $\mathcal{H}(u, u')$ in the case $c > 0$ and $e \in (-e_0, e_0)$ are shown on Fig. 1.

Since $2Q(u) = \partial_u \mathcal{H}(u, 0)$, the function $\mathcal{H}(u, 0)$ has three extremal points with two minima at u_* and u_{***} and one maximum at u_{**} . Let $h_* := \mathcal{H}(u_*, 0)$, $h_{**} := \mathcal{H}(u_{**}, 0)$, and $h_{***} := \mathcal{H}(u_{***}, 0)$. Recall that $\mathcal{H}(u, 0) = P(u) - d$, where $P(u)$ is given by (1.13). For every $d \in (d_1, d_2)$ with $d_1 = -h_{**}$ and $d_2 = -\max\{h_*, h_{***}\}$, the polynomial $P(u)$ has four real roots ordered as $u_4 < u_3 < u_2 < u_1$ also shown on Fig. 1. There exists two periodic solutions which correspond to closed orbits on the phase plane: one closed orbit surrounds the center point $(u_*, 0)$ and the other closed orbit surrounds the center point $(u_{***}, 0)$. These periodic solutions are shown on Fig. 2a, b.

For every $d \in (-\infty, d_1) \cup (d_2, d_3)$, where $d_3 := -\min\{h_*, h_{***}\}$, the polynomial $P(u)$ has only two real roots ordered as $b \leq a$ also shown on Fig. 1. There exists only one periodic solution which corresponds to either the closed orbit surrounding all three critical points $(u_*, 0)$, $(u_{**}, 0)$, and $(u_{***}, 0)$ if $d \in (-\infty, d_1)$ or the closed orbit surrounding only one center point if $d \in (d_2, d_3)$. Note that $d_2 < d_3$ if $e \neq 0$, whereas $d_2 = d_3$ if $e = 0$. The periodic solution surrounding all three critical points is shown on Fig. 2c.

If either $c \leq 0$ or $c > 0$ and $e \in (-\infty, -e_0] \cup [e_0, \infty)$, the cubic polynomial $Q(u)$ in (3.1) admits only one real root labeled as u_* , which corresponds to the global minimum of $\mathcal{H}(u, 0)$. For every $d \in (-\infty, d_1)$, where $d_1 := -\mathcal{H}(u_*, 0)$, there exists

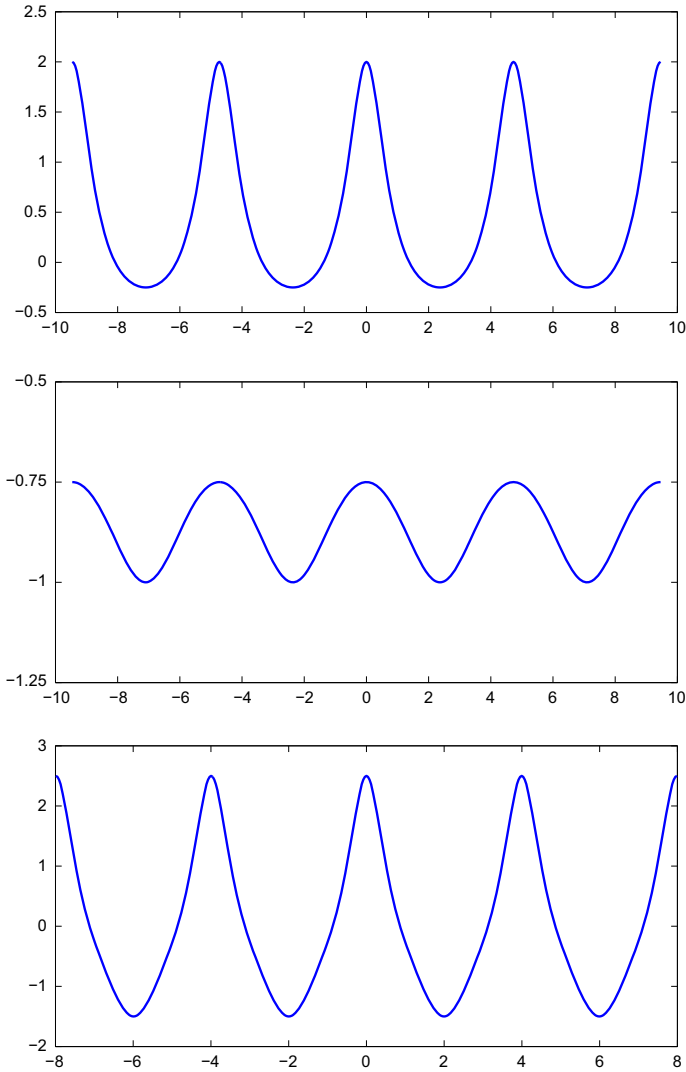


Fig. 2 Three periodic solutions for the phase plane on Fig. 1: (a) inside $[u_2, u_1]$, (b) inside $[u_4, u_3]$, (c) inside $[b, a]$

only one periodic solution which corresponds to the closed orbit surrounding the only center point $(u_*, 0)$ of the dynamical system.

In order to complete the proof of Theorem 1, it remains to verify the exact representations (1.14)–(1.15) and (1.16)–(1.17) of the periodic solutions as rational functions of the Jacobian elliptic functions. The following two lemmas state the corresponding exact representations.

Lemma 1 *Let $P(u)$ in (1.13) admit four real roots ordered as $u_4 \leq u_3 \leq u_2 \leq u_1$. The exact periodic solution to the first-order invariant (1.12) in the interval $[u_2, u_1]$ is given by*

$$u(x) = u_4 + \frac{(u_1 - u_4)(u_2 - u_4)}{(u_2 - u_4) + (u_1 - u_2)\operatorname{sn}^2(\nu x; \kappa)}, \quad (3.2)$$

where $\nu > 0$ and $\kappa \in (0, 1)$ are parameters given by

$$\begin{cases} 4\nu^2 = (u_1 - u_3)(u_2 - u_4), \\ 4\nu^2\kappa^2 = (u_1 - u_2)(u_3 - u_4). \end{cases} \quad (3.3)$$

The roots satisfy the constraint $u_1 + u_2 + u_3 + u_4 = 0$ and are related to the parameters (c, d, e) by

$$\begin{cases} c = -(u_1u_2 + u_1u_3 + u_1u_4 + u_2u_3 + u_2u_4 + u_3u_4), \\ 2e = u_1u_2u_3 + u_1u_2u_4 + u_1u_3u_4 + u_2u_3u_4, \\ d = u_1u_2u_3u_4. \end{cases} \quad (3.4)$$

Proof By factorizing $P(u)$ by its roots, we rewrite the first-order invariant (1.12) in the form:

$$\left(\frac{du}{dx}\right)^2 + (u - u_1)(u - u_2)(u - u_3)(u - u_4) = 0. \quad (3.5)$$

Expanding the quartic polynomial and comparing it with the coefficients in (1.12), we verify the constraint $u_1 + u_2 + u_3 + u_4 = 0$ and the relations (3.4). In order to prove that (3.2) with (3.3) is an exact solution of (3.5), we substitute $u(x) = u_4 + (u_1 - u_4)(u_2 - u_4)/v(x)$ into (3.5) and obtain the equivalent equation:

$$\begin{aligned} \left(\frac{dv}{dx}\right)^2 + [(u_1 - u_4) - v][(u_2 - u_4) - v] \\ [(u_1 - u_4)(u_2 - u_4) - (u_3 - u_4)v] = 0. \end{aligned} \quad (3.6)$$

Then, we substitute $v(x) = (u_2 - u_4) + (u_1 - u_2)w(x)$ into (3.6) and obtain the equivalent equation

$$\left(\frac{dw}{dx}\right)^2 = w(1 - w)[(u_1 - u_3)(u_2 - u_4) - (u_1 - u_2)(u_3 - u_4)w]. \quad (3.7)$$

Taking derivative of $w(x) = \operatorname{sn}^2(\nu x; \kappa)$ and defining ν and κ by (3.3) satisfies (3.7). Since the Jacobian elliptic function $\operatorname{sn}^2(\nu x; \kappa)$ is periodic in x with the period $L := 2K(\kappa)/\nu$, where $K(\kappa)$ is the complete elliptic integral, the solution u in (3.2) is L -periodic. It is unique up to the translation symmetry $u(x) \mapsto u(x + x_0)$, $x_0 \in \mathbb{R}$. Furthermore, it is defined in the interval $[u_2, u_1]$ with $u(0) = u_1$ and $u(L/2) = u_2$. \square

Remark 12 As follows from Fig. 1, there exists another periodic solution in the interval $[u_4, u_3]$ for the same combination of roots $\{u_1, u_2, u_3, u_4\}$. This solution is obtained by

the symmetry transformation (S2) in (1.18). The corresponding exact periodic solution is

$$u(x) = u_2 + \frac{(u_3 - u_2)(u_4 - u_2)}{(u_4 - u_2) + (u_3 - u_4)\text{sn}^2(vx; \kappa)}, \tag{3.8}$$

whereas the relations (3.3) remain the same. The new solution (3.8) is defined in the interval $[u_4, u_3]$ with $u(0) = u_3$ and $u(L/2) = u_4$. The new solution (3.8) is also L -periodic with the same period $L = 2K(\kappa)/v$.

Remark 13 The symmetry transformation (S1) in (1.18) generates another solution

$$u(x) = u_3 + \frac{(u_2 - u_3)(u_1 - u_3)}{(u_1 - u_3) + (u_2 - u_1)\text{sn}^2(vx; \kappa)}, \tag{3.9}$$

which also exists in the interval $[u_2, u_1]$ with $u(0) = u_2$ and $u(L/2) = u_1$. The symmetry transformation (S3) in (1.18) generates another solution

$$u(x) = u_1 + \frac{(u_4 - u_1)(u_3 - u_1)}{(u_3 - u_1) + (u_4 - u_3)\text{sn}^2(vx; \kappa)}, \tag{3.10}$$

which also exists in the interval $[u_4, u_3]$ with $u(0) = u_4$ and $u(L/2) = u_3$. By uniqueness of the closed orbits for periodic solutions on the phase plane, the solution (3.9) coincides with the periodic solution (3.2) translated by the half-period $L/2$, whereas the solution (3.10) coincides with the periodic solution (3.8) also translated by the half-period $L/2$.

Example 1 Let $e = 0$. Since $P(u)$ is even in (1.13) for $e = 0$, we have $u_4 = -u_1$ and $u_3 = -u_2$. It follows from (3.4) that $c = u_1^2 + u_2^2$ and $d = u_1^2 u_2^2$, whereas relations (3.3) yield $u_1 = v(1 + \kappa)$ and $u_2 = v(1 - \kappa)$, from which the exact solution (3.2) takes the equivalent form

$$u(x) = -u_1 + \frac{2u_1}{1 + \kappa\text{sn}^2(vx; \kappa)} = u_1 \frac{1 - \kappa\text{sn}^2(vx; \kappa)}{1 + \kappa\text{sn}^2(vx; \kappa)}. \tag{3.11}$$

From Table 8.152 of Gradshteyn and Ryzhik (2005), we have the transformation formula

$$\frac{1 - \kappa\text{sn}^2(vx; \kappa)}{1 + \kappa\text{sn}^2(vx; \kappa)} = \text{dn}(v(1 + \kappa)x; k), \quad k := \frac{2\sqrt{\kappa}}{1 + \kappa},$$

from which we derive

$$u(x) = u_1 \text{dn}(u_1 x; k), \quad k = \sqrt{1 - \frac{u_2^2}{u_1^2}}. \tag{3.12}$$

Setting $u_1 = 1$ and $u_2 = \sqrt{1 - k^2}$ yields the normalized dnoidal periodic wave (1.4).

Example 2 Let $u_1 = u_2$. The exact solution (3.2) becomes the constant solution $u(x) = u_1$.

Example 3 Let $u_2 = u_3$. It follows from (3.3) that $\kappa = 1$ so that the exact solution (3.2) becomes the exponential soliton on the constant background u_2 :

$$u(x) = u_4 + \frac{(u_1 - u_4)(u_2 - u_4)}{(u_2 - u_4) + (u_1 - u_2) \tanh^2(vx)}, \quad v = \frac{1}{2} \sqrt{(u_1 - u_2)(u_2 - u_4)}, \tag{3.13}$$

where roots satisfy the constraint $u_1 + 2u_2 + u_4 = 0$.

Example 4 Let $u_3 = u_4$. It follows from (3.3) that $\kappa = 0$ so that the exact solution (3.2) becomes

$$u(x) = u_4 + \frac{(u_1 - u_4)(u_2 - u_4)}{(u_2 - u_4) + (u_1 - u_2) \sin^2(vx)}, \quad v = \frac{1}{2} \sqrt{(u_1 - u_4)(u_2 - u_4)},$$

where roots satisfy the constraint $u_1 + u_2 + 2u_4 = 0$. This periodic wave solution is obtained in Akhmediev et al. (1987) and reviewed recently in Chowdury et al. (2016). Setting $u_1 = 1 + 2b$, $u_2 = 1 - 2b$, and $u_4 = -1$ with $b \in [0, 1]$ yields $v = \sqrt{1 - b^2}$ and transform the explicit solution to the form:

$$u(x) = -1 + \frac{2(1 - b^2)}{1 - b \cos(2\sqrt{1 - b^2}x)}. \tag{3.14}$$

It follows from (3.4) that $c = 2 + 4b^2$, $d = 1 - 4b^2$, and $e = 4b^2$. As $b \rightarrow 1$, the explicit solution (3.14) converges to the algebraic soliton (1.6) at $t = 0$ and after reflection $u \mapsto -u$.

Lemma 2 Let $P(u)$ in (1.13) admit two real roots ordered as $b \leq a$ and two complex conjugated roots labeled as $\alpha \pm i\beta$. The exact periodic solution to the first-order invariant (1.12) is given by

$$u(x) = a + \frac{(b - a)(1 - \text{cn}(vx; \kappa))}{1 + \delta + (\delta - 1)\text{cn}(vx; \kappa)}, \tag{3.15}$$

where $\delta > 0$, $v > 0$, and $\kappa \in (0, 1)$ are parameters given by

$$\begin{cases} \delta^2 = \frac{(b-\alpha)^2 + \beta^2}{(a-\alpha)^2 + \beta^2}, \\ v^2 = \sqrt{[(a - \alpha)^2 + \beta^2][(b - \alpha)^2 + \beta^2]}, \\ 2\kappa^2 = 1 - \frac{(a-\alpha)(b-\alpha) + \beta^2}{\sqrt{[(a-\alpha)^2 + \beta^2][(b-\alpha)^2 + \beta^2]}}. \end{cases} \tag{3.16}$$

The roots satisfy the constraint $a + b + 2\alpha = 0$ and are related to the parameters (c, d, e) by

$$\begin{cases} c = -(ab + 2\alpha(a + b) + \alpha^2 + \beta^2), \\ 2e = 2\alpha ab + (a + b)(\alpha^2 + \beta^2), \\ d = ab(\alpha^2 + \beta^2). \end{cases} \tag{3.17}$$

Proof By factorizing $P(u)$ by its roots, we rewrite the first-order invariant (1.12) in the form:

$$\left(\frac{du}{dx}\right)^2 + (u - a)(u - b) \left[(u - \alpha)^2 + \beta^2 \right] = 0. \tag{3.18}$$

Expanding the quartic polynomial and comparing it with the coefficients in (1.12), we verify the constraint $a + b + 2\alpha = 0$ and the relations (3.17). In order to prove that (3.15) with (3.16) is an exact solution of (3.18), we substitute $u(x) = a + (b - a)/v(x)$ into (3.18) and obtain the equivalent equation:

$$\left(\frac{dv}{dx}\right)^2 + (1 - v) \left[((a - \alpha)v + b - a)^2 + \beta^2 v^2 \right] = 0. \tag{3.19}$$

Then, we substitute

$$v(x) = 1 + \delta \frac{1 + \operatorname{cn}(vx; \kappa)}{1 - \operatorname{cn}(vx; \kappa)}$$

into (3.19) and obtain three equations on $\delta, v,$ and κ which yield (3.16). Since the Jacobian elliptic function $\operatorname{cn}(vx; \kappa)$ is periodic in x with the period $L := 4K(\kappa)/v,$ where $K(\kappa)$ is the complete elliptic integral, the solution u in (3.15) is L -periodic. It is unique up to the translation symmetry $u(x) \mapsto u(x + x_0), x_0 \in \mathbb{R}.$ Furthermore, it is defined in the interval $[b, a]$ with $u(0) = a$ and $u(L/2) = b.$ \square

Remark 14 The symmetry transformation (S0) in (1.19) generates another solution:

$$u(x) = b + \frac{(a - b)(1 - \operatorname{cn}(vx; \kappa))}{1 + \delta^{-1} + (\delta^{-1} - 1)\operatorname{cn}(vx; \kappa)}, \tag{3.20}$$

which also exists in the interval $[b, a]$ with $u(0) = b$ and $u(L/2) = a.$ By uniqueness of the closed orbits for periodic solutions on the phase plane, the solution (3.20) coincides with the periodic solution (3.15) translated by the half-period $L/2.$

Example 5 Let $e = 0.$ Since $P(u)$ is even in (1.13) for $e = 0,$ we have $b = -a, \alpha = 0,$ and $\delta = 1.$ The exact solution (3.15) becomes

$$u(x) = a \operatorname{cn}(vx; \kappa), \quad v = \sqrt{a^2 + \beta^2}, \quad \kappa = \frac{a}{\sqrt{a^2 + \beta^2}}.$$

Setting $a = k$ and $\beta = \sqrt{1 - k^2}$ yields the normalized cnoidal periodic wave (1.5).

4 Proof of Theorem 2

Combining (2.8), (2.40), and (2.41) yields the following system of algebraic equations

$$\begin{cases} c = 2E_0 + 4(\lambda_1^2 + \lambda_2^2), \\ d = E_0^2 - 4\lambda_1^2\lambda_2^2, \\ e^2 = 16\lambda_1^2\lambda_2^2(E_0 + \lambda_1^2 + \lambda_2^2). \end{cases} \quad (4.1)$$

Let us introduce $y := 4(\lambda_1^2 + \lambda_2^2)$ and $z := 4\lambda_1^2\lambda_2^2$. Substituting $y = c - 2E_0$ and $z = E_0^2 - d$ into the third equation of the system (4.1) yields the cubic equation for E_0 :

$$(E_0^2 - d)(2E_0 + c) = e^2. \quad (4.2)$$

There exists exactly three roots of the cubic equation (4.2) and each root defines uniquely (y, z) and hence $(\lambda_1^2, \lambda_2^2)$. In order to complete the proof of Theorem 2, it remains to verify the exact representations (1.21) and (1.22) for the eigenvalue pairs $\pm\lambda_1, \pm\lambda_2, \pm\lambda_3$. The following two lemmas give the corresponding exact representations of eigenvalues.

Lemma 3 *Assume $u_4 < u_3 < u_2 < u_1$ such that $u_1 + u_2 + u_3 + u_4 = 0$ and express (c, d, e) by (3.4). The cubic equation (4.2) admits three simple roots in the explicit form:*

$$\begin{aligned} (R1) \quad E_0 &= \frac{1}{2}(u_1u_4 + u_2u_3), & (R2) \quad E_0 &= \frac{1}{2}(u_1u_3 + u_2u_4), \\ (R3) \quad E_0 &= \frac{1}{2}(u_1u_2 + u_3u_4). \end{aligned} \quad (4.3)$$

Proof Due to three possible choices of two eigenvalues λ_1 and λ_2 from three admissible values, it is sufficient to consider one combination of the three eigenvalues, e.g., the expression (1.21) rewritten again in the form:

$$\lambda_1 = \frac{1}{2}(u_1 + u_2), \quad \lambda_2 = \frac{1}{2}(u_1 + u_3), \quad \lambda_3 = \frac{1}{2}(u_2 + u_3). \quad (4.4)$$

Here (λ_1, λ_2) are selected as eigenvalues of the algebraic method, whereas λ_3 is the third complementary eigenvalue. We prove validity of the root (R1) in (4.3) for E_0 . If $(\lambda_1, \lambda_2, \lambda_3)$ is given by (4.4), then root (R1) for E_0 is equivalent to the following representation:

$$\begin{aligned} E_0 &= \lambda_3^2 - \lambda_1^2 - \lambda_2^2 \\ &= \frac{1}{2}(u_2u_3 - u_1^2 - u_1u_2 - u_1u_3) \\ &= \frac{1}{2}(u_2u_3 + u_1u_4). \end{aligned} \quad (4.5)$$

Substituting (4.4) and (4.5) into (4.1) yields the following relations:

$$\begin{aligned}
 c &= 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \\
 &= (u_1 + u_2 + u_3)^2 - u_1u_2 - u_1u_3 - u_2u_3 \\
 &= -u_1u_4 - u_2u_4 - u_3u_4 - u_1u_2 - u_1u_3 - u_2u_3,
 \end{aligned} \tag{4.6}$$

$$\begin{aligned}
 d &= \lambda_1^4 + \lambda_2^4 + \lambda_3^4 - 2(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2) \\
 &= -u_1^2u_2u_3 - u_1u_2^2u_3 - u_1u_2u_3^2 \\
 &= u_1u_2u_3u_4,
 \end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
 e &= -4\lambda_1\lambda_2\lambda_3 \\
 &= \frac{1}{2}(u_1u_2u_3 - (u_1 + u_2 + u_3)(u_1u_2 + u_1u_3 + u_2u_3)) \\
 &= \frac{1}{2}(u_1u_2u_3 + u_1u_2u_4 + u_1u_3u_4 + u_2u_3u_4).
 \end{aligned} \tag{4.8}$$

These relations confirm validity of the parametrization in (3.4). Hence, the value (R1) in (4.3) gives one of the three roots of the cubic equation (4.2). The other two roots (R2) and (R3) in (4.3) are given by the interchanges $\lambda_2 \leftrightarrow \lambda_3$ and $\lambda_1 \leftrightarrow \lambda_3$ respectively in the list of three eigenvalues in (4.4). Since $u_4 < u_3 < u_2 < u_1$, the three roots (R1), (R2), and (R3) for E_0 in (4.3) are distinct and no other roots of the cubic equation (4.2) exists. \square

Remark 15 When $u_1 = u_2$ or $u_2 = u_3$ or $u_3 = u_4$ in Examples 2, 3, and 4, one eigenvalue in (4.4) is simple and the other eigenvalue is double. This corresponds to one simple and one double roots for E_0 in (4.3).

Remark 16 When $e = 0$ in Example 1, one eigenvalue is zero, e.g., $\lambda_3 = 0$ in (4.4). Setting $u_1 = 1$ and $u_2 = \sqrt{1 - k^2}$ for the normalized dn-periodic wave (1.4) yields the other two eigenvalues (λ_1, λ_2) in the form (1.7).

Lemma 4 Assume $\beta \neq 0$, $\alpha = -(a + b)/2$ and express (c, d, e) by (3.17). The cubic equation (4.2) admits three simple roots in the explicit form:

$$\begin{aligned}
 (R1) \quad E_0 &= \frac{1}{8}(a^2 + 6ab + b^2) + \frac{1}{2}\beta^2, \\
 (R2_{\pm}) \quad E_0 &= -\frac{1}{4}(a + b)^2 \pm \frac{i}{2}\beta(a - b).
 \end{aligned} \tag{4.9}$$

Proof Roots (4.9) are obtained from roots (4.3) with the formal correspondence: $u_1 = a$, $u_2 = \alpha + i\beta$, $u_3 = \alpha - i\beta$, $u_4 = b$. The constraint $u_1 + u_2 + u_3 + u_4 = 0$ is equivalent to $a + b + 2\alpha = 0$, which allows us to eliminate $\alpha = -(a + b)/2$ from all expressions and to obtain eigenvalues in the form:

$$\lambda_1 = \frac{1}{4}(a - b) + \frac{i}{2}\beta, \quad \lambda_2 = \frac{1}{4}(a - b) - \frac{i}{2}\beta, \quad \lambda_3 = \frac{1}{2}(a + b). \tag{4.10}$$

The three roots for E_0 in (4.9) and the three eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ in (4.10) are distinct if $\beta \neq 0$. \square

Remark 17 When $e = 0$ in Example 5, one eigenvalue is zero, e.g., $\lambda_3 = 0$ in (4.10). Setting $a = k$ and $\beta = \sqrt{1 - k^2}$ for the normalized cn-periodic wave (1.5) yields the other two eigenvalues (λ_1, λ_2) in the form (1.8).

Remark 18 The explicit expressions (4.4) of the three eigenvalues for the periodic wave (3.2) can be recovered from formulas (58) and (62) in Kamchatnov et al. (2012), where they are derived by a different technique. The explicit expressions (4.10) of the three eigenvalues for the periodic wave (3.15) are new to the best of our knowledge.

5 Proof of Theorems 3 and 4

Let us first establish algebraic relations on the squared eigenfunctions in the periodic spectral problem (1.2) with the periodic solution u to the mKdV equation (1.1). We rewrite Eqs. (2.1), (2.5), (2.7), and (2.9) as a system of linear equations for squared eigenfunctions:

$$\begin{cases} p_1^2 + q_1^2 + p_2^2 + q_2^2 = u, \\ 2\lambda_1(p_1^2 - q_1^2) + 2\lambda_2(p_2^2 - q_2^2) = u_x, \\ 4\lambda_1^2(p_1^2 + q_1^2) + 4\lambda_2^2(p_2^2 + q_2^2) = u_{xx} + 2u^3 - 2E_0u, \\ 8\lambda_1^3(p_1^2 - q_1^2) + 8\lambda_2^3(p_2^2 - q_2^2) = u_{xxx} + 6u^2u_x - 2E_0u_x, \end{cases} \quad (5.1)$$

where the subscripts denote the derivatives in x . Solving the algebraic system (5.1) with Cramer's rule yields the relations

$$p_1^2 + q_1^2 = \frac{u_{xx} + 2u^3 - 2E_0u - 4\lambda_2^2u}{4(\lambda_1^2 - \lambda_2^2)}, \quad (5.2)$$

$$p_1^2 - q_1^2 = \frac{u_{xxx} + 6u^2u_x - 2E_0u_x - 4\lambda_2^2u_x}{8\lambda_1(\lambda_1^2 - \lambda_2^2)} \quad (5.3)$$

and similar relations for $p_2^2 + q_2^2$ and $p_2^2 - q_2^2$ obtained by the transformation $\lambda_1 \leftrightarrow \lambda_2$. If u satisfies differential equations (2.30) and (2.36), the expressions (5.2) and (5.3) are simplified to the form

$$p_1^2 + q_1^2 = \frac{e + 4\lambda_1^2u}{4(\lambda_1^2 - \lambda_2^2)}, \quad (5.4)$$

$$p_1^2 - q_1^2 = \frac{2\lambda_1u_x}{4(\lambda_1^2 - \lambda_2^2)}. \quad (5.5)$$

We also rewrite Eqs. (2.6) and (2.10) as another system of linear equations for squared eigenfunctions:

$$\begin{cases} 4\lambda_1 p_1 q_1 + 4\lambda_2 p_2 q_2 = E_0 - u^2, \\ 16\lambda_1^3 p_1 q_1 + 16\lambda_2^3 p_2 q_2 = E_1 + E_0^2 + (u_x)^2 - 2uu_{xx} - 3u^4 + 2E_0 u^2. \end{cases} \tag{5.6}$$

Solving the algebraic system (5.6) with Cramer’s rule yields the relations

$$4\lambda_1 p_1 q_1 = \frac{E_1 + E_0^2 + (u_x)^2 - 2uu_{xx} - 3u^4 + 2E_0 u^2 + 4\lambda_2^2 u^2 - 4\lambda_2^2 E_0}{4(\lambda_1^2 - \lambda_2^2)} \tag{5.7}$$

and a similar relation for $4\lambda_2 p_2 q_2$ obtained by the transformation $\lambda_1 \leftrightarrow \lambda_2$. If u satisfies the differential equation (2.33), while d and E_1 are expressed by (2.40) and (2.42), then the expression (5.7) is simplified to the form

$$p_1 q_1 = \frac{\lambda_1(E_0 + 2\lambda_2^2 - u^2)}{4(\lambda_1^2 - \lambda_2^2)}. \tag{5.8}$$

By using the relations (4.5) and (4.6), the relation (5.8) can be rewritten in the equivalent form:

$$2p_1 q_1 = \frac{\lambda_1(c - 4\lambda_1^2 - 2u^2)}{4(\lambda_1^2 - \lambda_2^2)}. \tag{5.9}$$

Remark 19 Any eigenfunction $\varphi = (p_1, q_1)^t$ of the spectral problem (1.2) is defined up to the scalar multiplication, and so are the Darboux transformations (1.23), (1.24), and (1.25). Therefore, the denominators in (5.4), (5.5), and (5.9) can be canceled without loss of generality. On the other hand, the numerators in (5.4), (5.5), and (5.9) for the eigenfunction $\varphi = (p_1, q_1)^t$ can be extended to the other two eigenfunctions $\varphi = (p_2, q_2)^t$ and $\varphi = (p_3, q_3)^t$ by replacing $\lambda_1 \mapsto \lambda_2$ and $\lambda_1 \mapsto \lambda_3$ respectively.

We now proceed with the proof of Theorem 3. The following three lemmas represent the outcomes of the onefold, twofold, and threefold Darboux transformations for the periodic wave (1.14) with the periodic eigenfunctions of the spectral problem (1.2).

Lemma 5 Assume $u_4 < u_3 < u_2 < u_1$ such that $u_1 + u_2 + u_3 + u_4 = 0$, $u_1 + u_2 \neq 0$, $u_1 + u_3 \neq 0$, and $u_2 + u_3 \neq 0$. The onefold Darboux transformation (1.23) with the periodic eigenfunction for each eigenvalue in (1.21) transforms the periodic wave (1.14) to the periodic wave of the same period obtained after the corresponding symmetry transformation in (1.18) and the reflection $u \mapsto -u$.

Proof Substituting (5.4) and (5.9) into (1.23) yields the new solution to the mKdV equation (1.1) in the form:

$$\tilde{u} = \frac{eu + 2\lambda_1^2(c - 4\lambda_1^2)}{e + 4\lambda_1^2 u}. \tag{5.10}$$

The representation (5.10) is a linear fractional transformation between two solutions to the mKdV equation (1.1). By replacing $\lambda_1 \mapsto \lambda_2$ and $\lambda_1 \mapsto \lambda_3$ in (5.10), two more solutions \tilde{u} can be obtained from the representation (5.10).

We show with explicit computations that the transformation (5.10) applied to the periodic wave (3.2) with the eigenvalue λ_j produces the same periodic wave after the corresponding symmetry transformation (S_j) and the reflection $u \mapsto -u$.

By using (3.2), (3.4), and (4.4), we perform lengthy but straightforward computations to obtain the following expressions:

$$e + 4\lambda_1^2 u = \frac{1}{2}(u_1 + u_2)(u_1 - u_4)(u_2 - u_4) \times \frac{(u_1 - u_3) + (u_2 - u_1)\operatorname{sn}^2(vx; \kappa)}{(u_2 - u_4) + (u_1 - u_2)\operatorname{sn}^2(vx; \kappa)} \quad (5.11)$$

and

$$eu + 2\lambda_1^2(c - 4\lambda_1^2) = \frac{1}{2}(u_1 + u_2)(u_1 - u_4)(u_2 - u_4) \times \frac{(u_3 - u_1)u_2 + (u_1 - u_2)u_3\operatorname{sn}^2(vx; \kappa)}{(u_2 - u_4) + (u_1 - u_2)\operatorname{sn}^2(vx; \kappa)}, \quad (5.12)$$

where ν and κ are given by (3.3). Substituting (5.11) and (5.12) into (5.10) for $u_1 + u_2 \neq 0$ produces a new solution in the form:

$$\tilde{u}(x) = - \left[u_3 + \frac{(u_2 - u_3)(u_1 - u_3)}{(u_1 - u_3) + (u_2 - u_1)\operatorname{sn}^2(vx; \kappa)} \right]. \quad (5.13)$$

The new solution is obtained from the periodic wave (3.2) by the symmetry transformation (S₁) in (1.18) and the reflection $u \mapsto -u$, see the explicit form (3.9).

By repeating the previous computations for the eigenvalue λ_2 in (4.4), we obtain the following expressions:

$$e + 4\lambda_2^2 u = \frac{1}{2}(u_1 + u_3)(u_1 - u_4)(u_1 - u_2) \times \frac{(u_2 - u_4) + (u_4 - u_3)\operatorname{sn}^2(vx; \kappa)}{(u_2 - u_4) + (u_1 - u_2)\operatorname{sn}^2(vx; \kappa)} \quad (5.14)$$

and

$$eu + 2\lambda_2^2(c - 4\lambda_2^2) = \frac{1}{2}(u_1 + u_3)(u_1 - u_4)(u_1 - u_2) \times \frac{(u_4 - u_2)u_3 + (u_3 - u_4)u_2\operatorname{sn}^2(vx; \kappa)}{(u_2 - u_4) + (u_1 - u_2)\operatorname{sn}^2(vx; \kappa)}. \quad (5.15)$$

Substituting (5.14) and (5.15) into the onefold transformation for $u_1 + u_3 \neq 0$ produces a new solution in the form:

$$\tilde{u}(x) = - \left[u_2 + \frac{(u_4 - u_2)(u_3 - u_2)}{(u_4 - u_2) + (u_3 - u_4)\operatorname{sn}^2(vx; \kappa)} \right], \quad (5.16)$$

which is obtained from the periodic wave (3.2) by the symmetry transformation (S2) in (1.18) and the reflection $u \mapsto -u$, see the explicit form (3.8).

By repeating the previous computations for the eigenvalue λ_3 in (4.4), we obtain the following expressions:

$$e + 4\lambda_3^2 u = \frac{1}{2}(u_2 + u_3)(u_1 - u_2)(u_2 - u_4) \times \frac{(u_3 - u_1) + (u_4 - u_3)\text{sn}^2(vx; \kappa)}{(u_2 - u_4) + (u_1 - u_2)\text{sn}^2(vx; \kappa)} \tag{5.17}$$

and

$$eu + 2\lambda_3^2(c - 4\lambda_3^2) = \frac{1}{2}(u_2 + u_3)(u_1 - u_2)(u_2 - u_4) \times \frac{(u_1 - u_3)u_4 + (u_3 - u_4)u_1\text{sn}^2(vx; \kappa)}{(u_2 - u_4) + (u_1 - u_2)\text{sn}^2(vx; \kappa)}, \tag{5.18}$$

Substituting (5.17) and (5.18) into the onefold transformation for $u_2 + u_3 \neq 0$ produces a new solution in the form:

$$\tilde{u}(x) = - \left[u_1 + \frac{(u_3 - u_1)(u_4 - u_1)}{(u_3 - u_1) + (u_4 - u_3)\text{sn}^2(vx; \kappa)} \right], \tag{5.19}$$

which is obtained from the periodic wave (3.2) by the symmetry transformation (S3) in (1.18) and the reflection $u \mapsto -u$, see the explicit form (3.10). \square

Remark 20 If $e = 0$, the onefold transformation (5.10) simplifies to the form

$$\tilde{u} = \frac{c - 4\lambda_1^2}{2u}. \tag{5.20}$$

Setting $u_1 = 1$ and $u_2 = \sqrt{1 - k^2}$ yields the normalized dnoidal periodic wave (1.4) with the two nonzero eigenvalues (1.7). Substituting these expressions into (5.20) yields

$$\tilde{u}(x) = \mp \frac{\sqrt{1 - k^2}}{\text{dn}(x; k)} = \mp \text{dn}(x + K(k); k) = \mp u(x + L/2), \tag{5.21}$$

where $L = 2K(k)$ is the period of the periodic wave (1.4). Hence, the transformed solution (5.21) is again a translated and reflected version of the periodic wave (1.4).

Lemma 6 *Under the same assumptions as in Lemma 5, the twofold Darboux transformation (1.24) with the periodic eigenfunctions for any two eigenvalues from (1.21) transforms the periodic wave (1.14) to the periodic wave of the same period obtained after the complementary third symmetry transformation in (1.18).*

Proof Substituting (5.4), (5.5), and (5.8) into (1.24) yields the new solution to the mKdV equation (1.1) in the form:

$$\tilde{u} = u + \frac{4(\lambda_1^2 - \lambda_2^2)^2 [8\lambda_1^2 \lambda_2^2 u - e(E_0 - u^2)]}{8\lambda_1^2 \lambda_2^2 [(u_x)^2 + (E_0 + 2\lambda_1^2 - u^2)(E_0 + 2\lambda_2^2 - u^2)] - (\lambda_1^2 + \lambda_2^2)(e + 4\lambda_1^2 u)(e + 4\lambda_2^2 u)}.$$

By using the expressions (2.37), (2.40), and (2.41), we obtain

$$\tilde{u} = \frac{eE_0 - 4\lambda_1^2 \lambda_2^2 u}{eu + 4\lambda_1^2 \lambda_2^2}, \quad (5.22)$$

where $E_0 = \lambda_3^2 - \lambda_1^2 - \lambda_2^2$ with λ_3 being the complementary third eigenvalue to the pair (λ_1, λ_2) . Any pair of two eigenvalues from the list of three eigenvalues in (4.4) can be picked as (λ_1, λ_2) .

The representation (5.22) is another linear fractional transformation between two solutions of the mKdV equation (1.1). We show with explicit computations that this transformation applied to the periodic wave (3.2) with the eigenvalue pair (λ_1, λ_2) produces the same periodic wave after the complementary symmetry transformation (S3).

By using (3.2), (3.4), and (4.4), we obtain the following expressions:

$$\begin{aligned} eu + 4\lambda_1^2 \lambda_2^2 &= \frac{1}{4}(u_1 - u_2)(u_2 - u_4)(u_1 + u_2)(u_1 + u_3) \\ &\quad \times \frac{(u_1 - u_3) + (u_3 - u_4)\operatorname{sn}^2(vx; \kappa)}{(u_2 - u_4) + (u_1 - u_2)\operatorname{sn}^2(vx; \kappa)} \end{aligned}$$

and

$$\begin{aligned} eE_0 - 4\lambda_1^2 \lambda_2^2 u &= \frac{1}{4}(u_1 - u_2)(u_2 - u_4)(u_1 + u_2)(u_1 + u_3) \\ &\quad \times \frac{u_4(u_1 - u_3) + u_1(u_3 - u_4)\operatorname{sn}^2(vx; \kappa)}{(u_2 - u_4) + (u_1 - u_2)\operatorname{sn}^2(vx; \kappa)}, \end{aligned}$$

where ν and κ are given by (3.3). Substituting these expressions into (5.22) for $u_1 + u_2 \neq 0$ and $u_1 + u_3 \neq 0$ produces a new solution in the form:

$$\tilde{u}(x) = u_1 + \frac{(u_4 - u_1)(u_1 - u_3)}{(u_1 - u_3) + (u_3 - u_4)\operatorname{sn}^2(vx; \kappa)}, \quad (5.23)$$

The new solution is obtained from the periodic wave (3.2) by the symmetry transformation (S3) in (1.18). Repeating the same computations for the eigenvalue pair (λ_1, λ_3) in (4.4) for $u_1 + u_2 \neq 0$ and $u_2 + u_3 \neq 0$ produces the periodic wave (3.2) after the symmetry transformation (S2) in (1.18). Repeating the same computations for the eigenvalue pair (λ_2, λ_3) in (4.4) for $u_1 + u_3 \neq 0$ and $u_2 + u_3 \neq 0$ produces the periodic wave (3.2) after the symmetry transformation (S1) in (1.18). \square

Remark 21 The twofold transformation with two eigenvalues (λ_1, λ_2) can be thought as a composition of two onefold transformations with eigenvalues λ_1 and λ_2 . Indeed, by Lemma 5, the onefold transformation with λ_1 performs symmetry transformation (S1) and reflection, whereas the onefold transformation with λ_2 performs symmetry transformation (S2) and reflection. Composition of (S1) and (S2) yields (S3), whereas two reflections annihilate each other.

Remark 22 If $e = 0$, the twofold transformation (5.22) yields $\tilde{u} = -u$. Indeed, the onefold Darboux transformations of the dn-periodic wave (1.4) with the two nonzero eigenvalues in (1.7) produce (5.21), a composition of which yields just the reflection of u .

Lemma 7 Under the same assumptions as in Lemma 5, the threefold Darboux transformation (1.25) with the periodic eigenfunctions for all three eigenvalues in (1.21) transforms the periodic wave (1.14) to itself reflected with $u \mapsto -u$.

Proof By using (5.9) with $c = 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$ and neglecting the denominators, we write the product terms in the form:

$$\begin{cases} \lambda_1 p_1 q_1 \cong \lambda_1^2(\lambda_2^2 + \lambda_3^2 - \lambda_1^2 - u^2), \\ \lambda_2 p_2 q_2 \cong \lambda_2^2(\lambda_1^2 + \lambda_3^2 - \lambda_2^2 - u^2), \\ \lambda_3 p_3 q_3 \cong \lambda_3^2(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - u^2), \end{cases} \tag{5.24}$$

the sign \cong denotes the missing multiplication factor. By using (5.4) and (5.5) and neglecting the same denominators, we also express the squared components in the form:

$$\begin{cases} 2p_1^2 \cong e + 4\lambda_1^2 u + 2\lambda_1 u_x, & 2q_1^2 \cong e + 4\lambda_1^2 u - 2\lambda_1 u_x, \\ 2p_2^2 \cong e + 4\lambda_2^2 u + 2\lambda_2 u_x, & 2q_2^2 \cong e + 4\lambda_2^2 u - 2\lambda_2 u_x, \\ 2p_3^2 \cong e + 4\lambda_3^2 u + 2\lambda_3 u_x, & 2q_3^2 \cong e + 4\lambda_3^2 u - 2\lambda_3 u_x. \end{cases} \tag{5.25}$$

Only for succinctness in writing, let us make the notation

$$\Upsilon_1 = e + 4\lambda_1^2 u, \quad \Upsilon_2 = e + 4\lambda_2^2 u, \quad \Upsilon_3 = e + 4\lambda_3^2 u.$$

By substituting (5.24), and (5.25) back into the threefold transformation formula (1.25) we simplify the numerator N and the denominator D with the straightforward computations to the form:

$$\begin{aligned} N = & \lambda_3^2(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - u^2)[8\lambda_1^2\lambda_2^2(u^4 + d) \\ & + e^2(\lambda_1^2 + \lambda_2^2 - u^2) + 4eu(\lambda_1^2 - \lambda_2^2)^2] + \lambda_2^2(\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_3^2) \\ & (\lambda_1^2 + \lambda_3^2 - \lambda_2^2 - u^2)[8\lambda_1^2\lambda_3^2(u^4 + d) + e^2(\lambda_1^2 + \lambda_3^2 - u^2) + 4eu(\lambda_1^2 - \lambda_3^2)^2] \\ & + \lambda_1^2(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 + \lambda_3^2 - \lambda_1^2 - u^2) \\ & [8\lambda_2^2\lambda_3^2(u^4 + d) + e^2(\lambda_2^2 + \lambda_3^2 - u^2) + 4eu(\lambda_2^2 - \lambda_3^2)^2] \end{aligned}$$

$$-8\lambda_1^2\lambda_2^2\lambda_3^2(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - u^2)(\lambda_1^2 + \lambda_3^2 - \lambda_2^2 - u^2)(\lambda_2^2 + \lambda_3^2 - \lambda_1^2 - u^2) \\ \times (d + \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2),$$

and

$$D = (\lambda_1 + \lambda_2)^2(\lambda_1 + \lambda_3)^2(\lambda_2 - \lambda_3)^2 \left[\frac{1}{4}\Upsilon_1\Upsilon_2\Upsilon_3 + u_x^2(\lambda_2\lambda_3\Upsilon_1 - \lambda_1\lambda_2\Upsilon_3 - \lambda_1\lambda_3\Upsilon_2) \right] \\ + (\lambda_1 + \lambda_2)^2(\lambda_2 + \lambda_3)^2(\lambda_1 - \lambda_3)^2 \left[\frac{1}{4}\Upsilon_1\Upsilon_2\Upsilon_3 + u_x^2(\lambda_1\lambda_3\Upsilon_2 - \lambda_1\lambda_2\Upsilon_3 - \lambda_2\lambda_3\Upsilon_1) \right] \\ + (\lambda_1 + \lambda_3)^2(\lambda_2 + \lambda_3)^2(\lambda_1 - \lambda_2)^2 \left[\frac{1}{4}\Upsilon_1\Upsilon_2\Upsilon_3 + u_x^2(\lambda_1\lambda_2\Upsilon_3 - \lambda_1\lambda_3\Upsilon_2 - \lambda_2\lambda_3\Upsilon_1) \right] \\ + (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_3)^2 \left[\frac{1}{4}\Upsilon_1\Upsilon_2\Upsilon_3 + u_x^2(\lambda_1\lambda_3\Upsilon_2 + \lambda_1\lambda_2\Upsilon_3 + \lambda_2\lambda_3\Upsilon_1) \right] \\ - 8\lambda_1^2\lambda_2^2(\lambda_3^2 - \lambda_1^2)(\lambda_2^2 - \lambda_2^2)\Upsilon_3(\lambda_2^2 + \lambda_3^2 - \lambda_1^2 - u^2)(\lambda_1^2 + \lambda_3^2 - \lambda_2^2 - u^2) \\ - 8\lambda_1^2\lambda_3^2(\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_3^2)\Upsilon_2(\lambda_2^2 + \lambda_3^2 - \lambda_1^2 - u^2)(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - u^2) \\ - 8\lambda_2^2\lambda_3^2(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)\Upsilon_1(\lambda_1^2 + \lambda_3^2 - \lambda_2^2 - u^2)(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - u^2).$$

By using (2.37) to express $(u_x)^2$ and using (4.6), (4.7), and (4.8) to express (c, d, e) , we obtain

$$N = 32\lambda_1\lambda_2\lambda_3(\lambda_1^2 - \lambda_2^2)^2(\lambda_1^2 - \lambda_3^2)^2(\lambda_2^2 - \lambda_3^2)^2u = -\frac{1}{2}uD,$$

which yields $\tilde{u} = u - 2u = -u$. \square

We end this section with the proof of Theorem 4. The following three lemmas represent the outcomes of the onefold, twofold, and threefold Darboux transformations for the periodic wave (1.16) with the periodic eigenfunctions of the spectral problem (1.2). Only real-valued solutions \tilde{u} to the mKdV equation (1.1) are allowed as the outcomes of the Darboux transformations.

Lemma 8 Assume $a \neq \pm b$. The onefold Darboux transformation (1.23) with the periodic eigenfunction for eigenvalue λ_3 in (1.22) transforms the periodic wave (1.16) to the periodic wave of the same period obtained after the symmetry transformation (S0) in (1.19) and the reflection $u \mapsto -u$.

Proof By using (3.15), (3.17), and (4.10), we obtain the following expressions:

$$e + 4\lambda_3^2u = \frac{1}{2}(a + b)v^2 \frac{(1 + \delta) + (1 - \delta)\text{cn}(vx; \kappa)}{(1 + \delta) + (\delta - 1)\text{cn}(vx; \kappa)}$$

and

$$eu + 2\lambda_3^2(c - 4\lambda_3^2) = -\frac{1}{2}(a + b)v^2 \frac{(b + a\delta) + (b - a\delta)\text{cn}(vx; \kappa)}{(1 + \delta) + (\delta - 1)\text{cn}(vx; \kappa)},$$

where δ , ν , and κ are defined by (3.16) with $\alpha = -(a + b)/2$. Substituting these expressions into (5.10) with $\lambda_1 \mapsto \lambda_3$ for $a + b \neq 0$ produces a new solution

$$\tilde{u}(x) = - \left[b + \frac{(a - b)(1 - \operatorname{cn}(\nu x; \kappa))}{1 + \delta^{-1} + (\delta^{-1} - 1)\operatorname{cn}(\nu x; \kappa)} \right], \tag{5.26}$$

which is obtained from the periodic wave (3.15) by the symmetry transformation (S0) in (1.19) and the reflection $u \mapsto -u$, see the explicit form (3.20). \square

Remark 23 The onefold transformation (5.10) with $\lambda_1 \mapsto \lambda_3$ cannot be used for $e = 0$, because $\lambda_3 = 0$ and $(p_3, q_3) = (0, 0)$ if $e = 0$.

Lemma 9 Under the same assumptions as in Lemma 8, the twofold Darboux transformation (1.24) with the periodic eigenfunctions for two eigenvalues λ_1 and λ_2 from (1.22) transforms the periodic wave (1.16) to the periodic wave of the same period obtained after the symmetry transformation (S0) in (1.19).

Proof By using (3.15), (3.17), and (4.10), we obtain the following expressions:

$$eu + 4\lambda_1^2\lambda_2^2 = \frac{1}{4} \left[\frac{1}{4}(a - b)^2 + \beta^2 \right] v^2 \frac{(1 + \delta) + (1 - \delta)\operatorname{cn}(\nu x; \kappa)}{(1 + \delta) + (\delta - 1)\operatorname{cn}(\nu x; \kappa)}$$

and

$$eE_0 - 4\lambda_1^2\lambda_2^2u = \frac{1}{4} \left[\frac{1}{4}(a - b)^2 + \beta^2 \right] v^2 \frac{(b + a\delta) + (b - a\delta)\operatorname{cn}(\nu x; \kappa)}{(1 + \delta) + (\delta - 1)\operatorname{cn}(\nu x; \kappa)},$$

where δ , ν , and κ are defined by (3.16) with $\alpha = -(a + b)/2$. Substituting these expressions into (5.22) for $a + b \neq 0$ produces a new solution

$$\tilde{u}(x) = b + \frac{(a - b)(1 - \operatorname{cn}(\nu x; \kappa))}{1 + \delta^{-1} + (\delta^{-1} - 1)\operatorname{cn}(\nu x; \kappa)}, \tag{5.27}$$

which is obtained from the periodic wave (3.15) by the symmetry transformation (S0) in (1.19). \square

Remark 24 If $e = 0$, then the twofold transformation (5.22) produces $\tilde{u} = -u$, which is just a reflection of u .

Lemma 10 Under the same assumptions as in Lemma 8, the threefold Darboux transformation (1.25) with the periodic eigenfunctions for all three eigenvalues in (1.22) transforms the periodic wave (1.16) to itself reflected with $u \mapsto -u$.

Proof Computations in the proof of Lemma 7 are independent on the choice of $(\lambda_1, \lambda_2, \lambda_3)$, hence they extend to the eigenvalues in (1.22). \square

6 Second Solution of the Lax System

By Theorems 3 and 4, Darboux transformations with periodic eigenfunctions of the spectral problem (1.2) only generate symmetry transformations of the periodic solutions to the mKdV equation (1.1). In order to obtain new solutions to the mKdV equation (1.1), we construct the second linearly independent solution to the spectral problem (1.2) with the same eigenvalue λ . We also include the time evolution (1.3) in all expressions.

The following lemma gives the time evolution of the periodic eigenfunction φ satisfying the Lax equations (1.2)–(1.3) if u is the periodic travelling solution to the mKdV equation (1.1).

Lemma 11 *Let $u(x, t) = u(x - ct)$ be a periodic travelling wave of the mKdV equation (1.1) with the wave speed c , hence u satisfies the third-order differential equation (2.30). Let $\varphi = (p_1, q_1)^t$ be the periodic eigenfunction of the spectral problem (1.2) with $\lambda = \lambda_1$. Then, $\varphi(x, t) = \varphi(x - ct)$ satisfies the time evolution system (1.3).*

Proof Since the relation (2.1) holds for every t and $u(x, t) = u(x - ct)$, then $\varphi(x, t) = \varphi(x - ct)$. Alternatively, by using (2.36), (5.4), (5.5), and (5.8) in the time evolution problem (1.3), we obtain

$$\frac{\partial p_1}{\partial t} + c \frac{\partial p_1}{\partial x} = 0, \quad \frac{\partial q_1}{\partial t} + c \frac{\partial q_1}{\partial x} = 0,$$

hence $p_1(x, t) = p_1(x - ct)$ and $q_1(x, t) = q_1(x - ct)$. \square

Let $\varphi = (p_1, q_1)^t$ be the periodic eigenfunction of the spectral problem (1.2) with $\lambda = \lambda_1$ and denote the second linearly independent solution the same spectral problem (1.2) with the same $\lambda = \lambda_1$ by $\varphi = (\hat{p}_1, \hat{q}_1)^t$. Since the coefficient matrix in (1.2) has zero trace, the Wronskian determinant between the two solutions is constant in x and nonzero. To keep consistency with our previous work (Chen and Pelinovsky 2018), we normalize the Wronskian by 2, hence

$$p_1 \hat{q}_1 - \hat{p}_1 q_1 = 2. \quad (6.1)$$

In the previous work (Chen and Pelinovsky 2018), we constructed the second solution in the explicit form:

$$\hat{p}_1 = \frac{\theta_1 - 1}{q_1}, \quad \hat{q}_1 = \frac{\theta_1 + 1}{p_1}, \quad (6.2)$$

where θ_1 satisfies a certain scalar equation in x and t , which can be easily integrated. The corresponding expressions were used in Chen and Pelinovsky (2018) to construct the rogue waves on the periodic background; however, it was found that the expressions may be undefined if there exists a point of (x, t) for which either p_1 or q_1 vanishes.

Here we consider a different representation of the second solution which is free of the technical problem above. The following lemma represents the second solution in the explicit form.

Lemma 12 Let $\varphi = (p_1, q_1)^t$ be the periodic eigenfunction satisfying the Lax equations (1.2)–(1.3) with $\lambda = \lambda_1$ and $u(x, t) = u(x - ct)$ and let φ satisfy the normalization conditions (5.4), (5.5), and (5.9). The second linearly independent solution satisfying the normalization (6.1) can be written in the form:

$$\hat{p}_1 = p_1\phi_1 - \frac{2q_1}{p_1^2 + q_1^2}, \quad \hat{q}_1 = q_1\phi_1 + \frac{2p_1}{p_1^2 + q_1^2}, \tag{6.3}$$

with

$$\phi_1(x, t) = -16(\lambda_1^2 - \lambda_2^2) \left[\lambda_1^2 \int_0^{x-ct} \frac{c - 4\lambda_1^2 - 2u^2}{(e + 4\lambda_1^2 u)^2} dy + t + \psi_1 \right], \tag{6.4}$$

where ψ_1 is independent of (x, t) .

Proof By substituting the representation (6.3) to the spectral problem (1.2) with $\lambda = \lambda_1$ and using the same spectral problem (1.2) for $\varphi = (p_1, q_1)^t$, we obtain

$$\frac{\partial \phi_1}{\partial x} = -\frac{8\lambda_1 p_1 q_1}{(p_1^2 + q_1^2)^2}. \tag{6.5}$$

Thanks to (5.4) and (5.9), this equation can be integrated to the form

$$\phi_1 = -16(\lambda_1^2 - \lambda_2^2) \left[\lambda_1^2 \int_0^x \frac{c - 4\lambda_1^2 - 2u^2}{(e + 4\lambda_1^2 u)^2} dy + \psi_1 \right], \tag{6.6}$$

where ψ_1 is the constant of integration in x that may depend on t .

On the other hand, by substituting the representation (6.3) to the time evolution system (1.3) with λ_1 and using the same system (1.3) for $\varphi = (p_1, q_1)^t$, we obtain

$$\frac{\partial \phi_1}{\partial t} = \frac{8\lambda_1 p_1 q_1 (4\lambda_1^2 + 2u^2)}{(p_1^2 + q_1^2)^2} - \frac{8\lambda_1 u_x (p_1^2 - q_1^2)}{(p_1^2 + q_1^2)^2}. \tag{6.7}$$

Substituting (5.5), (5.9), and (6.5) into (6.7) yields the scalar equation

$$\frac{\partial \phi_1}{\partial t} + c \frac{\partial \phi_1}{\partial x} = -16(\lambda_1^2 - \lambda_2^2),$$

from which we obtain (6.6), where ψ_1 is now constant both in x and t . □

Remark 25 Without loss of generality, thanks to the translational invariance in (x, t) , we can set $\psi_1 = 0$ so that if u is even in x , then ϕ_1 is odd in x at $t = 0$.

Remark 26 The representation (6.3) is non-singular for every (x, t) for which $p_1^2 + q_1^2 \neq 0$. If $\lambda_1 \in \mathbb{R}$ with real $\varphi = (p_1, q_1)^t$, we have $p_1^2 + q_1^2 \neq 0$ everywhere because if φ vanishes at one point, then φ is identically zero everywhere since it satisfies the first-order systems (1.2) and (1.3).

Remark 27 If $\lambda_1 \notin \mathbb{R}$, the representation (6.3) is non-singular if $e \neq 0$ and singular if $e = 0$. This follows from the expression (5.4) which shows that $p_1^2 + q_1^2 \cong e + 4\lambda_1^2 u$ with $e \in \mathbb{R}$ and $u \in \mathbb{R}$. If $e = 0$, the cn -periodic solution (1.5) vanishes at some points of (x, t) , which lead to the singular behavior of \hat{p}_1 and \hat{q}_1 given by (6.3). Note in this case, the previous representation (6.2) is non-singular, as it follows from our previous work (Chen and Pelinovsky 2018).

7 Proof of Theorems 5 and 6

Here we use the Darboux transformation formulas (1.23), (1.24), and (1.25) with the second solutions of the Lax system (1.2) and (1.3) for the same eigenvalues λ_1, λ_2 , and λ_3 as in Theorem 2. Outcomes of the Darboux transformations are first represented graphically and then studied analytically.

Substituting (6.3) into the onefold transformation (1.23) yields the new solution to the mKdV equation (1.1) in the form:

$$\hat{u} = u + \frac{4\lambda_1 N_1}{D_1}, \quad (7.1)$$

where

$$N_1 := \frac{p_1 q_1}{p_1^2 + q_1^2} \left[(p_1^2 + q_1^2)^2 \phi_1^2 - 4 \right] + 2(p_1^2 - q_1^2) \phi_1,$$

$$D_1 := (p_1^2 + q_1^2)^2 \phi_1^2 + 4,$$

with ϕ_1 given by (6.4) for the choice of $\psi_1 = 0$.

Figure 3 shows three new solutions computed from the periodic wave (1.14) with the parameter values:

$$u_1 = 2, \quad u_2 = -0.25, \quad u_3 = -0.75, \quad u_4 = -1. \quad (7.2)$$

The three solutions are generated with three different choices for the eigenvalue λ_1 given by (1.21). Each solution displays an algebraic soliton propagating on the background of the periodic travelling wave.

Substituting (6.3) into the twofold transformation (1.24) yields the new solution to the mKdV equation (1.1) in the form:

$$\hat{u} = u + \frac{4(\lambda_1^2 - \lambda_2^2)(\lambda_1 N_1 D_2 - \lambda_2 N_2 D_1)}{(\lambda_1^2 + \lambda_2^2) D_1 D_2 - 8\lambda_1 \lambda_2 N_1 N_2 - 2\lambda_1 \lambda_2 S_1 S_2}, \quad (7.3)$$

where for $j = 1, 2$

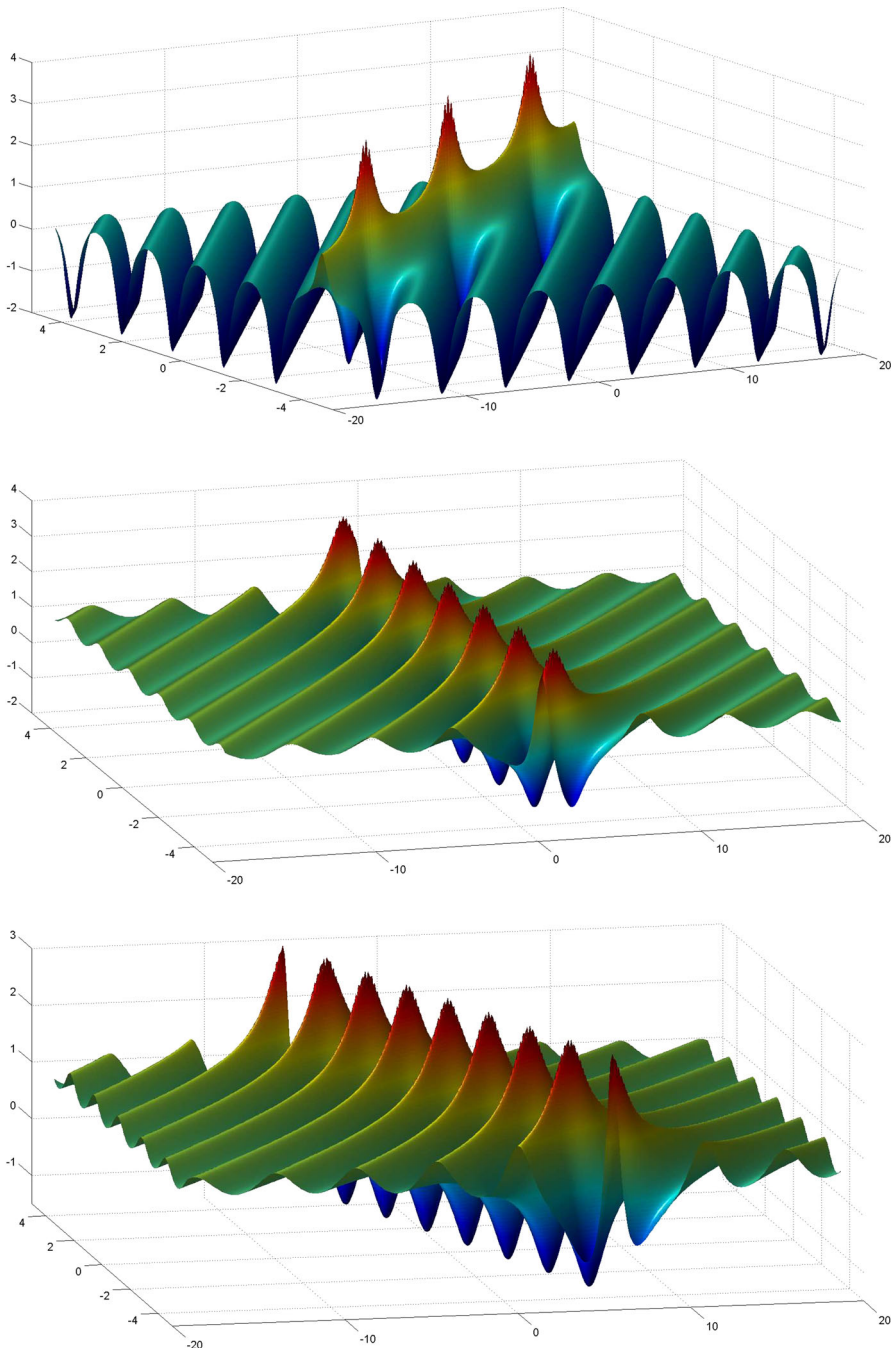


Fig. 3 Three outcomes of the onefold transformation of the periodic solution (1.14)

$$\begin{aligned}
 N_j &:= \frac{p_j q_j}{p_j^2 + q_j^2} \left[(p_j^2 + q_j^2)^2 \phi_j^2 - 4 \right] + 2(p_j^2 - q_j^2) \phi_j, \\
 S_j &:= \frac{(p_j^2 - q_j^2)}{p_j^2 + q_j^2} \left[(p_j^2 + q_j^2)^2 \phi_j^2 - 4 \right] - 8p_j q_j \phi_j, \\
 D_j &:= (p_j^2 + q_j^2)^2 \phi_j^2 + 4.
 \end{aligned}$$

The expression for ϕ_1 is given by (6.4) with $\psi_1 = 0$. Note that the numerical factor $(\lambda_1^2 - \lambda_2^2)$ in (6.4) cancels with the denominators in the expressions (5.4), (5.5), and (5.9) for the squared eigenfunctions. Therefore, the expression for ϕ_2 and ϕ_3 can be obtained with the transformation $\lambda_1 \mapsto \lambda_2$ and $\lambda_1 \mapsto \lambda_3$ respectively.

Three choices exist for an eigenvalue pair (λ_1, λ_2) from the three real eigenvalues in (1.21). Figure 4 shows three new solutions computed from the periodic wave (1.14) with the same choice of parameters as in (7.2). The three solutions display three different choices of two algebraic solitons propagating on the background of the periodic travelling wave.

Substituting (6.3) into the threefold transformation (1.25) yields a new solution in the explicit form that is similar to (7.1) and (7.3). There is only one choice for the three real eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ in (1.21). Figure 5 shows the new solution computed from the periodic wave (1.14) with the same choice of parameters as in (7.2). The new solution displays three algebraic solitons propagating on the background of the periodic travelling wave.

The proof of Theorem 5 relies on the following two lemmas.

Lemma 13 *Let $(p_{1,2,3}, q_{1,2,3})$ be the periodic solution to the Lax equations (1.2)–(1.3) for $\lambda_{1,2,3}$ and $(\hat{p}_{1,2,3}, \hat{q}_{1,2,3})$ be the second solution in Lemma 12. Let \tilde{u} be the Darboux transformations of u with the periodic solution and \hat{u} be the corresponding Darboux transformations of u with the second solution. Then, we have*

$$\lim_{|\phi_{1,2,3}| \rightarrow \infty} \hat{u} = \tilde{u}, \quad \lim_{|\phi_{1,2,3}| \rightarrow 0} \hat{u} = 2u - \tilde{u}. \tag{7.4}$$

Proof By using (5.10) and (7.1) for the onefold transformation, we obtain

$$\lim_{|\phi_1| \rightarrow \infty} \hat{u} = u + \frac{4\lambda_1 p_1 q_1}{p_1^2 + q_1^2} = \tilde{u}$$

and

$$\lim_{|\phi_1| \rightarrow 0} \hat{u} = u - \frac{4\lambda_1 p_1 q_1}{p_1^2 + q_1^2} = 2u - \tilde{u}.$$

By using (5.22) and (7.3) for the twofold transformation, we obtain similarly:

$$\lim_{|\phi_1|, |\phi_2| \rightarrow \infty} \hat{u} = u + \frac{4(\lambda_1^2 - \lambda_2^2) [\lambda_1 p_1 q_1 (p_2^2 + q_2^2) - \lambda_2 p_2 q_2 (p_1^2 + q_1^2)]}{(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 2\lambda_1 \lambda_2 [4p_1 q_1 p_2 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)]} = \tilde{u}$$

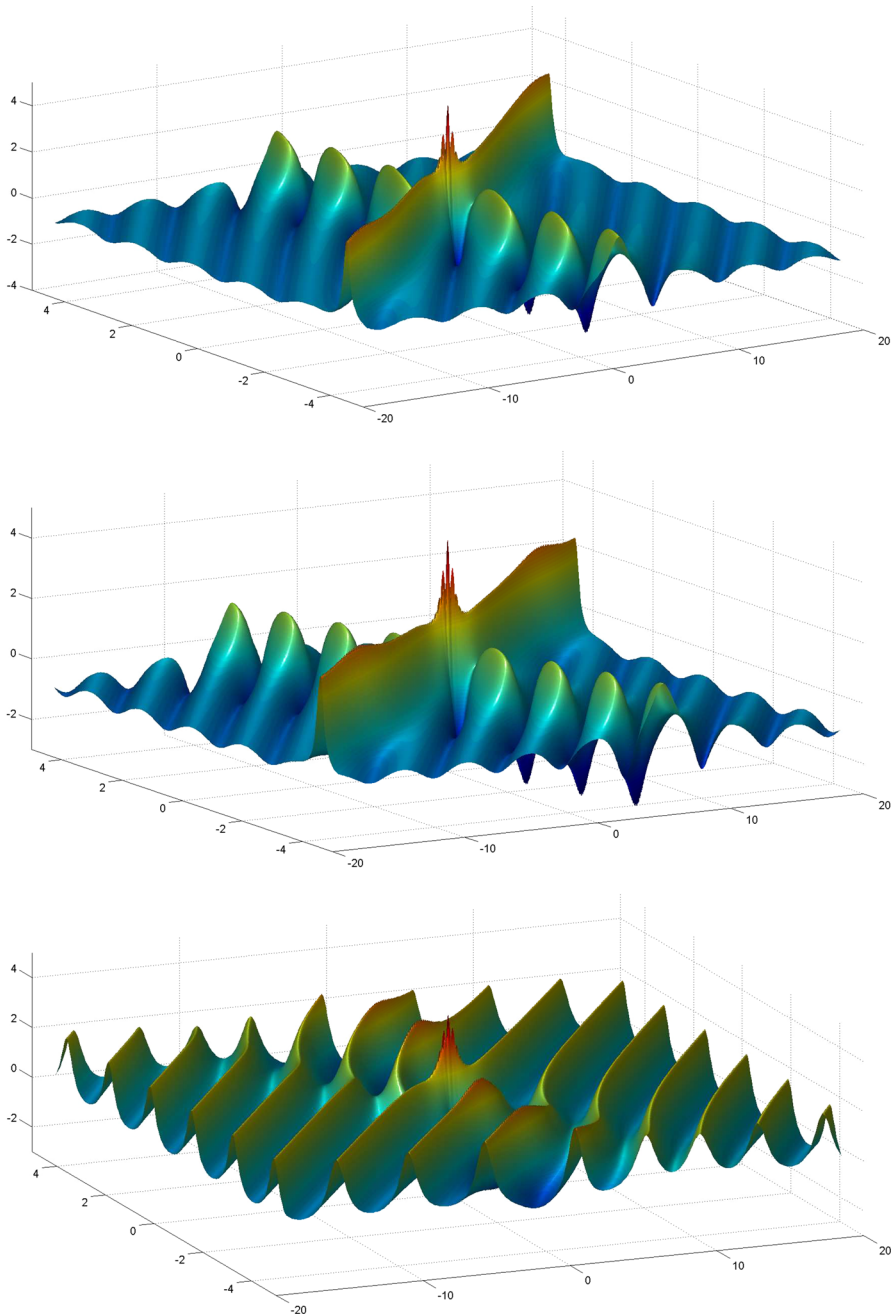


Fig. 4 Three outcomes of the twofold transformation of the periodic solution (1.14)

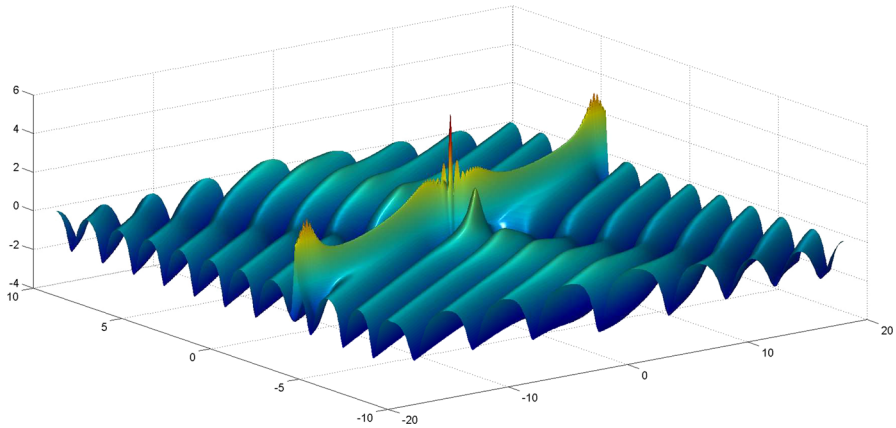


Fig. 5 Outcome of the threefold transformation of the periodic solution (1.14)

and

$$\lim_{|\phi_1|, |\phi_2| \rightarrow 0} \hat{u} = u - \frac{4(\lambda_1^2 - \lambda_2^2) [\lambda_1 p_1 q_1 (p_2^2 + q_2^2) - \lambda_2 p_2 q_2 (p_1^2 + q_1^2)]}{(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 2\lambda_1 \lambda_2 [4p_1 q_1 p_2 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)]} = 2u - \bar{u}.$$

The proof for the threefold Darboux transformation (1.25) is similar. □

Lemma 14 Assume $u_4 < u_3 < u_2 < u_1$ such that $u_1 + u_2 + u_3 + u_4 = 0$, $u_1 + u_2 \neq 0$, $u_1 + u_3 \neq 0$, and $u_2 + u_3 \neq 0$. Under the following three non-degeneracy conditions,

$$\oint \frac{\lambda_1^2(u_1 u_2 + u_3 u_4 + 2u^2)}{(e + 4\lambda_1^2 u)^2} dy \neq 0, \tag{7.5}$$

$$\oint \frac{\lambda_2^2(u_1 u_3 + u_2 u_4 + 2u^2)}{(e + 4\lambda_2^2 u)^2} dy \neq 0, \tag{7.6}$$

and

$$\oint \frac{\lambda_3^2(u_1 u_4 + u_2 u_3 + 2u^2)}{(e + 4\lambda_3^2 u)^2} dy \neq 0, \tag{7.7}$$

where \oint denotes the mean value integral, there exist $c_1, c_2, c_3 \neq c$ such that the second solutions to the Lax system (1.2)–(1.3) for u in (1.14) and eigenvalues $\lambda_1, \lambda_2, \lambda_3$ in (1.21) are linearly growing in x and t everywhere on the (x, t) plane except for the straight lines $x - c_{1,2,3}t = \xi_{1,2,3}$, where ξ_1, ξ_2, ξ_3 are phase parameters which are not uniquely defined.

Proof Thanks to the periodicity of $\varphi = (p_1, q_1)^t$ in (x, t) and the explicit representation (6.3)–(6.4) for the second solution $\varphi = (\hat{p}_1, \hat{q}_1)^t$, the proof follows from analysis of the factor

Table 1 Characteristics of rogue waves on the periodic background (1.14)

Transformation	$\tilde{u}(0)$	$\hat{u}(0)$	$\tilde{u}(L/2)$	$\hat{u}(L/2)$
Onefold with λ_1	$-u_2$	$2u_1 + u_2$	$-u_1$	$2u_2 + u_1$
Onefold with λ_2	$-u_3$	$2u_1 + u_3$	$-u_4$	$2u_2 + u_4$
Onefold with λ_3	$-u_4$	$2u_1 + u_4$	$-u_3$	$2u_2 + u_3$
Twofold with (λ_1, λ_2)	u_4	$2u_1 - u_4$	u_3	$2u_2 - u_3$
Twofold with (λ_1, λ_3)	u_3	$2u_1 - u_3$	u_4	$2u_2 - u_4$
Twofold with (λ_2, λ_3)	u_2	$2u_1 - u_2$	u_1	$2u_2 - u_1$
Threefold with $(\lambda_1, \lambda_2, \lambda_3)$	$-u_1$	$3u_1$	$-u_2$	$3u_2$

$$\tilde{\phi}_1(x, t) = \int_0^{x-ct} \frac{\lambda_1^2(c - 4\lambda_1^2 - 2u^2)}{(e + 4\lambda_1^2u)^2} dy + t + \psi_1. \tag{7.8}$$

Let M_1 be the mean value of the factor under the integration sign and assume that $M_1 \neq 0$. Then,

$$\tilde{\phi}_1(x, t) = M_1(x - ct) + t + \text{periodic function}. \tag{7.9}$$

Hence, $|\tilde{\phi}_1(x, t)| \rightarrow \infty$ as a linear function of (x, t) everywhere on the (x, t) plane except for the straight line $x - c_1t = \xi_1$, where $c_1 = c - M_1^{-1}$ and ξ_1 is not uniquely defined due to the phase parameter ψ_1 . Similar formulas are obtained for the second solutions (\hat{p}_2, \hat{q}_2) and (\hat{p}_3, \hat{q}_3) . The conditions $M_1, M_2, M_3 \neq 0$ are equivalent to the non-degeneracy conditions (7.5), (7.6), and (7.7) respectively, e.g., $c - 4\lambda_1^2 = 2(\lambda_2^2 + \lambda_3^2 - \lambda_1^2) = -(u_1u_2 + u_3u_4)$. \square

Remark 28 The first limit in (7.4) tells us that the new solution \hat{u} approaches the transformed periodic wave \tilde{u} almost everywhere on the (x, t) plane as $|x| + |t| \rightarrow \infty$ except for the lines $x - c_{1,2,3}t = \xi_{1,2,3}$. Algebraic solitons propagate along these lines with the wave speeds $c_{1,2,3}$.

Remark 29 The second limit in (7.4) tells us the amplitude of the algebraic soliton on the periodic background \tilde{u} . The periodic wave u in (3.2) has two extremal points $u(0) = u_1$ and $u(L/2) = u_2$ and our convention is $u_4 < u_3 < u_2 < u_1$. Table 1 shows the periodic background \tilde{u} and the new solution \hat{u} at two extremal points of u .

We next represent graphically the outcomes of the onefold, twofold, and threefold Darboux transformations for the periodic wave (1.16). For the onefold transformation (7.1), only one choice generates the real solution \hat{u} to the mKdV equation (1.1). This choice corresponds to the real eigenvalue λ_3 in (1.22). Figure 6 shows the new solution computed from the periodic wave (1.16) with the parameter values:

$$a = 1.5, \quad b = -0.5, \quad \alpha = -0.5, \quad \beta = 2. \tag{7.10}$$

The new solution displays the algebraic soliton propagating on the background of the periodic travelling wave.

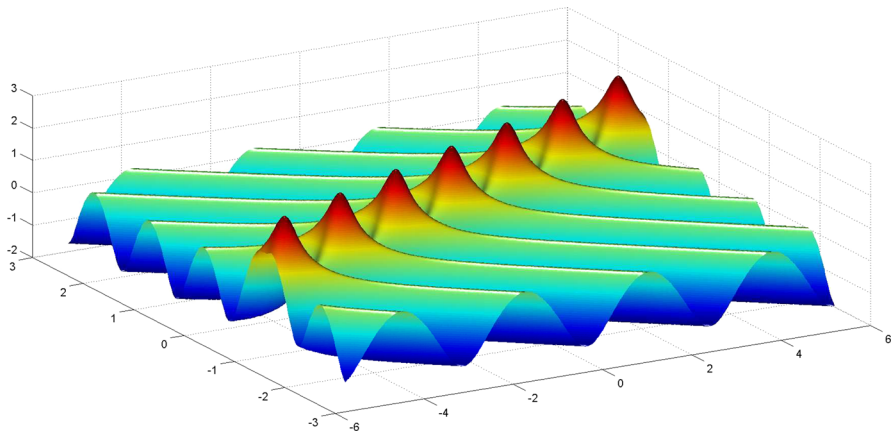


Fig. 6 Outcome of the onefold transformation of the periodic solution (1.16)

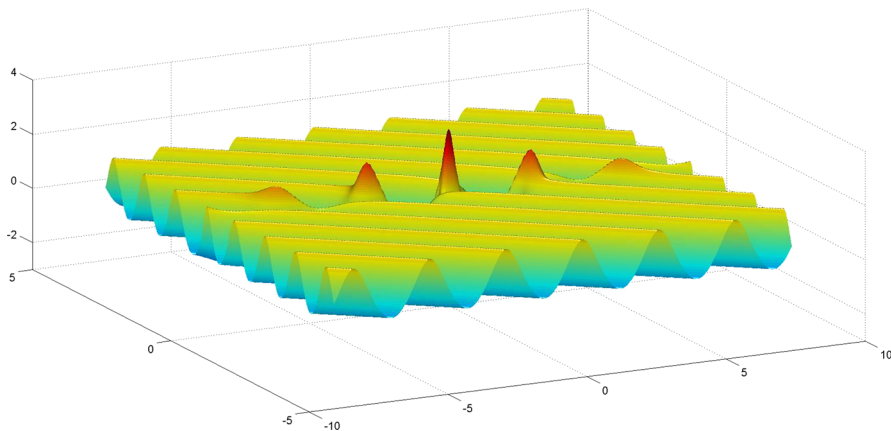


Fig. 7 Outcome of the twofold transformation of the periodic solution (1.16)

For the twofold transformation (7.3), only one choice generates the real solution \hat{u} to the mKdV equation (1.1). This choice corresponds to the complex-conjugate eigenvalues (λ_1, λ_2) in (1.22). Figure 7 shows the new solution computed from the periodic wave (1.16) with the same choice of parameters as in (7.10). The new solution displays a fully localized rogue wave on the background of the periodic travelling wave.

Figure 8 shows the outcome of the threefold transformation computed from the periodic wave (1.16) with the same choice of parameters as in (7.10). All three eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ are selected from the list (1.22). The new solution displays both the algebraic soliton propagating on the background of the periodic travelling wave and a localized rogue wave at the center.

The proof of Theorem 6 relies on Lemma 13 and the following lemma.

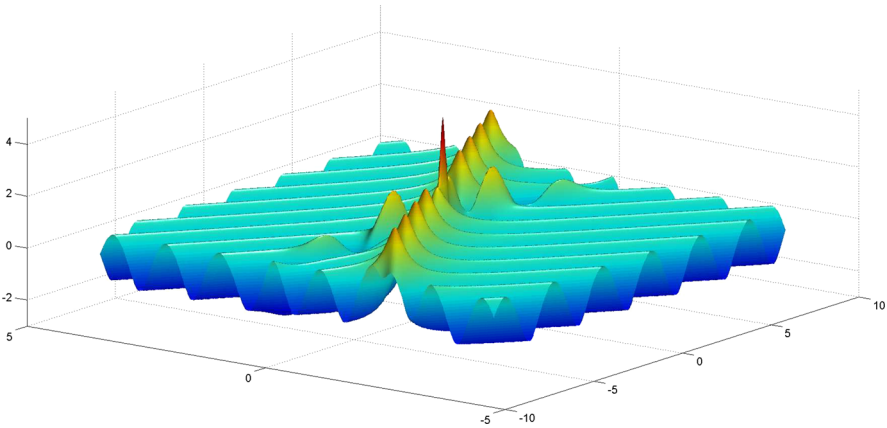


Fig. 8 Outcome of the threefold transformation of the periodic solution (1.16)

Lemma 15 Assume $a \neq \pm b$. Under the following two non-degeneracy conditions,

$$\text{Im} \oint \frac{\lambda_1^2(2i\beta(a - b) - (a + b)^2 + 4u^2)}{(e + 4\lambda_1^2u)^2} dy \neq 0 \tag{7.11}$$

and

$$\oint \frac{\lambda_3^2(4\beta^2 + a^2 + b^2 + 6ab + 8u^2)}{(e + 4\lambda_3^2u)^2} dy \neq 0, \tag{7.12}$$

the second solutions to the Lax system (1.2)–(1.3) for u in (1.16) and eigenvalues λ_1, λ_2 in (1.22) are linearly growing in x and t everywhere, whereas there exists $c_0 \neq c$ such that the second solution for λ_3 in (1.22) is linearly growing in x and t everywhere except for the straight line $x - c_0t = \xi_0$, where ξ_0 is the phase parameter which is not uniquely defined.

Proof In the representations (7.8) and (7.9), the value of M_1 is complex since λ_1 in (1.22) is complex. If $\text{Im}(M_1) \neq 0$, then $|\tilde{\phi}_1(x, t)| \rightarrow \infty$ as a linear function of (x, t) everywhere on the (x, t) plane. The same is true for the solution (\hat{p}_2, \hat{q}_2) since $\lambda_2 = \bar{\lambda}_1$. However, the solution (\hat{p}_3, \hat{q}_3) with real λ_3 is real and the representation (7.9) with $M_3 \neq 0$ shows that $|\tilde{\phi}_1(x, t)|$ remains bounded on the straight line $x - c_0t = \xi_0$, where $c_0 = c - M_3^{-1}$ and ξ_0 is not uniquely defined due to the phase parameter ψ_3 . The conditions $\text{Im}(M_1) \neq 0$ and $M_3 \neq 0$ are equivalent to the non-degeneracy conditions (7.11) and (7.12) respectively, e.g., $c - 4\lambda_3^2 = 2(\lambda_1^2 + \lambda_2^2 - \lambda_3^2) = -(\beta^2 + (a^2 + b^2 + 6ab)/4)$. \square

Remark 30 By Lemma 15, the new solution \hat{u} after the onefold transformation with eigenvalue λ_3 approaches the transformed periodic wave \tilde{u} everywhere on the (x, t) plane as $|x| + |t| \rightarrow \infty$ except for the line $x - c_0t = \xi_0$, where the algebraic soliton propagates. The periodic wave u in (3.15) has two extremal points $u(0) = a$ and

Table 2 Characteristics of rogue waves on the periodic background (1.16)

Transformation	$\tilde{u}(0)$	$\hat{u}(0)$	$\tilde{u}(L/2)$	$\hat{u}(L/2)$
Onefold with λ_3	$-b$	$2a + b$	$-a$	$2b + a$
Twofold with (λ_1, λ_2)	b	$2a - b$	a	$2b - a$
Threefold with $(\lambda_1, \lambda_2, \lambda_3)$	$-a$	$3a$	$-b$	$3b$

$u(L/2) = b$ and our convention is $b < a$. Table 2 shows the periodic background \tilde{u} and the new solution \hat{u} at two extremal points of u .

Remark 31 By Lemma 15, the new solution \hat{u} after the twofold transformation with eigenvalues (λ_1, λ_2) approaches the transformed periodic wave \tilde{u} everywhere on the (x, t) plane as $|x| + |t| \rightarrow \infty$. Hence, this is the proper rogue wave in the sense of Definition 1. Table 2 shows the magnification factors of the rogue wave defined as a ratio between the extremal values of \hat{u} and \tilde{u} , e.g., $M = \max\{|2a - b|/|b|, |2b - a|/|a|\}$.

Remark 32 The threefold transformation with all three eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ produces both the rogue wave and the algebraic soliton with the wave speed c_0 . Table 2 shows that the new wave \hat{u} has triple magnification compared to the periodic background \tilde{u} .

8 Conclusion

We have addressed the most general periodic travelling wave solutions to the mKdV equation (1.1) and obtained the periodic solutions to the Lax system (1.2)–(1.3) for three particular pairs of eigenvalues λ away from the imaginary axis. For the family of periodic waves (1.14) generalizing the dn -periodic wave (1.4), the three pairs of eigenvalues $\pm\lambda_1, \pm\lambda_2, \pm\lambda_3$ are real. For the family of periodic waves (1.16) generalizing the cn -periodic wave (1.5), one pair $\pm\lambda_3$ is real and two pairs $\pm\lambda_1, \pm\lambda_2$ form a quadruplet of complex eigenvalues. By using the Darboux transformations, we have showed that transformations involving the periodic eigenfunctions remain in the class of the same periodic wave solutions, whereas transformations involving second non-periodic solutions to the Lax system (1.2)–(1.3) for the same eigenvalues generate new solutions on the background of the periodic waves. Among new solutions, one solution is a rogue wave on the periodic background satisfying (1.9), whereas all others are algebraic solitons propagating on the periodic background. The rogue wave exist on the background of the periodic wave (1.16) which is expected to be modulationally unstable with respect to perturbations of long periods.

Let us summarize the outcomes of the algebraic method for periodic solutions to the mKdV equation (1.1) expressed by the Riemann Theta functions of genus g with $g = 0, 1, 2$. These solutions are obtained by degeneration of the three Dubrovin variables μ_1, μ_2 , and μ_3 related to the periodic solution u in (2.25), (2.26), and (2.27).

$g = 0$ If $\mu_1 = \mu_2 = \mu_3 = 0$, then $u(x, t) = u_1$ is a constant wave with $u_1 \in \mathbb{R}$. This corresponds to only one pairs of real eigenvalues $\pm\lambda_1$ with $\lambda_1 = u_1$ and the constant solution to the Lax system (1.2)–(1.3).

$g = 1$ If $\mu_2 = \mu_3 = 0$, then $u(x, t) = u(x - ct)$ satisfies the second-order equation

$$\frac{d^2u}{dx^2} + 2u^3 - cu = 0,$$

which is solved by two families of the solutions (1.14) and (1.16) in the case $e = 0$. One solution corresponds to $u_4 = -u_1$, $u_3 = -u_2$ and it generalizes the dn -periodic wave (1.4), whereas the other solution corresponds to $b = -a$, $\alpha = 0$, $\beta \neq 0$ and it generalizes the cn -periodic wave (1.5). As is shown in Chen and Pelinovsky (2018), the algebraic method produces only two pairs of eigenvalues $\pm\lambda_1, \pm\lambda_2$ with the periodic eigenfunctions of the Lax equations (1.2)–(1.3), where

$$\lambda_1 = \frac{1}{2}(u_1 + u_2), \quad \lambda_2 = \frac{1}{2}(u_1 - u_2)$$

for the periodic solution (1.14) and

$$\lambda_1 = \frac{1}{2}(a + i\beta), \quad \lambda_2 = \frac{1}{2}(a - i\beta)$$

for the periodic solution (1.16).

$g = 2$ If $\mu_3 = 0$, then $u(x, t) = u(x - ct)$ satisfies the third-order equation

$$\frac{d^3u}{dx^3} + 6u^2 \frac{du}{dx} - c \frac{du}{dx} = 0,$$

which is solved by two families of the solutions (1.14) and (1.16) in the general case $e \neq 0$. As is shown here, the algebraic method produces only three pairs of eigenvalues $\pm\lambda_1, \pm\lambda_2, \pm\lambda_3$ with the periodic eigenfunctions of the Lax equations (1.2)–(1.3).

Based on the summary above, it is natural to conjecture that the solution u to the mKdV equation (1.1) expressed by quasi-periodic Riemann Theta function of genus g is related to exactly $g + 1$ pairs of eigenvalues in the spectral problem (1.2) with the quasi-periodic eigenfunctions of the same periods. Moreover, location of these eigenvalues is related to parameters of the Riemann Theta functions. It is also natural to conjecture that no other eigenvalues λ with the quasi-periodic eigenfunctions exist away from the imaginary axis. To the best of our knowledge, these mathematical questions have not been solved in the literature, in spite of the large amount of publications on the mKdV equation. Solving these problems in future looks an interesting question of fundamental significance with many potential applications.

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