Rogue periodic waves of the modified KdV equation

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Abstract

Rogue periodic waves stand for rogue waves on a periodic background. Two families of travelling periodic waves of the modified Korteweg–de Vries (mKdV) equation in the focusing case are expressed by the Jacobian elliptic functions *dn* and *cn*. By using one-fold and two-fold Darboux transformations of the travelling periodic waves, we construct new explicit solutions for the mKdV equation. Since the *dn*-periodic wave is modulationally stable with respect to long-wave perturbations, the new solution constructed from the *dn*-periodic wave is a nonlinear superposition of an algebraically decaying soliton and the *dn*-periodic wave. On the other hand, since the *cn*-periodic wave is modulationally unstable with respect to long-wave perturbations, the new solution constructed from the *cn*-periodic wave is a rogue wave on the *cn*-periodic background, which generalizes the classical rogue wave (the so-called Peregrine's breather) of the nonlinear Schrödinger equation. We compute the magnification factor for the rogue *cn*-periodic wave of the mKdV equation and show that it remains constant for all amplitudes. As a by-product of our work, we find explicit expressions for the periodic eigenfunctions of the spectral problem associated with the *dn* and *cn* periodic waves of the mKdV equation.

(Some figures may appear in colour only in the online journal)

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1. Introduction

The simplest models for nonlinear waves in fluids such as the nonlinear Schrödinger equation (NLS), the Korteweg–de Vries equation (KdV), and the modified Korteweg–de Vries equation (mKdV) have many things in common. First, they appear to be integrable by using the inverse scattering transform method for the same AKNS (Ablowitz–Kaup–Newell–Segur) spectral problem [1]. Second, there exist asymptotic transformations of one nonlinear evolution equation to another nonlinear evolution equation, e.g. from defocusing NLS to KdV and from KdV and focusing mKdV to the defocusing and focusing NLS respectively [37].

The modulation instability of the constant wave background in the focusing NLS equation has been a paramount concept in modern nonlinear physics [38]. More recently, the spectral instability of the periodic waves expressed by the Jacobian elliptic functions *dn* and *cn* has been investigated in the focusing NLS [18] (see also [23, 25]). Regarding periodic waves in the focusing mKdV equation, it was found that the *dn*-periodic waves are modulationally stable with respect to the long-wave perturbations, whereas the *cn*-periodic waves are modulationally unstable [7, 8] (see also [17]).

The outcome of the modulation instability in the focusing NLS equation is the emergence of localized spatially temporal patterns on the background of the unstable periodic or quasi-periodic waves (see review in [9]). Such spatially temporal patterns are known under the generic name of *rogue waves* [26].

In the simplest setting of the constant wave background, the rogue waves are expressed as rational solutions of the NLS equation. Explicit expressions for such rational solutions have been obtained by using available algebraic constructions such as applications of the multi-fold Darboux transformations [2, 19, 29]. For example, if the focusing NLS equation is set in the form

$$i\psi_t + \psi_{xx} + 2(|\psi|^2 - 1)\psi = 0, \tag{1.1}$$

then the *classical rogue wave* up to the translations in (x, t) is given by

$$\psi(x, t) = 1 - \frac{4(1 + 4it)}{1 + 4x^2 + 16t^2}. \tag{1.2}$$

As $|t| + |x| \to \infty$, the rogue wave (1.2) approaches the constant wave background $\psi_0(x, t) = 1$. On the other hand, at $(x, t) = (0, 0)$, the rogue wave reaches a maximum at $|\psi(0, 0)| = 3$, from which we define the *magnification factor* of the constant wave background to be $M_0 = 3$. The rogue wave (1.2) was derived by Peregrine [32] as an outcome of the modulation instability of the constant wave background, and is sometimes referred to as Peregrine's breather.

Rogue waves over nonconstant backgrounds (e.g. periodic waves or two-phase solutions) were addressed only recently in the context of the focusing NLS equation (1.1). Computations of such rogue waves rely on the numerical implementation of the Darboux transformation to the periodic waves [27] or the two-phase solutions [10] of the NLS. Further analytical work to characterize the general two-phase solutions of the NLS can be found in [36] and in [5, 6].

The purpose of this work is to obtain exact solutions for the rogue waves on the periodic background, which we name here as rogue periodic waves. Computations of such rogue waves are developed by an analytical algorithm with a precise characterization of the periodic and nonperiodic eigenfunctions of the AKNS spectral problem at the periodic wave. Although our computations are reported in the context of the focusing mKdV equation, the algorithm can be applied to other nonlinear evolution equations associated with the AKNS spectral problem such as the focusing NLS.

Hence we consider the focusing mKdV equation written in the normalized form

$$u_t + 6u^2 u_x + u_{xxx} = 0. \tag{1.3}$$

Some particular rational and trigonometric solutions of the mKdV were recently constructed in [15] and discussed in connection with the rogue waves of the NLS. In comparison with [15], the novelty of our work is to obtain the rogue periodic waves expressed by the Jacobian elliptic functions and to investigate how these rogue periodic waves generalize the classical rogue wave in the small-amplitude limit (1.2). In particular, we shall compute explicitly the magnification factor for the rogue periodic waves that depends on the elliptic modulus of the Jacobian elliptic functions.

There are two particular travelling periodic wave solutions of the mKdV. One solution is strictly positive and is given by the *dn* elliptic function. The other solution is sign-indefinite and is given by the *cn* elliptic function. Up to the translations in (x, t) as well as the scaling transformation, a positive solution is given by

$$u_{dn}(x, t) = dn(x - ct; k), \quad c = c_{dn}(k) := 2 - k^2, \tag{1.4}$$

whereas the sign-indefinite solution is given by

$$u_{cn}(x, t) = k cn(x - ct; k), \quad c = c_{cn}(k) := 2k^2 - 1. \tag{1.5}$$

In both cases, $k \in (0, 1)$ is the elliptic modulus, which defines two different asymptotic limits. As $k \to 0$, we obtain the following Stokes expansions of the two periodic waves:

$$u_{dn}(x, t) \sim 1 - \frac{1}{2}k^2 \sin^2(x - 2t) \tag{1.6}$$

and

$$u_{cn}(x, t) \sim k \cos(x + t). \tag{1.7}$$

As is well known [22, 30], the mKdV equation can be reduced asymptotically to the NLS equation in the small-amplitude limit. The *cn*-periodic wave of the mKdV is reduced asymptotically at the order $\mathcal{O}(k)$ to the constant wave background ψ_0 of the NLS equation (1.1), which is modulationally unstable with respect to the long-wave perturbations. Hence, the *cn*-periodic wave for the mKdV generalizes the constant wave background of the NLS and inherits the modulation instability with respect to long-wave perturbations. The *dn*-periodic wave has a nonzero mean value, which is large enough to make the *dn*-periodic wave modulationally stable [22].

In the limit $k \to 1$, both Jacobian elliptic functions (1.4) and (1.5) converge to the normalized mKdV soliton

$$u_{dn}(x, t), u_{cn}(x, t) \to u_{\text{soliton}}(x, t) = \text{sech}(x - t). \tag{1.8}$$

Very recently, the rogue waves of the mKdV built from a superposition of slowly interacting nearly identical solitons were considered numerically [33] and analytically [34]. It was found in these studies that the magnification factor of the rogue waves built from N nearly identical solitons is exactly N.

In our work, we construct new solutions to the mKdV equation (1.3) from the *dn* and *cn* Jacobian elliptic functions. For a given travelling periodic wave $u_{\text{per}}(x, t) = U(x - ct)$ with the period L, we say that the new solution u is a rogue periodic wave if

$$\inf_{x_0 \in [0, L]} \sup_{x \in \mathbb{R}} |u(x, t) - U(x - ct - x_0)| \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{1.9}$$

This definition corresponds to the common understanding of rogue waves as waves that appear from nowhere and disappear without a trace as time evolves [3].

We construct new solutions to the mKdV equation (1.3) by means of the following algorithm. First, by using the algebraic technique based on the nonlinearization of the Lax pair [11], we obtain explicit expressions for the eigenvalues λ with $\mathrm{Re}(\lambda) > 0$ and the associated periodic eigenfunctions in the AKNS spectral problem associated with the Jacobian elliptic functions. These eigenvalues correspond to the branch points of the continuous bands, when the AKNS spectral problem with the periodic potentials is considered on the real line with the help of the Floquet–Bloch transform [9]. For each periodic eigenfunction, we construct the second, linearly independent solution of the AKNS spectral problem, which is not periodic but linearly growing in (x, t). The latter solution is expressed in terms of integrals of the Jacobian elliptic functions and is hence not explicit. Finally, by using the one-fold and two-fold Darboux transformations [24] with nonperiodic solutions of the AKNS spectral problem, we obtain new solutions to the mKdV equation (1.3). Although the resulting solutions are not explicit, we prove that the new solution constructed from the *cn*-periodic wave approaches the *cn*-periodic wave in the sense of definition (1.9), hence it is a rogue *cn*-periodic wave. The magnification factor is computed in the explicit form from the maximum of the new solution at the origin $(x, t) = (0, 0)$. Regarding the new solution constructed from the *dn*-periodic wave, it approaches the *dn*-periodic wave almost everywhere on the (x, t) plane, but excluding a particular direction $x = c_* t$ with some $c_* > c$; therefore, it is not a rogue wave in the sense of definition (1.9).

Figure 1 shows a new solution obtained from the *dn*-periodic wave for $k = 0.5$ (left) and $k = 0.99$ (right). The new solution describes a nonlinear superposition of the algebraically decaying soliton of the mKdV [31] and the *dn*-periodic wave. The maximal amplitude at the origin is brought by the algebraic soliton from infinity. Hence it is not a rogue wave in the sense of the definition (1.9). This outcome of our algorithm is related to the fact that the *dn*-periodic wave in the mKdV is modulationally stable with respect to the long-wave perturbations [8]. Indeed, it is argued in [4] with several examples involving the constant wave background that the rogue wave solutions exist only in the parameter regions where the constant wave background is modulationally unstable.

Figure 2 shows a new solution obtained from the *cn*-periodic wave for $k = 0.5$ (left) and $k = 0.99$ (right). This solution is a rogue wave on the periodic background in the sense of definition (1.9). The existence of such a rogue periodic wave is related to the fact that the *cn*-periodic wave in the mKdV is modulationally unstable with respect to the long-wave perturbations [8].

The magnification factors of the new solutions can be computed in the explicit form:

$$M_{dn}(k) = 2 + \sqrt{1 - k^2}, \quad M_{cn}(k) = 3, \quad k \in [0, 1]. \tag{1.10}$$

It is remarkable that the magnification factor $M_{cn}(k) = 3$ is independent of the wave amplitude in agreement with $M_0 = 3$ for the classical rogue wave (1.2). At the same time $M_{dn}(k) \in [2, 3]$ and $M_{dn}(k) \to 3$ as $k \to 0$ thanks to the limit $\lim_{k \to 0} u_{dn}(x, t) = 1$, which gives the same potential to the AKNS spectral problem as the constant wave background $\psi_0(x, t) = 1$ of the NLS equation (1.1).

In the soliton limit (1.8), $M_{dn}(k) \to 2$ as $k \to 1$ in agreement with the recent results in [33, 34]. Indeed, the new solution obtained from the *dn*-periodic wave degenerates as $k \to 1$ to the two-soliton solutions constructed of two nearly identical solitons. Such solutions are constructed by the one-fold Darboux transformation from the one-soliton solutions, when the eigenfunction of the AKNS spectral problem is nondecaying (exponentially growing) [28].
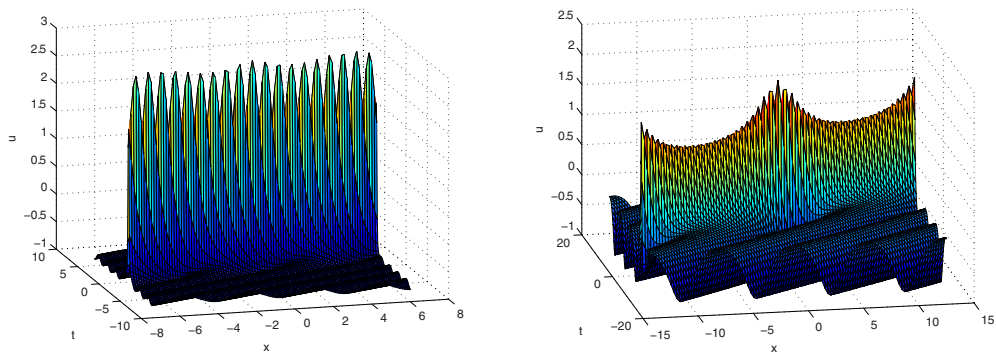
Figure 1. Nonlinear superposition of an algebraically decaying soliton and the *dn*-periodic wave for $k = 0.5$ (left) and $k = 0.99$ (right).
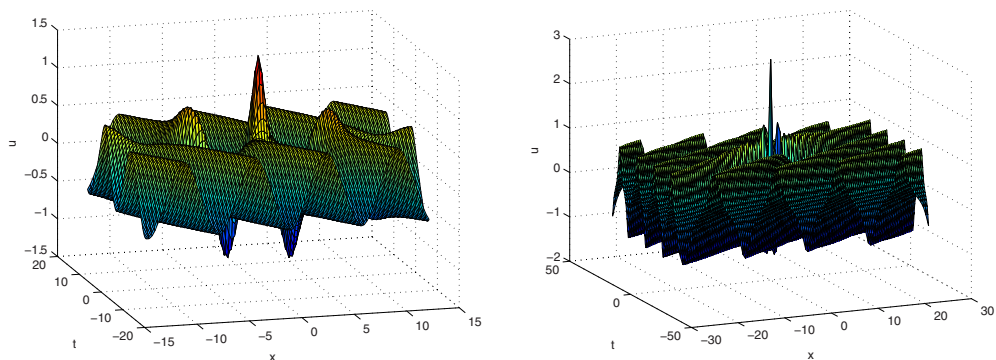

Figure 2. The rogue *cn*-periodic wave for $k = 0.5$ (left) and $k = 0.99$ (right).

Therefore, the magnification factor $M_{dn}(1) = 2$ is explained by the weak interaction between two nearly identical solitons. On the other hand, $M_{cn}(1) = 3$ is explained by the fact that the rogue *cn*-periodic wave is built from the two-fold Darboux transformation; hence it degenerates as $k \to 1$ to the three-soliton solutions constructed of three nearly identical solitons [34].

The paper is organized as follows. Section 2 gives details of the periodic eigenfunctions of the AKNS spectral problem associated with the *dn* and *cn* Jacobian elliptic functions. The nonperiodic solutions of the AKNS spectral problem are computed in section 3. Section 4 presents the general *N*-fold Darboux transformation for the mKdV equation and the explicit formulas for the one-fold and two-fold Darboux transformations. The new solutions are constructed in sections 5 and 6 from the *dn*-periodic and *cn*-periodic waves respectively. The appendix gives a proof of the *N*-fold Darboux transformations in the explicit form.

2. Periodic eigenfunctions of the AKNS spectral problem

The mKdV equation (1.3) is obtained as a compatibility condition of the following Lax pair of two linear equations for the vector $\varphi = (\varphi_1, \varphi_2)^t$:

$$\varphi_x = U(\lambda, u)\varphi, \quad U(\lambda, u) = \begin{pmatrix} \lambda & u \\ -u & -\lambda \end{pmatrix}, \tag{2.1}$$

and

$$\varphi_t = V(\lambda, u)\varphi, \quad V(\lambda, u) = \begin{pmatrix} -4\lambda^3 - 2\lambda u^2 & -4\lambda^2 u - 2\lambda u_x - 2u^3 - u_{xx} \\ 4\lambda^2 u - 2\lambda u_x + 2u^3 + u_{xx} & 4\lambda^3 + 2\lambda u^2 \end{pmatrix}. \tag{2.2}$$

The first linear equation (2.1) is referred to as the AKNS spectral problem, as it defines the spectral parameter λ for a given potential $u(x, t)$ at a frozen time t, e.g. at $t = 0$. By using the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we can rewrite $U(\lambda, u)$ and $V(\lambda, u)$ in (2.1) and (2.2) in the form

$$U(\lambda, u) = \lambda\sigma_3 + u\sigma_3\sigma_1, \tag{2.3}$$

$$V(\lambda, u) = -(4\lambda^3 + 2\lambda u^2)\sigma_3 - 4\lambda^2 u\sigma_3\sigma_1 - 2\lambda u_x\sigma_1 - (2u^3 + u_{xx})\sigma_3\sigma_1. \tag{2.4}$$

If u is either *dn* or *cn* Jacobian elliptic functions (1.4) and (1.5), the potentials are L-periodic in x with the period $L = 2K(k)$ for *dn*-functions and $L = 4K(k)$ for *cn*-functions, where $K(k)$ is the complete elliptic integral. If the AKNS spectral problem (2.1) is considered in the space of L-periodic functions, then the admissible set for the spectral parameter λ is discrete, as the AKNS spectral problem has a purely point spectrum.

In the case of periodic or quasi-periodic potentials u, the algebraic technique based on the nonlinearization of the Lax pair [11] (see also applications in [12–14, 20]) can be used to obtain explicit solutions for the eigenfunctions of the AKNS spectral problem related to the particular eigenvalues λ with $\text{Re}(\lambda) > 0$. Below we simplify the general method in order to obtain particular solutions of the AKNS spectral problem for the periodic waves in the focusing mKdV equation (1.3). The following two propositions represent the explicit expressions for eigenvalues and periodic eigenfunctions of systems (2.1) and (2.2) related to the travelling periodic wave solution of the mKdV.

Proposition 1. *Let u be a travelling wave solution of the mKdV equation (1.3) satisfying*

$$\frac{d^2 u}{dx^2} + 2u^3 = cu, \quad \left(\frac{du}{dx}\right)^2 + u^4 = cu^2 + d, \tag{2.5}$$

where c and d are real constants parameterized by

$$c = 4\lambda_1^2 + 2E_0, \quad d = -E_0^2 \tag{2.6}$$

with possibly complex λ_1 and E_0. Then, there exists a solution $\varphi = (\varphi_1, \varphi_2)^t$ to the AKNS spectral problem (2.1) with $\lambda = \lambda_1$ such that

$$\varphi_1^2 + \varphi_2^2 = u, \quad \varphi_1^2 - \varphi_2^2 = \frac{1}{2\lambda_1}\frac{du}{dx}, \quad 4\lambda_1\varphi_1\varphi_2 = E_0 - u^2. \tag{2.7}$$

In particular, if u is periodic in x, then φ is periodic in x.

Proof. Following [11], we set $u = \varphi_1^2 + \varphi_2^2$ and consider a nonlinearization of the AKNS spectral problem (2.1) given by the Hamiltonian system

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} = \frac{\partial H}{\partial \varphi_2}, \quad \frac{\mathrm{d}\varphi_2}{\mathrm{d}x} = -\frac{\partial H}{\partial \varphi_1}, \tag{2.8}$$

which is related to the Hamiltonian function

$$H(\varphi_1, \varphi_2) = \frac{1}{4}(\varphi_1^2 + \varphi_2^2)^2 + \lambda_1 \varphi_1 \varphi_2 = \frac{1}{4}E_0, \tag{2.9}$$

where E_0 is constant in x. It follows from (2.9) that $4\lambda_1 \varphi_1 \varphi_2 = E_0 - u^2$. Also note that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2\left(\varphi_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + \varphi_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\right) = 2\lambda_1(\varphi_1^2 - \varphi_2^2),$$

so that all three equations in (2.7) are satisfied by the construction.

Let us introduce

$$Q(\lambda) = \begin{pmatrix} \lambda & \varphi_1^2 + \varphi_2^2 \\ -\varphi_1^2 - \varphi_2^2 & -\lambda \end{pmatrix}, \quad W(\lambda) = \begin{pmatrix} W_{11}(\lambda) & W_{12}(\lambda) \\ W_{12}(-\lambda) & -W_{11}(-\lambda) \end{pmatrix}, \tag{2.10}$$

with

$$W_{11}(\lambda) = 1 - \frac{\varphi_1 \varphi_2}{\lambda - \lambda_1} + \frac{\varphi_1 \varphi_2}{\lambda + \lambda_1} = 1 - \frac{E_0 - u^2}{2(\lambda^2 - \lambda_1^2)}, \tag{2.11}$$

$$W_{12}(\lambda) = \frac{\varphi_1^2}{\lambda - \lambda_1} + \frac{\varphi_2^2}{\lambda + \lambda_1} = \frac{2\lambda u + u_x}{2(\lambda^2 - \lambda_1^2)}, \tag{2.12}$$

where the constraints (2.7) have been used. One can check directly that the Lax equation

$$\frac{\mathrm{d}}{\mathrm{d}x} W(\lambda) = Q(\lambda)W(\lambda) - W(\lambda)Q(\lambda), \tag{2.13}$$

is satisfied for every $\lambda \in \mathbb{C}$ if and only if (φ_1, φ_2) satisfies the Hamiltonian system (2.8) with (2.9). In particular, the $(1,2)$-entry in the above relations yields the equation

$$\frac{\mathrm{d}}{\mathrm{d}x} W_{12}(\lambda) = 2\lambda W_{12}(\lambda) - 2(\varphi_1^2 + \varphi_2^2)W_{11}(\lambda). \tag{2.14}$$

Substituting (2.11) and (2.12) into (2.14) yields the differential equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + 2u^3 = (4\lambda_1^2 + 2E_0)u.$$

This equation yields $c = 4\lambda_1^2 + 2E_0$ in comparison with the first equation in (2.5). It follows from (2.10) with the first representations in (2.11) and (2.12) that

$$\begin{aligned} \det[W(\lambda)] &= -[W_{11}(\lambda)]^2 - W_{12}(\lambda)W_{21}(\lambda) \\ &= -1 + \frac{4\lambda_1 \varphi_1 \varphi_2 + (\varphi_1^2 + \varphi_2^2)^2}{\lambda^2 - \lambda_1^2} \\ &= -1 + \frac{E_0}{\lambda^2 - \lambda_1^2}, \end{aligned}$$

hence $\det[W(\lambda)]$ has only simple poles at $\lambda = \pm\lambda_1$. On the other hand, computing $\det[W(\lambda)]$ with the second representations in (2.11) and (2.12) yields double poles at $\lambda = \pm\lambda_1$ unless the following constraint is satisfied:

$$\left(\frac{du}{dx}\right)^2 + u^4 - (4\lambda_1^2 + 2E_0)u^2 + E_0^2 = 0.$$

This equation yields $d = -E_0^2$ in comparison with the second equation in (2.5). Hence, the compatibility condition of the Lax equation (2.13) yields the differential constraints on u given by the two differential equations in (2.5) with parameters c and d related to λ_1 and E_0 by the constraints (2.6). $\qquad\square$

Proposition 2. *Let u, $\varphi = (\varphi_1, \varphi_2)^t$ and λ_1 be the same as in proposition 1. Then $\varphi(x - ct)$ satisfies the linear system (2.2) with $\lambda = \lambda_1$ and $u(x - ct)$.*

Proof. By using (2.5), we rewrite the first equation of system (2.2) with $\lambda = \lambda_1$ as

$$\partial_t\varphi_1 = -(4\lambda_1^3 + 2\lambda_1 u^2)\varphi_1 - (4\lambda_1^2 u + 2\lambda_1 u_x + cu)\varphi_2. \tag{2.15}$$

By using (2.7), we note that

$$(4\lambda_1^2 + 2u^2)\varphi_1 + (4\lambda_1 u + 2u_x)\varphi_2 = (4\lambda_1^2 + 2u^2 + 8\lambda_1\varphi_1\varphi_2)\varphi_1 = (4\lambda_1^2 + 2E_0)\varphi_1.$$

By using (2.6) and the first equation in system (2.1), equation (2.15) becomes

$$\partial_t\varphi_1 = -\lambda_1 c\varphi_1 - cu\varphi_2 = -c\partial_x\varphi_1,$$

hence $\varphi_1(x - ct)$ is a solution of system (2.1) and (2.2) with $\lambda = \lambda_1$ and $u(x - ct)$. Similar computations hold for φ_2 by symmetry from the second equations in systems (2.1) and (2.2). $\qquad\square$

For the *dn* Jacobian elliptic functions (1.4), we have $c = 2 - k^2$ and $d = k^2 - 1 \leqslant 0$. Since $u(x) > 0$ for every $x \in \mathbb{R}$, the periodic eigenfunction $\varphi = (\varphi_1, \varphi_2)^t$ in proposition 1 is real, with parameters $E_0 = \pm\sqrt{1 - k^2}$ and

$$\lambda_1^2 = \frac{1}{4}\left[2 - k^2 \mp 2\sqrt{1 - k^2}\right]. \tag{2.16}$$

Taking the positive square root of (2.16), we obtain two particular real points

$$\lambda_\pm(k) := \frac{1}{2}\left(1 \pm \sqrt{1 - k^2}\right), \tag{2.17}$$

such that $0 < \lambda_-(k) < \lambda_+(k) < 1$ for every $k \in (0, 1)$. As $k \to 0$, we have $\lambda_-(k) \to 0$ and $\lambda_+(k) \to 1$, whereas as $k \to 1$, we have $\lambda_-(k), \lambda_+(k) \to 1/2$.

For the *cn* Jacobian elliptic functions (1.5), we have $c = 2k^2 - 1$ and $d = k^2(1 - k^2) \geqslant 0$. Since $u(x)$ is sign-indefinite, the periodic eigenfunction $\varphi = (\varphi_1, \varphi_2)^t$ in proposition 1 is complex-valued with parameters $E_0 = \pm ik\sqrt{1 - k^2}$ and

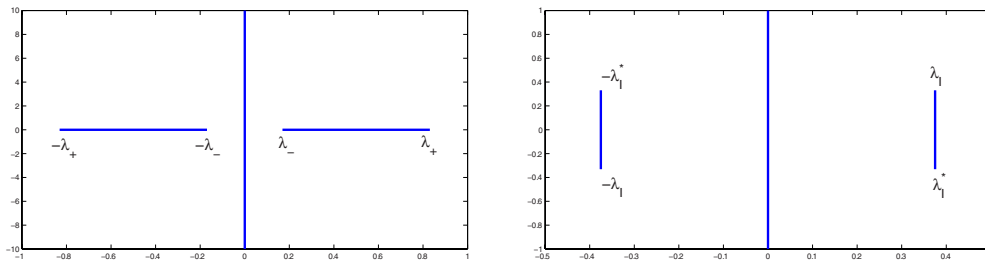$$\lambda_1^2 = \frac{1}{4}\left[2k^2 - 1 \mp 2ik\sqrt{1 - k^2}\right]. \tag{2.18}$$

Figure 3. The spectral plane for the *dn*-periodic (left) and *cn*-periodic (right) waves with $k = 0.75$.

Defining the square root of (2.18) in the first quadrant of the complex plane, we obtain

$$\lambda_I(k) := \frac{1}{2}\left(k + i\sqrt{1-k^2}\right). \tag{2.19}$$

As $k \to 0$, we have $\lambda_I(k) \to i/2$, whereas as $k \to 1$, we have $\lambda_I(k) \to 1/2$.

Figure 3 shows the spectral plane of λ with the schematic representation of the Floquet–Bloch spectrum for the *dn*-periodic wave with $k = 0.75$ (left) and the *cn*-periodic wave with $k = 0.75$ (right). The branch points $\lambda_\pm(k)$ and $\lambda_I(k)$ obtained in (2.17) and (2.19) are marked explicitly as the end points of the Floquet–Bloch spectral bands away from the imaginary axis.

3. Nonperiodic solutions of the AKNS spectral problem

Here we construct the second linearly independent solution to the AKNS spectral problem (2.1) with $\lambda = \lambda_1$ and extend it to satisfy the linear system (2.2). The second solution is no longer periodic in variables (x, t). The following two propositions represent the corresponding solutions.

Proposition 3. *Let u, λ_1, E_0 and $\varphi = (\varphi_1, \varphi_2)^t$ be the same as in proposition 1. Assume that $u(x)^2 - E_0 \neq 0$ for every x. The second linearly independent solution of the AKNS spectral problem (2.1) with $\lambda = \lambda_1$ is given by $\psi = (\psi_1, \psi_2)^t$, where*

$$\psi_1 = \frac{\theta - 1}{\varphi_2}, \quad \psi_2 = \frac{\theta + 1}{\varphi_1}, \tag{3.1}$$

and

$$\theta(x) = -4\lambda_1(u(x)^2 - E_0)\int_0^x \frac{u(y)^2}{(u(y)^2 - E_0)^2}\mathrm{d}y. \tag{3.2}$$

In particular, if u is periodic in x, then θ grows linearly in x as $|x| \to \infty$, so that ψ_1 and ψ_2 are not periodic in x.

Proof. Since the AKNS spectral problem (2.1) is related to the traceless matrix, the Wronskian of the two linearly independent solutions $\varphi = (\varphi_1, \varphi_2)^t$ and $\psi = (\psi_1, \psi_2)^t$ is independent of x. Normalizing it by two, we write the relation

$$\varphi_1\psi_2 - \varphi_2\psi_1 = 2,$$

from which the representation (3.1) follows with arbitrary θ. If $u(x)^2 - E_0 \neq 0$ for every x, then $\varphi_1(x) \neq 0$ and $\varphi_2(x) \neq 0$ for every x. Substituting (3.1) into (2.1), we obtain the follow-

ing scalar linear differential equation for θ:

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = u\theta\frac{\varphi_2^2 - \varphi_1^2}{\varphi_1\varphi_2} + u\frac{\varphi_1^2 + \varphi_2^2}{\varphi_1\varphi_2}.$$

By using the relations (2.7), we rewrite it in the equivalent forms:

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \theta\frac{2uu'}{u^2 - E_0} - \frac{4\lambda_1 u^2}{u^2 - E_0} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\theta}{u^2 - E_0}\right] = -\frac{4\lambda_1 u^2}{(u^2 - E_0)^2}.$$

Integrating the last equation with the boundary condition $\theta(0) = 0$, we obtain (3.2). □

Proposition 4. *Let u, λ_1, E_0, $\varphi = (\varphi_1, \varphi_2)^t$ and $\psi = (\psi_1, \psi_2)^t$ be the same as in proposition 3. Then, $\psi = (\psi_1, \psi_2)^t$ expressed by (3.1) satisfies the linear system (2.2) with $\lambda = \lambda_1$ and $u(x - ct)$ if θ is expressed by*

$$\theta(x, t) = -4\lambda_1(u(x - ct)^2 - E_0)\left[\int_0^{x-ct} \frac{u(y)^2}{(u(y)^2 - E_0)^2}\mathrm{d}y - t\right]. \tag{3.3}$$

Proof. By using (2.5), we rewrite the first equation of system (2.2) with $\lambda = \lambda_1$ as

$$\partial_t\psi_1 = -(4\lambda_1^3 + 2\lambda_1 u^2)\psi_1 - (4\lambda_1^2 u + 2\lambda_1 u_x + cu)\psi_2. \tag{3.4}$$

By using (2.7) and (3.1), and expressing $\partial_t\varphi_2$ from the second equation of system (2.2), we obtain from (3.4):

$$\begin{aligned}
\partial_t\theta &= \frac{(4\lambda_1^2 u - 2\lambda_1 u_x + cu)\varphi_1(\theta - 1)}{\varphi_2} - \frac{(4\lambda_1^2 u + 2\lambda_1 u_x + cu)\varphi_2(\theta + 1)}{\varphi_1} \\
&= -16\lambda_1^2\varphi_1\varphi_2 - \frac{cu}{\varphi_1\varphi_2}\left[\theta(\varphi_2^2 - \varphi_1^2) + \varphi_1^2 + \varphi_2^2\right] \\
&= 4\lambda_1(u^2 - E_0) - c\partial_x\theta.
\end{aligned}$$

Let us represent $\theta = -4\lambda_1(u^2 - E_0)\chi$ so that χ satisfies

$$\partial_t\chi = -c\partial_x\chi - 1.$$

Hence $\chi(x, t) = -t + f(x - ct)$, where f is obtained from (3.2) in the form

$$f(x) = \int_0^x \frac{u(y)^2}{(u(y)^2 - E_0)^2}\mathrm{d}y$$

to yield the representation (3.3). Similar computations hold for ψ_2 by symmetry from the second equations in systems (2.1) and (2.2). □

Note that a more general solution for $\psi = (\psi_1, \psi_2)^t$ is defined arbitrary up to an addition to the first solution $\varphi = (\varphi_1, \varphi_2)^t$. However, this addition is equivalent to the arbitrary choice of the lower limit in the integral (3.3), which is then equivalent to the translation in time t. Thus, the second linearly independent solution in the form (3.1) and (3.3) is unique up to the translation in x and t.

4. One-fold, two-fold and *N*-fold Darboux transformations

Here we give the explicit formulas for the one-fold and two-fold Darboux transformations for the focusing mKdV equation (1.3), as well as the general formula for the *N*-fold Darboux transformation. Although the formal derivation of the *N*-fold Darboux transformation can be found in several sources, e.g. in book [24] or the original papers [21, 35], we find it useful to derive the explicit transformation formulas by using purely algebraic calculations.

By definition, we say that $T(\lambda)$ is a Darboux transformation if

$$\widetilde{\varphi} = T(\lambda)\varphi, \tag{4.1}$$

where φ satisfies (2.1) and (2.2) for a particular potential u and $\widetilde{\varphi}$ satisfies (2.1) and (2.2) for a new potential $\widetilde{u}$, which is related to u. The transformation formulas between φ and $\widetilde{\varphi}$ follow from the Darboux equations

$$\partial_x T(\lambda) + T(\lambda)U(\lambda, u) = U(\lambda, \widetilde{u})T(\lambda) \tag{4.2}$$

and

$$\partial_t T(\lambda) + T(\lambda)V(\lambda, u) = V(\lambda, \widetilde{u})T(\lambda). \tag{4.3}$$

In many derivations, e.g. in [21, 24, 35], the *N*-fold Darboux transformation is deduced formally from a linear system of equations imposed on the coefficients of the polynomial representation of $T(\lambda)$ without checking all the constraints arising from the Darboux equations (4.2) and (4.3). In order to avoid such formal computations, in the appendix we give a rigorous derivation of the *N*-fold Darboux transformation in the explicit form and show how the Darboux equations (4.2) and (4.3) are satisfied. Our derivation relies on a particular implementation of the dressing method [39, 40], which was recently reviewed in the context of the cubic NLS equation in [16].

The general *N*-fold Darboux transformation is given by the following theorem.

Theorem 1. *Let u be a smooth solution of the mKdV equation (1.3). Let $\varphi^{(k)} = (p_k, q_k)^t$, $1 \leqslant k \leqslant N$ be a particular smooth nonzero solution of system (2.1) and (2.2) with fixed $\lambda = \lambda_k \in \mathbb{C} \backslash \{0\}$ and potential u. Assume that $\lambda_k \neq \pm\lambda_j$ for every $k \neq j$. Let $\{\widetilde{\varphi}^{(k)}\}_{1 \leqslant k \leqslant N}$ be a solution of the linear algebraic system*

$$\sigma_3 \sigma_1 \varphi^{(j)} = \sum_{k=1}^{N} \frac{\langle \varphi^{(j)}, \varphi^{(k)} \rangle}{\lambda_j + \lambda_k} \widetilde{\varphi}^{(k)}, \quad 1 \leqslant j \leqslant N, \tag{4.4}$$

where $\langle \varphi^{(j)}, \varphi^{(k)} \rangle := p_j p_k + q_j q_k$ is the inner vector product. Assume that the linear system (4.4) has a unique solution. Then, $\widetilde{\varphi}^{(k)} = (\widetilde{p}_k, \widetilde{q}_k)^t$, $1 \leqslant k \leqslant N$ is a particular solution of system (2.1) and (2.2) with $\lambda = \lambda_k$ and the new potential $\widetilde{u}$ given by

$$\widetilde{u} = u + 2 \sum_{j=1}^{N} \widetilde{p}_j p_j = u - 2 \sum_{j=1}^{N} \widetilde{q}_j q_j. \tag{4.5}$$

Consequently, $\widetilde{u}$ is a new solution of the mKdV equation (1.3).

The proof of theorem 1 is given in the appendix. The following two propositions represent the one-fold and two-fold Darboux transformation formulas deduced from theorem 1 for $N = 1$ and $N = 2$ respectively.

Proposition 5. *Let u be a smooth solution of the mKdV equation (1.3). Let $\varphi = (p, q)^t$ be a particular smooth nonzero solution of systems (2.1) and (2.2) with fixed $\lambda = \lambda_1 \in \mathbb{C} \backslash \{0\}$. Then,*

$$\widetilde{u} = u + \frac{4\lambda_1 pq}{p^2 + q^2} \tag{4.6}$$

is a new solution of the mKdV equation (1.3).

Proof. Solving the linear system (4.4) for $\widetilde{\varphi} = (\widetilde{p}, \widetilde{q})^t$ yields

$$\widetilde{p} = \frac{2\lambda_1 q}{p^2 + q^2}, \quad \widetilde{q} = \frac{-2\lambda_1 p}{p^2 + q^2}. \tag{4.7}$$

Substituting (4.7) into (4.5) for $N = 1$ results in the transformation formula (4.6).   □

Proposition 6. *Let u be a smooth solution of the mKdV equation (1.3). Let $\varphi^{(k)} = (p_k, q_k)^t$ be a particular smooth nonzero solution of system (2.1) and (2.2) with fixed $\lambda = \lambda_k \in \mathbb{C} \backslash \{0\}$ for k = 1,2 such that $\lambda_1 \neq \pm \lambda_2$. Then,*

$$\tilde{u} = u + \frac{4(\lambda_1^2 - \lambda_2^2)\left[\lambda_1 p_1 q_1(p_2^2 + q_2^2) - \lambda_2 p_2 q_2(p_1^2 + q_1^2)\right]}{(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 2\lambda_1 \lambda_2 \left[4p_1 q_1 p_2 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)\right]} \tag{4.8}$$

is a new solution of the mKdV equation (1.3).

Proof. The linear system (4.4) is generated by the matrix A with the entries

$$A_{jk} = \frac{\langle \varphi^{(j)}, \varphi^{(k)} \rangle}{\lambda_j + \lambda_k}, \quad 1 \leqslant j, k \leqslant N. \tag{4.9}$$

For $N = 2$, we compute the determinant of this matrix as

$$\det(A) = \frac{1}{4\lambda_1 \lambda_2(\lambda_1 + \lambda_2)^2}\left[(\lambda_1 + \lambda_2)^2(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 4\lambda_1 \lambda_2(p_1 p_2 + q_1 q_2)^2\right]$$

$$= \frac{1}{4\lambda_1 \lambda_2(\lambda_1 + \lambda_2)^2}\left[(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2)\right.$$

$$\left. - 2\lambda_1 \lambda_2\left(4p_1 p_2 q_1 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)\right)\right].$$

Solving the linear system (4.4) with Cramer's rule yields the components

$$\widetilde{p}_1 = \frac{(\lambda_1 + \lambda_2)q_1(p_2^2 + q_2^2) - 2\lambda_2 q_2(p_1 p_2 + q_1 q_2)}{2\lambda_2(\lambda_1 + \lambda_2)\det(A)}$$

and

$$\widetilde{p}_2 = \frac{(\lambda_1 + \lambda_2)q_2(p_1^2 + q_1^2) - 2\lambda_1 q_1(p_1 p_2 + q_1 q_2)}{2\lambda_1(\lambda_1 + \lambda_2)\det(A)}.$$

Substituting these formulas into the representation (4.5) with $N = 2$ and reordering the similar terms results in the transformation formula (4.8).   □

5. New solution obtained from the *dn*-periodic wave

Here we apply the one-fold Darboux transformation (4.6) to the Jacobian elliptic function *dn* in (1.4) in order to obtain a new solution to the mKdV equation (1.3). The new solution represents a nonlinear superposition of an algebraically decaying soliton and the *dn*-periodic wave, hence the maximal amplitude is brought to the origin by the algebraic soliton from infinity. The rogue *dn*-periodic wave in the sense of the definition (1.9) does not exist in the mKdV equation (1.3), because the *dn*-periodic wave is modulationally stable.

Let u be the *dn*-periodic wave (1.4), and $\varphi = (\varphi_1, \varphi_2)^t$ be the periodic eigenfunction of the linear systems (2.1) and (2.2) with $\lambda = \lambda_1$ defined by propositions 1 and 2. Since the connection formulas (2.7) are satisfied for every $t \in \mathbb{R}$, substituting $p = \varphi_1$ and $q = \varphi_2$ into the one-fold Darboux transformation (4.6) yields another solution of the mKdV equation in the form

$$\tilde{u} = u + \frac{4\lambda_1 \varphi_1 \varphi_2}{\varphi_1^2 + \varphi_2^2} = \frac{E_0}{u},$$

where $E_0 = \pm\sqrt{1 - k^2}$. However, since

$$dn(x + K(k); k) = \frac{\sqrt{1 - k^2}}{dn(x; k)},$$

the new solution $\tilde{u}$ to the mKdV equation (1.3) is obtained trivially by the spatial translation of the *dn*-periodic wave on the half-period $\frac{1}{2}L = K(k)$. This computation explains why we need to use the second nonperiodic solution ψ instead of the periodic eigenfunction φ.

Let u be the *dn*-periodic wave (1.4), and $\psi = (\psi_1, \psi_2)^t$ be the nonperiodic solution to the linear systems (2.1) and (2.2) with $\lambda = \lambda_1$ defined by propositions 3 and 4. Recall that there exist two choices for λ_1 in (2.17). However, for the choice $\lambda_1 = \lambda_-(k)$, we have $E_0 = \sqrt{1 - k^2}$ and $u(x)^2 - E_0 = 0$ for some values of x in $[-K(k), K(k)]$; therefore, the assumption of proposition 3 is not satisfied. For the choice $\lambda_1 = \lambda_+(k)$, we have $E_0 = -\sqrt{1 - k^2}$ and $u(x)^2 - E_0 > 0$ for every x, therefore, the assumption of proposition 3 is satisfied. Substituting $p = \psi_1$ and $q = \psi_2$ given by (3.1) into the one-fold Darboux transformation (4.6) with $\lambda_1 = \lambda_+(k)$ and $E_0 = -\sqrt{1 - k^2}$ yields another solution of the mKdV equation in the form

$$\tilde{u} = u + \frac{4\lambda_1 \psi_1 \psi_2}{\psi_1^2 + \psi_2^2} = u + \frac{4\lambda_1 \varphi_1 \varphi_2 (\theta^2 - 1)}{(\varphi_1^2 + \varphi_2^2)(1 + \theta^2) - 2(\varphi_1^2 - \varphi_2^2)\theta}.$$

By using the relations (2.7) again, we finally write the new solution in the form

$$u_{dn-\text{alg}} = u_{dn} + \frac{(1 - \theta_{dn}^2)(u_{dn}^2 + \sqrt{1 - k^2})}{(1 + \theta_{dn}^2)u_{dn} - \lambda_1^{-1}\theta_{dn}u'_{dn}} \tag{5.1}$$

where

$$\theta_{dn}(x, t) = -4\lambda_1(u_{dn}(x - ct)^2 + \sqrt{1 - k^2})\left[\int_0^{x-ct} \frac{u_{dn}(y)^2}{(u_{dn}(y)^2 + \sqrt{1 - k^2})^2}dy - t\right]. \tag{5.2}$$

If $k = 0$, then $u_{dn}(x, t) = 1$, $\lambda_1 = 1$, $c = 2$, $\theta_{dn}(x, t) = -2(x - 6t)$ and

$$k = 0: \quad u_{dn-\text{alg}}(x, t) = -1 + \frac{4}{1 + 4(x - 6t)^2}.$$

Although this expression is an analogue of the rogue wave of the NLS on the constant wave background [2, 15], it corresponds to the algebraically decaying soliton of the mKdV [31].

If $k = 1$, then $u_{dn}(x, t) = \text{sech}(x - t)$, $\lambda_1 = \frac{1}{2}$, $c = 1$,

$$\theta_{dn}(x, t) = -(x - 3t)\text{sech}^2(x - t) - \tanh(x - t),$$

and

$$k = 1: \quad u_{dn-\text{alg}}(x, t) = 2\text{sech}(x - t)\frac{1 - (x - 3t)\tanh(x - t)}{1 + (x - 3t)^2\text{sech}^2(x - t)}$$

in agreement with the two-soliton solutions of the mKdV for two nearly identical solitons [33, 34].

Next, we show that for every $k \in [0, 1)$, there exists a particular line $x = c_* t$ with $c_* > c$ such that $\theta_{dn}(x, t)$ given by (5.2) remains bounded as $|x| + |t| \to \infty$. This value of c_* gives the speed of the algebraically decaying soliton propagating on the dn-periodic wave background. For instance, if $k = 0$ then $c_* = 6 > 2 = c$. To show the claim above, we inspect the expression

$$\int_0^{x-ct} \frac{u_{dn}(y)^2}{(u_{dn}(y)^2 + \sqrt{1 - k^2})^2}\,dy - t.$$

Since the integrand is a positive $L = 2K(k)$-periodic function with a positive mean value denoted by $I(k)$, then the expression can be written as

$$I(k)(x - ct) - t + \text{a periodic function of } (x, t).$$

Therefore, $\theta_{dn}(x, t)$ is bounded at $x = c_* t$, where $c_* = c + [I(k)]^{-1} > c$.

Except for the line $x = c_* t$, the function $\theta_{dn}(x, t)$ given by (5.2) grows linearly in x and t as $|x| + |t| \to \infty$ for every $k \in [0, 1)$. Hence the representation (5.1) yields asymptotic behaviour

$$u_{dn-\text{alg}}(x, t) \sim -\frac{\sqrt{1 - k^2}}{dn(x - ct; k)} = -dn(x - ct + K(k); k) = -u_{dn}(x - ct + K(k)).$$

The maximal value of $u_{dn-\text{alg}}(x, t)$ as $|x| + |t| \to \infty$, except for the line $x = c_* t$, coincides with the maximal value of $u_{dn}(x, t) = dn(x - ct; k)$.

For $t = 0$, $u_{dn}(x, 0)$ is even in x and $\theta_{dn}(x, 0)$ is odd in x, hence $u_{dn-\text{alg}}(x, 0)$ is even in x. The maximal value of $u_{dn}(x, 0)$ occurs at $u_{dn}(0, 0) = 1$. Since $u_{dn-\text{alg}}(x, 0)$ is even in x, then $x = 0$ is an extremal point of $u_{dn-\text{alg}}(x, 0)$. Moreover, $\partial_x^2 u_{dn-\text{alg}}(0, 0) < 0$, which follows from the expansions $u_{dn}(x, 0) = 1 - \frac{1}{2}k^2x^2 + \mathcal{O}(x^4)$, $\theta_{dn}(x, 0) = -4\lambda_1(1 + \sqrt{1 - k^2})^{-1}x + \mathcal{O}(x^3)$, and

$$u_{dn-\text{alg}}(x, 0) = 2 + \sqrt{1 - k^2} - \left[8 - 3k^2 + 8\sqrt{1 - k^2} - \frac{1}{2}k^2\sqrt{1 - k^2}\right]x^2 + \mathcal{O}(x^4).$$

Hence $x = 0$ is the point of the maximum of $u_{dn-\text{alg}}(x, 0)$. Defining the magnification number as

$$M_{dn}(k) = \frac{u_{dn-\text{alg}}(0, 0)}{\max\limits_{x \in [-K(k), K(k)]} u_{dn}(x, 0)} = 2 + \sqrt{1 - k^2},$$

we obtain the expression in (1.10). The value $M_{dn}(k)$ corresponds to the amplitude of the algebraically decaying soliton propagating on the background of the dn-periodic wave.

6. New solution obtained from the *cn*-periodic wave

Here we apply the one-fold and two-fold Darboux transformations (4.6) and (4.8) to the Jacobian elliptic function *cn* in (1.5) in order to obtain a new solution to the mKdV equation (1.3). This is a proper rogue wave on the *cn*-periodic background in the sense of the definition (1.9) because the *cn*-periodic wave is modulationally unstable.

Let u be the *cn*-periodic wave (1.5), by $\varphi = (\varphi_1, \varphi_2)^t$ be the periodic solution to the linear systems (2.1) and (2.2) with $\lambda = \lambda_1$ defined by propositions 1 and 2. Without loss of generality, we choose $\lambda_1 = \lambda_I(k)$, where $\lambda_I(k)$ is given by (2.19), so that $E_0 = -ik\sqrt{1-k^2}$. Since the periodic solution φ is complex, the one-fold Darboux transformation (4.6) produces a complex-valued solution to the mKdV; hence we should use the two-fold Darboux transformation (4.8).

By virtue of relations (2.7), substituting $(p_1, q_1) = (\varphi_1, \varphi_2)$ with $\lambda_1 = \lambda_I$ and $(p_2, q_2) = (\overline{\varphi}_1, \overline{\varphi}_2)$ with $\lambda_2 = \overline{\lambda}_I$ into the two-fold Darboux transformation (4.8) yields another solution of the mKdV equation in the form

$$\tilde{u} = u + \frac{4k^2(1-k^2)u}{(2k^2-1)u^2 - u^4 - k^2(1-k^2) - (u')^2} = -u,$$

where the first-order invariant in (2.5) is used in the second identity with $c = 2k^2 - 1$ and $d = k^2(1-k^2)$. Thus, the new solution $\tilde{u}$ in the two-fold transformation (4.8) is trivially related to the previous solution u if the functions (p_1, q_1) and (p_2, q_2) are periodic.

Let us now consider the nonperiodic solution $\psi = (\psi_1, \psi_2)^t$ to the linear systems (2.1) and (2.2) with $\lambda = \lambda_I$. The assumption of proposition 3 is satisfied because $E_0 = -ik\sqrt{1-k^2} \neq 0$ for $k \in (0, 1)$ and $u(x)^2 - E_0 \neq 0$ for every x. Therefore, the nonperiodic solution ψ in propositions 3 and 4 is well defined. Substituting $(p_1, q_1) = (\psi_1, \psi_2)$ with $\lambda_1 = \lambda_I$ and $(p_2, q_2) = (\overline{\psi}_1, \overline{\psi}_2)$ with $\lambda_2 = \overline{\lambda}_I$ into the two-fold Darboux transformation (4.8) yields another solution of the mKdV in the form

$$\tilde{u} = u + \frac{4(\lambda_I^2 - \overline{\lambda}_I^2)\left[\lambda_I\psi_1\psi_2(\overline{\psi}_1^2 + \overline{\psi}_2^2) - \overline{\lambda}_I\overline{\psi}_1\overline{\psi}_2(\psi_1^2 + \psi_2^2)\right]}{(\lambda_I^2 + \overline{\lambda}_I^2)|\psi_1^2 + \psi_2^2|^2 - 2|\lambda_I|^2\left[4|\psi_1|^2|\psi_2|^2 + |\psi_1^2 - \psi_2^2|^2\right]} = u + \frac{F_1}{F_2},$$

where

$$F_1 = 8\text{Im}(\lambda_I^2)\text{Im}\left[\lambda_I\varphi_1\varphi_2(1-\theta^2)[(1+\overline{\theta}^2)(\overline{\varphi}_1^2 + \overline{\varphi}_2^2) - 2\overline{\theta}(\overline{\varphi}_1^2 - \overline{\varphi}_2^2)]\right],$$

$$F_2 = \text{Re}(\lambda_I^2)|(1+\theta^2)(\varphi_1^2 + \varphi_2^2) - 2\theta(\varphi_1^2 - \varphi_2^2)|^2$$
$$- |\lambda_I|^2\left(4|1-\theta^2|^2|\varphi_1|^2|\varphi_2|^2 + |(1+\theta^2)(\varphi_1^2 - \varphi_2^2) - 2\theta(\varphi_1^2 + \varphi_2^2)|^2\right).$$

By using relations (2.7) and (2.19), we finally write the new solution in the form

$$u_{cn-\text{rogue}} = u_{cn} + \frac{G_1}{G_2}, \tag{6.1}$$

where

$$G_1 = 4k\sqrt{1-k^2}\text{Im}\left[(u_{cn}^2 + ik\sqrt{1-k^2})(1-\theta_{cn}^2)[(1+\overline{\theta}_{cn}^2)u_{cn} - \overline{\lambda}_I^{-1}\overline{\theta}_{cn}u_{cn}']\right],$$

$$G_2 = (1-2k^2)|(1+\theta_{cn}^2)u_{cn} - \lambda_I^{-1}\theta_{cn}u_{cn}'|^2$$
$$+ |1-\theta_{cn}^2|^2\left[u_{cn}^4 + k^2(1-k^2)\right] + |(1+\theta_{cn}^2)(2\lambda_I)^{-1}u_{cn}' - 2\theta_{cn}u_{cn}|^2,$$

and

$$\theta_{cn}(x,t) = -4\lambda_I(u_{cn}(x-ct)^2 + ik\sqrt{1-k^2})\left[\int_0^{x-ct}\frac{u_{cn}(y)^2}{(u_{cn}(y)^2+ik\sqrt{1-k^2})^2}\mathrm{d}y - t\right].$$
(6.2)

As $k \to 0$, then $u_{cn}(x,t) \to 0$, $\lambda_I \to i/2$, $\theta_{cn}(x,t) \to 0$ and $u_{cn-\mathrm{rogue}}(x,t) \to 0$. Although the limit is zero, one can derive asymptotic expansions at the order of $\mathcal{O}(k)$, recovering the rogue wave of the NLS equation (1.1), according to the asymptotic transformation of the focusing mKdV to the focusing NLS in the small-amplitude limit [22]. The rogue cn-periodic wave generalizes the rogue wave (1.2) on the constant wave background.

As $k \to 1$, then $u_{cn}(x,t) \to \mathrm{sech}(x-t)$, $\lambda_I \to 1/2$, and it may first seem that the second term in (6.1) vanishes. However, $G_1 = \mathcal{O}(1-k^2)$ and $G_2 = \mathcal{O}(1-k^2)$, hence a nontrivial limit exists to yield a three-soliton solution to the mKdV with three nearly identical solitons [34].

Let us inspect the expression

$$\int_0^{x-ct}\frac{u_{cn}(y)^2}{(u_{cn}(y)^2+ik\sqrt{1-k^2})^2}\mathrm{d}y - t = \int_0^{x-ct}\frac{u_{cn}(y)^2(u_{cn}(y)^2-ik\sqrt{1-k^2})^2}{(u_{cn}(y)^4+k^2(1-k^2))^2}\mathrm{d}y - t.$$

For every $k \in (0,1)$, the imaginary part in the integrand is a negative $L = 4K(k)$-periodic function with a negative mean value. It is only bounded on the line $x = ct$, however, the real part of the last term in the expression grows linearly in t. Therefore, for every $k \in (0,1)$, $|\theta_{cn}(x,t)|$ grows linearly in x and t as $|x| + |t| \to \infty$ everywhere on the (x,t) plane. Hence the representation (6.1) yields the asymptotic behaviour

$$u_{cn-\mathrm{rogue}}(x,t) \sim u_{cn}(x,t) + \frac{4k^2(1-k^2)u_{cn}(x,t)}{(2k^2-1)u_{cn}(x,t)^2 - (\partial_x u_{cn}(x,t))^2 - u_{cn}(x,t)^4 - k^2(1-k^2)}$$
$$= -u_{cn}(x,t),$$

where the first-order invariant in (2.5) is used for the last identity with $c = 2k^2 - 1$ and $d = k^2(1-k^2)$. The maximal value of $u_{cn-\mathrm{rogue}}(x,t)$ as $|x| + |t| \to \infty$ coincides with the maximal value of $u_{cn}(x,t) = kcn(x-ct;k)$.

For $t = 0$, $u_{cn}(x,0)$ is even in x and $\theta_{cn}(x,0)$ is odd in x, hence $u_{cn-\mathrm{rogue}}(x,0)$ is even in x. The maximal value of $u_{cn}(x,0)$ occurs at $u_{cn}(0,0) = k$. Since $u_{cn-\mathrm{rogue}}(x,0)$ is even in x, then $x = 0$ is an extremal point of $u_{cn-\mathrm{rogue}}(x,0)$. Moreover, $\partial_x^2 u_{cn-\mathrm{rogue}}(0,0) < 0$, which follows from the expansions $u_{cn}(x,0) = k - \frac{1}{2}kx^2 + \mathcal{O}(x^4)$, $\theta_{cn}(x,0) = -4\lambda_I(k^2 - ik\sqrt{1-k^2})x + \mathcal{O}(x^3)$, and

$$u_{cn-\mathrm{rogue}}(x,0) = 3k - \left[\frac{3}{2}k + 16k^3\right]x^2 + \mathcal{O}(x^4).$$

Hence $x = 0$ is the point of the maximum of $u_{cn-\mathrm{rogue}}(x,0)$. Defining the magnification number as

$$M_{cn}(k) = \frac{u_{cn-\mathrm{rogue}}(0,0)}{\max\limits_{x\in[-2K(k),2K(k)]}|u_{cn}(x,0)|} = 3,$$

we obtain the expression in (1.10). The magnification factor is independent of the amplitude of the cn-periodic wave.

Acknowledgments

Appendix. Proof of *N*-fold Darboux transformation

Here we prove theorem 1 with explicit algebraic computations. The Darboux transformation matrix $T(\lambda)$ in (4.1) is sought in the following explicit form:

$$T(\lambda) = I + \sum_{k=1}^{N} \frac{1}{\lambda - \lambda_k} T_k, \quad T_k = \widetilde{\varphi}^{(k)} \otimes (\varphi^{(k)})^t \sigma_1 \sigma_3, \tag{A.1}$$

where the sign $\otimes$ denotes the outer vector product and I denotes an identity 2×2 matrix.

We note that $\varphi^{(k)} \in \ker(T_k)$ and $\widetilde{\varphi}^{(k)} \in \mathrm{ran}(T_k)$. It is assumed in theorem 1 that $\varphi^{(k)} = (p_k, q_k)^t$, $1 \leqslant k \leqslant N$ is a particular smooth nonzero solution to system (2.1) and (2.2) with fixed $\lambda = \lambda_k \in \mathbb{C}\backslash\{0\}$ satisfying $\lambda_k \neq \pm\lambda_j$ for every $k \neq j$, whereas $\{\widetilde{\varphi}^{(k)}\}_{1 \leqslant k \leqslant N}$ is a unique solution of the linear algebraic system (4.4). Deeper in the proof, we will be able to show that $\widetilde{\varphi}^{(k)} = (\widetilde{p}_k, \widetilde{q}_k)^t$, $1 \leqslant k \leqslant N$ is a particular solution to system (2.1) and (2.2) with $\lambda = \lambda_k$ and the new potential $\widetilde{u}$ is given by the transformation formula (4.5).

First, let us show that the two lines in the definition (4.5) are identical. Let us define entries of the matrix A by (4.9). Each entry is finite, moreover, $A_{jk} = A_{kj}$. The linear system (4.4) can be split into two parts as follows:

$$\sum_{k=1}^{N} \frac{\langle \varphi^{(j)}, \varphi^{(k)} \rangle}{\lambda_j + \lambda_k} \widetilde{p}_k = q_j, \quad \sum_{k=1}^{N} \frac{\langle \varphi^{(j)}, \varphi^{(k)} \rangle}{\lambda_j + \lambda_k} \widetilde{q}_k = -p_j. \tag{A.2}$$

Thanks to the symmetry of A, we obtain from (A.2):

$$\sum_{j=1}^{N} \widetilde{q}_j q_j = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\langle \varphi^{(j)}, \varphi^{(k)} \rangle}{\lambda_j + \lambda_k} \widetilde{p}_k \widetilde{q}_j = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\langle \varphi^{(j)}, \varphi^{(k)} \rangle}{\lambda_j + \lambda_k} \widetilde{p}_j \widetilde{q}_k = -\sum_{j=1}^{N} \widetilde{p}_j p_j. \tag{A.3}$$

This proves that the two lines in the definition (4.5) are identical. For further use, let us also derive another relation from the system (A.2):

$$\sum_{j=1}^{N} \lambda_j \widetilde{q}_j q_j - \sum_{j=1}^{N} \lambda_j \widetilde{p}_j p_j = \sum_{j=1}^{N} \sum_{k=1}^{N} \langle \varphi^{(j)}, \varphi^{(k)} \rangle \widetilde{p}_k \widetilde{q}_j$$

$$= \left(\sum_{j=1}^{N} \widetilde{p}_j p_j \right) \left(\sum_{k=1}^{N} p_k \widetilde{q}_k \right) + \left(\sum_{j=1}^{N} \widetilde{q}_j q_j \right) \left(\sum_{k=1}^{N} \widetilde{p}_k q_k \right)$$

$$= \frac{1}{2}(\widetilde{u} - u) \left(\sum_{k=1}^{N} p_k \widetilde{q}_k - \sum_{k=1}^{N} \widetilde{p}_k q_k \right). \tag{A.4}$$

Next, we show the validity of the Darboux equation (4.2) under the transformation formula (A.1). Substituting (A.1) into (4.2) yields the following equations at the simple poles:

$$\partial_x T_k + T_k U(\lambda_k, u) = U(\lambda_k, \widetilde{u}) T_k, \quad 1 \leqslant k \leqslant N, \tag{A.5}$$

and the following equation at the constant term

$$\widetilde{u}\sigma_3\sigma_1 = u\sigma_3\sigma_1 + \sum_{k=1}^{N} T_k\sigma_3 - \sigma_3 T_k. \tag{A.6}$$

Equation (A.6) yields (4.5) due to representation (A.1).

Let us show that equation (A.5) is satisfied if $\varphi^{(k)}$ solves (2.1) with $\lambda = \lambda_k$ and u, whereas $\widetilde{\varphi}^{(k)}$ solves (2.1) with $\lambda = \lambda_k$ and $\widetilde{u}$. Recall that $\sigma_1\sigma_3 = -\sigma_3\sigma_1$ and $\sigma_1\sigma_1 = \sigma_3\sigma_3 = I$. Substituting (A.1) on both sides of (A.5) yields

$$\left[\partial_x\widetilde{\varphi}^{(k)}\right] \otimes (\varphi^{(k)})^t\sigma_1\sigma_3 + \widetilde{\varphi}^{(k)} \otimes \left[\partial_x(\varphi^{(k)})^t\right]\sigma_1\sigma_3 + \widetilde{\varphi}^{(k)} \otimes (\varphi^{(k)})^t\sigma_1\sigma_3 U(\lambda_1, u)$$

$$= \left[\partial_x\widetilde{\varphi}^{(k)}\right] \otimes (\varphi^{(k)})^t\sigma_1\sigma_3 + \widetilde{\varphi}^{(k)} \otimes \left[\partial_x(\varphi^{(k)})^t\right]\sigma_1\sigma_3 - \widetilde{\varphi}^{(k)} \otimes (\varphi^{(k)})^t U(\lambda_1, u)^t\sigma_1\sigma_3$$

$$= \left[\partial_x\widetilde{\varphi}^{(k)}\right] \otimes (\varphi^{(k)})^t\sigma_1\sigma_3$$

and

$$U(\lambda_1, \widetilde{u})\widetilde{\varphi}^{(k)} \otimes (\varphi^{(k)})^t\sigma_1\sigma_3 = \left[\partial_x\widetilde{\varphi}^{(k)}\right] \otimes (\varphi^{(k)})^t\sigma_1\sigma_3,$$

hence equation (A.5) is satisfied.

We show now that if $\{\varphi^{(k)}\}_{1\leqslant k\leqslant N}$ solve (2.1) with $\{\lambda_k\}_{1\leqslant k\leqslant N}$ and u and $\{\widetilde{\varphi}^{(k)}\}_{1\leqslant k\leqslant N}$ are obtained from the linear algebraic system (4.4), then $\{\widetilde{\varphi}^{(k)}\}_{1\leqslant k\leqslant N}$ solve (2.1) with $\{\lambda_k\}_{1\leqslant k\leqslant N}$ and $\widetilde{u}$. We note from the linear system (2.1) that

$$\partial_x\langle\varphi^{(j)}, \varphi^{(k)}\rangle = (\lambda_j + \lambda_k)\langle\varphi^{(j)}, \sigma_3\varphi^{(k)}\rangle. \tag{A.7}$$

Differentiating (4.4) in x and substituting (2.3) and (A.7) yields

$$\sum_{k=1}^{N} \frac{\langle\varphi^{(j)}, \varphi^{(k)}\rangle}{\lambda_j + \lambda_k} \left[\partial_x\widetilde{\varphi}^{(k)} - \lambda_k\sigma_3\widetilde{\varphi}^{(k)} - \widetilde{u}\sigma_3\sigma_1\widetilde{\varphi}^{(k)}\right]$$

$$= (\widetilde{u} - u)\varphi^{(j)} - \sum_{k=1}^{N} \left[\langle\varphi^{(j)}, \sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)}, \varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\right] = 0, \tag{A.8}$$

where the last equality is due to the transformation formula (4.5). Thus, if the linear system (4.4) is assumed to admit a unique solution, then $\widetilde{\varphi}^{(k)}$ solves (2.1) with $\lambda = \lambda_k$ and $\widetilde{u}$.

It remains to show the validity of the Darboux equation (4.3) under the transformation formula (A.1). Substituting (A.1) into (4.3) yields the following equations at the simple poles:

$$\partial_t T_k + T_k V(\lambda_k, u) = V(\lambda_k, \widetilde{u}) T_k, \quad 1 \leqslant k \leqslant N, \tag{A.9}$$

the same equation (A.6) at λ^2 and the following two equations at λ^1 and λ^0 respectively:

$$\widetilde{u}^2\sigma_3 + \widetilde{u}_x\sigma_1 + 2\widetilde{u}\sum_{k=1}^{N}\sigma_3\sigma_1 T_k = u^2\sigma_3 + u_x\sigma_1 + 2u\sum_{k=1}^{N}T_k\sigma_3\sigma_1 + 2\sum_{k=1}^{N}\lambda_k(T_k\sigma_3 - \sigma_3 T_k) \tag{A.10}$$

and

$$(2\widetilde{u}^3 + \widetilde{u}_{xx})\sigma_3\sigma_1 + 2\widetilde{u}^2\sum_{k=1}^{N}\sigma_3 T_k + 2\widetilde{u}_x\sum_{k=1}^{N}\sigma_1 T_k + 4\widetilde{u}\sum_{k=1}^{N}\lambda_k\sigma_3\sigma_1 T_k + 4\sum_{k=1}^{N}\lambda_k^2\sigma_3 T_k$$
$$= (2u^3 + u_{xx})\sigma_3\sigma_1 + 2u^2\sum_{k=1}^{N}T_k\sigma_3 + 2u_x\sum_{k=1}^{N}T_k\sigma_1 + 4u\sum_{k=1}^{N}\lambda_k T_k\sigma_3\sigma_1 + 4\sum_{k=1}^{N}\lambda_k^2 T_k\sigma_3.$$
(A.11)

Let us show that equation (A.9) is satisfied if $\varphi^{(k)}$ solves (2.2) with $\lambda = \lambda_k$ and u, whereas $\widetilde{\varphi}^{(k)}$ solves (2.2) with $\lambda = \lambda_k$ and $\widetilde{u}$. Substituting (A.1) into both sides of (A.9) yields

$$\left[\partial_t\widetilde{\varphi}^{(k)}\right]\otimes(\varphi^{(k)})^t\sigma_1\sigma_3 + \widetilde{\varphi}^{(k)}\otimes\left[\partial_t(\varphi^{(k)})^t\right]\sigma_1\sigma_3 + \widetilde{\varphi}^{(k)}\otimes(\varphi^{(k)})^t\sigma_1\sigma_3 V(\lambda_1, u)$$
$$= \left[\partial_t\widetilde{\varphi}^{(k)}\right]\otimes(\varphi^{(k)})^t\sigma_1\sigma_3 + \widetilde{\varphi}^{(k)}\otimes\left[\partial_t(\varphi^{(k)})^t\right]\sigma_1\sigma_3 - \widetilde{\varphi}^{(k)}\otimes(\varphi^{(k)})^t V(\lambda_1, u)^t\sigma_1\sigma_3$$
$$= \left[\partial_t\widetilde{\varphi}^{(k)}\right]\otimes(\varphi^{(k)})^t\sigma_1\sigma_3$$

and

$$V(\lambda_1, \widetilde{u})\widetilde{\varphi}^{(k)}\otimes(\varphi^{(k)})^t\sigma_1\sigma_3 = \left[\partial_t\widetilde{\varphi}^{(k)}\right]\otimes(\varphi^{(k)})^t\sigma_1\sigma_3,$$

hence equation (A.9) is satisfied.

In order to show the validity of equation (A.10), we differentiate (A.6) in x and substitute (A.5) to obtain

$$(\widetilde{u}_x - u_x)\sigma_1 = 2\sum_{k=1}^{N}\lambda_k(T_k\sigma_3 - \sigma_3 T_k) + \widetilde{u}\sum_{k=1}^{N}(\sigma_1 T_k\sigma_3 - \sigma_3\sigma_1 T_k) + u\sum_{k=1}^{N}(\sigma_3 T_k\sigma_1 + T_k\sigma_3\sigma_1).$$
(A.12)

Substituting (A.12) into (A.10) yields a simplified form of the equation:

$$(\widetilde{u}^2 - u^2)\sigma_3 + \widetilde{u}\sum_{k=1}^{N}\sigma_1 T_k\sigma_3 + \sigma_3\sigma_1 T_k + u\sum_{k=1}^{N}\sigma_3 T_k\sigma_1 - T_k\sigma_3\sigma_1 = 0. \quad \text{(A.13)}$$

Further substituting (A.6) into (A.13) yields

$$\sum_{k=1}^{N}(\sigma_1 T_k\sigma_3 + \sigma_3\sigma_1 T_k + T_k\sigma_3\sigma_1 - \sigma_3 T_k\sigma_1) = 0. \quad \text{(A.14)}$$

The validity of equation (A.14) is satisfied thanks again to equation (A.6):

$$(\widetilde{u} - u)\sigma_3 = \sum_{k=1}^{N}(T_k\sigma_3\sigma_1 - \sigma_3 T_k\sigma_1), \quad (u - \widetilde{u})\sigma_3 = \sum_{k=1}^{N}(\sigma_1 T_k\sigma_3 + \sigma_3\sigma_1 T_k). \quad \text{(A.15)}$$

Hence, equation (A.10) is satisfied.

In order to show the validity of equation (A.11), we differentiate (A.12) in x and substitute (A.5) to obtain

$$(\tilde{u}_{xx} - u_{xx})\sigma_3\sigma_1 = 4\sum_{k=1}^{N}\lambda_k^2(T_k\sigma_3 - \sigma_3 T_k) + 2\tilde{u}\sum_{k=1}^{N}\lambda_k(\sigma_1 T_k\sigma_3 - \sigma_3\sigma_1 T_k)$$

$$+ 2u\sum_{k=1}^{N}\lambda_k(\sigma_3 T_k\sigma_1 + T_k\sigma_3\sigma_1) + \tilde{u}_x\sum_{k=1}^{N}(\sigma_3\sigma_1 T_k\sigma_3 - \sigma_1 T_k)$$

$$+ u_x\sum_{k=1}^{N}(T_k\sigma_1 + \sigma_3 T_k\sigma_3\sigma_1) + (\tilde{u}^2 + u^2)\sum_{k=1}^{N}(\sigma_3 T_k - T_k\sigma_3)$$

$$+ 2u\tilde{u}\sum_{k=1}^{N}(\sigma_3\sigma_1 T_k\sigma_1 + \sigma_1 T_k\sigma_3\sigma_1). \tag{A.16}$$

Substituting (A.6) and (A.16) into (A.11) yields a simplified form of the equation:

$$2\tilde{u}\sum_{k=1}^{N}\lambda_k(\sigma_1 T_k\sigma_3 + \sigma_3\sigma_1 T_k) + 2u\sum_{k=1}^{N}\lambda_k(\sigma_3 T_k\sigma_1 - T_k\sigma_3\sigma_1)$$

$$+ \tilde{u}_x\sum_{k=1}^{N}(\sigma_3\sigma_1 T_k\sigma_3 + \sigma_1 T_k) + u_x\sum_{k=1}^{N}(\sigma_3 T_k\sigma_3\sigma_1 - T_k\sigma_1) + (\tilde{u}^2 - u^2)\sum_{k=1}^{N}(T_k\sigma_3 + \sigma_3 T_k)$$

$$+ 2\tilde{u}u\sum_{k=1}^{N}(\sigma_3\sigma_1 T_k\sigma_1 + \sigma_1 T_k\sigma_3\sigma_1 + T_k\sigma_3 - \sigma_3 T_k) = 0. \tag{A.17}$$

The last term on the left-hand side of (A.17) is identically zero thanks to equation (A.14) after multiplication by σ_1 on the right. Multiplication of equation (A.14) by σ_3 on the right allows us to group the terms containing u_x and $\tilde{u}_x$. As a result, we rewrite (A.17) in the equivalent form

$$2\tilde{u}\sum_{k=1}^{N}\lambda_k(\sigma_1 T_k\sigma_3 + \sigma_3\sigma_1 T_k) + 2u\sum_{k=1}^{N}\lambda_k(\sigma_3 T_k\sigma_1 - T_k\sigma_3\sigma_1)$$

$$+ (u_x - \tilde{u}_x)\sum_{k=1}^{N}(\sigma_3 T_k\sigma_3\sigma_1 - T_k\sigma_1) + (\tilde{u}^2 - u^2)\sum_{k=1}^{N}(T_k\sigma_3 + \sigma_3 T_k) = 0. \tag{A.18}$$

Multiplying (A.12) by σ_1 from the left and from the right, we obtain

$$(\tilde{u}_x - u_x)I = 2\sum_{k=1}^{N}\lambda_k(\sigma_1 T_k\sigma_3 + \sigma_3\sigma_1 T_k) + \tilde{u}\sum_{k=1}^{N}(T_k\sigma_3 + \sigma_3 T_k) + u\sum_{k=1}^{N}(\sigma_1\sigma_3 T_k\sigma_1 + \sigma_1 T_k\sigma_3\sigma_1)$$

and

$$(\tilde{u}_x - u_x)I = 2\sum_{k=1}^{N}\lambda_k(T_k\sigma_3\sigma_1 - \sigma_3 T_k\sigma_1) + \tilde{u}\sum_{k=1}^{N}(\sigma_1 T_k\sigma_3\sigma_1 - \sigma_3\sigma_1 T_k\sigma_1) + u\sum_{k=1}^{N}(\sigma_3 T_k + T_k\sigma_3),$$

from which one can rewrite (A.18) in the equivalent form

$$(\tilde{u}_x - u_x)(\tilde{u} - u)I - (\tilde{u}_x - u_x)\sum_{k=1}^{N}(\sigma_3 T_k\sigma_3\sigma_1 - T_k\sigma_1) = 0,$$

which is satisfied thanks to equation (A.15). Hence, equation (A.11) is satisfied.

Finally, we show that if $\{\varphi^{(k)}\}_{1\leqslant k\leqslant N}$ solves (2.2) with $\{\lambda_k\}_{1\leqslant k\leqslant N}$ and u and $\{\widetilde{\varphi}^{(k)}\}_{1\leqslant k\leqslant N}$ are obtained from the linear algebraic system (4.4), then $\{\widetilde{\varphi}^{(k)}\}_{1\leqslant k\leqslant N}$ solves (2.2) with $\{\lambda_k\}_{1\leqslant k\leqslant N}$ and $\widetilde{u}$. We note from the linear system (2.2) that

$$\partial_t\langle\varphi^{(j)},\varphi^{(k)}\rangle = -(\lambda_j+\lambda_k)\left[4(\lambda_j^2-\lambda_j\lambda_k+\lambda_k^2)+2u^2\right]\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle$$
$$+4(\lambda_j^2-\lambda_k^2)u\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle - 2(\lambda_j+\lambda_k)u_x\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle. \tag{A.19}$$

Differentiating (4.4) in t and substituting (2.4) and (A.19) yields

$$\sum_{k=1}^N \frac{\langle\varphi^{(j)},\varphi^{(k)}\rangle}{\lambda_j+\lambda_k}\left[\partial_t\widetilde{\varphi}^{(k)}+(4\lambda_k^3+2\lambda_k\widetilde{u}^2)\sigma_3\widetilde{\varphi}^{(k)}+4\lambda_k^2\widetilde{u}\sigma_3\sigma_1\widetilde{\varphi}^{(k)}+2\lambda_k\widetilde{u}_x\sigma_1\widetilde{\varphi}^{(k)}+(2\widetilde{u}^3+\widetilde{u}_{xx})\sigma_3\sigma_1\widetilde{\varphi}^{(k)}\right]$$

$$=\sum_{k=1}^N\left[(4\lambda_j^2-4\lambda_j\lambda_k+4\lambda_k^2+2u^2)\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle+4(\lambda_k-\lambda_j)u\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle+2u_x\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\right]\widetilde{\varphi}^{(k)}$$

$$+\sum_{k=1}^N\langle\varphi^{(j)},\varphi^{(k)}\rangle\left[(4\lambda_k^2-4\lambda_k\lambda_j+4\lambda_j^2+2\widetilde{u}^2)\sigma_3\widetilde{\varphi}^{(k)}+4(\lambda_k-\lambda_j)\widetilde{u}\sigma_3\sigma_1\widetilde{\varphi}^{(k)}+2\widetilde{u}_x\sigma_1\widetilde{\varphi}^{(k)}\right]$$

$$+2\lambda_j(u^2-\widetilde{u}^2)\sigma_1\varphi^{(j)}+4\lambda_j^2(u-\widetilde{u})\varphi^{(j)}-2\lambda_j(u_x-\widetilde{u}_x)\sigma_3\varphi^{(j)}+(2u^3+u_{xx}-2\widetilde{u}^3-\widetilde{u}_{xx})\varphi^{(j)}. \tag{A.20}$$

The terms proportional to $4\lambda_j^2$ cancel out due to the same relation (A.8). The terms proportional to $2\lambda_j$ cancel out if the following relation is true:

$$(u^2-\widetilde{u}^2)\sigma_1\varphi^{(j)}+(\widetilde{u}_x-u_x)\sigma_3\varphi^{(j)}=2\sum_{k=1}^N\lambda_k\left[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\right]$$

$$+2\widetilde{u}\sum_{k=1}^N\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}+2u\sum_{k=1}^N\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}. \tag{A.21}$$

The other λ_j-independent terms cancel out if the following relation is true:

$$(2\widetilde{u}^3+\widetilde{u}_{xx}-2u^3-u_{xx})\varphi^{(j)}=4\sum_{k=1}^N\lambda_k^2\left[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}+\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\right]$$

$$+2\widetilde{u}^2\sum_{k=1}^N\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}+2u^2\sum_{k=1}^N\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}$$

$$+4\widetilde{u}\sum_{k=1}^N\lambda_k\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}+4u\sum_{k=1}^N\lambda_k\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}$$

$$+2\widetilde{u}_x\sum_{k=1}^N\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}+2u_x\sum_{k=1}^N\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}. \tag{A.22}$$

Provided equations (A.21) and (A.22) are satisfied, the right-hand side of equation (A.20) is zero. If the linear system (4.4) is assumed to admit a unique solution, then $\widetilde{\varphi}^{(k)}$ solves (2.2) with $\{\lambda_k\}_{1\leqslant k\leqslant N}$ and $\widetilde{u}$.

Finally, we show the validity of equations (A.21) and (A.22). In order to show (A.21), we first obtain the relation

$$\partial_x \langle \varphi^{(j)}, \sigma_3 \varphi^{(k)} \rangle = (\lambda_j + \lambda_k) \langle \varphi^{(j)}, \varphi^{(k)} \rangle + 2u \langle \varphi^{(j)}, \sigma_1 \varphi^{(k)} \rangle, \tag{A.23}$$

in addition to the relation (A.7). Then, we differentiate (A.8) in x, substitute (2.3), (A.7) and (A.23), and obtain

$$(\widetilde{u}_x - u_x)\varphi^{(j)} + (\widetilde{u} - u)u\sigma_3\sigma_1\varphi^{(j)} = 2 \sum_{k=1}^{N} \lambda_k \left[\langle \varphi^{(j)}, \sigma_3 \varphi^{(k)} \rangle \sigma_3 \widetilde{\varphi}^{(k)} + \langle \varphi^{(j)}, \varphi^{(k)} \rangle \widetilde{\varphi}^{(k)} \right]$$

$$+ \widetilde{u} \sum_{k=1}^{N} \left[\langle \varphi^{(j)}, \sigma_3 \varphi^{(k)} \rangle \sigma_3 \sigma_1 \widetilde{\varphi}^{(k)} + \langle \varphi^{(j)}, \varphi^{(k)} \rangle \sigma_1 \widetilde{\varphi}^{(k)} \right] + 2u \sum_{k=1}^{N} \langle \varphi^{(j)}, \sigma_1 \varphi^{(k)} \rangle \widetilde{\varphi}^{(k)}, \tag{A.24}$$

where the relation (A.8) was used to cancel the λ_j term. By using the transformation formulas (4.5), we verify that

$$(\widetilde{u} - u)\sigma_1 \varphi^{(j)} = \sum_{k=1}^{N} \left[\langle \varphi^{(j)}, \sigma_1 \varphi^{(k)} \rangle \sigma_3 \widetilde{\varphi}^{(k)} - \langle \varphi^{(j)}, \sigma_3 \sigma_1 \varphi^{(k)} \rangle \widetilde{\varphi}^{(k)} \right]. \tag{A.25}$$

This allows us to simplify (A.24) to the form

$$(\widetilde{u}_x - u_x)\varphi^{(j)} = 2 \sum_{k=1}^{N} \lambda_k \left[\langle \varphi^{(j)}, \sigma_3 \varphi^{(k)} \rangle \sigma_3 \widetilde{\varphi}^{(k)} + \langle \varphi^{(j)}, \varphi^{(k)} \rangle \widetilde{\varphi}^{(k)} \right]$$

$$+ \widetilde{u} \sum_{k=1}^{N} \left[\langle \varphi^{(j)}, \sigma_3 \varphi^{(k)} \rangle \sigma_3 \sigma_1 \widetilde{\varphi}^{(k)} + \langle \varphi^{(j)}, \varphi^{(k)} \rangle \sigma_1 \widetilde{\varphi}^{(k)} \right]$$

$$+ u \sum_{k=1}^{N} \left[\langle \varphi^{(j)}, \sigma_1 \varphi^{(k)} \rangle \widetilde{\varphi}^{(k)} + \langle \varphi^{(j)}, \sigma_3 \sigma_1 \varphi^{(k)} \rangle \sigma_3 \widetilde{\varphi}^{(k)} \right]. \tag{A.26}$$

Substituting (A.26) into (A.21) yields the following equation:

$$(u^2 - \widetilde{u}^2)\sigma_1 \varphi^{(j)} = \widetilde{u} \sum_{k=1}^{N} \left[\langle \varphi^{(j)}, \varphi^{(k)} \rangle \sigma_3 \sigma_1 \widetilde{\varphi}^{(k)} - \langle \varphi^{(j)}, \sigma_3 \varphi^{(k)} \rangle \sigma_1 \widetilde{\varphi}^{(k)} \right]$$

$$+ u \sum_{k=1}^{N} \left[\langle \varphi^{(j)}, \sigma_3 \sigma_1 \varphi^{(k)} \rangle \widetilde{\varphi}^{(k)} - \langle \varphi^{(j)}, \sigma_1 \varphi^{(k)} \rangle \sigma_3 \widetilde{\varphi}^{(k)} \right]. \tag{A.27}$$

Thanks to the relations (A.8) and (A.25), equation (A.27) is satisfied, and so is equation (A.21).

In order to show (A.22), we first obtain the relations

$$\partial_x \langle \varphi^{(j)}, \sigma_1 \varphi^{(k)} \rangle = (\lambda_j - \lambda_k) \langle \varphi^{(j)}, \sigma_3 \sigma_1 \varphi^{(k)} \rangle - 2u \langle \varphi^{(j)}, \sigma_3 \varphi^{(k)} \rangle \tag{A.28}$$

and

$$\partial_x \langle \varphi^{(j)}, \sigma_3 \sigma_1 \varphi^{(k)} \rangle = (\lambda_j - \lambda_k) \langle \varphi^{(j)}, \sigma_1 \varphi^{(k)} \rangle. \tag{A.29}$$

Then, we differentiate (A.26) in x, substitute (2.3), (A.7), (A.23), (A.28) and (A.29), and obtain

$$(\widetilde{u}_{xx} - u_{xx})\varphi^{(j)} + u(\widetilde{u}_x - u_x)\sigma_3\sigma_1\varphi^{(j)} = 4\sum_{k=1}^{N}\lambda_k^2\left[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\right]$$

$$+ 2\widetilde{u}\sum_{k=1}^{N}\lambda_k\left[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}\right] + 4u\sum_{k=1}^{N}\lambda_k\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}$$

$$+ \widetilde{u}_x\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]$$

$$+ u_x\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\right]$$

$$+ \widetilde{u}u\sum_{k=1}^{N}\left[3\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]$$

$$- \widetilde{u}^2\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\right] - 2u^2\sum_{k=1}^{N}\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}, \qquad (A.30)$$

where the relation (A.26) was used to cancel the λ_j term. Substituting (A.30) into (A.22) and using (A.8) and (A.25) yields

$$2(\widetilde{u}^3 - u^3)\varphi^{(j)} = (2u - \widetilde{u})(\widetilde{u}_x - u_x)\sigma_3\sigma_1\varphi^{(j)} + \widetilde{u}^2\sum_{k=1}^{N}\left[3\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}\right]$$

$$+ 4u^2\sum_{k=1}^{N}\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} - \widetilde{u}u\sum_{k=1}^{N}\left[3\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]$$

$$+ 2\widetilde{u}\sum_{k=1}^{N}\lambda_k\left[\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)} - \langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]$$

$$+ 4u\sum_{k=1}^{N}\lambda_k\left[\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} - \langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)}\right]. \qquad (A.31)$$

Substituting (A.26)–(A.31) yields

$$2(\widetilde{u} - u)(\widetilde{u}^2 + \widetilde{u}u + u^2)\varphi^{(j)} = 2\widetilde{u}^2\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)}\right]$$

$$+ 2\widetilde{u}u\sum_{k=1}^{N}\left[\langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)} - \langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} - 2\langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}\right]$$

$$+ 2u^2\sum_{k=1}^{N}\left[2\langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)} - \langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]$$

$$+ 4u\sum_{k=1}^{N}\lambda_k\left[\langle\varphi^{(j)},\sigma_3\sigma_1\varphi^{(k)}\rangle\widetilde{\varphi}^{(k)} + \langle\varphi^{(j)},\varphi^{(k)}\rangle\sigma_3\sigma_1\widetilde{\varphi}^{(k)}\right.$$

$$\left. - \langle\varphi^{(j)},\sigma_1\varphi^{(k)}\rangle\sigma_3\widetilde{\varphi}^{(k)} - \langle\varphi^{(j)},\sigma_3\varphi^{(k)}\rangle\sigma_1\widetilde{\varphi}^{(k)}\right]. \qquad (A.32)$$

By using the relations (A.3) and explicit computations, we obtain

$$\sum_{k=1}^{N} \left[\langle \varphi^{(j)}, \varphi^{(k)} \rangle \sigma_3 \widetilde{\varphi}^{(k)} - \langle \varphi^{(j)}, \sigma_3 \varphi^{(k)} \rangle \widetilde{\varphi}^{(k)} - 2 \langle \varphi^{(j)}, \sigma_1 \varphi^{(k)} \rangle \sigma_3 \sigma_1 \widetilde{\varphi}^{(k)} \right]$$

$$= (\widetilde{u} - u) \varphi^{(j)} - 2 \left(\sum_{k=1}^{N} p_k \widetilde{q}_k - \sum_{k=1}^{N} \widetilde{p}_k q_k \right) \sigma_1 \varphi^{(j)}, \tag{A.33}$$

$$\sum_{k=1}^{N} \left[2 \langle \varphi^{(j)}, \sigma_3 \varphi^{(k)} \rangle \widetilde{\varphi}^{(k)} + \langle \varphi^{(j)}, \sigma_1 \varphi^{(k)} \rangle \sigma_3 \sigma_1 \widetilde{\varphi}^{(k)} - \langle \varphi^{(j)}, \sigma_3 \sigma_1 \varphi^{(k)} \rangle \sigma_1 \widetilde{\varphi}^{(k)} \right]$$

$$= (\widetilde{u} - u) \varphi^{(j)} + 2 \left(\sum_{k=1}^{N} p_k \widetilde{q}_k - \sum_{k=1}^{N} \widetilde{p}_k q_k \right) \sigma_1 \varphi^{(j)} \tag{A.34}$$

and

$$\sum_{k=1}^{N} \lambda_k \left[\langle \varphi^{(j)}, \sigma_3 \sigma_1 \varphi^{(k)} \rangle \widetilde{\varphi}^{(k)} + \langle \varphi^{(j)}, \varphi^{(k)} \rangle \sigma_3 \sigma_1 \widetilde{\varphi}^{(k)} \right.$$

$$\left. - \langle \varphi^{(j)}, \sigma_1 \varphi^{(k)} \rangle \sigma_3 \widetilde{\varphi}^{(k)} - \langle \varphi^{(j)}, \sigma_3 \varphi^{(k)} \rangle \sigma_1 \widetilde{\varphi}^{(k)} \right]$$

$$= -2 \left(\sum_{k=1}^{N} \lambda_k \widetilde{p}_k p_k - \sum_{k=1}^{N} \lambda_k \widetilde{q}_k q_k \right) \sigma_1 \varphi^{(j)}. \tag{A.35}$$

Substituting (A.33)–(A.35) in (A.32) cancels all terms thanks to the relations (A.4) and (A.8). Therefore, equation (A.32) is satisfied and so is equation (A.22).

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