

## Rogue waves on the background of periodic standing waves in the derivative nonlinear Schrödinger equation

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The derivative nonlinear Schrödinger (DNLS) equation is the canonical model for the dynamics of nonlinear waves in plasma physics and optics. We study exact solutions describing rogue waves on the background of periodic standing waves in the DNLS equation. We show that the space-time localization of a rogue wave is only possible if the periodic standing wave is modulationally unstable. If the periodic standing wave is modulationally stable, the rogue wave solutions degenerate into algebraic solitons propagating along the background and interacting with the periodic standing waves. Maximal amplitudes of rogue waves are found analytically and confirmed numerically.

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### I. INTRODUCTION

Fundamental models for the dynamics of waves in fluids, plasmas, and optical systems are written in terms of integrable systems such as the nonlinear Schrödinger (NLS) equation, the derivative nonlinear Schrödinger (DNLS) equation, and their multicomponent generalizations. These models simplify the complicated dynamics by accounting for only two mechanisms in the wave evolution: dispersion and nonlinearity.

The focusing NLS equation is the most studied model of this class. Periodic waves in complex physical systems are modeled by the constant-amplitude waves of the NLS equation, which are known to be modulationally unstable [1]. Related to the modulational instability, rogue waves (localized waves of large amplitude appearing from nowhere and disappearing without a trace) emerge on the constant-amplitude wave background. These rogue waves are described by the exact solutions of the NLS equation [2–5].

In the context of the multicomponent NLS models, it was discovered in Refs. [6,7] that rogue waves only emerge on the modulationally unstable constant-amplitude wave background. If waves are modulationally stable in a subset of the parameter region, no rogue wave solutions can be constructed, and numerical simulations do not show the occurrence of localized waves of large amplitude [8].

The concept of *modulational instability* is introduced to describe the instability of the constant-amplitude wave with respect to perturbations of increasingly large periods [1]. If the constant-amplitude wave is unstable with respect to perturbations of smaller periods but stable with respect to perturbations of increasingly larger periods, it is said to be *modulationally stable*, even though it is still unstable in the time evolution of the governing model [9]. Note that in [6,7] the concept of modulational instability was dubbed “baseband modulational instability.”

Multiperiodic wave patterns in complex physical systems are modeled by the periodic standing wave solutions of the NLS equation. These periodic standing waves are also modulationally unstable [10], and rogue waves in their background exist as exact solutions of the NLS equation [11,12] and are observed in numerical simulations [13] and optical and hydrodynamical experiments [14].

A special relation between the modulational instability of periodic standing waves and the existence of rogue wave solutions was discovered in [15]. If the unstable spectral band intersects the origin in the complex spectral plane tangentially to the imaginary axis, the corresponding rogue wave solution degenerates into a propagating algebraic soliton on the periodic standing wave background. Similarly, it was shown for the sine-Gordon equation [16] that rogue waves are localized in space-time for modulationally unstable librational waves but degenerate into propagating algebraic solitons for modulationally stable rotational waves (which are still unstable with respect to perturbations of shorter periods).

*The purpose of this work* is to study rogue waves in the DNLS equation, which models long weakly nonlinear Alfvén waves propagating along the constant magnetic field in cold plasmas [17,18]. Rogue waves on the constant-amplitude wave background were constructed recently in Ref. [19]. Here we investigate the properties of the rogue waves arising in the background of the periodic standing waves.

The precise characterization of modulational instability of periodic standing waves in the DNLS equation was completed in our previous work [20], where we implemented the algebraic method of nonlinearization of Lax equations after the previous works in [21–25]. Compared to the previous studies of periodic standing waves in the DNLS equation in [26–29], the results of [20] gave a full picture of three possible types of periodic standing waves. The first type is modulationally unstable with two figure-8 unstable bands, the second type is modulationally unstable with one figure-8 unstable band,

and the third type is modulationally stable (in fact, stable with respect to perturbations of any period).

The main outcomes of this work are summarized as follows:

(i) The periodic standing waves with two figure-8 unstable bands admit two rogue waves localized in space-time.

(ii) The periodic standing waves with one figure-8 unstable band admit one rogue wave and two algebraic solitons propagating along its background.

(iii) The periodic standing waves with no unstable bands admit four algebraic solitons and no rogue waves.

Our main results are in agreement with the previous observations in [6,7] and in [15,16] that the space-time localization of rogue waves on the constant-amplitude and periodic standing wave background is possible if and only if the background wave is modulationally unstable (once again, unstable with respect to perturbations of increasingly long periods). There is currently no mathematical proof of this result for a general integrable system.

The regular way to construct the rogue wave solutions on the background of the periodic standing waves is to use the Darboux transformation. Darboux transformations for the DNLS equation (1) are well-known after the previous works in [30–33]. Applications of the Darboux transformation to generate breathers on the background of constant-amplitude solutions can be found in [34]. Darboux transformations are also useful to add solitons in the mathematical analysis of the existence of solutions to the initial-value problem [35].

We use the Darboux transformation with the nonperiodic solutions of the Lax equations associated with the periodic standing waves of the DNLS equation. Although the algebraic method from [20] only gives periodic solutions of the Lax equations, the nonperiodic solutions can be explicitly characterized in terms of integrals of the periodic standing wave solutions, as we show here. Since the computations are long and technical, our presentation includes only main results and numerical visualizations, whereas computational details are given as Appendixes.

In addition to numerical visualization of rogue wave solutions (either localized in space-time or propagating as algebraic solitons), we also compute maximal amplitudes of the rogue waves. These maximal amplitudes are important for future experiments with the rogue waves on the periodic standing wave background. Some progress in the analysis of maximal amplitudes for more general quasiperiodic solutions of the DNLS equation was recently obtained in [36]. Other recent studies of quasiperiodic solutions of the DNLS equation can be found in [25] and also in [37,38].

The paper is organized as follows. Section II reviews the construction of periodic standing waves based on our previous work [20]. We show in Sec. III that Darboux transformations with the periodic eigenfunctions of the Lax equations transform the class of periodic standing waves into itself. Section IV reports on rogue waves obtained after Darboux transformations with the nonperiodic solutions of the Lax equations. Section V concludes the paper with a discussion of future directions.

## II. PERIODIC STANDING WAVES

We consider the DNLS equation in the following normalized form:

$$i\psi_t + \psi_{xx} + i(|\psi|^2\psi)_x = 0, \tag{1}$$

where  $i = \sqrt{-1}$  and  $\psi(x, t) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$ . As is shown in [39], the DNLS equation is a compatibility condition  $\phi_{xt} = \phi_{tx}$  of the following Lax equations:

$$\phi_x = U(\psi, \lambda)\phi, \quad \phi_t = V(\psi, \lambda)\phi, \tag{2}$$

where

$$U(\psi, \lambda) = \begin{pmatrix} -i\lambda^2 & \lambda\psi \\ -\lambda\bar{\psi} & i\lambda^2 \end{pmatrix} \tag{3}$$

and

$$V(\psi, \lambda) = \begin{pmatrix} -2i\lambda^4 + i\lambda^2|\psi|^2 & 2\lambda^3\psi + \lambda(i\psi_x - |\psi|^2\psi) \\ -2\lambda^3\bar{\psi} + \lambda(i\bar{\psi}_x + |\psi|^2\bar{\psi}) & 2i\lambda^4 - i\lambda^2|\psi|^2 \end{pmatrix}. \tag{4}$$

The standing wave reduction of the DNLS equation (1) takes the form

$$\psi(x, t) = u(x + 2ct)e^{Aibr}, \tag{5}$$

where  $b$  and  $c$  are real parameters, and  $u$  satisfies the second-order equation:

$$\frac{d^2u}{dx^2} + i\frac{d}{dx}(|u|^2u) + 2ic\frac{du}{dx} - 4bu = 0. \tag{6}$$

The second-order equation (6) is integrable with the following two first-order invariants:

$$2i\left(\bar{u}\frac{du}{dx} - u\frac{d\bar{u}}{dx}\right) - 3|u|^4 - 4c|u|^2 = 4a \tag{7}$$

and

$$2\left|\frac{du}{dx}\right|^2 - |u|^6 - 2c|u|^4 - 4(a + 2b)|u|^2 = 8d, \tag{8}$$

where  $a$  and  $d$  are real parameters. As shown in [26] (see also [20]), the standing wave solutions of the differential equations (6)–(8) are related to four pairs of roots of the following polynomial:

$$P(\lambda) = \lambda^8 - 2c\lambda^6 + \lambda^4(a + 2b + c^2) + \lambda^2[d - c(a + 2b)] + b^2. \tag{9}$$

In the remainder of this section, we review details in the construction of periodic standing waves based on our previous work [20].

### A. Eigenvalues and eigenvectors of the Lax equations

The roots of  $P(\lambda)$  given by (9) determine the eigenvalues of the Kaup-Newell (KN) spectral problem

$$\varphi_x = U(u, \lambda)\varphi, \quad (10)$$

for either periodic or antiperiodic eigenvectors  $\varphi = (p, q)^T$  related to the periodic potential  $u$ . Eigenvector  $\varphi$  arises in the separation of variable

$$\phi(x, t) = e^{2ibt\sigma_3}\varphi(x + 2ct, t), \quad (11)$$

where  $\phi$  is a solution of the Lax equations (2) associated with the standing wave  $\psi$  of the DNLS equation (1) given by (5) and  $\sigma_3 = \text{diag}(1, -1)$ . Therefore, the eigenvector  $\varphi$  of the KN spectral problem (10) also satisfies the time-evolution equation

$$\varphi_t + 2c\varphi_x + 2ib\sigma_3\varphi = V(u, \lambda)\varphi. \quad (12)$$

The roots of  $P(\lambda)$  arise either as complex quadruplets or double real pairs or purely imaginary pairs [20]:

(i) If  $P(\lambda)$  admits a quadruplet of complex roots  $\{\lambda_1, \bar{\lambda}_1, -\lambda_1, -\bar{\lambda}_1\}$ , then the standing wave  $u$  and the eigenvector  $\varphi_1 = (p_1, q_1)^T$  for the eigenvalue  $\lambda_1$  are related by

$$u = \lambda_1 p_1^2 + \bar{\lambda}_1 q_1^2. \quad (13)$$

(ii) If  $P(\lambda)$  admits a pair of double real roots  $\{\lambda_1, -\lambda_1\}$  with two linearly independent eigenvectors  $\varphi_1 = (p_1, q_1)^T$  and  $\varphi_2 = (\bar{q}_1, -\bar{p}_1)^T$  for the same eigenvalue  $\lambda_1$ , then the same expression (13) holds.

(iii) If  $P(\lambda)$  admits two pairs of purely imaginary eigenvalues  $\{i\beta_1, -i\beta_1, i\beta_2, -i\beta_2\}$ , then the standing wave  $u$  and the eigenvectors  $\varphi_1 = (p_1, -i\bar{p}_1)^T$  and  $\varphi_2 = (p_2, -i\bar{p}_2)^T$  for the eigenvalues  $\lambda_1 = i\beta_1$  and  $\lambda_2 = i\beta_2$  are related by

$$u = i\beta_1 p_1^2 + i\beta_2 p_2^2. \quad (14)$$

### B. Complex Hamiltonian systems

Eigenvectors  $\varphi_1 = (p_1, q_1)^T$  and  $\varphi_2 = (p_2, q_2)^T$  of the KN spectral problem (10) with eigenvalues  $\lambda_1$  and  $\lambda_2$  satisfy a finite-dimensional complex Hamiltonian system, which is integrable [25]. This complex Hamiltonian system is equivalent to the Lax equation

$$\frac{d}{dx}\Psi = [\mathcal{U}, \Psi], \quad (15)$$

where  $\mathcal{U}$  is obtained from  $U(u, \lambda)$  in (3) with  $u$  given by either (13) or (14). The 2-by-2 Lax matrix  $\Psi$  is given by

$$\Psi := \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & -\Psi_{11} \end{pmatrix}, \quad (16)$$

where

$$\Psi_{11} = -i - \frac{\lambda_1^2 p_1 q_1}{\lambda^2 - \lambda_1^2} - \frac{\lambda_2^2 p_2 q_2}{\lambda^2 - \lambda_2^2},$$

$$\Psi_{12} = \lambda \left( \frac{\lambda_1 p_1^2}{\lambda^2 - \lambda_1^2} + \frac{\lambda_2 p_2^2}{\lambda^2 - \lambda_2^2} \right),$$

$$\Psi_{21} = -\lambda \left( \frac{\lambda_1 q_1^2}{\lambda^2 - \lambda_1^2} + \frac{\lambda_2 q_2^2}{\lambda^2 - \lambda_2^2} \right).$$

Entries of the Lax matrix  $\Psi$  can be rewritten in terms of  $u$  by

$$\begin{aligned} \Psi_{11} &= \frac{-i}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)} \left[ \lambda^4 - \lambda^2 \left( c + \frac{1}{2}|u|^2 \right) + b \right], \\ \Psi_{12} &= \frac{\lambda}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)} \left[ \lambda^2 u + \frac{i}{2} \frac{du}{dx} - \frac{1}{2} u|u|^2 - cu \right], \\ \Psi_{21} &= \frac{-\lambda}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)} \left[ \lambda^2 \bar{u} - \frac{i}{2} \frac{d\bar{u}}{dx} - \frac{1}{2} \bar{u}|u|^2 - c\bar{u} \right], \end{aligned}$$

so that

$$\det \Psi = -\Psi_{11}^2 - \Psi_{12}\Psi_{21} = \frac{P(\lambda)}{(\lambda^2 - \lambda_1^2)^2 (\lambda^2 - \lambda_2^2)^2}, \quad (17)$$

where  $P(\lambda)$  is given by (9). Eigenvalues  $\lambda_1$  and  $\lambda_2$  arising in the poles of  $\Psi$  must be chosen from the roots of  $P(\lambda)$ .

Let the four pairs of roots of the polynomial  $P(\lambda)$  in (9) be denoted by  $\{\pm\lambda_1, \pm\lambda_2, \pm\lambda_3, \pm\lambda_4\}$ . The polynomial can be factorized by its roots:

$$P(\lambda) = (\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)(\lambda^2 - \lambda_3^2)(\lambda^2 - \lambda_4^2). \quad (18)$$

Comparison of (9) and (18) yields the relations

$$\begin{aligned} \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 &= 2c, \\ (\lambda_1^2 + \lambda_2^2)(\lambda_3^2 + \lambda_4^2) + \lambda_1^2 \lambda_2^2 + \lambda_3^2 \lambda_4^2 &= a + 2b + c^2, \\ \lambda_1^2 \lambda_2^2 (\lambda_3^2 + \lambda_4^2) + \lambda_3^2 \lambda_4^2 (\lambda_1^2 + \lambda_2^2) &= ac + 2bc - d, \\ \lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_4^2 &= b^2, \end{aligned} \quad (19)$$

which allow us to express parameter  $(a, b, c, d)$  in terms of  $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$  explicitly:

$$\begin{aligned} a &= -\frac{1}{4}[(\lambda_1 + \lambda_2)^2 - (\lambda_3 + \lambda_4)^2] \\ &\quad \times [(\lambda_1 - \lambda_2)^2 - (\lambda_3 - \lambda_4)^2], \\ b &= \lambda_1 \lambda_2 \lambda_3 \lambda_4, \\ c &= \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2), \\ d &= -\frac{1}{8}(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2)(\lambda_1^2 - \lambda_2^2 + \lambda_3^2 - \lambda_4^2) \\ &\quad \times (\lambda_1^2 - \lambda_2^2 - \lambda_3^2 + \lambda_4^2). \end{aligned} \quad (20)$$

### C. Characterization of the periodic standing waves

Explicit solutions for the periodic standing waves satisfying (6)–(8) are obtained after using the polar form  $u(x) = R(x)e^{i\theta(x)}$  with

$$\frac{d\theta}{dx} = -\frac{a}{R^2} - \frac{3}{4}R^2 - c \quad (21)$$

and

$$\begin{aligned} \left( \frac{dR}{dx} \right)^2 + \frac{a^2}{R^2} + \frac{1}{16}R^6 + \frac{c}{2}R^4 \\ + R^2 \left( c^2 - 4b - \frac{a}{2} \right) + 2ac - 4d = 0. \end{aligned} \quad (22)$$

The transformation  $\rho = \frac{1}{2}R^2$  brings (22) to the form

$$\left( \frac{d\rho}{dx} \right)^2 + Q(\rho) = 0, \quad (23)$$

where

$$Q(\rho) = \rho^4 + 4c\rho^3 + 2(2c^2 - a - 8b)\rho^2 + 4(ac - 2d)\rho + a^2. \tag{24}$$

Let the four roots of the polynomial  $Q(\rho)$  be denoted by  $\{u_1, u_2, u_3, u_4\}$  so that

$$Q(\rho) = (\rho - u_1)(\rho - u_2)(\rho - u_3)(\rho - u_4). \tag{25}$$

Comparison of (24) with (25) yields

$$\begin{aligned} u_1 + u_2 + u_3 + u_4 &= -4c, \\ u_1u_2 + u_1u_3 + u_1u_4 + u_2u_3 + u_2u_4 + u_3u_4 \\ &= 2(2c^2 - a - 8b), \\ u_1u_2u_3 + u_1u_2u_4 + u_1u_3u_4 + u_2u_3u_4 &= 4(2d - ac), \\ u_1u_2u_3u_4 &= a^2. \end{aligned} \tag{26}$$

The roots  $\{u_1, u_2, u_3, u_4\}$  of  $Q$  are related to the roots  $\{\pm\lambda_1, \pm\lambda_2, \pm\lambda_3, \pm\lambda_4\}$  of  $P$  due to (20) and (26). It was shown in [26] and more recently in [20] that the relations are expressed by

$$\begin{aligned} u_1 &= -\frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)^2, \\ u_2 &= -\frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)^2, \\ u_3 &= -\frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^2, \\ u_4 &= -\frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^2. \end{aligned} \tag{27}$$

Two families of periodic standing waves are obtained from the quadrature (23) with (25).

**1. Four roots of  $Q$  are real**

Assume the following ordering for the four real roots of  $Q$ :

$$u_4 \leq u_3 \leq u_2 \leq u_1. \tag{28}$$

Periodic solutions to the quadrature (23) are expressed explicitly (see, e.g., [40]) by

$$\rho(x) = u_4 + \frac{(u_1 - u_4)(u_2 - u_4)}{(u_2 - u_4) + (u_1 - u_2)\text{sn}^2(vx; k)}, \tag{29}$$

where positive parameters  $v$  and  $k$  are uniquely expressed by

$$\begin{aligned} v &= \frac{1}{2}\sqrt{(u_1 - u_3)(u_2 - u_4)}, \\ k &= \frac{\sqrt{(u_1 - u_2)(u_3 - u_4)}}{\sqrt{(u_1 - u_3)(u_2 - u_4)}}. \end{aligned} \tag{30}$$

The periodic solution  $\rho$  is located in the interval  $[u_2, u_1]$  and has the period  $L = 2K(k)v^{-1}$ . The solution (29) with (30) is meaningful for  $\rho = \frac{1}{2}R^2 \geq 0$  in two cases:

$$\lambda_1 = \bar{\lambda}_3 = \alpha_1 + i\beta_1, \quad \lambda_2 = \bar{\lambda}_4 = \alpha_2 + i\beta_2 \tag{31}$$

and

$$\lambda_1 = i\beta_1, \quad \lambda_2 = i\beta_2, \quad \lambda_3 = i\beta_3, \quad \lambda_4 = i\beta_4. \tag{32}$$

The roots ordered as (28) satisfy the more precise ordering

$$u_4 \leq u_3 \leq 0 \leq u_2 \leq u_1 \tag{33}$$

in the case of (31) and

$$0 \leq u_4 \leq u_3 \leq u_2 \leq u_1 \tag{34}$$

in the case of (32). Note that in the case of (34), another periodic solution is obtained from (29) by exchanging  $u_1$  with  $u_3$  and  $u_2$  with  $u_4$ :

$$\rho(x) = u_2 - \frac{(u_2 - u_3)(u_2 - u_4)}{(u_2 - u_4) - (u_3 - u_4)\text{sn}^2(vx; k)}, \tag{35}$$

where the values of  $v$  and  $k$  are the same as in (30).

In the case of (31), if  $\alpha_1, \alpha_2, \beta_1, \beta_2$  are all positive, it follows from (27) that

$$\begin{aligned} \alpha_1 &= \frac{1}{2\sqrt{2}}(\sqrt{-u_4} + \sqrt{-u_3}), \\ \alpha_2 &= \frac{1}{2\sqrt{2}}(\sqrt{-u_4} - \sqrt{-u_3}), \\ \beta_1 &= \frac{1}{2\sqrt{2}}(\sqrt{u_1} + \sqrt{u_2}), \\ \beta_2 &= \frac{1}{2\sqrt{2}}(\sqrt{u_1} - \sqrt{u_2}), \end{aligned} \tag{36}$$

so that  $0 \leq \alpha_2 \leq \alpha_1$  and  $0 \leq \beta_2 \leq \beta_1$ .

In the case of (32), it follows from (27) that

$$\begin{aligned} \beta_1 &= \frac{1}{2\sqrt{2}}(\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4}), \\ \beta_2 &= \frac{1}{2\sqrt{2}}(-\sqrt{u_1} - \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4}), \\ \beta_3 &= \frac{1}{2\sqrt{2}}(\sqrt{u_1} - \sqrt{u_2} - \sqrt{u_3} + \sqrt{u_4}), \\ \beta_4 &= \frac{1}{2\sqrt{2}}(-\sqrt{u_1} + \sqrt{u_2} - \sqrt{u_3} + \sqrt{u_4}), \end{aligned} \tag{37}$$

so that  $\beta_2 \leq \beta_4 \leq \beta_3 \leq \beta_1$ .

**2. Two roots of  $Q$  are real and one pair of roots is complex-conjugate**

Assume that the two real roots are ordered as  $u_2 \leq u_1$  and the complex-conjugate roots are given by  $u_{3,4} = \gamma \pm i\eta$  so that

$$0 \leq u_2 \leq u_1, \quad u_3 = \gamma + i\eta, \quad u_4 = \gamma - i\eta. \tag{38}$$

Periodic solutions to the quadrature (23) are expressed explicitly (see, e.g., [40]) by

$$\rho(x) = u_1 + \frac{(u_2 - u_1)[1 - \text{cn}(\mu x; k)]}{1 + \delta + (\delta - 1)\text{cn}(\mu x; k)}, \tag{39}$$

where positive parameters  $\delta, \mu,$  and  $k$  are uniquely expressed by

$$\delta = \frac{\sqrt{(u_2 - \gamma)^2 + \eta^2}}{\sqrt{(u_1 - \gamma)^2 + \eta^2}}, \tag{40}$$

$$\mu = \sqrt{[(u_1 - \gamma)^2 + \eta^2][(u_2 - \gamma)^2 + \eta^2]},$$

and

$$2k^2 = 1 - \frac{(u_1 - \gamma)(u_2 - \gamma) + \eta^2}{\sqrt{[(u_1 - \gamma)^2 + \eta^2][(u_2 - \gamma)^2 + \eta^2]}}. \tag{41}$$

The periodic solution  $\rho$  is located in the interval  $[u_2, u_1]$  and has the period  $L = 4K(k)\mu^{-1}$ . The periodic solution (39) with

(40) and (41) is meaningful for  $\rho = \frac{1}{2}R^2 \geq 0$  if

$$\lambda_1 = \bar{\lambda}_2 = \alpha_1 + i\beta_1, \quad \lambda_3 = i\beta_3, \quad \lambda_4 = i\beta_4 \quad (42)$$

with the following relations:

$$\begin{aligned} \alpha_1 &= \frac{1}{2}\sqrt{\sqrt{\gamma^2 + \eta^2} - \gamma}, \\ \beta_1 &= \frac{1}{2\sqrt{2}}(\sqrt{u_1} + \sqrt{u_2}), \\ \beta_3 &= \frac{\eta}{2\sqrt{\sqrt{\gamma^2 + \eta^2} - \gamma}} + \frac{1}{2\sqrt{2}}(\sqrt{u_1} - \sqrt{u_2}), \\ \beta_4 &= \frac{\eta}{2\sqrt{\sqrt{\gamma^2 + \eta^2} - \gamma}} - \frac{1}{2\sqrt{2}}(\sqrt{u_1} - \sqrt{u_2}). \end{aligned} \quad (43)$$

where  $\alpha_1 \geq 0$ ,  $\beta_1 \geq 0$ , and  $\beta_4 \leq \beta_3$ .

### III. DARBOUX TRANSFORMATIONS

Darboux transformations for the DNLS equation (1) were constructed previously in [30–33]. Here we use the exact formulas for the onefold and twofold Darboux transformations. The validity of the transformation formulas is verified in Appendixes A and B.

Let  $\psi$  be a solution of the DNLS equation (1) and let  $\phi$  be a solution of the Lax equations (2) for the potential  $\psi$  with a fixed value of  $\lambda$ . If  $\psi$  is the standing wave solution in the form (5), then the solution  $\phi$  can be expressed in the form (11). We denote eigenvectors  $\varphi_1 = (p_1, q_1)^T$  and  $\varphi_2 = (p_2, q_2)^T$  for eigenvalues  $\lambda_1$  and  $\lambda_2$ , respectively.

Darboux transformations generate new solutions to the DNLS equation (1) in the form

$$\hat{\psi}(x, t) = \hat{u}(x + 2ct, t)e^{4ibt}. \quad (44)$$

The following three basic Darboux transformations will be used:

(i) If  $\lambda_1 = i\beta_1$  and  $q_1 = -i\bar{p}_1$ , then the new solution is given by

$$\hat{u} = -\frac{\bar{p}_1^2}{p_1^2} \left[ u + 2i\beta_1 \frac{p_1}{\bar{p}_1} \right] e^{-8ibt}. \quad (45)$$

(ii) If  $\lambda_{1,2} = i\beta_{1,2}$  and  $q_{1,2} = -i\bar{p}_{1,2}$ , then the new solution is given by

$$\hat{u} = \left( \frac{\beta_1 \bar{p}_1 p_2 - \beta_2 \bar{p}_2 p_1}{\beta_1 \bar{p}_2 p_1 - \beta_2 \bar{p}_1 p_2} \right)^2 \left( u + \frac{2i(\beta_1^2 - \beta_2^2)p_1 p_2}{\beta_1 \bar{p}_1 p_2 - \beta_2 \bar{p}_2 p_1} \right). \quad (46)$$

(iii) If  $\lambda_1 \in \mathbb{C}$  with  $\text{Re}(\lambda_1), \text{Im}(\lambda_1) \neq 0$ , then the new solution is given by

$$\hat{u} = \left( \frac{\bar{\lambda}_1 |p_1|^2 + \lambda_1 |q_1|^2}{\lambda_1 |p_1|^2 + \bar{\lambda}_1 |q_1|^2} \right)^2 \left[ u - \frac{2i(\lambda_1^2 - \bar{\lambda}_1^2)p_1 \bar{q}_1}{\bar{\lambda}_1 |p_1|^2 + \lambda_1 |q_1|^2} \right]. \quad (47)$$

Here we consider eigenvectors of the KN spectral problem (10) found from the complex Hamiltonian system (15). We show that the Darboux transformations with these eigenvectors produce new solutions of the DNLS equation that are translated versions of the same periodic standing waves.

Comparing expressions for  $\Psi_{12}$  and  $\Psi_{21}$  yields the following relations for the squared components of the eigenvectors:

$$\begin{aligned} \lambda_1 p_1^2 &= \frac{1}{\lambda_1^2 - \lambda_2^2} \left[ \frac{1}{2} \left( i \frac{du}{dx} - |u|^2 u \right) + (\lambda_1^2 - c)u \right], \\ \lambda_1 q_1^2 &= \frac{1}{\lambda_1^2 - \lambda_2^2} \left[ -\frac{1}{2} \left( i \frac{d\bar{u}}{dx} + |u|^2 \bar{u} \right) + (\lambda_1^2 - c)\bar{u} \right], \end{aligned} \quad (48)$$

whereas comparing expressions for  $\Psi_{11}$  yields

$$\lambda_1^2 p_1 q_1 = \frac{i}{\lambda_1^2 - \lambda_2^2} \left[ b - c\lambda_1^2 + \lambda_1^4 - \frac{1}{2}\lambda_1^2 |u|^2 \right]. \quad (49)$$

By using the polar form decompositions

$$\begin{aligned} u(x) &= R(x)e^{i\theta(x)}, \\ p_1(x) &= P_1(x)e^{\frac{i}{2}\theta(x)}, \\ q_1(x) &= Q_1(x)e^{-\frac{i}{2}\theta(x)} \end{aligned} \quad (50)$$

and the phase equation (21), we can rewrite relations (48) and (49) in the form

$$\begin{aligned} \lambda_1 P_1^2 &= \frac{1}{\lambda_1^2 - \lambda_2^2} \left[ \frac{i}{2} \frac{dR}{dx} - \frac{1}{8}R^3 + \frac{a}{2R} + \left( \lambda_1^2 - \frac{c}{2} \right)R \right], \\ \lambda_1 Q_1^2 &= \frac{1}{\lambda_1^2 - \lambda_2^2} \left[ -\frac{i}{2} \frac{dR}{dx} - \frac{1}{8}R^3 + \frac{a}{2R} + \left( \lambda_1^2 - \frac{c}{2} \right)R \right], \end{aligned} \quad (51)$$

and

$$\lambda_1^2 P_1 Q_1 = \frac{i}{\lambda_1^2 - \lambda_2^2} \left[ b - c\lambda_1^2 + \lambda_1^4 - \frac{1}{2}\lambda_1^2 R^2 \right]. \quad (52)$$

The periodic standing waves are given by either (29) or (39) for  $\rho := \frac{1}{2}R^2$ . Depending on the parameters  $(a, b, c, d)$ , the roots of  $P(\lambda)$  include either pairs of purely imaginary eigenvalues or complex quadruplets.

#### A. Onefold transformation (45) for the periodic wave (29)

Let us consider the case (32) for the periodic wave (29) with  $0 \leq u_4 \leq u_3 \leq u_2 \leq u_1$ . Four pairs of purely imaginary eigenvalues exist. Without loss of generality, we pick one eigenvalue  $\lambda_1 = i\beta_1$ . It is shown in Appendix C that the new solution can be expressed in the form  $\hat{\rho} := \frac{1}{2}|\hat{u}|^2$  with

$$\hat{\rho} = \frac{\beta_1^2 a(b - c\beta_1^2 - \beta_1^4) + 2d\beta_1^4 + [b^2 + \beta_1^4(2b + a - c^2) + \beta_1^8]\rho + \beta_1^2(b - c\beta_1^2 - \beta_1^4)\rho^2}{(b + c\beta_1^2 + \beta_1^4 + \beta_1^2\rho)^2}. \quad (53)$$

For the periodic wave (29), this expression reduces to the form

$$\hat{\rho}(x) = v_1 - \frac{(v_1 - v_3)(v_1 - v_4)}{(v_1 - v_3) + (v_3 - v_4)\text{sn}^2(vx; k)}, \quad (54)$$

where the values of  $v$  and  $k$  are the same as in (30), and the new roots  $0 \leq v_4 \leq v_3 \leq v_2 \leq v_1$  are given by

$$\begin{aligned} v_1 &= \frac{1}{4}(\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} - \sqrt{u_4})^2, \\ v_2 &= \frac{1}{4}(\sqrt{u_1} + \sqrt{u_2} - \sqrt{u_3} + \sqrt{u_4})^2, \\ v_3 &= \frac{1}{4}(\sqrt{u_1} - \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4})^2, \\ v_4 &= \frac{1}{4}(-\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4})^2. \end{aligned} \quad (55)$$

The new solution (54) is obtained from the periodic solution (29) after  $u_{1,2,3,4}$  are replaced by  $v_{1,2,3,4}$  and the transformation  $v_1 \leftrightarrow v_4$  and  $v_2 \leftrightarrow v_3$  is used. The new periodic solution has four pairs of purely imaginary eigenvalues  $\{\pm i\tilde{\beta}_1, \pm i\tilde{\beta}_2, \pm i\tilde{\beta}_3, \pm i\tilde{\beta}_4\}$  related to  $v_{1,2,3,4}$  by (37) after the transformation (55). It is easy to verify that the location of the purely imaginary eigenvalues is invariant under the transformation (55) with  $\tilde{\beta}_1 = \beta_1$  and  $\tilde{\beta}_{2,3,4} = -\beta_{2,3,4}$  if  $2\sqrt{v_4} = -\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4} \geq 0$ .

The new periodic solution (54) satisfies the same differential equation (23) with (24) having parameters  $\tilde{a}, \tilde{b}, \tilde{c}$ , and  $\tilde{d}$  related to eigenvalues  $\{\pm i\tilde{\beta}_1, \pm i\tilde{\beta}_2, \pm i\tilde{\beta}_3, \pm i\tilde{\beta}_4\}$  by (20) and to turning points  $v_{1,2,3,4}$  by (26). Since  $\tilde{\beta}_1 = \beta_1$  and  $\tilde{\beta}_{2,3,4} = -\beta_{2,3,4}$ , it follows from (20) and (26) that

$$\tilde{d} = d, \quad \tilde{c} = c, \quad \tilde{b} = -b, \quad \text{and} \quad \tilde{a} = a + 4b. \quad (56)$$

Thus, the onefold Darboux transformation (45) transforms one periodic solution (29) of the differential equation (23) with given parameters  $(a, b, c, d)$  to another periodic solution (54) of the same equation with different parameters  $(a + 4b, -b, c, d)$ . The new and old solutions are related to the same four pairs of purely imaginary eigenvalues  $\{\pm i\beta_1, \pm i\beta_2, \pm i\beta_3, \pm i\beta_4\}$ . Note that the transformation  $\tilde{b} = -b$  also follows from a comparison of (44) and (45).

**B. Twofold transformation (46) for the periodic wave (29)**

Let us now pick two eigenvalues  $\lambda_{1,2} = i\beta_{1,2}$  in the case (32) with  $0 \leq u_4 \leq u_3 \leq u_2 \leq u_1$ . The new solution (46) can be reduced after long symbolic computations to the form  $\hat{\rho} := \frac{1}{2}|\hat{u}|^2$  with

$$\hat{\rho}(x) = u_3 + \frac{(u_1 - u_3)(u_2 - u_3)}{(u_1 - u_3) - (u_1 - u_2)\text{sn}^2(vx; k)}. \quad (57)$$

The expression (57) is obtained from the expression (29) by means of the transformation  $u_1 \leftrightarrow u_2$  and  $u_3 \leftrightarrow u_4$ . This transformation generates the same periodic wave (29) but translated by a half-period:

$$\begin{aligned} &\hat{\rho}(x + K(k)v^{-1}) \\ &= \frac{u_2(u_1 - u_3) - u_3(u_1 - u_2)\text{sn}^2(vx + K(k); k)}{(u_1 - u_3) - (u_1 - u_2)\text{sn}^2(vx + K(k); k)} \\ &= \frac{u_1(u_2 - u_4) + u_4(u_1 - u_2)\text{sn}^2(vx; k)}{(u_2 - u_4) + (u_1 - u_2)\text{sn}^2(vx; k)} = \rho(x), \end{aligned}$$

where we have used formulas

$$\begin{aligned} \text{sn}(x + K(k); k) &= \frac{\text{cn}(x; k)}{\text{dn}(x; k)}, \\ \text{dn}^2(x; k) &= 1 - k^2\text{sn}^2(x; k), \\ \text{cn}^2(x; k) &= 1 - \text{sn}^2(x; k) \end{aligned}$$

together with the definition of  $k$  in (30).

This recurrence of the periodic solution (29) after the twofold transformation (46) can be explained as follows. It follows from (55) that

$$\begin{aligned} \sqrt{v_1} + \sqrt{v_2} &= \sqrt{u_1} + \sqrt{u_2}, \\ \sqrt{v_1} - \sqrt{v_2} &= \sqrt{u_3} - \sqrt{u_4}, \\ \sqrt{v_3} + \sqrt{v_4} &= \sqrt{u_3} + \sqrt{u_4}, \\ \sqrt{v_3} - \sqrt{v_4} &= \sqrt{u_1} - \sqrt{u_2}. \end{aligned}$$

A composition of two transformations (55) restores the original roots  $u_{1,2,3,4}$ :

$$\begin{aligned} u_1 &= \frac{1}{4}(\sqrt{v_1} + \sqrt{v_2} + \sqrt{v_3} - \sqrt{v_4})^2, \\ u_2 &= \frac{1}{4}(\sqrt{v_1} + \sqrt{v_2} - \sqrt{v_3} + \sqrt{v_4})^2, \\ u_3 &= \frac{1}{4}(\sqrt{v_1} - \sqrt{v_2} + \sqrt{v_3} + \sqrt{v_4})^2, \\ u_4 &= \frac{1}{4}(-\sqrt{v_1} + \sqrt{v_2} + \sqrt{v_3} + \sqrt{v_4})^2, \end{aligned} \quad (58)$$

Similarly, parameters  $(a, b, c, d)$  are invariant after the composition of two transformations (56):

$$\begin{aligned} (a, b, c, d) &\mapsto (a + 4b, -b, c, d) \\ &\mapsto (a + 4b + 4(-b), -(-b), c, d) \\ &= (a, b, c, d). \end{aligned} \quad (59)$$

As a result, the new solution (57) satisfies the quadrature (23) with the same values of parameters  $(a, b, c, d)$  as in (24).

**C. Twofold transformation (47) for the periodic wave (29)**

Let us consider the case (31) for the periodic wave (29) with  $u_4 \leq u_3 \leq 0 \leq u_2 \leq u_1$ . Two complex quadruplets exist. We pick one eigenvalue  $\lambda_1$  from the two quadruplets. It is shown in Appendix D that the new solution (47) can be written in the form  $\hat{\rho} := \frac{1}{2}|\hat{u}|^2$  with

$$\hat{\rho}(x) = u_3 + \frac{(u_1 - u_3)(u_2 - u_3)}{(u_1 - u_3) - (u_1 - u_2)\text{sn}^2(vx; k)}, \quad (60)$$

which is the same as (57). Thus, the twofold transformation (47) with the complex quadruplet produces the same result as the twofold transformation (46) with two purely imaginary eigenvalues.

**D. Onefold transformation (45) for the periodic wave (39)**

Let us consider the case (42) for the periodic wave (39) with  $0 \leq u_2 \leq u_1$  and  $u_{3,4} = \gamma \pm i\eta$ . There exist two pairs of purely imaginary eigenvalues and a quadruplet of complex quadruplets. Without loss of generality, we pick one eigenvalue  $\lambda_3 = i\beta_3$ . It is shown in Appendix E that the new solution (45) can be written in the form  $\hat{\rho} := \frac{1}{2}|\hat{u}|^2$  with

$$\hat{\rho}(x) = \tilde{v}_2 + \frac{(\tilde{v}_1 - \tilde{v}_2)[1 - \text{cn}(\mu x; k)]}{1 + \tilde{\delta} + (\tilde{\delta} - 1)\text{cn}(\mu x; k)}, \quad (61)$$

where the new roots  $0 \leq \tilde{v}_2 \leq \tilde{v}_1$  and  $\tilde{v}_{3,4} = \tilde{\gamma} \pm i\tilde{\eta}$  are given by

$$\begin{aligned}\tilde{v}_1 &= \frac{1}{4}(\sqrt{u_1} + \sqrt{u_2} + \sqrt{2(\sqrt{\gamma^2 + \eta^2} + \gamma)})^2, \\ \tilde{v}_2 &= \frac{1}{4}(\sqrt{u_1} + \sqrt{u_2} - \sqrt{2(\sqrt{\gamma^2 + \eta^2} + \gamma)})^2, \\ \tilde{v}_3 &= \frac{1}{4}(\sqrt{u_1} - \sqrt{u_2} + i\sqrt{2(\sqrt{\gamma^2 + \eta^2} - \gamma)})^2, \\ \tilde{v}_4 &= \frac{1}{4}(\sqrt{u_1} - \sqrt{u_2} - i\sqrt{2(\sqrt{\gamma^2 + \eta^2} - \gamma)})^2,\end{aligned}\quad (62)$$

the values of  $\mu$  and  $k$  are the same as in (40) and (41), and

$$\tilde{\delta} = \frac{\sqrt{(\tilde{v}_1 - \tilde{v}_3)(\tilde{v}_1 - \tilde{v}_4)}}{\sqrt{(\tilde{v}_2 - \tilde{v}_3)(\tilde{v}_2 - \tilde{v}_4)}}. \quad (63)$$

Note that the new periodic solution (61) coincides with the periodic solution (39) after  $u_{1,2,3,4}$  are replaced by  $\tilde{v}_{1,2,3,4}$  and  $\tilde{v}_1$  is exchanged with  $\tilde{v}_2$ . We have also confirmed the validity of the transformation (56) for the case of (61). Thus, the onefold Darboux transformation (45) transforms the periodic solution (39) of the differential equation (23) with given parameters  $(a, b, c, d)$  to another solution (61) of the same equation with different parameters  $(a + 4b, -b, c, d)$ .

#### E. Twofold transformation (46) for the periodic wave (39)

Let us pick now two purely imaginary eigenvalues  $\lambda_{3,4} = i\beta_{3,4}$  in the case of (42). The new solution (46) can be reduced after long symbolic computations to the form  $\hat{\rho} := \frac{1}{2}|\hat{u}|^2$  with

$$\hat{\rho}(x) = u_2 + \frac{(u_1 - u_2)[1 - \text{cn}(\mu x; k)]}{[1 - \text{cn}(\mu x; k)] + \delta^{-1}[1 + \text{cn}(\mu x; k)]}, \quad (64)$$

which is the periodic wave (39) after the transformation  $u_1 \leftrightarrow u_2$ . The latter transformation yields the same periodic wave (39) but translated by a half-period:

$$\begin{aligned}\hat{\rho}(x + 2K(k)\mu^{-1}) &= u_2 + \frac{(u_1 - u_2)[1 + \text{cn}(\mu x; k)]}{1 + \text{cn}(\mu x; k) + \delta^{-1}[1 - \text{cn}(\mu x; k)]} \\ &= u_1 + \frac{(u_2 - u_1)[1 - \text{cn}(\mu x; k)]}{1 + \delta + (\delta - 1)\text{cn}(\mu x; k)} = \rho(x),\end{aligned}$$

where we have used the relation  $\text{cn}(x + 2K(k); k) = -\text{cn}(x; k)$ . Thus, the twofold transformation (46) applied to the periodic solution (39) produces a translation of the same periodic solution (39). This is explained again by the fact that the composition of two transformations (56) in (59) returns the original parameters  $(a, b, c, d)$  of the quadrature (23).

#### F. Twofold transformation (47) for the periodic wave (39)

Finally, we pick eigenvalue  $\lambda_1$  from the complex quadruplet in the case (42). It is shown in Appendix F that the new solution (47) can be written in the form  $\hat{\rho} := \frac{1}{2}|\hat{u}|^2$  with

$$\hat{\rho} = u_2 + \frac{(u_1 - u_2)[1 - \text{cn}(\mu x; k)]}{1 - \text{cn}(\mu x; k) + \delta^{-1}[1 + \text{cn}(\mu x; k)]}, \quad (65)$$

which is the same as (64). Thus, the twofold transformation (47) with the complex eigenvalue produces again the same outcome as the twofold transformation (46) with two purely imaginary eigenvalues.

## IV. ROGUE WAVE SOLUTIONS

Here we construct rogue wave solutions to the DNLS equation (1) by using Darboux transformations with the second solution of the Lax equations (2) for the same eigenvalues given by roots of the polynomial  $P(\lambda)$ . The first solution of the Lax equations is periodic for these eigenvalues, whereas the second solution is generally nonperiodic. We use transformations (45) and (46) if  $P(\lambda)$  admits pairs of purely imaginary roots, and transformation (47) if  $P(\lambda)$  admits quadruplets of complex roots.

It is known from [20] that two pairs of purely imaginary roots of  $P(\lambda)$  are related to the stable spectrum in the linearization of the DNLS equation (1) at the periodic standing wave solution (5), whereas a quadruplet of complex roots of  $P(\lambda)$  is related to the modulationally unstable spectrum. In full agreement with the modulational stability analysis, we show that the new solutions related to two pairs of purely imaginary roots describe algebraic solitons propagating on the background of the periodic standing wave, whereas the new solutions related to a quadruplet of complex eigenvalues describe rogue waves localized in space and time on the background of the periodic standing wave.

From a technical point of view, we construct the second solution to the Lax equations differently for the purely imaginary roots and for the complex roots. Similar differences were previously discovered for the periodic traveling waves in the sine-Gordon equation [16].

#### A. Periodic wave (29) with $\lambda_1$ being purely imaginary

Let  $\lambda_1 = i\beta_1 \in i\mathbb{R}$  be an eigenvalue of the KN spectral problem (10). We use the decomposition (5) and (11) with the eigenvector  $\varphi_1 = (p_1, q_1)^T$ , where  $q_1 = -i\bar{p}_1$  and  $p_1$  satisfies the linear system

$$\frac{\partial p_1}{\partial x} = i\beta_1^2 p_1 + \beta_1 u \bar{p}_1 \quad (66)$$

and

$$\begin{aligned}\frac{\partial p_1}{\partial t} + 2c \frac{\partial p_1}{\partial x} + 2ibp_1 &= -i(2\beta_1^4 + \beta_1^2 |u|^2) p_1 - 2\beta_1^3 u \bar{p}_1 \\ &\quad + \beta_1(iu_x - |u|^2 u) \bar{p}_1.\end{aligned}\quad (67)$$

This system follows from the Lax equations (10) and (12) due to the reduction  $q_1 = -i\bar{p}_1$  for  $\lambda_1 = i\beta_1$ . The second, linearly independent solution  $\hat{\varphi}_1 = (\hat{p}_1, \hat{q}_1)^T$  of the system (66) and (67) can be written in the form

$$\hat{p}_1 = p_1 \chi_1 - \frac{1}{2q_1}, \quad \hat{q}_1 = q_1 \chi_1 + \frac{1}{2p_1}, \quad (68)$$

where  $\chi_1$  is assumed to be a real-valued function of  $x$  and  $t$ . Wronskian between the two solutions is normalized by  $p_1 \hat{q}_1 - \hat{p}_1 q_1 = 1$ . If  $q_1 = -i\bar{p}_1$  and  $\chi_1$  is real, then  $\hat{q}_1 = -i\bar{\hat{p}}_1$ .

Substituting (68) into (66) and (67) written for  $\hat{p}_1$  and using the same equations for  $p_1$  yields the following equations for  $\chi_1$ :

$$\frac{\partial \chi_1}{\partial x} = \frac{i\beta_1}{2|p_1|^4} (u \bar{p}_1^2 - \bar{u} p_1^2) \quad (69)$$

and

$$\begin{aligned} \frac{\partial \chi_1}{\partial t} + 2c \frac{\partial \chi_1}{\partial x} &= \frac{i\beta_1}{2|p_1|^4} (\bar{u}p_1^2 - u\bar{p}_1^2)(|u|^2 + 2\beta_1^2) \\ &\quad - \frac{\beta_1}{2|p_1|^4} (\bar{u}_x p_1^2 + u_x \bar{p}_1^2). \end{aligned} \quad (70)$$

In particular, we confirm that the function  $\chi_1(x, t)$  is real.

$$\frac{\partial \chi_1}{\partial t} = - \frac{2\beta_1^4(\beta_1^2 - \beta_2^2)[a(c + \beta_1^2) - d - 2(c\beta_1^2 + \beta_1^4 + b)\rho - \beta_1^2\rho^2]}{(b + c\beta_1^2 + \beta_1^4 + \beta_1^2\rho)^2}, \quad (72)$$

where we have used (21), (23), and (24) in order to express  $u_x$ . Equations (71) and (72) are compatible with  $\chi_{1xt} = \chi_{1tx}$  if and only if the right-hand side of (72) is constant because  $\rho$  depends on  $x$  only. It is shown in Appendix G that substituting (20), (29), and (37) into (72) yields the following simple equation:

$$\frac{\partial \chi_1}{\partial t} = 2\beta_1^2(\beta_1^2 - \beta_2^2), \quad (73)$$

from which we obtain

$$\chi_1(x, t) = c_1 + k_1x + f(x) + 2\beta_1^2(\beta_1^2 - \beta_2^2)t, \quad (74)$$

where  $c_1 \in \mathbb{R}$  is an arbitrary constant of integration,

$$k_1 = \frac{\nu\beta_1^4(\beta_1^2 - \beta_2^2)}{4K(k)} \int_0^{2K(k)\nu^{-1}} \frac{[a - 2(c + 2\beta_1^2)\rho - \rho^2]}{(b + c\beta_1^2 + \beta_1^4 + \beta_1^2\rho)^2} dx$$

is the mean value of  $\frac{\partial \chi_1}{\partial x}$  over the period  $L = 2\nu^{-1}K(k)$  of the periodic wave  $\rho$ , and  $f$  is the  $L$ -periodic function with the zero mean. The function  $\chi_1$  remains bounded on the  $(x, t)$  plane along the line

$$k_1(x + 2ct) + 2\beta_1^2(\beta_1^2 - \beta_2^2)t = 0, \quad (75)$$

where we have recalled the transformation (11). The function  $\chi_1(x, t)$  grows linearly in  $|x| + |t| \rightarrow \infty$  in the direction transversal to the line (75).

Recall that the eigenvector  $\varphi_1 = (p_1, q_1)^T$  defines the transformed periodic wave in the form  $\psi_{tr}(x, t) = u_{tr}(x + 2ct)e^{4ibt}$  with

$$u_{tr} = -\frac{\bar{p}_1^2}{p_1^2} \left[ u + 2i\beta_1 \frac{p_1}{\bar{p}_1} \right] e^{-8ibt}. \quad (76)$$

By using the second solution  $\hat{\varphi}_1 = (\hat{p}_1, \hat{q}_1)^T$  given by (68), we define a new solution to the DNLS equation in the form  $\hat{\psi}(x, t) = \hat{u}(x + 2ct)e^{4ibt}$  with

$$\hat{u} = -\frac{\bar{\hat{p}}_1^2}{\hat{p}_1^2} \left[ u + 2i\beta_1 \frac{\hat{p}_1}{\bar{\hat{p}}_1} \right] e^{-8ibt}. \quad (77)$$

To illustrate the two solutions  $u_{tr}$  and  $\hat{u}$ , we consider the periodic standing wave (29) with the particular choice of

$$u_1 = 2, \quad u_2 = 1, \quad u_3 = 0.5, \quad u_4 = 0.$$

This choice corresponds to parameters

$$a = 0, \quad b = -\frac{7}{256}, \quad c = -\frac{7}{8}, \quad d = \frac{1}{8}$$

By using the decomposition (50) and the representations (51) and (52), we deduce from (69) that

$$\frac{\partial \chi_1}{\partial x} = \frac{\beta_1^4(\beta_1^2 - \beta_2^2)[a - 2(c + 2\beta_1^2)\rho - \rho^2]}{2(b + c\beta_1^2 + \beta_1^4 + \beta_1^2\rho)^2}, \quad (71)$$

where we substituted  $\lambda_2 = i\beta_2$  for another purely imaginary eigenvalue. Similarly, we deduce from (70) that

in the quadrature (23) with (24). In particular, parameters satisfy  $c^2 - 4b > 0$  [20].

Two periodic waves  $\rho$  exist for the same values of parameters: the sign-definite wave in  $[u_2, u_1]$  is given by (29) and the sign-indefinite wave in  $[u_4, u_3]$  is given by (35). The sign-definite wave has the period  $L = 2\nu^{-1}K(k)$ , whereas the sign-indefinite wave has the double period  $L = 4\nu^{-1}K(k)$ . The sign-indefinite wave is smoothly represented in the original variable  $R$  in the form

$$R(x) = \frac{\sqrt{2u_2u_3} \operatorname{cn}(\nu x; k)}{\sqrt{u_2 - u_3 \operatorname{sn}^2(\nu x; k)}}. \quad (78)$$

Figure 1 shows the plot of  $\rho$  and  $\rho_{tr} := \frac{1}{2}|u_{tr}|^2$  versus  $x$ . Panel (a) shows the transformation of the sign-definite wave (29) and panel (b) shows the same for the sign-indefinite wave (78).

As is explained in Sec. III, the transformed wave  $\rho_{tr}$  in (76) is different from a translation of the original wave  $\rho$ . In particular,  $\rho$  has turning points  $u_{1,2,3,4}$ , but the transformed wave  $\rho_{tr}$  has turning points  $\nu_{1,2,3,4}$  given by (55). For the sign-definite wave,  $\rho$  changes between  $u_2 = 1$  and  $u_1 = 2$ , whereas  $\rho_{tr}$  changes between  $\nu_4 \approx 0.02$  and  $\nu_3 \approx 0.31$ . For the sign-indefinite wave,  $\rho$  changes between  $u_4 = 0$  and  $u_3 = 0.5$ , whereas  $\rho_{tr}$  changes between  $\nu_2 \approx 0.73$  and  $\nu_1 \approx 2.44$ .

Figure 2 shows the surface plots of  $\hat{\rho} := \frac{1}{2}|\hat{u}|^2$  (a,c) and the contour plots (b,d). We always use the choice  $c_1 = 0$  in (74). The red line on the contour plots shows the line  $x + 2ct = 0$ , whereas the black line shows the line (75).

The new solution in Fig. 2 describes propagation of the algebraic soliton along the direction (75) on the background of the periodic standing wave propagating along the line  $x + 2ct = 0$ . The background periodic wave is given in the limit  $|\chi| \rightarrow \infty$ , where

$$\lim_{|\chi| \rightarrow \infty} \hat{u} = -\frac{\bar{p}_1^2}{p_1^2} \left[ u + 2i\beta_1 \frac{p_1}{\bar{p}_1} \right] e^{-8ibt} = u_{tr}, \quad (79)$$

which coincides with (76). The maximum of the algebraic soliton is located at  $\chi_1 = 0$ , where

$$\lim_{\chi_1 \rightarrow 0} \hat{u} = -\frac{\bar{\hat{p}}_1^2}{\hat{p}_1^2} \left[ u - 2i\beta_1 \frac{\hat{p}_1}{\bar{\hat{p}}_1} \right] e^{-8ibt}. \quad (80)$$

For the sign-definite periodic wave (29), it is shown in Appendix H that the algebraic soliton reaches its maximal



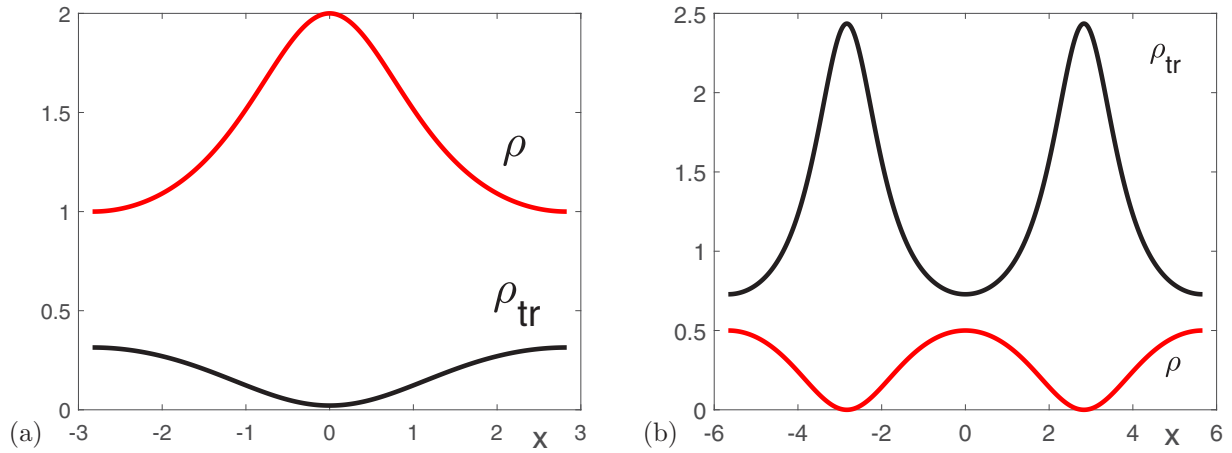


FIG. 1. The periodic standing wave  $\rho$  (red) and its transformed version  $\rho_{tr}$  (black) vs  $x$  for the sign-definite wave (29) (a) and the sign-indefinite wave (78) (b).

value given by

$$\hat{\rho}_{max} = \frac{1}{4}(3\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4})^2. \quad (81)$$

We have computed  $\hat{\rho}_{max} \approx 8.85$ , which coincides with the numerical values in Fig. 2(a).

For the sign-indefinite periodic wave (78), similar computations give

$$\hat{\rho}_{max} = \frac{1}{4}(\sqrt{u_1} + \sqrt{u_2} + 3\sqrt{u_3} + \sqrt{u_4})^2. \quad (82)$$

We have computed  $\hat{\rho}_{max} \approx 5.14$ , which coincides with the numerical values in Fig. 2(c).

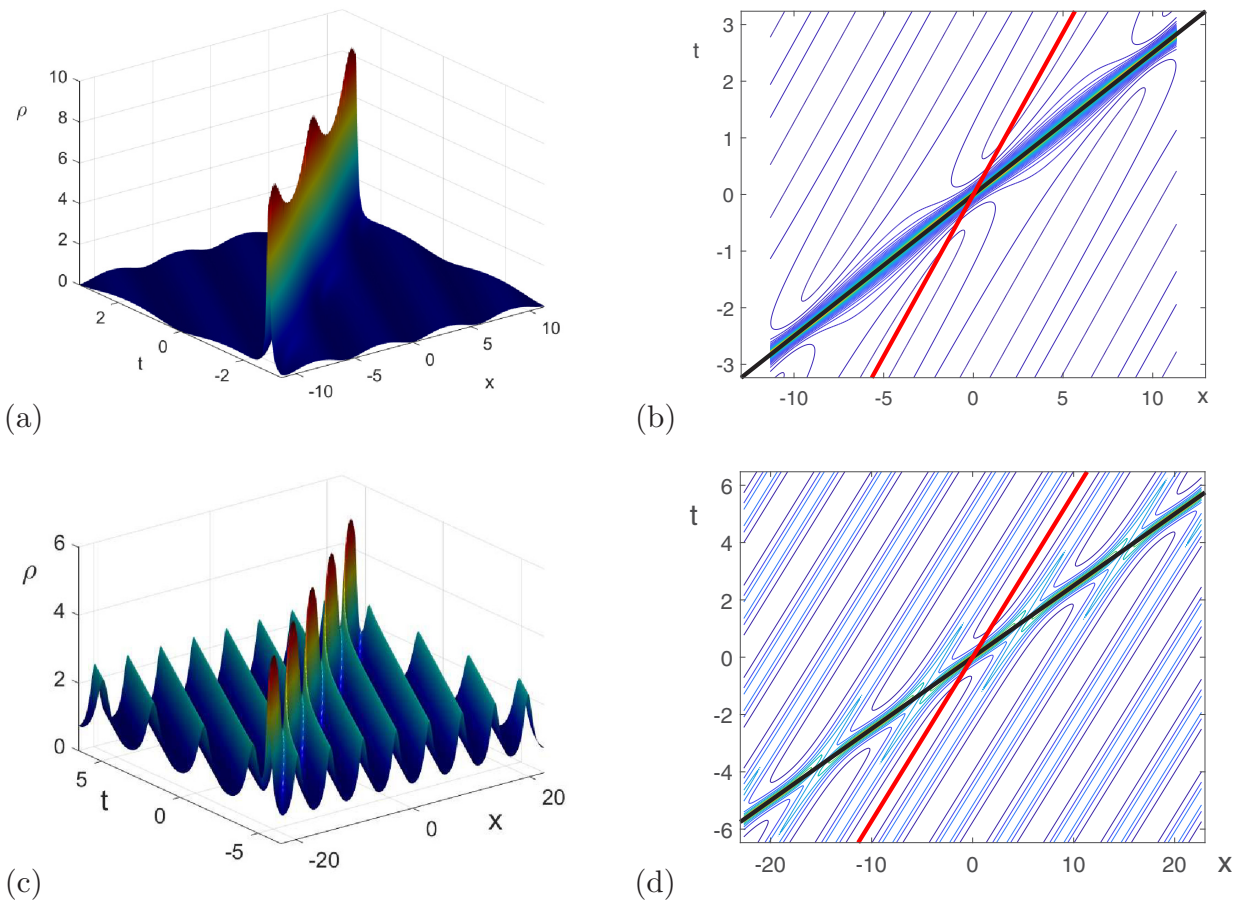


FIG. 2. New solutions to the DNLS equation in variable  $\hat{\rho}$  which describe propagation of an algebraic soliton on the background of the periodic standing wave: solution surfaces (a), (c) and contour plots (b), (d) for the sign-definite wave (29) (a), (b) and the sign-indefinite wave (78) (c), (d).

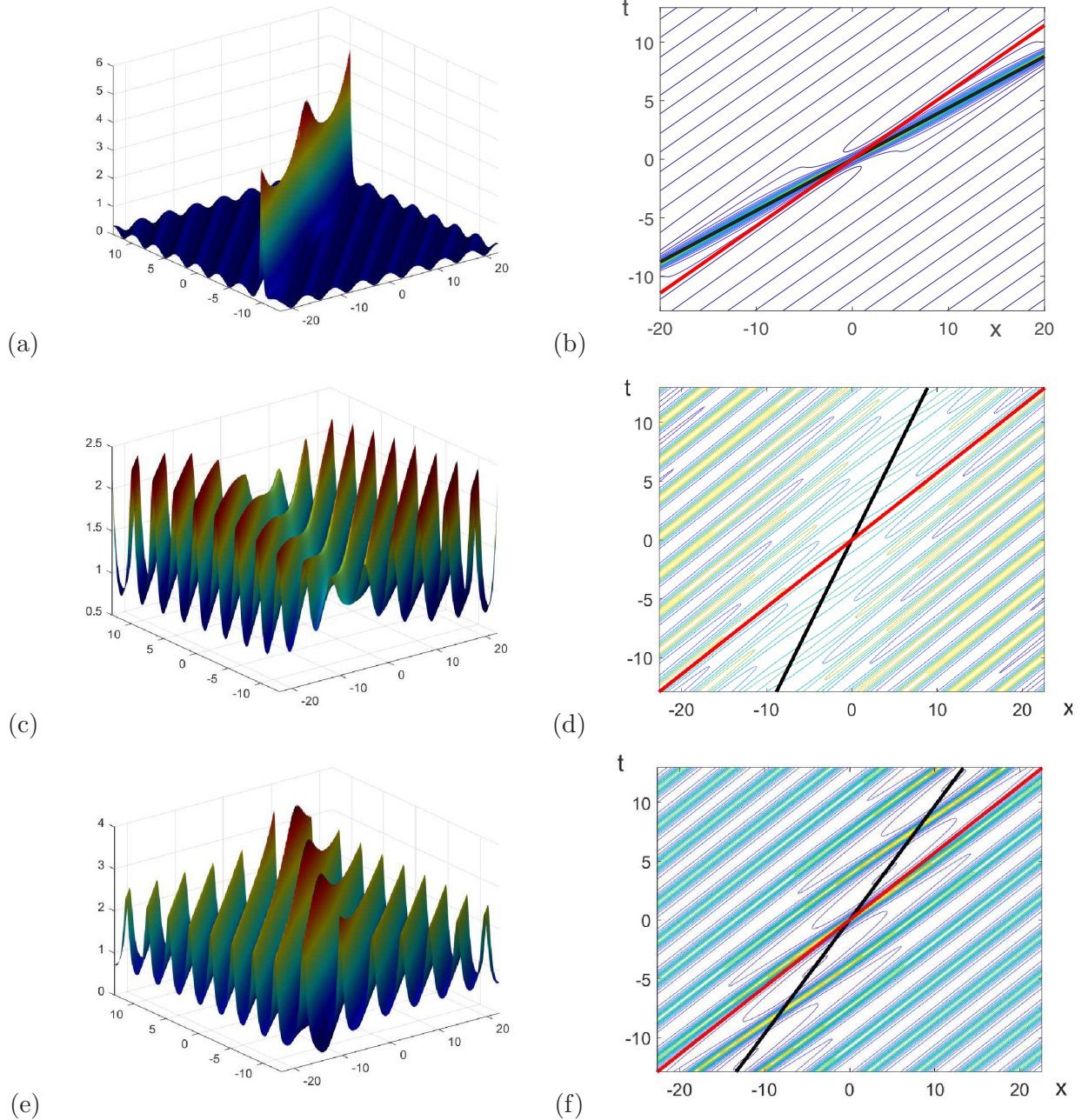


FIG. 3. New solutions to the DNLS equation in variable  $\hat{\rho}$  obtained after the onefold transformation of the periodic wave (29) associated with the eigenvalues  $i\beta_2$  (a), (b),  $i\beta_3$  (c), (d), and  $i\beta_4$  (e), (f).

Figure 2 corresponds to the case of  $\lambda_1 = i\beta_1$ . Figure 3 presents similar results for the sign-definite periodic wave (29) when the onefold transformation is used with the eigenvalues  $\lambda_2 = i\beta_2$  (a), (b),  $\lambda_3 = i\beta_3$  (c), (d), and  $\lambda_4 = i\beta_4$  (e), (f). For the sign-definite periodic wave (29), the transformed wave (76) is the same for  $\lambda_1$  and  $\lambda_2$ ; it is still located between  $v_4 \approx 0.02$  and  $v_3 \approx 0.31$  but it is translated by half-period between the two cases. For the same sign-definite periodic wave (29), the transformed wave (76) is the same for  $\lambda_3$  and  $\lambda_4$ ; it is located between  $v_2 \approx 0.73$  and  $v_1 \approx 2.44$  but it is translated by half-period between the two cases. Note that  $\beta_1 \approx 1.1$ ,  $\beta_2 \approx -0.6$ ,  $\beta_3 \approx -0.1$ , and  $\beta_4 \approx -0.4$ . The algebraic soliton is largest in the case of the largest eigenvalue

$\beta_1$  and smallest in the case of the smallest (in absolute value) eigenvalue  $\beta_3$ . In fact, it is a depression wave in the case of  $\beta_3$ .

The maximal values of  $\hat{\rho}$  are computed similarly to (81) and (82). They correspond to  $2\hat{\beta}_{1,2,3,4}^2$  for each eigenvalue  $\lambda_{1,2,3,4} = i\beta_{1,2,3,4}$ , where  $\hat{\beta}_{1,2,3,4}$  is obtained from  $\beta_{1,2,3,4}$  after  $\sqrt{u_1}$  is replaced by  $3\sqrt{u_1}$  as in (81) and  $\sqrt{u_3}$  is replaced by  $3\sqrt{u_3}$  as in (82). As a result, we obtain  $\hat{\rho}_{\max} \approx 5.14$  for  $i\beta_2$ ,  $\hat{\rho}_{\max} \approx 1.61$  for  $i\beta_3$ , and  $\hat{\rho}_{\max} \approx 3.90$  for  $i\beta_4$ , in agreement with Fig. 3.

Figure 4 presents the new solution to the DNLS equation (1) obtained with the twofold transformation (46) for  $\hat{\rho}_1$  and  $\hat{\rho}_2$ . Here we take one eigenvalue as  $\lambda_1 = i\beta_1$  and the other eigenvalue being  $\lambda_2 = i\beta_2$  (a), (b),  $\lambda_3 = i\beta_3$  (c), (d), and  $\lambda_4 =$

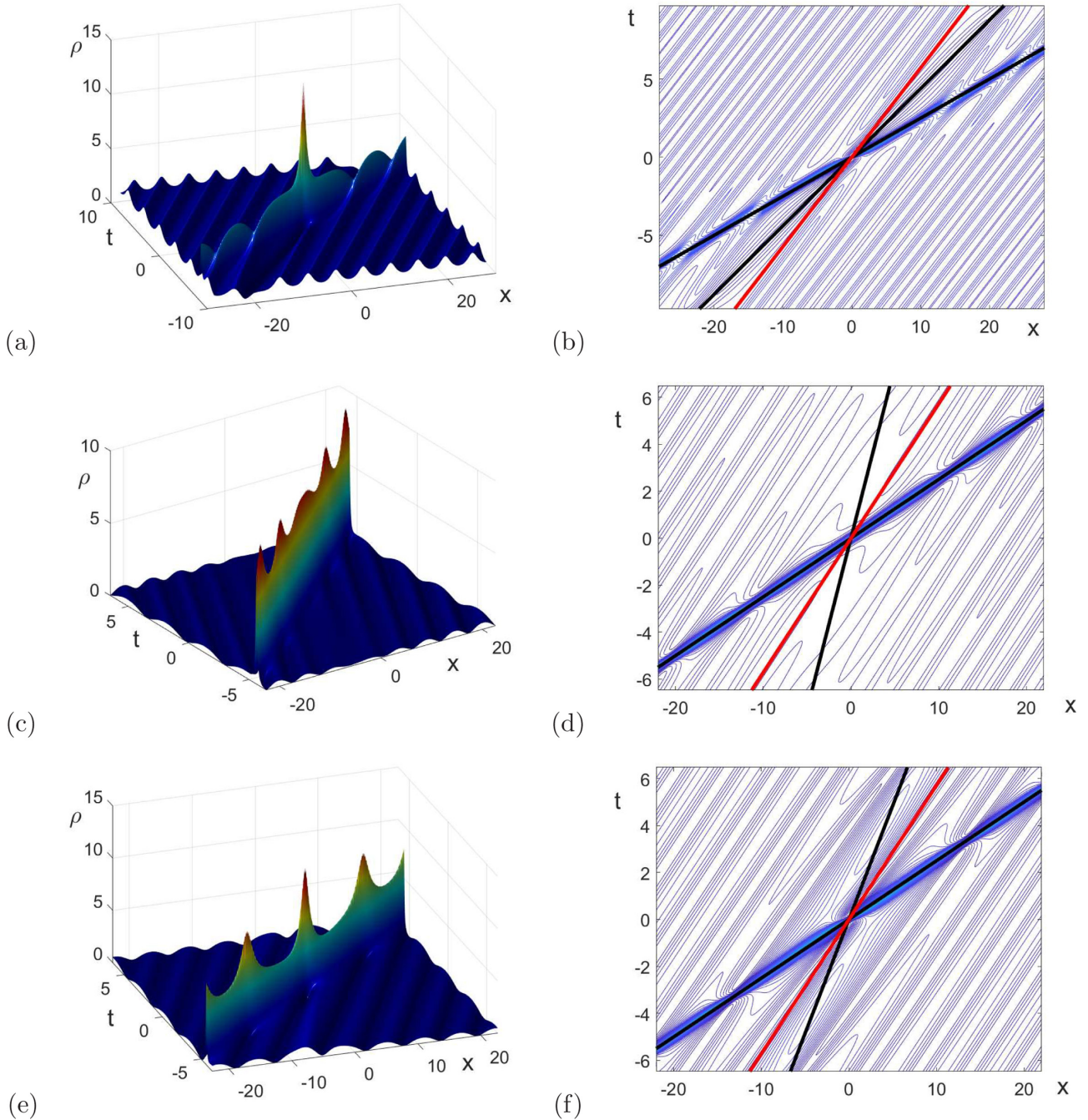


FIG. 4. New solutions to the DNLS equation in variable  $\hat{\rho}$  obtained after the twofold transformation of the periodic wave (29). One eigenvalue is  $i\beta_1$ , whereas the other eigenvalue is  $i\beta_2$  (a), (b),  $i\beta_3$  (c), (d), and  $i\beta_4$  (e), (f).

$i\beta_4$  (e), (f). The second solution  $\hat{\varphi}_1 = (\hat{\rho}_1, \hat{q}_1)^T$  is defined by (68) with  $\chi_1$  given by (74). We always take  $c_1 = 0$  in (74) so that the algebraic solitons propagate along the corresponding lines (75). The line  $x + 2ct = 0$  is shown by the red curve and the two lines (75) for the two eigenvalues are shown by the black curves in Figs. 4(b), 4(d) and 4(f).

The new solution describes interaction of the two algebraic solitons on the background of the transformed wave obtained with the twofold transformation (46) for  $p_1$  and  $p_2$ . As was established earlier, for  $\lambda_1$  and  $\lambda_2$ , the background wave is the same as the original sign-definite wave (29) but translated by a half-period with  $\rho_{tr} \in [u_2, u_1]$ . For  $\lambda_1$  and either  $\lambda_3$  or  $\lambda_4$ , the

background wave corresponds to the sign-indefinite wave (78) with  $\rho_{tr} \in [u_4, u_3]$ . In the latter cases, the second algebraic soliton is almost invisible in Figs. 4(c) and 4(e). We have checked by taking nonzero  $c_1$  in (74) that the two algebraic solitons become visible when they overlap at a point on the  $(x, t)$  plane away from the origin. However, when we take  $c_1 = 0$  in (74) for both eigenvalues, the two algebraic solitons overlap at the origin.

#### B. Periodic wave (29) with $\lambda_1$ in a complex quadruplet

Let  $\lambda_1 \in \mathbb{C} \setminus i\mathbb{R}$  be an eigenvalue of the KN spectral problem (10). We use the decomposition (5) and (11) with the

eigenvector  $\varphi_1 = (p_1, q_1)^T$  satisfying the linear system

$$\begin{aligned} \frac{\partial p_1}{\partial x} &= -i\lambda_1^2 p_1 + \lambda_1 u q_1, \\ \frac{\partial q_1}{\partial x} &= i\lambda_1^2 q_1 - \lambda_1 \bar{u} p_1, \end{aligned} \tag{83}$$

and

$$\begin{aligned} \frac{\partial p_1}{\partial t} + 2c \frac{\partial p_1}{\partial x} + 2ibp_1 &= i(-2\lambda_1^4 + \lambda_1^2 |u|^2) p_1 + 2\lambda_1^3 u q_1 + \lambda_1 (iu_x - |u|^2 u) q_1, \\ \frac{\partial q_1}{\partial t} + 2c \frac{\partial q_1}{\partial x} - 2ibq_1 &= i(2\lambda_1^4 - \lambda_1^2 |u|^2) q_1 - 2\lambda_1^3 \bar{u} p_1 + \lambda_1 (i\bar{u}_x + |u|^2 \bar{u}) p_1. \end{aligned} \tag{84}$$

The second, linearly independent solution  $\hat{\varphi}_1 = (\hat{p}_1, \hat{q}_1)^T$  of the system (83) and (84) can be written in the form

$$\begin{aligned} \hat{p}_1 &= p_1 \chi_1 - \frac{\bar{q}_1}{|p_1|^2 + |q_1|^2}, \\ \hat{q}_1 &= q_1 \chi_1 + \frac{\bar{p}_1}{|p_1|^2 + |q_1|^2}, \end{aligned} \tag{85}$$

where  $\chi_1$  is a complex-valued function of  $x$  and  $t$ . Wronskian between the two solutions is normalized by  $p_1 \hat{q}_1 - \hat{p}_1 q_1 = 1$ .

Substituting (85) into (83) and (84) written for  $\hat{\varphi}_1 = (\hat{p}_1, \hat{q}_1)^T$  and using the same equations for  $\varphi_1 = (p_1, q_1)^T$  yields the following equations for  $\chi_1$ :

$$\frac{\partial \chi_1}{\partial x} = \frac{2i(\lambda_1^2 - \bar{\lambda}_1^2) \bar{p}_1 \bar{q}_1 + (\lambda_1 - \bar{\lambda}_1)(u \bar{p}_1^2 + \bar{u} \bar{q}_1^2)}{(|p_1|^2 + |q_1|^2)^2} \tag{86}$$

and

$$\begin{aligned} \frac{\partial \chi_1}{\partial t} + 2c \frac{\partial \chi_1}{\partial x} &= \frac{1}{(|p_1|^2 + |q_1|^2)^2} [2i \bar{p}_1 \bar{q}_1 (\lambda_1^2 - \bar{\lambda}_1^2) [2(\lambda_1^2 + \bar{\lambda}_1^2) - |u|^2] + 2(\lambda_1^3 - \bar{\lambda}_1^3)(u \bar{p}_1^2 + \bar{u} \bar{q}_1^2) \\ &\quad + (\lambda_1 - \bar{\lambda}_1) [i(u_x \bar{p}_1^2 - \bar{u}_x \bar{q}_1^2) - |u|^2 (u \bar{p}_1^2 + \bar{u} \bar{q}_1^2)]]. \end{aligned} \tag{87}$$

Substituting (51) and (52) into (86) yields

$$\frac{\partial \chi_1}{\partial x} = \frac{\bar{\lambda}_1 [\rho^2 + 2(c - 2\bar{\lambda}_1^2) \rho - a] - 2(\lambda_1 + \bar{\lambda}_1)(b - c\bar{\lambda}_1^2 + \bar{\lambda}_1^4 - \bar{\lambda}_1^2 \rho)}{\bar{\lambda}_1^2 (\lambda_1 + \bar{\lambda}_1) (|p_1|^2 + |q_1|^2)^2}. \tag{88}$$

Regarding (87), it must again be constant. It is shown in Appendix G that

$$\frac{\partial \chi_1}{\partial t} = 2\lambda_1^2 (\lambda_1^2 - \bar{\lambda}_1^2). \tag{89}$$

We obtain from (88) and (89) that

$$\chi_1(x, t) = c_1 + k_1 x + f(x) + 2\lambda_1^2 (\lambda_1^2 - \bar{\lambda}_1^2) t, \tag{90}$$

where  $c_1 \in \mathbb{C}$  is an arbitrary constant of integration,  $k_1$  is the mean value of  $\frac{\partial \chi_1}{\partial x}$  over the period  $L = 2\nu^{-1}K(k)$  of the periodic wave  $\rho$ , and  $f$  is the  $L$ -periodic function with the zero mean. The line equation

$$k_1(x + 2ct) + 2\lambda_1^2 (\lambda_1^2 - \bar{\lambda}_1^2) t = 0 \tag{91}$$

is now complex-valued, hence it defines two straight lines on the  $(x, t)$  plane. If the straight lines have different slopes, they only intersect at  $(x, t) = (0, 0)$  and this implies that the function  $\chi_1(x, t)$  grows linearly as  $|x| + |t| \rightarrow \infty$  everywhere in the  $(x, t)$  plane. Consequently, the new solution obtained with the Darboux transformation (47) at the second solution  $\hat{\varphi}_1 = (\hat{p}_1, \hat{q}_1)^T$  given by (85) displays the rogue wave localized on the transformed periodic wave. The transformed periodic wave is obtained with the Darboux transformation (47) at the first solution  $\varphi_1 = (p_1, q_1)^T$ .

To illustrate the two solutions, we consider the periodic standing wave (29) with the particular choice of

$$u_1 = 2, \quad u_2 = 1, \quad u_3 = 0, \quad u_4 = -0.5.$$

This choice corresponds to parameters

$$a = 0, \quad b = \frac{17}{256}, \quad c = -\frac{5}{8}, \quad d = -\frac{1}{8}$$

in the quadrature (23) with (24). Again, we preserve the constraint  $c^2 - 4b > 0$ .

Figure 5 shows the periodic standing wave  $\rho$  (red) and its transformed version  $\rho_{tr}$  (black) after the twofold transformation (47) with  $\varphi_1 = (p_1, q_1)^T$ . In agreement with (60), the transformed wave is a half-period translation of the original wave. Moreover, the same translation is true for both quadruplets in (31).

Figure 6 shows the rogue wave  $\hat{\rho}$  on the background of the periodic standing wave  $\rho$  in (29) with the same parameters as in Fig. 5 after the twofold transformation (47) with  $\hat{\varphi}_1 = (\hat{p}_1, \hat{q}_1)^T$ . Panel (a) corresponds to the quadruplet with  $\lambda_1$  and panel (b) corresponds to the quadruplet with  $\lambda_2$ . Although the

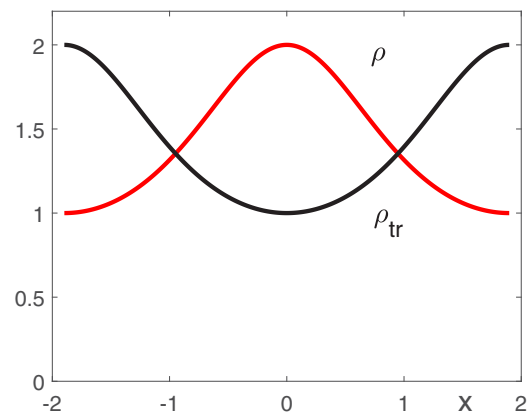


FIG. 5. The periodic standing wave  $\rho$  (red) and its transformed version  $\rho_{tr}$  (black).

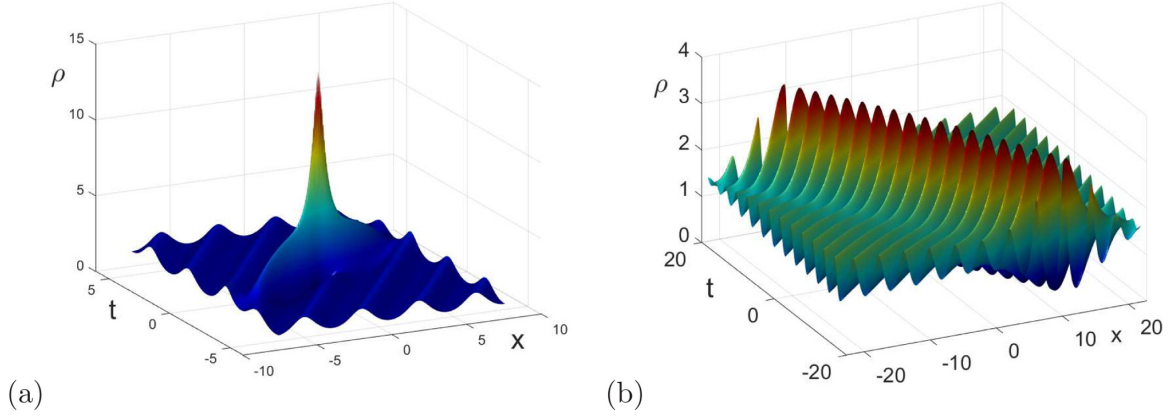


FIG. 6. New solutions to the DNLS equation in variable  $\hat{\rho}$ , which corresponds to the rogue waves on the background of the periodic standing wave. Panels (a) and (b) correspond to (29) after the Darboux transformation (47) with  $\hat{\varphi}_1 = (\hat{\rho}_1, \hat{q}_1)^T$  for eigenvalues  $\lambda_1$  and  $\lambda_2$ , respectively.

surface plot in Fig. 6(b) does not show localization of the rogue wave on the scale displayed, we have checked that the real and imaginary part of Eq. (91) give two different lines intersecting at (0,0), but the slopes of the two lines are close to each other. As a result,  $\text{Re}(\chi)$  and  $\text{Im}(\chi)$  are bounded along two different directions, hence the rogue wave  $\hat{\rho}$  is still localized in space and time.

It is shown in Appendix I that the maximum of the rogue wave occurs at the point (0,0) if  $c_1 = 0$  and it is given by

$$\hat{\rho}_{\max} = (2\sqrt{u_1} + \sqrt{u_2})^2 \quad (92)$$

for eigenvalue  $\lambda_1$  and

$$\hat{\rho}_{\max} = (2\sqrt{u_1} - \sqrt{u_2})^2 \quad (93)$$

for eigenvalue  $\lambda_2$ . We have  $\hat{\rho}_{\max} \approx 14.66$  for eigenvalue  $\lambda_1$  and  $\hat{\rho}_{\max} \approx 3.34$  for eigenvalue  $\lambda_2$ , which agree with the numerical values in Fig. 6.

### C. Darboux transformations for the periodic wave (39)

We end this section with an example of the periodic standing wave (39) for the particular choice

$$u_1 = 2, \quad u_2 = 0, \quad \gamma = 0.2, \quad \eta = 0.1.$$

This choice corresponds to parameters

$$a = 0, \quad b = 0.036, \quad c = -0.6, \quad d = 0.0125$$

in the quadrature (23) and (24) satisfying  $c^2 - 4b > 0$ . The expression (39) with  $u_2 = 0$  can be written as

$$\rho(x) = u_1 \delta \frac{1 + \text{cn}(\mu x; k)}{1 + \delta + (\delta - 1)\text{cn}(\mu x; k)}. \quad (94)$$

The solution (94) is sign-indefinite since  $\rho \in [0, u_1]$ . Extracting the square root analytically yields the exact expression

$$R(x) = \frac{\sqrt{2u_1} \delta \text{cn}(\frac{1}{2}\mu x; k)}{\sqrt{\delta \text{cn}^2(\frac{1}{2}\mu x; k) + \text{sn}^2(\frac{1}{2}\mu x; k) \text{dn}^2(\frac{1}{2}\mu x; k)}}. \quad (95)$$

The period of the periodic wave is  $L = 8K(k)\mu^{-1}$ .

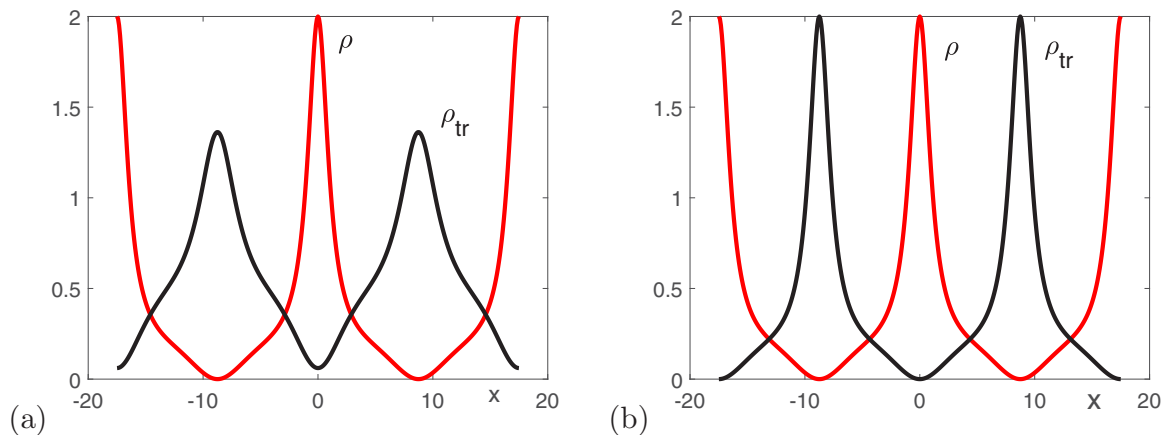


FIG. 7. The periodic standing wave  $\rho$  (red) and its transformed version  $\rho_{\text{tr}}$  (black) vs  $x$  for the periodic standing wave (94). Panels (a) and (b) correspond to the eigenvalues  $\lambda_3 = i\beta_3$  and  $\lambda_1 = \alpha_1 + i\beta_1$ , respectively.

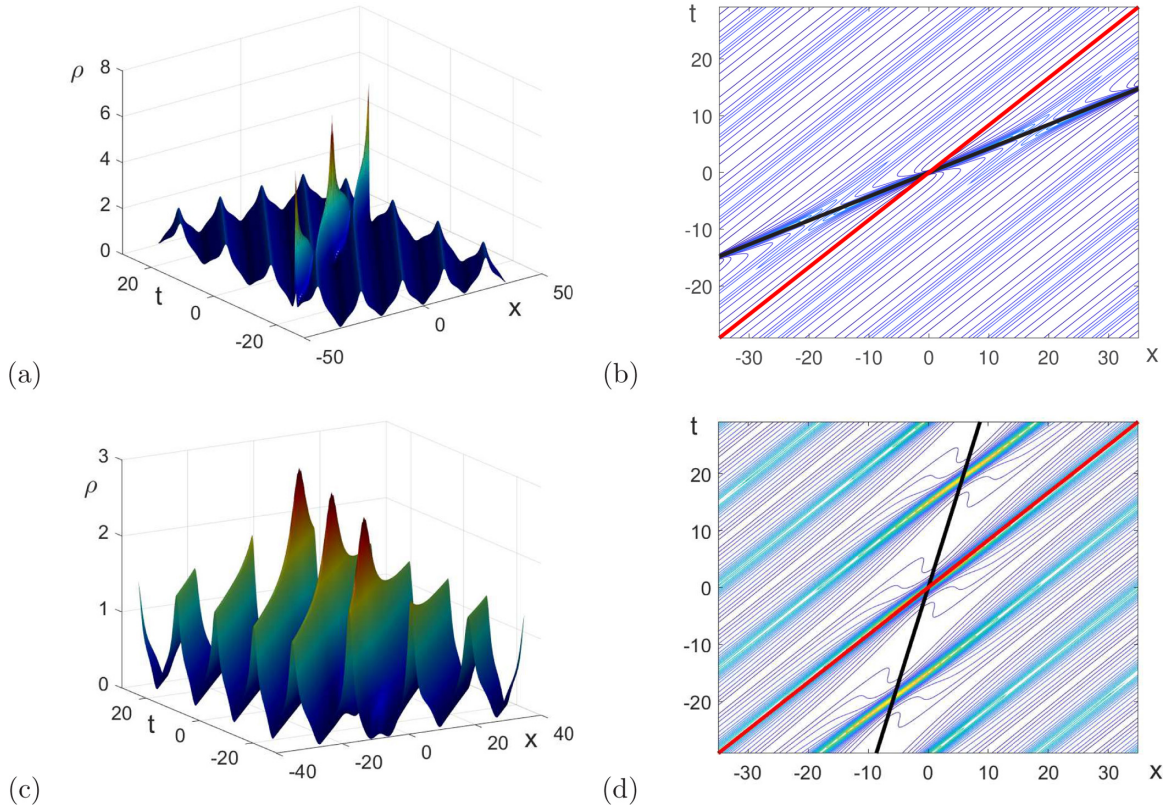


FIG. 8. New solutions to the DNLS equation which corresponds to the propagation of an algebraic soliton on the background of the periodic standing wave (94). Solutions are obtained by the transformation (77) with the eigenvalues  $\lambda_3 = i\beta_3$  (a,b) and  $\lambda_4 = i\beta_4$  (c,d).

The periodic wave (94) corresponds to the configuration (42) with one complex quadruplet  $\{\lambda_1, \bar{\lambda}_1, -\lambda_1, -\bar{\lambda}_1\}$  and two pairs of purely imaginary eigenvalues  $\{\pm i\beta_3, \pm i\beta_4\}$ .

Figure 7(a) shows the plot of  $\rho$  and  $\rho_{tr} := \frac{1}{2}|u_{tr}|^2$  versus  $x$  after the transformation (76) for the eigenvalue  $\lambda_3 = i\beta_3$ . The transformed wave for the eigenvalue  $\lambda_4 = i\beta_4$  is a half-period translation of  $\rho_{tr}$  for the eigenvalue  $\lambda_3 = i\beta_3$ . Figure 7(b) shows  $\rho$  and  $\rho_{tr}$  after the twofold transformation (47) with the complex eigenvalue  $\lambda_1 = \alpha_1 + i\beta_1$ .

Figure 8 shows the surface plots of  $\hat{\rho} := \frac{1}{2}|\hat{u}|^2$  (a), (c) and the contour plots (b), (d) after the transformation (77) associated with the eigenvalues  $\lambda_3 = i\beta_3$  (a), (b) and  $\lambda_4 = i\beta_4$

(c), (d). The red line on the contour plots shows the line  $x + 2ct = 0$  and the black line shows the line (75). We can see that the algebraic soliton propagates along this direction and is modulated due to interaction with the periodic standing wave.

The maxima of algebraic solitons are given by

$$\hat{\rho}_{\max} = \frac{1}{4} \left( 3\sqrt{u_1} - \sqrt{u_2} \pm \sqrt{2(\sqrt{\gamma^2 + \eta^2} + \gamma)} \right)^2, \quad (96)$$

where the upper sign is for the eigenvalue  $\lambda_3 = i\beta_3$  and the lower sign is for the eigenvalue  $\lambda_4 = i\beta_4$ . For the parameters in Fig. 8, we have excellent agreement with  $\hat{\rho}_{\max} \approx 6.66$  for  $\lambda_3 = i\beta_3$  and  $\hat{\rho}_{\max} \approx 2.76$  for  $\lambda_4 = i\beta_4$ .

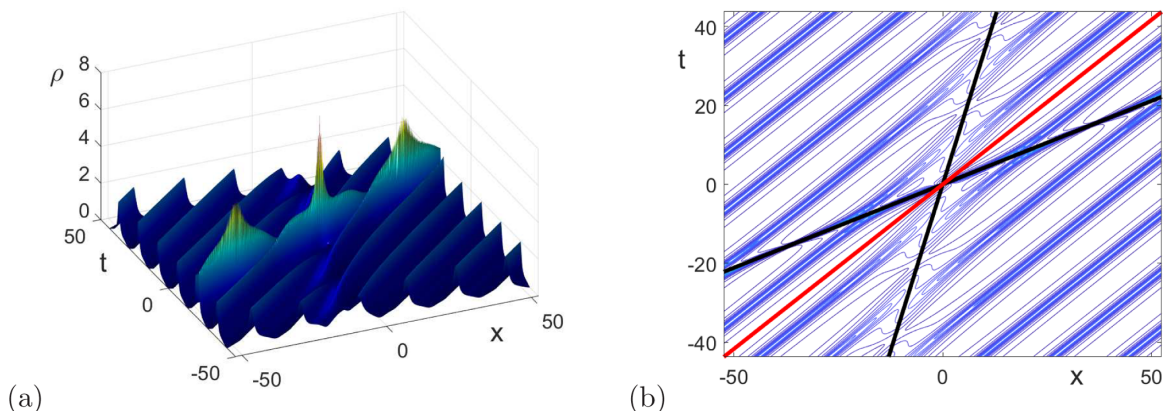


FIG. 9. New solutions to the DNLS equation obtained by the twofold transformation with the two eigenvalues  $\lambda_3 = i\beta_3$  and  $\lambda_4 = i\beta_4$ . Two algebraic solitons propagate along the lines shown by black curves.

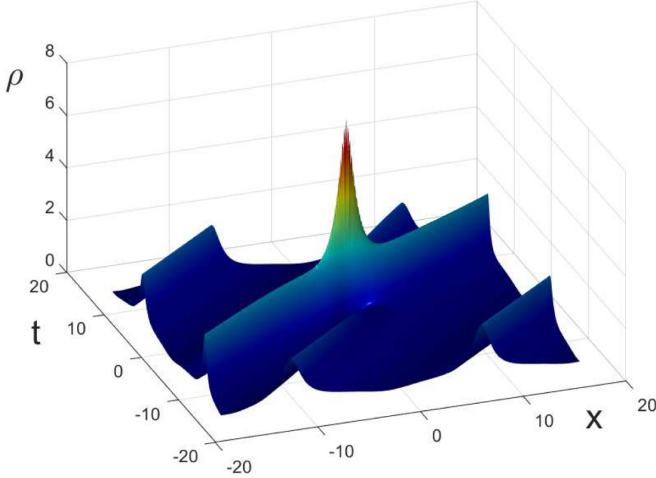


FIG. 10. New solution to the DNLS equation in variable  $\hat{\rho}$  obtained by the twofold transformation with the quadruplet  $\lambda_1 = \alpha_1 + i\beta_1$ . A rogue wave is formed on the background of the periodic standing wave (94).

Figure 9 shows the result of the twofold transformation (46) associated with the two eigenvalues  $\lambda_3 = i\beta_3$  and  $\lambda_4 = i\beta_4$ . The background wave is the same as in Fig. 7(b), that is, it is a half-period translation of the periodic standing wave (39). Two algebraic solitons propagate along the lines (75) shown by black curves in Fig. 9(b) together with the line  $x + 2ct = 0$  shown by the red curve.

Finally, Fig. 10 shows the result of the twofold transformation (47) associated with the quadruplet of complex eigenvalue  $\lambda_1 = \alpha_1 + i\beta_1$ . The surface plot of  $\hat{\rho}$  indicates that the rogue wave is fully localized on the background of the periodic standing waves. This is explained again by the fact that the real and imaginary parts of the complex-valued equation (91) give two lines intersecting at the only point (0,0). The maximal amplitude is given by the formula (92) with  $\hat{\rho}_{\max} = 8$ .

## V. CONCLUSION

We have studied the rogue waves and algebraic solitons arising on the background of periodic standing waves in the derivative NLS equation. By using comprehensive analysis and numerical visualizations for selected parameter values, we showed that the modulationally stable periodic standing waves support propagation of algebraic solitons along the background, whereas the modulationally unstable periodic standing waves are associated with the rogue waves localized in space and time. Although only very few numerical experiments were displayed, our analysis ensures that the conclusion extends to all periodic standing waves of the DNLS equation (both sign-definite and sign-indefinite). We have also derived the exact expressions for the maximal amplitudes of the rogue waves and the associated algebraic solitons.

This work paves the way to an understanding of the rogue wave phenomenon in the DNLS equation, which is one of the canonical models for the dynamics of waves in plasma physics and optics. The recent results complement the study of rogue waves on the constant-amplitude wave background

[19], maximal amplitudes of hyperelliptic solutions [36], and modulational instability of periodic standing waves [20]. The next tasks would involve setting up physical experiments and confirming the modulational instability of the periodic standing waves and the maximal amplitudes of rogue waves as was done in [14] with hydrodynamical and optical experiments. It is also interesting to understand rogue wave phenomena in a more general setting of quasiperiodic solutions of the DNLS equation studied in [25].

## ACKNOWLEDGMENTS

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## APPENDIX A: PROOF OF ONEFOLD DARBOUX TRANSFORMATION

For convenience, let us denote the solution of the DNLS equation (1) by  $\psi = u$  and the solution of the Lax equations (2) for the eigenvalue  $\lambda = \lambda_1$  by  $\phi_1 = (p_1, q_1)^T$ . Let  $T(u, \lambda)$  be the gauge transformation in the form

$$T(u, \lambda) = \lambda \begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix} + \begin{pmatrix} 0 & \lambda_1 \\ \lambda_1 & 0 \end{pmatrix}, \quad (\text{A1})$$

where

$$A_1 = -\frac{q_1}{p_1}, \quad D_1 = -\frac{p_1}{q_1}, \quad (\text{A2})$$

so that  $A_1 D_1 = 1$  and  $\det T(u, \lambda) = \lambda^2 - \lambda_1^2$ . We impose the constraints on  $T(u, \lambda)$  from the condition that if  $\phi$  is a solution of the Lax equations (2) with  $u$ , then  $\hat{\phi} = T(u, \lambda)\phi$  is also a solution of the Lax equations (2) with a new solution of the DNLS equation (1), which we denote by  $\hat{u}$ . This condition yields the Darboux equations:

$$\begin{aligned} U(\hat{u}, \lambda) &= (T_x + TU)T^{-1}, \\ V(\hat{u}, \lambda) &= (T_t + TV)T^{-1}, \end{aligned} \quad (\text{A3})$$

where  $U = U(u, \lambda)$  and  $T = T(u, \lambda)$ . By using (A1) and (A3), we obtain the following system in different powers of  $\lambda$ :

$$\begin{aligned} A_{1,x} - \lambda_1 \bar{u} &= \lambda_1 \hat{u}, \\ uA_1 + 2i\lambda_1 &= \hat{u}D_1, \\ \bar{u}D_1 + 2i\lambda_1 &= \bar{\hat{u}}A_1, \\ D_{1,x} + \lambda_1 u &= -\lambda_1 \bar{\hat{u}} \end{aligned} \quad (\text{A4})$$

and

$$\begin{aligned} iA_1|u|^2 - 2\lambda_1 \bar{u} &= iA_1|\hat{u}|^2 + 2\lambda_1 \hat{u}, \\ A_{1,t} + \lambda_1(i\bar{u}_x + |u|^2 \bar{u}) &= \lambda_1(i\hat{u}_x - |\hat{u}|^2 \hat{u}), \\ A_1(iu_x - |u|^2 u) - i\lambda_1|u|^2 &= D_1(i\hat{u}_x - |\hat{u}|^2 \hat{u}) + i\lambda_1|\hat{u}|^2, \\ D_1(i\bar{u}_x + |u|^2 \bar{u}) + i\lambda_1|u|^2 &= A_1(i\bar{\hat{u}}_x + |\hat{u}|^2 \bar{\hat{u}}) - i\lambda_1|\hat{u}|^2, \\ -iD_1|u|^2 + 2\lambda_1 u &= -iD_1|\hat{u}|^2 - 2\lambda_1 \bar{\hat{u}}, \\ D_{1,t} + \lambda_1(iu_x - |u|^2 u) &= \lambda_1(i\bar{\hat{u}}_x + |\hat{u}|^2 \bar{\hat{u}}). \end{aligned} \quad (\text{A5})$$

It follows from (A2) and the Lax equations (2) that

$$\begin{aligned} A_{1,x} &= \lambda_1 \bar{u} + \lambda_1 u A_1^2 + 2i\lambda_1^2 A_1, \\ D_{1,x} &= -\lambda_1 u - \lambda_1 \bar{u} D_1^2 - 2i\lambda_1^2 D_1, \end{aligned} \tag{A6}$$

and

$$\begin{aligned} A_{1,t} &= 2\lambda_1^3 \bar{u} - \lambda_1 (i\bar{u}_x + |u|^2 \bar{u}) + 4i\lambda_1^4 A_1 \\ &\quad - 2i\lambda_1^2 A_1 |u|^2 + 2\lambda_1^3 A_1^2 u + \lambda_1 A_1^2 (iu_x - |u|^2 u), \\ D_{1,t} &= -2\lambda_1^3 u - \lambda_1 (iu_x - |u|^2 u) - 4i\lambda_1^4 D_1 + 2i\lambda_1^2 D_1 |u|^2 \\ &\quad - 2\lambda_1^3 D_1^2 \bar{u} + \lambda_1 D_1^2 (i\bar{u}_x + |u|^2 \bar{u}). \end{aligned} \tag{A7}$$

Substituting (A6) into (A4) shows that the first and fourth equations of system (A4) are redundant, whereas the other two

equations produce the following transformation formulas:

$$\hat{u} = u \frac{q_1^2}{p_1^2} - 2i\lambda_1 \frac{q_1}{p_1}, \quad \hat{\bar{u}} = \bar{u} \frac{p_1^2}{q_1^2} - 2i\lambda_1 \frac{p_1}{q_1}. \tag{A8}$$

The complex-conjugate reduction in (A8) is satisfied if  $|p_1| = |q_1|$  and  $\lambda_1 + \bar{\lambda}_1 = 0$ . Hence,  $\lambda_1 = i\beta_1 \in i\mathbb{R}$ . Selecting  $q_1 = -i\bar{p}_1$  as in the reduction (14) gives the onefold transformation formula (45) from system (A8).

In view of relations (A8), the first and fifth equations in system (A5) are redundant. Substituting (A7) into (A5) shows that the second and sixth equations of system (A5) are redundant. Finally, the third and fourth equations of system (A5) are satisfied when we substitute derivatives of (A8) in  $x$  and use relations (A6).

**APPENDIX B: PROOF OF TWOFOLD DARBOUX TRANSFORMATION**

Let  $\phi_1 = (p_1, q_1)^T$  and  $\phi_2 = (p_2, q_2)^T$  be two solutions of the Lax equations (2) for eigenvalues  $\lambda = \lambda_1$  and  $\lambda = \lambda_2$  satisfying  $\lambda_1^2 \neq \lambda_2^2$ . Let  $T(u, \lambda)$  be the gauge transformation in the form

$$T(u, \lambda) = \begin{pmatrix} \mathcal{T}_{11}(\lambda) & \mathcal{T}_{12}(\lambda) \\ \mathcal{T}_{21}(\lambda) & \mathcal{T}_{22}(\lambda) \end{pmatrix} = \lambda^2 \mathcal{T}_2 + \lambda \mathcal{T}_1 + \mathcal{T}_0, \tag{B1}$$

where

$$\mathcal{T}_2 = \begin{pmatrix} A_2 & 0 \\ 0 & D_2 \end{pmatrix}, \quad \mathcal{T}_1 = \begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix}, \quad \mathcal{T}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix}, \tag{B2}$$

with

$$A_2 = \frac{\lambda_1 \alpha_1 - \lambda_2 \alpha_2}{\lambda_1 \alpha_2 - \lambda_2 \alpha_1}, \quad B_1 = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_1 \alpha_2 - \lambda_2 \alpha_1}, \quad C_1 = \frac{\alpha_1 \alpha_2 (\lambda_2^2 - \lambda_1^2)}{\lambda_1 \alpha_1 - \lambda_2 \alpha_2}$$

and

$$A_0 = \lambda_1 \lambda_2, \quad A_2 D_2 = 1, \quad \alpha_1 = \frac{q_1}{p_1}, \quad \alpha_2 = \frac{q_2}{p_2}.$$

It follows from (2) that  $\alpha_k$  for  $k = 1, 2$  satisfy the following Riccati equations:

$$\alpha_{kx} = -\lambda_k \bar{u} + 2i\lambda_k^2 \alpha_k - \lambda_k u \alpha_k^2, \tag{B3}$$

$$\alpha_{kt} = -2\lambda_k^3 \bar{u} + \lambda_k (i\bar{u}_x + |u|^2 \bar{u}) + 2i\lambda_k^2 \alpha_k (2\lambda_k^2 - |u|^2) - \alpha_k^2 [2\lambda_k^3 u + \lambda_k (iu_x - |u|^2 u)]. \tag{B4}$$

In addition, we check that

$$\det T(u, \lambda) = (\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2). \tag{B5}$$

By using the Darboux equations (A3), we show that the new solution  $\hat{u}$  is expressed by

$$\hat{u} = A_2^2 u + 2iA_2 B_1, \quad \hat{\bar{u}} = D_2^2 \bar{u} + 2iC_1 D_2. \tag{B6}$$

Let  $T^* := T^{-1} \det T$  be the adjugate matrix of  $T(u, \lambda)$  and

$$(T_x + TU)T^* = \begin{pmatrix} \mathcal{F}_{11}(\lambda, u) & \mathcal{F}_{12}(\lambda, u) \\ \mathcal{F}_{21}(\lambda, u) & \mathcal{F}_{22}(\lambda, u) \end{pmatrix}. \tag{B7}$$

It follows from (3) and (B1) that  $\mathcal{F}_{11}(\lambda, u)$  and  $\mathcal{F}_{22}(\lambda, u)$  are the sixth-order polynomials in  $\lambda$ , and  $\mathcal{F}_{12}(\lambda, u)$  and  $\mathcal{F}_{21}(\lambda, u)$  are the fifth-order polynomials in  $\lambda$ . It follows from (B1) and (B2) that for any  $k = 1, 2$ ,

$$\mathcal{T}_{11}(\lambda_k) = -\mathcal{T}_{12}(\lambda_k) \alpha_k, \quad \mathcal{T}_{11,x}(\lambda_k) = -\mathcal{T}_{12,x}(\lambda_k) \alpha_k - \mathcal{T}_{12}(\lambda_k) \alpha_{kx}, \tag{B8}$$

$$\mathcal{T}_{21}(\lambda_k) = -\mathcal{T}_{22}(\lambda_k) \alpha_k, \quad \mathcal{T}_{21,x}(\lambda_k) = -\mathcal{T}_{22,x}(\lambda_k) \alpha_k - \mathcal{T}_{22}(\lambda_k) \alpha_{kx}, \tag{B9}$$

and

$$\begin{aligned} \mathcal{T}_{11}(-\lambda_k) &= \mathcal{T}_{11}(\lambda_k), & \mathcal{T}_{12}(-\lambda_k) &= -\mathcal{T}_{12}(\lambda_k), \\ \mathcal{T}_{21}(-\lambda_k) &= -\mathcal{T}_{21}(\lambda_k), & \mathcal{T}_{22}(-\lambda_k) &= \mathcal{T}_{22}(\lambda_k). \end{aligned} \tag{B10}$$



It follows from (B3) that  $\pm\lambda_1$  and  $\pm\lambda_2$  are the roots of  $\mathcal{F}_{ij}(\lambda, u)$  for every  $i, j = 1, 2$ . For instance, since

$$\mathcal{F}_{11}(\lambda, u) = \mathcal{T}_{22}(-i\lambda^2\mathcal{T}_{11} - \lambda\bar{u}\mathcal{T}_{12} + \mathcal{T}_{11,x}) - \mathcal{T}_{21}(\lambda u\mathcal{T}_{11} + i\lambda^2\mathcal{T}_{12} + \mathcal{T}_{12,x}),$$

we confirm

$$\mathcal{F}_{11}(\pm\lambda_k, u) = \mathcal{T}_{12}(\lambda_k)\mathcal{T}_{22}(\lambda_k)(-\alpha_{kx} - \lambda_k\bar{u} + 2i\lambda_k^2\alpha_k - \lambda_k u\alpha_k^2) = 0.$$

Dividing both sides of (B7) by  $\det T$  and using (B5) yields

$$U(\hat{u}, \lambda) := \lambda^2\mathcal{U}_2 + \lambda\mathcal{U}_1,$$

where the matrices  $\mathcal{U}_1$  and  $\mathcal{U}_2$  are independent of  $\lambda$ . Substituting these expressions into the first Darboux equation (A3) and using  $U(u, \lambda) = \lambda^2U_2 + \lambda U_1$ , we obtain

$$\begin{aligned} &\lambda^4\mathcal{T}_2U_2 + \lambda^3(\mathcal{T}_1U_2 + \mathcal{T}_2U_1) + \lambda^2(\mathcal{T}_1U_1 + \mathcal{T}_0U_2 + \mathcal{T}_{2x}) + \lambda(\mathcal{T}_0U_1 + \mathcal{T}_{1x}) \\ &= \lambda^4\mathcal{U}_2\mathcal{T}_2 + \lambda^3(\mathcal{U}_2\mathcal{T}_1 + \mathcal{U}_1\mathcal{T}_2) + \lambda^2(\mathcal{U}_2\mathcal{T}_0 + \mathcal{U}_1\mathcal{T}_1) + \lambda\mathcal{U}_1\mathcal{T}_0. \end{aligned} \tag{B11}$$

The comparison of  $\lambda^4$  and  $\lambda^3$  on both sides of (B11) delivers

$$\mathcal{U}_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \mathcal{U}_1 = \begin{pmatrix} 0 & A_2^2u + 2iA_2B_1 \\ -D_2^2\bar{u} - 2iD_2C_1 & 0 \end{pmatrix} =: \begin{pmatrix} 0 & \hat{u} \\ -\bar{\hat{u}} & 0 \end{pmatrix},$$

which confirms the claim in (B6). Also, the coefficients of  $\lambda^2$  and  $\lambda$  on both sides of (B11) yield four identities,

$$\begin{aligned} A_{2x} &= B_1\bar{u} + C_1(A_2^2u + 2iA_2B_1), \\ D_{2x} &= -C_1u - B_1(D_2^2\bar{u} + 2iD_2C_1), \\ B_{1x} &= A_0(A_2^2u + 2iA_2B_1 - u), \\ C_{1x} &= A_0(\bar{u} - D_2^2\bar{u} - 2iD_2C_1), \end{aligned} \tag{B12}$$

which can be verified by using the formulas (B2) and (B3).

For the time-evolution equation, let us denote

$$(T_t + TV)T^* = \begin{pmatrix} \mathcal{G}_{11}(\lambda, u) & \mathcal{G}_{12}(\lambda, u) \\ \mathcal{G}_{21}(\lambda, u) & \mathcal{G}_{22}(\lambda, u) \end{pmatrix}. \tag{B13}$$

It is seen from (4) and (B1) that  $\mathcal{G}_{11}(\lambda, u)$  and  $\mathcal{G}_{22}(\lambda, u)$  are the eighth-order polynomials in  $\lambda$ , and  $\mathcal{G}_{12}(\lambda, u)$  and  $\mathcal{G}_{21}(\lambda, u)$  are the seventh-order polynomials in  $\lambda$ . It follows from (B8) and (B9) that

$$\begin{aligned} \mathcal{T}_{11,t}(\lambda_k) &= -\mathcal{T}_{12,t}(\lambda_k)\alpha_k - \mathcal{T}_{12}(\lambda_k)\alpha_{kt}, \\ \mathcal{T}_{21,t}(\lambda_k) &= -\mathcal{T}_{22,t}(\lambda_k)\alpha_k - \mathcal{T}_{22}(\lambda_k)\alpha_{kt}, \end{aligned}$$

which, together with (B4) and (B10), indicates that  $\pm\lambda_1$  and  $\pm\lambda_2$  are the roots of the polynomials  $\mathcal{G}_{ij}(\lambda, u)$  for all  $i, j = 1, 2$ . For instance, using

$$\begin{aligned} \mathcal{G}_{11}(\lambda, u) &= \mathcal{T}_{11}\mathcal{T}_{22}(-2i\lambda^4 + i\lambda^2|u|^2) + \mathcal{T}_{12}\mathcal{T}_{22}[-2\lambda^3\bar{u} + \lambda(i\bar{u}_x + |u|^2\bar{u})] + \mathcal{T}_{11,t}\mathcal{T}_{22} \\ &\quad - \mathcal{T}_{11}\mathcal{T}_{21}[2\lambda^3u + \lambda(iu_x - |u|^2u)] - \mathcal{T}_{12}\mathcal{T}_{21}(2i\lambda^4 - i\lambda^2|u|^2) - \mathcal{T}_{21}\mathcal{T}_{12,t}, \end{aligned}$$

we confirm

$$\mathcal{G}_{11}(\pm\lambda_k, u) = \mathcal{T}_{12}(\lambda_k)\mathcal{T}_{22}(\lambda_k)[2\alpha_k(2i\lambda_k^4 - i\lambda_k^2|u|^2) - \alpha_k^2[2\lambda_k^3u + \lambda_k(iu_x - |u|^2u)] - 2\lambda_k^3\bar{u} + \lambda_k(i\bar{u}_x + |u|^2\bar{u}) - \alpha_{kt}] = 0.$$

Dividing both sides of (B13) by  $\det T$  and using (B5) yields

$$V(\hat{u}, \lambda) := \lambda^4\mathcal{V}_4 + \lambda^3\mathcal{V}_3 + \lambda^2\mathcal{V}_2 + \lambda\mathcal{V}_1,$$

where the matrices  $\mathcal{V}_{1,2,3,4}$  are independent of  $\lambda$ . Substituting these expressions into the second Darboux equation (A3) and using  $V(u, \lambda) = \lambda^4V_4 + \lambda^3V_3 + \lambda^2V_2 + \lambda V_1$ , we obtain

$$\begin{aligned} &\lambda^6\mathcal{T}_2V_4 + \lambda^5(\mathcal{T}_2V_3 + \mathcal{T}_1V_4) + \lambda^4(\mathcal{T}_2V_2 + \mathcal{T}_1V_3 + \mathcal{T}_0V_4) + \lambda^3(\mathcal{T}_2V_1 + \mathcal{T}_1V_2 + \mathcal{T}_0V_3) + \lambda^2(\mathcal{T}_1V_1 + \mathcal{T}_0V_2 + \mathcal{T}_{2t}) + \lambda(\mathcal{T}_0V_1 + \mathcal{T}_{1t}) \\ &= \lambda^6\mathcal{V}_4\mathcal{T}_2 + \lambda^5(\mathcal{V}_3\mathcal{T}_2 + \mathcal{V}_4\mathcal{T}_1) + \lambda^4(\mathcal{V}_2\mathcal{T}_2 + \mathcal{V}_3\mathcal{T}_1 + \mathcal{V}_4\mathcal{T}_0) + \lambda^3(\mathcal{V}_1\mathcal{T}_2 + \mathcal{V}_2\mathcal{T}_1 + \mathcal{V}_3\mathcal{T}_0) + \lambda^2(\mathcal{V}_1\mathcal{T}_1 + \mathcal{V}_2\mathcal{T}_0) + \lambda\mathcal{V}_1\mathcal{T}_0. \end{aligned} \tag{B14}$$

The comparison of  $\lambda^6$ ,  $\lambda^5$ ,  $\lambda^4$ , and  $\lambda^3$  on both sides of (B14) yields

$$\mathcal{V}_4 = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}, \quad \mathcal{V}_3 = \begin{pmatrix} 0 & 2A_2^2u + 4iA_2B_1 \\ -2D_2^2\bar{u} - 4iD_2C_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2\hat{u} \\ -2\bar{\hat{u}} & 0 \end{pmatrix}, \quad \mathcal{V}_2 = \begin{pmatrix} \mathcal{V}_2^{(11)} & 0 \\ 0 & \mathcal{V}_2^{(22)} \end{pmatrix} = \begin{pmatrix} i|\hat{u}|^2 & 0 \\ 0 & -i|\hat{u}|^2 \end{pmatrix},$$

and

$$\mathcal{V}_1 = \begin{pmatrix} 0 & \mathcal{V}_1^{(12)} \\ \mathcal{V}_1^{(21)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\hat{u}_x - |\hat{u}|^2\hat{u} \\ i\bar{u}_x + |\hat{u}|^2\bar{u} & 0 \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{V}_2^{(11)} &= i|u|^2 - 2D_2B_1\bar{u} - C_1(2uA_2 + 4iB_1), \\ \mathcal{V}_2^{(22)} &= -i|u|^2 + 2uC_1A_2 + B_1(2D_2\bar{u} + 4iC_1), \\ \mathcal{V}_1^{(12)} &= A_2^2(iu_x - |u|^2u) - 2A_2(iB_1|u|^2 - A_0u) + 2B_1^2\bar{u} + 2(B_1C_1 - A_0A_2)(uA_2^2 + 2iA_2B_1), \\ \mathcal{V}_1^{(21)} &= D_2^2(i\bar{u}_x + |u|^2\bar{u}) + 2D_2(iC_1|u|^2 - A_0\bar{u}) - 2C_1^2u - 2(B_1C_1 - A_0D_2)(D_2^2\bar{u} + 2iC_1D_2). \end{aligned}$$

Note that the equality for  $\mathcal{V}_1$  can be confirmed in view of (B6) and (B12). Furthermore, the comparison of  $\lambda^2$  and  $\lambda$  on both sides of (B14) gives rise to four identities,

$$\begin{aligned} iB_1\bar{u}_x + B_1|u|^2\bar{u} + iA_0|u|^2 + A_{2t} &= iA_0|u|^2 - 2D_2B_1A_0\bar{u} - A_0C_1(2A_2u + 4iB_1) + C_1\mathcal{V}_1^{(12)}, \\ iC_1u_x - C_1|u|^2u - iA_0|u|^2 + D_{2t} &= -iA_0|u|^2 + 2A_0A_2C_1u + A_0B_1(2D_2\bar{u} + 4iC_1) + B_1\mathcal{V}_1^{(21)}, \end{aligned}$$

and

$$iA_0u_x - A_0|u|^2u + B_{1t} = A_0\mathcal{V}_1^{(12)}, \quad iA_0\bar{u}_x + A_0|u|^2\bar{u} + C_{1t} = A_0\mathcal{V}_1^{(21)},$$

which could be directly verified by using (B2) and (B4). Once again, the time-dependent problem confirms the validity of the Darboux transformation formula (B6).

If we restrict  $\lambda_{1,2} = i\beta_{1,2}$  and  $q_{1,2} = -i\bar{p}_{1,2}$  in (B6), then we obtain the twofold Darboux transformation (46). If we restrict  $\lambda_2 = \bar{\lambda}_1$ ,  $q_2 = -\bar{p}_1$ , and  $p_2 = \bar{q}_1$  in (B6), then we obtain the twofold Darboux transformation (47).

**APPENDIX C: PROOF OF (54)**

For  $\lambda_1 = i\beta_1$  with  $\beta_1 \in \mathbb{R}$ , we substitute (50), (51), and (52) with  $Q_1 = -i\bar{P}_1$  into (45) and obtain

$$\hat{u} = -\frac{\bar{P}_1^2 b - \beta_1^4 + \frac{1}{2}\beta_1^2\rho + \frac{\beta_1^2}{2\rho}(i\rho' + a)}{P_1^2 b + c\beta_1^2 + \beta_1^4 + \beta_1^2\rho} Re^{-i\theta} e^{-8ibt}. \tag{C1}$$

Expressing  $\rho'$  from (23) and (24) for  $\hat{\rho} = \frac{1}{2}|\hat{u}|^2$ , we reduce (C1) to (53), which we can write as  $\mathcal{N}_1/\mathcal{D}_1$ . Substituting parameters  $(a, b, c, d)$  from (20), the periodic wave  $\rho$  from (29), and the eigenvalue  $\beta_1$  from (37) into (53) yields

$$\begin{aligned} \mathcal{D}_1 &= [(u_2 - u_4)(b + c\beta_1^2 + \beta_1^4 + u_1\beta_1^2) + (u_1 - u_2)(b + c\beta_1^2 + \beta_1^4 + u_4\beta_1^2)\text{sn}^2(vx; k)]^2 \\ &= \frac{1}{8}\beta_1^2(\sqrt{u_1} + \sqrt{u_2})^2(\sqrt{u_1} + \sqrt{u_4})^2(\sqrt{u_2} + \sqrt{u_4})^2 \\ &\quad \times [(\sqrt{u_1} + \sqrt{u_3})(\sqrt{u_2} - \sqrt{u_4}) + (\sqrt{u_1} - \sqrt{u_2})(\sqrt{u_3} + \sqrt{u_4})\text{sn}^2(vx; k)]^2 \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}_1 &= \text{sn}^4(vx; k)(u_1 - u_2)^2\{\beta_1^2(a + u_4^2)(b - c\beta_1^2 - \beta_1^4) + 2d\beta_1^4 + u_4[b^2 + \beta_1^4(2b + a - c^2) + \beta_1^8]\} \\ &\quad + \text{sn}^2(vx; k)(u_1 - u_2)(u_2 - u_4)\{2\beta_1^2(a + u_1u_4)(b - c\beta_1^2 - \beta_1^4) + 4d\beta_1^4 \\ &\quad + (u_1 + u_4)[b^2 + \beta_1^4(2b + a - c^2) + \beta_1^8]\} + (u_2 - u_4)^2\{\beta_1^2(a + u_1^2)(b - c\beta_1^2 - \beta_1^4) \\ &\quad + 2d\beta_1^4 + u_1[b^2 + \beta_1^4(2b + a - c^2) + \beta_1^8]\} \\ &= \frac{\beta_1^2}{32}(\sqrt{u_1} + \sqrt{u_2})^2(\sqrt{u_1} + \sqrt{u_4})^2(\sqrt{u_2} + \sqrt{u_4})^2 \\ &\quad \times [\text{sn}^2(vx; k)(\sqrt{u_1} - \sqrt{u_2})(\sqrt{u_3} + \sqrt{u_4}) + (\sqrt{u_2} - \sqrt{u_4})(\sqrt{u_1} + \sqrt{u_3})] \\ &\quad \times [\text{sn}^2(vx; k)(\sqrt{u_1} - \sqrt{u_2})(\sqrt{u_3} + \sqrt{u_4})(\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} - \sqrt{u_4})^2 \\ &\quad + (\sqrt{u_2} - \sqrt{u_4})(\sqrt{u_1} + \sqrt{u_3})(\sqrt{u_1} - \sqrt{u_2} - \sqrt{u_3} - \sqrt{u_4})^2]. \end{aligned}$$

Canceling the common factors of  $\mathcal{D}_1$  and  $\mathcal{N}_1$  yields the quotient

$$\hat{\rho} = \frac{1}{4}(\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} - \sqrt{u_4})^2 - \frac{(\sqrt{u_1} + \sqrt{u_3})(\sqrt{u_2} - \sqrt{u_4})(\sqrt{u_1} - \sqrt{u_4})(\sqrt{u_2} + \sqrt{u_3})}{(\sqrt{u_1} + \sqrt{u_3})(\sqrt{u_2} - \sqrt{u_4}) + (\sqrt{u_1} - \sqrt{u_2})(\sqrt{u_3} + \sqrt{u_4})\text{sn}^2(vx; k)},$$

which is equivalent to (54) with notations in (55).

APPENDIX D: PROOF OF (60)

Substituting (50) into (47) yields

$$\hat{u} = \left( \frac{\bar{\lambda}_1 |P_1|^2 + \lambda_1 |Q_1|^2}{\lambda_1 |P_1|^2 + \bar{\lambda}_1 |Q_1|^2} \right)^2 \left[ R - \frac{2i(\lambda_1^2 - \bar{\lambda}_1^2) P_1 \bar{Q}_1}{\bar{\lambda}_1 |P_1|^2 + \lambda_1 |Q_1|^2} \right] e^{i\theta} \tag{D1}$$

so that

$$\hat{\rho} = \frac{1}{2} \left| R - \frac{2i(\lambda_1^2 - \bar{\lambda}_1^2) P_1 \bar{Q}_1}{\bar{\lambda}_1 |P_1|^2 + \lambda_1 |Q_1|^2} \right|^2, \tag{D2}$$

where  $\hat{\rho} := \frac{1}{2} |\hat{u}|^2$ . In what follows, we obtain explicit expressions for  $|P_1|^2$ ,  $|Q_1|^2$ , and  $P_1 \bar{Q}_1$ .

By substituting (22) into (51) and (52), we obtain

$$2|\lambda_1|^2 |\lambda_1^2 - \bar{\lambda}_1^2|^2 (|P_1|^4 + |Q_1|^4) = 4d - 4ac + 2a(\lambda_1^2 + \bar{\lambda}_1^2) + [4b + 4|\lambda_1|^4 - 2c(\lambda_1^2 + \bar{\lambda}_1^2)]R^2 - \frac{1}{2}(\lambda_1^2 + \bar{\lambda}_1^2)R^4$$

and

$$\begin{aligned} |\lambda_1|^4 |\lambda_1^2 - \bar{\lambda}_1^2|^2 |P_1|^2 |Q_1|^2 &= b^2 - cb(\lambda_1^2 + \bar{\lambda}_1^2) + b(\lambda_1^4 + \bar{\lambda}_1^4) + c^2 |\lambda_1|^4 - c|\lambda_1|^4 (\lambda_1^2 + \bar{\lambda}_1^2) + |\lambda_1|^8 \\ &\quad - \frac{1}{2}b(\lambda_1^2 + \bar{\lambda}_1^2)R^2 + c|\lambda_1|^4 R^2 - \frac{1}{2}|\lambda_1|^4 (\lambda_1^2 + \bar{\lambda}_1^2)R^2 + \frac{1}{4}|\lambda_1|^4 R^4. \end{aligned}$$

Combining these two expressions together yields

$$\begin{aligned} 4|\lambda_1|^2 |\lambda_1^2 - \bar{\lambda}_1^2|^2 (|P_1|^2 + |Q_1|^2)^2 &= 8d - 8ac + 4a(\lambda_1^2 + \bar{\lambda}_1^2) + 8|\lambda_1|^{-2} b [b - c(\lambda_1^2 + \bar{\lambda}_1^2) + (\lambda_1^4 + \bar{\lambda}_1^4)] \\ &\quad + 8c^2 |\lambda_1|^2 - 8c|\lambda_1|^2 (\lambda_1^2 + \bar{\lambda}_1^2) + 8|\lambda_1|^6 + [8b + 8|\lambda_1|^4 - 4c(\lambda_1^2 + \bar{\lambda}_1^2)]R^2 \\ &\quad - 4b|\lambda_1|^{-2} (\lambda_1^2 + \bar{\lambda}_1^2)R^2 + 8c|\lambda_1|^2 R^2 - 4|\lambda_1|^2 (\lambda_1^2 + \bar{\lambda}_1^2)R^2 \\ &\quad - (\lambda_1^2 + \bar{\lambda}_1^2)R^4 + 2|\lambda_1|^2 R^4. \end{aligned}$$

When  $a$ ,  $b$ ,  $c$ , and  $d$  are expressed from (20) and (31), it shows that each term on the right-hand side has a common factor  $(\lambda_1 - \bar{\lambda}_1)^2$ . When the common factor is canceled, we obtain the following compact expression:

$$\begin{aligned} 4|\lambda_1|^2 (\lambda_1 + \bar{\lambda}_1)^2 (|P_1|^2 + |Q_1|^2)^2 &= (\lambda_1^2 + \bar{\lambda}_1^2 - \lambda_2^2 - \bar{\lambda}_2^2 + 2|\lambda_1|^2 - 2|\lambda_2|^2)^2 \\ &\quad + 2(\lambda_1^2 + \bar{\lambda}_1^2 + \lambda_2^2 + \bar{\lambda}_2^2 + 2|\lambda_1|^2 + 2|\lambda_2|^2)R^2 + R^4. \end{aligned}$$

Expressing  $\lambda_1$  and  $\lambda_2$  by (31) and (36) and using  $\rho := \frac{1}{2}R^2$ , we rewrite this expression in the following form:

$$|\lambda_1|^2 (\lambda_1 + \bar{\lambda}_1)^2 (|P_1|^2 + |Q_1|^2)^2 = (\rho - u_3)(\rho - u_4). \tag{D3}$$

Similarly, we obtain

$$\begin{aligned} 4|\lambda_1|^2 |\lambda_1^2 - \bar{\lambda}_1^2|^2 (|P_1|^2 - |Q_1|^2)^2 &= 8d - 8ac + 4a(\lambda_1^2 + \bar{\lambda}_1^2) - 8|\lambda_1|^{-2} b [b - c(\lambda_1^2 + \bar{\lambda}_1^2) + (\lambda_1^4 + \bar{\lambda}_1^4)] \\ &\quad - 8c^2 |\lambda_1|^2 + 8c|\lambda_1|^2 (\lambda_1^2 + \bar{\lambda}_1^2) - 8|\lambda_1|^6 + [8b + 8|\lambda_1|^4 - 4c(\lambda_1^2 + \bar{\lambda}_1^2)]R^2 \\ &\quad + 4b|\lambda_1|^{-2} (\lambda_1^2 + \bar{\lambda}_1^2)R^2 - 8c|\lambda_1|^2 R^2 + 4|\lambda_1|^2 (\lambda_1^2 + \bar{\lambda}_1^2)R^2 \\ &\quad - (\lambda_1^2 + \bar{\lambda}_1^2)R^4 - 2|\lambda_1|^2 R^4, \end{aligned}$$

where each term on the right-hand side now has a common factor  $(\lambda_1 + \bar{\lambda}_1)^2$ . When the common factor is canceled, we obtain the following compact expression:

$$\begin{aligned} 4|\lambda_1|^2 (\lambda_1 - \bar{\lambda}_1)^2 (|P_1|^2 - |Q_1|^2)^2 &= (\lambda_1^2 + \bar{\lambda}_1^2 - \lambda_2^2 - \bar{\lambda}_2^2 - 2|\lambda_1|^2 + 2|\lambda_2|^2)^2 \\ &\quad + 2(\lambda_1^2 + \bar{\lambda}_1^2 + \lambda_2^2 + \bar{\lambda}_2^2 - 2|\lambda_1|^2 - 2|\lambda_2|^2)R^2 + R^4, \end{aligned}$$

from which we obtain with the help of (31) and (36):

$$|\lambda_1|^2 (\lambda_1 - \bar{\lambda}_1)^2 (|P_1|^2 - |Q_1|^2)^2 = (\rho - u_1)(\rho - u_2). \tag{D4}$$

By substituting (29) and (30) into (D3) and (D4) and extracting the square root, we derive the following expressions:

$$|\lambda_1| (\lambda_1 + \bar{\lambda}_1) (|P_1|^2 + |Q_1|^2) = \frac{(u_2 - u_4) \sqrt{(u_1 - u_3)(u_1 - u_4)} \operatorname{dn}(vx; k)}{(u_2 - u_4) + (u_1 - u_2) \operatorname{sn}^2(vx; k)} \tag{D5}$$

and

$$i|\lambda_1|(\lambda_1 - \bar{\lambda}_1)(|P_1|^2 - |Q_1|^2) = \frac{(u_1 - u_2)\sqrt{(u_1 - u_4)(u_2 - u_4)}\text{sn}(vx; k)\text{cn}(vx; k)}{(u_2 - u_4) + (u_1 - u_2)\text{sn}^2(vx; k)}, \tag{D6}$$

where  $\alpha_1$  and  $\beta_1$  are assumed to be positive. The particular sign in (D6) has been chosen due to the following expression obtained from (51):

$$i|\lambda_1|^2(\lambda_1^2 - \bar{\lambda}_1^2)(|P_1|^4 - |Q_1|^4) = -\frac{d\rho}{dx},$$

which implies that the sign of  $i(\lambda_1^2 - \bar{\lambda}_1^2)(|P_1|^4 - |Q_1|^4)$  is the same as the sign of  $\text{sn}(vx; k)\text{cn}(vx; k)$ .

By adding and subtracting (D5) and (D6), we obtain the desired expressions:

$$2|\lambda_1|(\lambda_1|P_1|^2 + \bar{\lambda}_1|Q_1|^2) = \sqrt{(u_1 - u_4)(u_2 - u_4)}\frac{2v\text{dn}(vx; k) - i(u_1 - u_2)\text{sn}(vx; k)\text{cn}(vx; k)}{(u_2 - u_4) + (u_1 - u_2)\text{sn}^2(vx; k)} \tag{D7}$$

and

$$2|\lambda_1|(\bar{\lambda}_1|P_1|^2 + \lambda_1|Q_1|^2) = \sqrt{(u_1 - u_4)(u_2 - u_4)}\frac{2v\text{dn}(vx; k) + i(u_1 - u_2)\text{sn}(vx; k)\text{cn}(vx; k)}{(u_2 - u_4) + (u_1 - u_2)\text{sn}^2(vx; k)}. \tag{D8}$$

It follows from (51) that

$$4|\lambda_1|^2|\lambda_1^2 - \bar{\lambda}_1^2|^2 P_1^2 \bar{Q}_1^2 = -2\left(\frac{dR}{dx}\right)^2 + 2i\frac{dR}{dx}\left((\lambda_1^2 + \bar{\lambda}_1^2 - c)R - \frac{1}{4}R^3 + \frac{a}{R}\right) + 4d - 4ac + 2a(\lambda_1^2 + \bar{\lambda}_1^2) + [4b + 4|\lambda_1|^4 - 2c(\lambda_1^2 + \bar{\lambda}_1^2)]R^2 - \frac{1}{2}(\lambda_1^2 + \bar{\lambda}_1^2)R^4.$$

Substituting (20) and (36) and expressing  $\rho = \frac{1}{2}R^2$  yields

$$4|\lambda_1|^2|\lambda_1^2 - \bar{\lambda}_1^2|^2 P_1^2 \bar{Q}_1^2 = -\frac{1}{\rho}\left(\frac{d\rho}{dx}\right)^2 - i\frac{1}{\rho}\frac{d\rho}{dx}(\rho + \sqrt{u_1 u_2})(\rho - \sqrt{u_3 u_4}) - (\lambda_1 + \bar{\lambda}_1)^2(\rho - u_1)(\rho - u_2) - (\lambda_1 - \bar{\lambda}_1)^2(\rho - u_3)(\rho - u_4).$$

To see that the right-hand side is a complete square, we use (36) and write

$$4|\lambda_1|^2|\lambda_1^2 - \bar{\lambda}_1^2|^2 P_1^2 \bar{Q}_1^2 = \frac{(u_1 - u_4)(u_2 - u_4)Z}{[u_1(u_2 - u_4) + u_4(u_1 - u_2)\text{sn}^2(vx; k)][(u_2 - u_4) + (u_1 - u_2)\text{sn}^2(vx; k)]^3},$$

where

$$\begin{aligned} Z = & -(u_1 - u_3)(u_1 - u_4)(u_1 - u_2)^2(u_2 - u_4)^2\text{dn}^2(vx; k)\text{sn}^2(vx; k)\text{cn}^2(vx; k) \\ & + i\sqrt{(u_1 - u_3)(u_2 - u_4)}(u_1 - u_2)\text{dn}(vx; k)\text{sn}(vx; k)\text{cn}(vx; k) \\ & \times [(u_1 + \sqrt{u_1 u_2})(u_2 - u_4) + (u_4 + \sqrt{u_1 u_2})(u_1 - u_2)\text{sn}^2(vx; k)] \\ & \times [(u_1 - \sqrt{u_3 u_4})(u_2 - u_4) + (u_4 - \sqrt{u_3 u_4})(u_1 - u_2)\text{sn}^2(vx; k)] \\ & + \frac{1}{2}(\sqrt{-u_4} + \sqrt{-u_3})^2(u_1 - u_2)^2\text{sn}^2(vx; k)\text{cn}^2(vx; k) \\ & \times [u_1(u_2 - u_4) + u_4(u_1 - u_2)\text{sn}^2(vx; k)][(u_2 - u_4) + (u_1 - u_2)\text{sn}^2(vx; k)] \\ & + \frac{1}{2}(\sqrt{u_1} + \sqrt{u_2})^2(u_1 - u_3)(u_2 - u_4)\text{dn}^2(vx; k) \\ & \times [u_1(u_2 - u_4) + u_4(u_1 - u_2)\text{sn}^2(vx; k)][(u_2 - u_4) + (u_1 - u_2)\text{sn}^2(vx; k)]. \end{aligned}$$

Long but straightforward computations yield

$$\begin{aligned} Z = & \frac{1}{2}(\sqrt{u_1} + \sqrt{u_2})^2\{2v\text{dn}(vx; k)[\sqrt{u_1}(u_2 - u_4)\text{cn}^2(vx; k) + \sqrt{u_2}(u_1 - u_4)\text{sn}^2(vx; k)] \\ & + i(\sqrt{u_1} - \sqrt{u_2})\text{sn}(vx; k)\text{cn}(vx; k)[(u_1 - \sqrt{u_3 u_4})(u_2 - u_4) + (u_4 - \sqrt{u_3 u_4})(u_1 - u_2)\text{sn}^2(vx; k)]\}^2. \end{aligned}$$

Extracting the negative square root yields the final expression:

$$2|\lambda_1|(\lambda_1 + \bar{\lambda}_1)P_1 \bar{Q}_1 = -\sqrt{(u_1 - u_4)(u_2 - u_4)}\frac{\mathcal{N}_2}{\mathcal{D}_2}, \tag{D9}$$

where

$$\begin{aligned} \mathcal{N}_2 := & 2v\text{dn}(vx; k)[\sqrt{u_1}(u_2 - u_4)\text{cn}^2(vx; k) + \sqrt{u_2}(u_1 - u_4)\text{sn}^2(vx; k)] \\ & + i(\sqrt{u_1} - \sqrt{u_2})\text{sn}(vx; k)\text{cn}(vx; k)[(u_1 - \sqrt{u_3 u_4})(u_2 - u_4) + (u_4 - \sqrt{u_3 u_4})(u_1 - u_2)\text{sn}^2(vx; k)] \end{aligned}$$

and

$$\mathcal{D}_2 := \sqrt{[u_1(u_2 - u_4) + u_4(u_1 - u_2)\text{sn}^2(vx; k)][(u_2 - u_4) + (u_1 - u_2)\text{sn}^2(vx; k)]^3}.$$

We chose the negative root by the continuity argument from the degenerate case  $u_1 = u_2$  ( $k = 0$ ), for which the periodic wave (29) becomes the constant-amplitude solution  $\rho(x) = u_1$ . Indeed, it follows from (D5) and (D6) that if  $u_1 = u_2$  ( $k = 0$ ), then

$$2|\lambda_1|(\lambda_1 + \bar{\lambda}_1)|P_1|^2 = 2|\lambda_1|(\lambda_1 + \bar{\lambda}_1)|Q_1|^2 = \sqrt{(u_1 - u_3)(u_1 - u_4)},$$

which coincides with the expression

$$2|\lambda_1|(\lambda_1 + \bar{\lambda}_1)P_1\bar{Q}_1 = -\sqrt{(u_1 - u_3)(u_1 - u_4)}$$

obtained from (D9) with the negative sign. Note that  $P_1 = -Q_1$  is  $x$ -independent in the degenerate case.

Combining together (D8) and (D9) yields

$$\rho - \frac{i(\lambda_1^2 - \bar{\lambda}_1^2)RP_1\bar{Q}_1}{\bar{\lambda}_1|P_1|^2 + \lambda_1|Q_1|^2} = \frac{-\sqrt{u_1u_2}\sqrt{(u_1 - u_3)(u_2 - u_4)}\text{dn}(vx; k) + i\sqrt{u_3u_4}(u_1 - u_2)\text{sn}(vx; k)\text{cn}(vx; k)}{\sqrt{(u_1 - u_3)(u_2 - u_4)}\text{dn}(vx; k) + i(u_1 - u_2)\text{sn}(vx; k)\text{cn}(vx; k)}.$$

Substituting this expression into (D2) yields

$$\begin{aligned} \hat{\rho}(x) &= \frac{u_1u_2(u_1 - u_3)(u_2 - u_4)\text{dn}^2(vx; k) + u_3u_4(u_1 - u_2)^2\text{sn}^2(vx; k)\text{cn}^2(vx; k)}{\rho[(u_1 - u_3)(u_2 - u_4)\text{dn}^2(vx; k) + (u_1 - u_2)^2\text{sn}^2(vx; k)\text{cn}^2(vx; k)]} \\ &= \frac{u_2(u_1 - u_3) - u_3(u_1 - u_2)\text{sn}^2(vx; k)}{(u_1 - u_3) - (u_1 - u_2)\text{sn}^2(vx; k)}, \end{aligned} \tag{D10}$$

which is simply Eq. (60).

**APPENDIX E: PROOF OF (61)**

Substituting parameters ( $a, b, c, d$ ) from (20), the periodic wave  $\rho$  from (39), and the eigenvalue  $\beta_3$  from (43) into (53) yields

$$\hat{\rho} = \frac{\mathcal{N}_3}{\mathcal{D}_3}, \tag{E1}$$

where

$$\mathcal{D}_3 = [(b + c\beta_3^2 + \beta_3^4)[1 + \delta + (\delta - 1)\text{cn}(\mu x; k)] + \beta_3^2[u_1\delta + u_2 + (u_1\delta - u_2)\text{cn}(\mu x; k)]]^2 \tag{E2}$$

and

$$\begin{aligned} \mathcal{N}_3 &= [\beta_3^2a(b - c\beta_3^2 - \beta_3^4) + 2d\beta_3^4][1 + \delta + (\delta - 1)\text{cn}(\mu x; k)]^2 + [b^2 + \beta_3^4(2b + a - c^2) + \beta_3^8][1 + \delta + (\delta - 1)\text{cn}(\mu x; k)] \\ &\quad \times [u_1\delta + u_2 + (u_1\delta - u_2)\text{cn}(\mu x; k)] + \beta_3^2(b - c\beta_3^2 - \beta_3^4)[u_1\delta + u_2 + (u_1\delta - u_2)\text{cn}(\mu x; k)]^2. \end{aligned} \tag{E3}$$

After canceling the common factors in  $\mathcal{D}_3$  and  $\mathcal{N}_3$ , we arrive at the expression

$$\hat{\rho} = \frac{1}{4}(\sqrt{u_1} + \sqrt{u_2} - \sqrt{2(\gamma + \sqrt{\gamma^2 + \eta^2})})^2 + \frac{A(1 - \text{cn}(\mu x; k))}{B(1 - \text{cn}(\mu x; k)) + C(1 + \text{cn}(\mu x; k))}, \tag{E4}$$

where

$$\begin{aligned} A &= \sqrt{2(\gamma + \sqrt{\gamma^2 + \eta^2})(\sqrt{u_1} + \sqrt{u_2})(u_2 - \sqrt{2u_2(\gamma + \sqrt{\gamma^2 + \eta^2}) + \sqrt{\gamma^2 + \eta^2}})}, \\ B &= u_2 - \sqrt{2u_2(\gamma + \sqrt{\gamma^2 + \eta^2}) + \sqrt{\gamma^2 + \eta^2}}, \\ C &= \delta(u_1 + \sqrt{2u_1(\gamma + \sqrt{\gamma^2 + \eta^2}) + \sqrt{\gamma^2 + \eta^2}}). \end{aligned}$$

This expression yields (61) with (62) and (63).

**APPENDIX F: PROOF OF (65)**

The explicit expressions for  $\lambda_1|P_1|^2 + \bar{\lambda}_1|Q_1|^2$  and  $P_1\bar{Q}_1$  at the periodic wave (39) are obtained similarly to the derivation explained in Appendix D. By using (39)–(41), (43), (D3), and (D4), we obtain

$$|\lambda_1|^2(\lambda_1 + \bar{\lambda}_1)^2(|P_1|^2 + |Q_1|^2)^2 = \frac{4(u_2 - \gamma)^2 + \eta^2\text{dn}^2(\mu x; k)}{[1 + \delta + (\delta - 1)\text{cn}(\mu x; k)]^2}$$

and

$$|\lambda_1|^2(\lambda_1 - \bar{\lambda}_1)^2(|P_1|^2 - |Q_1|^2)^2 = \frac{-\delta(u_1 - u_2)^2 \text{sn}^2(\mu x; k)}{[1 + \delta + (\delta - 1)\text{cn}(\mu x; k)]^2},$$

which result in

$$|\lambda_1|(\lambda_1 + \bar{\lambda}_1)(|P_1|^2 + |Q_1|^2) = \frac{2\sqrt{(u_2 - \gamma)^2 + \eta^2} \text{dn}(\mu x; k)}{1 + \delta + (\delta - 1)\text{cn}(\mu x; k)} \quad (\text{F1})$$

and

$$i|\lambda_1|(\lambda_1 - \bar{\lambda}_1)(|P_1|^2 - |Q_1|^2) = \frac{\sqrt{\delta}(u_1 - u_2)\text{sn}(\mu x; k)}{1 + \delta + (\delta - 1)\text{cn}(\mu x; k)}, \quad (\text{F2})$$

where we have extracted the negative squared root analogous to Eq. (D6). With the help of (F1) and (F2), we come to the required expressions for the periodic standing waves (39),

$$2|\lambda_1|(\lambda_1|P_1|^2 + \bar{\lambda}_1|Q_1|^2) = \frac{2\sqrt{(u_2 - \gamma)^2 + \eta^2} \text{dn}(\mu x; k) - i\sqrt{\delta}(u_1 - u_2)\text{sn}(\mu x; k)}{1 + \delta + (\delta - 1)\text{cn}(\mu x; k)} \quad (\text{F3})$$

and

$$2|\lambda_1|(\bar{\lambda}_1|P_1|^2 + \lambda_1|Q_1|^2) = \frac{2\sqrt{(u_2 - \gamma)^2 + \eta^2} \text{dn}(\mu x; k) + i\sqrt{\delta}(u_1 - u_2)\text{sn}(\mu x; k)}{1 + \delta + (\delta - 1)\text{cn}(\mu x; k)}. \quad (\text{F4})$$

On the other hand, we obtain from (51) after a lengthy calculation

$$\sqrt{2}|\lambda_1|[\lambda_1^2 - \bar{\lambda}_1^2|P_1\bar{Q}_1] = -\frac{\mathcal{N}_4}{\mathcal{D}_4}, \quad (\text{F5})$$

where

$$\begin{aligned} \mathcal{N}_4 &= \text{dn}(\mu x; k)(\sqrt{u_1} + \sqrt{u_2})\sqrt{(u_2 - \gamma)^2 + \eta^2}[(\delta\sqrt{u_1} - \sqrt{u_2})\text{cn}(\mu x; k) + \delta\sqrt{u_1} + \sqrt{u_2}] \\ &\quad + \frac{i}{2}(u_1 - u_2)\text{sn}(\mu x; k)\sqrt{\delta}\{u_2 + u_1\delta - (1 + \delta)\sqrt{\gamma^2 + \eta^2} + \text{cn}(\mu x; k)[u_1\delta - u_2 + (1 - \delta)\sqrt{\gamma^2 + \eta^2}]\}, \\ \mathcal{D}_4 &= [1 + \delta + (\delta - 1)\text{cn}(\mu x; k)]^{\frac{3}{2}}[u_2(1 - \text{cn}(\mu x; k)) + u_1\delta(1 + \text{cn}(\mu x; k))]^{\frac{1}{2}}, \end{aligned}$$

where we have extracted the negative squared root due to the same reason as (D9).

By applying (F4) and (F5), we obtain

$$\rho - \frac{i(\lambda_1^2 - \bar{\lambda}_1^2)RP_1\bar{Q}_1}{\bar{\lambda}_1|P_1|^2 + \lambda_1|Q_1|^2} = \frac{-2\sqrt{u_1u_2}[(u_2 - \gamma)^2 + \eta^2]\text{dn}(\mu x; k) + i(u_1 - u_2)\sqrt{\delta(\gamma^2 + \eta^2)}\text{sn}(\mu x; k)}{2\sqrt{(u_2 - \gamma)^2 + \eta^2}\text{dn}(\mu x; k) + i\sqrt{\delta}(u_1 - u_2)\text{sn}(\mu x; k)},$$

which yields

$$\begin{aligned} \hat{\rho}(x) &= \frac{4u_1u_2[(u_2 - \gamma)^2 + \eta^2]\text{dn}^2(\mu x; k) + \delta(u_1 - u_2)^2(\gamma^2 + \eta^2)\text{sn}^2(\mu x; k)}{4[(u_2 - \gamma)^2 + \eta^2]\text{dn}^2(\mu x; k) + \delta(u_1 - u_2)^2\text{sn}^2(\mu x; k)} \\ &= \frac{u_1\delta(1 - \text{cn}(\mu x; k)) + u_2(1 + \text{cn}(\mu x; k))}{1 + \text{cn}(\mu x; k) + \delta(1 - \text{cn}(\mu x; k))}, \end{aligned}$$

and hence the explicit expression (65).

#### APPENDIX G: PROOF OF (73) AND (89)

For the derivation of (73), it follows from (20) and (37) that

$$\beta_1^2(d - ac - a\beta_1^2) = (b + c\beta_1^2 + \beta_1^4)^2.$$

Substituting this relation into (72) yields

$$\begin{aligned} \frac{\partial \chi_1}{\partial t} &= \frac{2\beta_1^2(\beta_1^2 - \beta_2^2)[\beta_1^2d - a\beta_1^2(c + \beta_1^2) + 2\beta_1^2(c\beta_1^2 + \beta_1^4 + b)\rho + \beta_1^4\rho^2]}{(b + c\beta_1^2 + \beta_1^4 + \beta_1^2\rho)^2} \\ &= \frac{2\beta_1^2(\beta_1^2 - \beta_2^2)[(b + c\beta_1^2 + \beta_1^4)^2 + 2\beta_1^2(c\beta_1^2 + \beta_1^4 + b)\rho + \beta_1^4\rho^2]}{(b + c\beta_1^2 + \beta_1^4 + \beta_1^2\rho)^2} \\ &= 2\beta_1^2(\beta_1^2 - \beta_2^2), \end{aligned} \quad (\text{G1})$$

which coincides with (73).

For the derivation of (89), we substitute (20), (51), and (52) into (87) and obtain

$$\frac{\partial \chi_1}{\partial t} = \frac{\mathcal{N}_5}{\mathcal{D}_5}, \tag{G2}$$

where

$$\begin{aligned} \mathcal{D}_5 &= (|p_1|^2 + |q_1|^2)^2, \\ \mathcal{N}_5 &= \frac{4}{\bar{\lambda}_1^2} (b - c\bar{\lambda}_1^2 + \bar{\lambda}_1^4 - \bar{\lambda}_1^2 \rho) (\rho + c - \lambda_1^2 - \bar{\lambda}_1^2) \\ &\quad + \frac{1}{\bar{\lambda}_1(\lambda_1 + \bar{\lambda}_1)} \{2(\lambda_1^2 + \bar{\lambda}_1^2 + |\lambda_1|^2) [\rho^2 + 2(c - 2\bar{\lambda}_1^2)\rho - a] + 2\bar{\lambda}_1^2 \rho^2 + 4(c\bar{\lambda}_1^2 - 2b)\rho + 4ac - 2a\bar{\lambda}_1^2 - 4d\}. \end{aligned}$$

It follows from (D5) that

$$\mathcal{D}_5 = \frac{2(u_2 - u_4)^2(u_1 - u_4)(u_1 - u_3) \operatorname{dn}^2(\nu x; k)}{|\lambda_1|^2(\sqrt{-u_3} + \sqrt{-u_4})^2}. \tag{G3}$$

On the other hand, it follows from (29) and (36) that

$$\begin{aligned} \mathcal{N}_5 &= \frac{2i(\sqrt{u_1} + \sqrt{u_2})(u_1 - u_4)(u_2 - u_4)[\sqrt{u_1} + \sqrt{u_2} - i(\sqrt{-u_3} + \sqrt{-u_4})]}{(\sqrt{-u_3} + \sqrt{-u_4})[\sqrt{u_1} + \sqrt{u_2} + i(\sqrt{-u_3} + \sqrt{-u_4})]} \\ &\quad \times [u_2 u_3 \operatorname{cn}^2(\nu x; k) + u_4(u_2 \operatorname{sn}^2(\nu x; k) - u_3) - u_1(u_2 - u_3 \operatorname{sn}^2(\nu x; k) - u_4 \operatorname{cn}^2(\nu x; k))] \\ &= \frac{2i(\sqrt{u_1} + \sqrt{u_2})(u_1 - u_4)(u_2 - u_4)^2[\sqrt{u_1} + \sqrt{u_2} - i(\sqrt{-u_3} + \sqrt{-u_4})]}{(\sqrt{-u_3} + \sqrt{-u_4})[\sqrt{u_1} + \sqrt{u_2} + i(\sqrt{-u_3} + \sqrt{-u_4})]} (u_1 - u_3) \left[ 1 + \frac{(u_1 - u_2)(-u_3 + u_4)}{(u_1 - u_3)(u_2 - u_4)} \operatorname{sn}^2(\nu x; k) \right] \\ &= \frac{2i(\sqrt{u_1} + \sqrt{u_2})(u_1 - u_3)(u_1 - u_4)(u_2 - u_4)^2}{(\sqrt{-u_3} + \sqrt{-u_4})[\sqrt{u_1} + \sqrt{u_2} + i(\sqrt{-u_3} + \sqrt{-u_4})]} [\sqrt{u_1} + \sqrt{u_2} - i(\sqrt{-u_3} + \sqrt{-u_4})] \operatorname{dn}^2(\nu x; k). \end{aligned} \tag{G4}$$

A simple calculation for  $\mathcal{N}_5/\mathcal{D}_5$  via (G3) and (G4) yields (89).

**APPENDIX H: PROOF OF (81) AND (82)**

We select  $c_1 = 0$  so that  $\chi_1(0, 0) = 0$  in (74). Resorting to (20), (29), (37), and (50)–(52), we have

$$\begin{aligned} |\hat{u}(0, 0)| &= \left| u(0, 0) - 2i\beta_1 \frac{p_1(0, 0)}{\bar{p}_1(0, 0)} \right| \\ &= \left| \frac{-i\beta_1^2 \rho'(0) + 3\beta_1^2 \rho^2(0) + 2\rho(0)(3\beta_1^4 + 2c\beta_1^2 + b) - a\beta_1^2}{\sqrt{2\rho(0)}[b + c\beta_1^2 + \beta_1^4 + \beta_1^2 \rho(0)]} \right|, \end{aligned} \tag{H1}$$

which yields (81) since  $\rho(0) = u_1$ ,  $\rho'(0) = 0$ , and  $\hat{\rho} = \frac{1}{2}|\hat{u}|^2$ .

Since the expression (H1) applies to other purely imaginary eigenvalues, we can replace  $\rho(0) = u_1$  and  $\beta_1$  with  $\rho(0) = u_3$  and  $\beta_3$  in (H1) and obtain (82).

**APPENDIX I: PROOF OF (92) AND (93)**

We select  $c_1 = 0$  so that  $\chi_1(0, 0) = 0$  in (90). We obtain from (47), (50), and (85) that

$$|\hat{u}(0, 0)| = \left| R(0) + \frac{2i(\lambda_1^2 - \bar{\lambda}_1^2)P_1(0, 0)\bar{Q}_1(0, 0)}{\bar{\lambda}_1|P_1(0, 0)|^2 + \lambda_1|Q_1(0, 0)|^2} \right|. \tag{I1}$$

It follows from (D7) and (D9) that

$$2|\lambda_1| [|\lambda_1| |P_1(0, 0)|^2 + \bar{\lambda}_1 |Q_1(0, 0)|^2] = \sqrt{(u_1 - u_3)(u_1 - u_4)} \tag{I2}$$

and

$$2|\lambda_1| (\lambda_1 + \bar{\lambda}_1) P_1(0, 0)\bar{Q}_1(0, 0) = -\sqrt{(u_1 - u_3)(u_1 - u_4)}. \tag{I3}$$

Substituting  $\lambda_1$  from (36),  $R(0) = \sqrt{2u_1}$ , and relations (I2) and (I3) into (I1) yields (92).

Repeating computations for eigenvalue  $\lambda_2$  in (36) yields

$$2|\lambda_2|[\lambda_2|P_2(0, 0)|^2 + \bar{\lambda}_2|Q_2(0, 0)|^2] = \sqrt{(u_1 - u_3)(u_1 - u_4)} \quad (14)$$

and

$$2|\lambda_2|(\lambda_2 + \bar{\lambda}_2)P_2(0, 0)\bar{Q}_2(0, 0) = -\sqrt{(u_1 - u_3)(u_1 - u_4)}, \quad (15)$$

which yields (93).

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- [1] V. E. Zakharov and L. A. Ostrovsky, Modulation instability: The beginning, *Physica D* **238**, 540 (2009).
- [2] N. Akhmediev, A. Ankiewicz, and J. M. Soto-Crespo, Rogue waves and rational solutions of the nonlinear Schrödinger equation, *Phys. Rev. E* **80**, 026601 (2009).
- [3] P. Dubard and V. B. Matveev, Multi-rogue waves solutions: from the NLS to the KP-I equation, *Nonlinearity* **26**, R93 (2013).
- [4] Y. Ohta and J. Yang, General high-order rogue waves and their dynamics in the nonlinear Schrödinger equation, *Proc. R. Soc. London A* **468**, 1716 (2012).
- [5] B. Yang and J. Yang, Rogue wave patterns in the nonlinear Schrödinger equation, *Physica D* **419**, 132850 (2021).
- [6] F. Baronio, M. Conforti, A. Degasperis, S. Lombardo, M. Onorato, and S. Wabnitz, Vector Rogue Waves and Baseband Modulational Instability in the Defocusing Regime, *Phys. Rev. Lett.* **113**, 034101 (2014).
- [7] F. Baronio, S. Chen, P. Grelu, S. Wabnitz, and M. Conforti, Baseband modulation instability as the origin of rogue waves, *Phys. Rev. A* **91**, 033804 (2015).
- [8] A. Mancic, F. Baronio, L. Hadzievski, S. Wabnitz, and A. Maluckov, Statistics of vector Manakov rogue waves, *Phys. Rev. E* **98**, 012209 (2018).
- [9] J. C. Bronski, V. M. Hur, and M. A. Johnson, Modulational instability in equations of KdV type, in *New Approaches to Nonlinear Waves*, Lecture Notes in Physics (Springer, Cham, 2016), Vol. 908, pp. 83–133.
- [10] B. Deconinck and B. L. Segal, The stability spectrum for elliptic solutions to the focusing NLS equation, *Physica D* **346**, 1 (2017).
- [11] J. Chen and D. E. Pelinovsky, Rogue periodic waves in the focusing nonlinear Schrödinger equation, *Proc. R. Soc. London A* **474**, 20170814 (2018).
- [12] B. F. Feng, L. M. Ling, and D. A. Takahashi, Multi-breathers and high order rogue waves for the nonlinear Schrödinger equation on the elliptic function background, *Stud. Appl. Math.* **144**, 46 (2020).
- [13] D. S. Agafontsev and V. E. Zakharov, Integrable turbulence generated from modulational instability of cnoidal waves, *Nonlinearity* **29**, 3551 (2016).
- [14] G. Xu, A. Chabchoub, D. E. Pelinovsky, and B. Kibler, Observation of modulation instability and rogue breathers on stationary periodic waves, *Phys. Rev. Research* **2**, 033528 (2020).
- [15] J. Chen, D. E. Pelinovsky, and R. E. White, Periodic standing waves in the focusing nonlinear Schrödinger equation: Rogue waves and modulation instability, *Physica D* **405**, 132378 (2020).
- [16] D. E. Pelinovsky and R. E. White, Localized structures on librational and rotational travelling waves in the sine-Gordon equation, *Proc. R. Soc. London A* **476**, 20200490 (2020).
- [17] W. Mio, T. Ogino, K. Minami, and S. Takeda, Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas, *J. Phys. Soc. Jpn.* **41**, 265 (1976).
- [18] E. Mjølhus, On the modulational instability of hydromagnetic waves parallel to the magnetic field, *J. Plasma Phys.* **16**, 321 (1976).
- [19] B. Yang, J. Chen, and J. Yang, Rogue waves in the generalized derivative nonlinear Schrödinger equations, *J. Nonlin. Sci.* **30**, 3027 (2020).
- [20] J. Chen, D. E. Pelinovsky, and J. Upsal, Modulational instability of periodic standing waves in the derivative nonlinear Schrödinger equation, *J. Nonlin. Sci.* **31**, 58 (2021).
- [21] Z. Qiao, A new completely integrable Liouville’s system produced by the Kaup-Newell eigenvalue problem, *J. Math. Phys.* **34**, 3110 (1993).
- [22] W. X. Ma and R. G. Zhou, On the relationship between classical Gaudin models and BC-type Gaudin models, *J. Phys. A* **34**, 867 (2001).
- [23] R. G. Zhou, An integrable decomposition of the derivative nonlinear Schrödinger equation, *Chin. Phys. Lett.* **24**, 589 (2007).
- [24] C. W. Cao and X. Yang, A (2+1)-dimensional derivative Toda equation in the context of the Kaup-Newell spectral problem, *J. Phys. A* **41**, 025203 (2008).
- [25] J. Chen and R. Zhang, The complex Hamiltonian systems and quasi-periodic solutions in the derivative nonlinear Schrödinger equations, *Stud. Appl. Math.* **145**, 153 (2020).
- [26] A. M. Kamchatnov, On improving the effectiveness of periodic solutions of the NLS and DNLS equations, *J. Phys. A* **23**, 2945 (1990).
- [27] A. M. Kamchatnov, New approach to periodic solutions of integrable equations and nonlinear theory of modulational instability, *Phys. Rep.* **286**, 199 (1997).
- [28] J. Upsal and B. Deconinck, Real Lax spectrum implies spectral stability, *Stud. Appl. Math.* **145**, 765 (2020).
- [29] S. Hakkaev, M. Stanislavova, and A. Stefanov, On the stability of periodic waves for the cubic derivative NLS and the quintic NLS, *J. Nonlin. Sci.* **31**, 54 (2021).
- [30] K. Imai, Generalization of Kaup-Newell inverse scattering formulation and Darboux transformation, *J. Phys. Soc. Jpn.* **68**, 355 (1999).
- [31] H. Steudel, The hierarchy of multi-soliton solutions of the derivative nonlinear Schrödinger equation, *J. Phys. A* **36**, 1931 (2003).



- [32] X. Xu, J. He, and L. Wang, The Darboux transformation of the derivative nonlinear Schrödinger equation, *J. Phys. A* **44**, 305203 (2011).
- [33] B. L. Guo, L. M. Ling, and Q. P. Liu, High-order solutions and generalized Darboux transformations of derivative nonlinear Schrödinger equation, *Stud. Appl. Math.* **130**, 317 (2013).
- [34] B. Xue, J. Shen and X. G. Geng, Breathers and breather-rogue waves on a periodic background for the derivative nonlinear Schrödinger equation, *Phys. Scr.* **95**, 055216 (2020).
- [35] D. E. Pelinovsky, A. Saalman, and Y. Shimabukuro, The derivative NLS equation: Global existence with solitons, *Dynam. PDEs* **14**, 271 (2017).
- [36] O. C. Wright, Maximal amplitudes of hyperelliptic solutions of the derivative nonlinear Schrödinger equation, *Stud. Appl. Math.* **144**, 327 (2020).
- [37] X. G. Geng, Z. Li, B. Xue, and L. Guan, Explicit quasi-periodic solutions of the Kaup-Newell hierarchy, *J. Math. Anal. Appl.* **425**, 1097 (2015).
- [38] P. Zhao and E. G. Fan, Finite gap integration of the derivative nonlinear Schrödinger equation: A Riemann-Hilbert method, *Physica D* **402**, 132213 (2020).
- [39] D. J. Kaup and A. C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, *J. Math. Phys.* **19**, 798 (1978).
- [40] J. Chen and D. E. Pelinovsky, Periodic travelling waves of the modified KdV equation and rogue waves on the periodic background, *J. Nonlin. Sci.* **29**, 2797 (2019).