On the impossibility of solitary Rossby waves in meridionally unbounded domains

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Evolution of weakly nonlinear and slowly varying Rossby waves in planetary atmospheres and oceans is considered within the quasi-geostrophic equation on unbounded domains. When the mean flow profile has a jump in the ambient potential vorticity, localized eigenmodes are trapped by the mean flow with a non-resonant speed of propagation. We discuss amplitude equations for these modes. Whereas the linear problem is suggestive of a two-dimensional Zakharov-Kuznetsov equation, we found that the dynamics of Rossby waves are effectively linear and confined to zonal waveguides of the mean flow. This eliminates even the ubiquitous Korteweg-de Vries equations as the underlying model for spatially localized coherent structures in these geophysical flows. *Published by AIP Publishing.*
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I. INTRODUCTION

Planetary atmospheres and oceans are strongly turbulent media. However, highly ordered coherent structures arise in a process of self-organization and dominate the dynamics on slow temporal and large spatial scales. Rapidly rotating geophysical flows with small variations in stratification compared to the background stratification are well described by the quasi-geostrophic equation

$$\frac{Dq}{Dt} = 0, \tag{1}$$

where $q = \nabla^2 \psi + \beta y - F\psi$ is the shallow-water potential vorticity, $D/Dt = \partial_t + u\partial_x + v\partial_y$ denotes the material derivative, ψ is the stream function for the horizontal velocities given by $(u, v) = (-\psi_y, \psi_x), f = f_0 + \beta y$ describes the ambient rotation of the planet at latitude y, and $F = 1/L_R^2$ is defined in terms of the Rossby radius of deformation $L_R = \sqrt{g\bar{H}}/f_0$ with earth's gravitational acceleration g and typical height $\bar{H}$.[1–3] This equation was first derived by Charney[4] and independently by Obukhov.[5] In the context of low-frequency drift waves in magnetized plasmas, the quasi-geostrophic equation (1) is known as the Hasegawa–Mima equation.[6] In the presence of an ambient meridional mean flow U that depends on latitude y, we employ the decomposition

$$\psi = -\int_0^y U(y)dy + \tilde{\psi}$$

and $q = q^{(0)} + q^{(1)}$ with

$$q^{(0)} = \beta y + F\int_0^y U(y)dy - U', \quad q^{(1)} = \nabla^2\tilde{\psi} - F\tilde{\psi},$$

after which the quasi-geostrophic equation (1) is expressed as

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\nabla^2\tilde{\psi} - F\tilde{\psi}\right) + (\beta + FU - U'')\frac{\partial\tilde{\psi}}{\partial x} + J\left(\tilde{\psi}, \nabla^2\tilde{\psi}\right) = 0, \tag{2}$$

where $J(a, b) = a_x b_y - a_y b_x$ is the Jacobian encapsulating the nonlinearity coming from the material derivative and the term $\beta + FU - U'' \equiv q_y^{(0)}$ represents the leading order potential vorticity gradient.

The main purpose of this work is to discuss the underlying evolution equations for the stream function $\tilde{\psi}(x, y)$ on long spatial and slow temporal scales in a weakly nonlinear analysis. Such equations typically support solitary Rossby waves and are found in numerous contexts ranging from engineering,[7] tidal flows,[8] topographic forcing[9] to rotating vortices.[10] In the above context of geophysical waves, one-dimensional solitary Rossby waves were discussed in Refs. 11 and 12 where the Korteweg-de Vries (KdV) equation and the modified KdV equation were formally derived to describe the persistence of the Great Red Spot in the Jovian atmosphere. The associated linear problem was stated in these works without much analysis. The works of Refs. 11 and 12 spawned a huge activity in deriving KdV equations[13–20] and the (bidirectional) Boussinesq equation[21,22] in the geophysical context to describe and identify mechanisms for atmospheric blocking, cyclogenesis and the meandering of oceanic streams. Most of these works considered laterally bounded domains allowing for one-dimensional propagation in the zonal direction of large-scale solitary waves.

In the present consideration, we investigate the possibility of solitary Rossby waves in the unbounded domain and whether one can derive two-dimensional extensions of the KdV models such as the Zakharov-Kuznetsov (ZK) equation[23] which supports stable lump solitary waves. We establish a

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number of rigorous results on the characterization of eigenvalues of the associated Rayleigh–Kuo spectral problem, which allow us to characterize localization in the meridional direction. In particular, we prove under some natural conditions that no localized eigenmodes with speeds below the wave continuum exist for a smooth flow U. If the potential vorticity of the mean flow has a jump, however, we show that localized eigenmodes do exist and are trapped by the jump of the mean flow.

A formal derivation of amplitude equations, however, reveals that nonlinear solitary wave equations are not possible in the geophysical situation, neither one- nor two-dimensional. Rather, the dynamics of small large-scale localized perturbations are governed by linear dispersion and wave propagation is confined to zonal wave guides prescribed by the linear localized eigenmodes of the associated Rayleigh-Kuo problem. We corroborate this prediction of the asymptotic analysis by direct numerical simulation of the quasi-geostrophic equation (2).

The paper is organized as follows. In Sec. II, we develop the linear theory for the quasi-geostrophic equation (2). In Sec. III, we present the formal multiple scale analysis demonstrating the impossibility of non-linear solitary waves in unbounded domains. Section IV presents numerical simulations of localized initial conditions, which disperse away in the time evolution. Section V concludes with a discussion.

II. LINEAR THEORY

Before we consider the weakly nonlinear and slowly varying approximation in Sec. III, it is necessary to discuss some properties of the linearized version of the quasi-geostrophic equation (2) with a non-constant mean flow in terms of normal mode analysis.[1,3] Linearization of Eq. (2) yields

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(\nabla^2\tilde{\psi} - F\tilde{\psi}) + (\beta + FU - U'')\frac{\partial\tilde{\psi}}{\partial x} = 0, \quad (3)$$

where U depends on y only. Separating variables with the normal mode $\tilde{\psi}(x, y, t) = e^{ik(x-ct)}\phi(y)$, where $k \in \mathbb{R}$ is the zonal wave number, c is the phase speed, and ϕ is the meridional profile, we obtain the Rayleigh–Kuo spectral problem

$$(U - c)(\phi'' - (F + k^2)\phi) + (\beta + FU - U'')\phi = 0, \quad (4)$$

where c is the spectral parameter and ϕ is an eigenfunction to be found.

If $U(y) = \bar{U}$ is a constant mean flow, the spectral problem (4) admits only the continuous spectrum located at

$$c = \bar{U} - \frac{\beta + F\bar{U}}{F + k^2 + \ell^2}, \quad (5)$$

where $\ell \in \mathbb{R}$ is the meridional wave number for the Fourier mode $\phi(y) = e^{i\ell y}$. Expanding (5) in the long-wave limit as

$$c = -\frac{\beta}{F} + \frac{\beta + F\bar{U}}{F^2}(k^2 + \ell^2) + \mathcal{O}((k^2 + \ell^2)^2), \quad (6)$$

we obtain the phase speed of the linearized ZK equation[23] for the two-dimensional perturbations on the constant background $\bar{U}$. However, it is impossible to justify the quadratic nonlinearity of the ZK equation if we start with the quasi-geostrophic

equation (2) for the constant mean flow $U = \bar{U}$ because the limiting Fourier mode $\phi = 1$ corresponds to $\ell = 0$. This prompts us to look at the y-dependent mean flow U such that $U(y) \to \bar{U}$ as $|y| \to \infty$ and to seek a localized eigenfunction ϕ such that $\phi(y) \to 0$ as $|y| \to \infty$ for an eigenvalue c outside the continuous spectrum $[-\beta/F, \bar{U}]$, where we assume that β, F, and $\bar{U}$ are all positive. To avoid resonances and critical layers, we assume $U(y) + \beta/F > 0$ for all y.

In particular, we are looking at the eigenvalues c located below the continuous spectrum, that is, $c < -\beta/F$. Although the dispersion relation (6) suggests that the ZK equation may be the appropriate two-dimensional nonlinear wave model for localized perturbations of the y-dependent mean flow U, we will show in Sec. III that this is not the case. To preempt our results, we will show that the amplitude equation contains neither the quadratic nonlinearity nor the meridional component of the dispersion.

In order to formulate rigorous results of the linear theory, we place the Rayleigh–Kuo spectral problem (4) in a functional-analytic setting. Because we consider x-dependent perturbations in the long-wave limit, we set $k = 0$ and rewrite the limiting spectral problem in the form

$$\mathcal{L}(c)\phi = 0, \quad (7)$$

where

$$\mathcal{L}(c) := (U - c)(\partial_y^2 - F) + \beta + FU - U''.$$

If $U, U'' \in L^\infty(\mathbb{R})$, then for every $c \in \mathbb{R}$, $\mathcal{L}(c) : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is an unbounded non-selfadjoint operator with bounded coefficients. Since $F > 0$, the self-adjoint Helmholtz operator $(F - \partial_y^2) : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is invertible. Hence, introducing $\varphi := (F - \partial_y^2)\phi$, the limiting spectral problem (7) can be formulated as a standard eigenvalue problem

$$\mathcal{M}\varphi = c\varphi \quad (8)$$
$$\text{with} \quad \mathcal{M} := U - (\beta + FU - U'')(F - \partial_y^2)^{-1},$$

where $\mathcal{M} : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a bounded non-selfadjoint operator. The adjoint eigenvalue problem

$$\mathcal{M}^*\theta = c\theta \quad (9)$$
$$\text{with} \quad \mathcal{M}^* := U - (F - \partial_y^2)^{-1}(\beta + FU - U'')$$

coincides with the adjoint spectral problem

$$\mathcal{L}^*(c)\theta = 0, \quad (10)$$

if $(U - c)\theta \in H^2(\mathbb{R})$, where $\mathcal{L}^*(c) : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is the adjoint operator to $\mathcal{L}(c)$ given by

$$\mathcal{L}^*(c) := (\partial_y^2 - F)(U - c) + \beta + FU - U''.$$

The condition $(U - c)\theta \in H^2(\mathbb{R})$ is satisfied for any $\theta \in L^2(\mathbb{R})$ satisfying (9) because if $\theta \in L^2(\mathbb{R})$, then $(F - \partial_y^2)^{-1}(\beta + FU - U'')\theta \in H^2(\mathbb{R})$, which implies due to the spectral problem $\mathcal{M}^*\theta = c\theta$ that $(U - c)\theta \in H^2(\mathbb{R})$. Hence, the eigenfunction θ of the adjoint problems for the bounded operator $\mathcal{M}^*$ and for the unbounded operator $\mathcal{L}^*(c)$ in (9) and (10), respectively, coincide.

The following result is concerned with the Fredholm theory for a simple isolated eigenvalue.

Lemma 1. Assume $U, U'' \in L^\infty(\mathbb{R})$ and let $c_0 \in \mathbb{R}$ be a simple isolated eigenvalue of the spectral problem (8) with the eigenfunction $\varphi_0 \in L^2(\mathbb{R})$. Then, c_0 is also the simple isolated eigenvalue of the adjoint spectral problem (9) with the eigenfunction $\theta_0 \in L^2(\mathbb{R})$ and the inner product $\langle \theta_0, \varphi_0 \rangle_{L^2}$ is nonzero.

Proof. The result follows by Fredholm theory for bounded operators since the reformulation of the spectral problem (7) into the form (8) involves the bounded operator $\mathcal{M}$. □

Remark 1. For each eigenfunction $\varphi_0 \in L^2(\mathbb{R})$ of the spectral problem (8), one can define the eigenfunction of the spectral problem (7) by $\phi_0 := (F - \partial_y^2)^{-1}\varphi_0 \in H^2(\mathbb{R})$ and rewrite the inner product in the form

$$0 \neq \langle \theta_0, \varphi_0 \rangle_{L^2} = \left\langle \theta_0, (F - \partial_y^2)\phi_0 \right\rangle_{L^2}. \tag{11}$$

In what follows, we normalize θ_0 from the condition that the inner product (11) is one.

The following result states that no eigenvalues c generally exist below $-\beta/F$ if U is smooth.

Lemma 2. Assume that U is a smooth bounded function of y and that $U(y) + \beta/F > 0$ for every $y \in \mathbb{R}$. Then, the spectral problem (7) admits no eigenvalues c with $c < -\beta/F$.

Proof. Let $c = -\beta/F + \gamma$ and rewrite the spectral problem (7) in the equivalent form

$$-(U - c)\phi'' + U''\phi = \gamma F\phi.$$

We are looking for eigenvalues $\gamma < 0$ with eigenfunction $\phi \in H^2(\mathbb{R})$. By using the quotient rule, the spectral problem can be rewritten in the form

$$-\frac{d}{dy}(U - c)^2\frac{d}{dy}\frac{\phi}{U - c} = \gamma F\phi,$$

thanks to the fact that $U(y) - c > 0$ for every $y \in \mathbb{R}$ under the assumptions of the lemma. Assume that $\phi \in H^2(\mathbb{R})$ is an eigenfunction for an eigenvalue $\gamma < 0$. The existence of this eigenfunction contradicts the Green's first identity

$$\gamma F \int_\mathbb{R} \frac{\phi^2}{U - c}dy = \int_\mathbb{R}(U - c)^2\left[\frac{d}{dy}\frac{\phi}{U - c}\right]^2 dy,$$

where the right-hand side is positive; hence $\gamma \geq 0$. □

The following result also eliminates the possibility of eigenvalues for convex U.

Lemma 3. Assume that $U, U'' \in L^\infty(\mathbb{R})$ with $U(y) + \beta/F > 0$ and $U''(y) \geq 0$ for every $y \in \mathbb{R}$. Then, the spectral problem (7) admits no eigenvalues c with $c < -\beta/F$.

Proof. Under assumptions of the lemma, we rewrite the spectral problem (7) in another equivalent form as

$$-\phi'' = \frac{cF + \beta - U''}{U - c}\phi.$$

It follows again from the Green's first identity that

$$\int_\mathbb{R}(\phi')^2 dy = \int_\mathbb{R}\frac{cF + \beta - U''}{U - c}\phi^2 dy,$$

where the left-hand side is positive, whereas the right-hand side is negative and well-defined under the assumptions of the lemma. The contradiction excludes eigenvalues with $c < -\beta/F$. □

Because of the negative results in Lemmas 2 and 3, we have to consider piecewise smooth mean flows U. In particular, we consider examples of symmetric localized jets on a constant mean flow of the form

$$U(y) = \bar{U}[1 - a\exp(-b|y|)], \quad y \in \mathbb{R}, \tag{12}$$

with positive parameters $\bar{U}, a$, and b, and asymmetric localized jets of the form

$$U(y) = \begin{cases} \bar{U}_-[1 - a_- \exp(b_-y)], & y < 0, \\ \bar{U}_+[1 - a_+ \exp(-b_+y)], & y > 0, \end{cases} \tag{13}$$

where $\bar{U}_\pm$, $a_\pm$, and $b_\pm$ are positive parameters. To assure the continuity of U and U'' at $y = 0$ for (13), we require $\bar{U}_-(1 - a_-) = \bar{U}_+(1 - a_+)$ and $b_- = b_+\sqrt{a_-\bar{U}_-/a_+\bar{U}_+}$.

Both the symmetric and asymmetric flows (12) and (13) exhibit a discontinuity of their first derivative at $y = 0$, which causes the leading order potential vorticity gradient in $q_y^{(0)} = \beta + FU - U''$ to have a δ-function singularity. It is standard to consider the linear problem for the regions $y < 0$ and $y > 0$ separately and to treat $y = 0$ as a boundary; see Refs. 24 (Chap. 9.3.3) and 1 (Chap. 9.2). After removal of the δ-function singularity, U'' is continuous across the point $y = 0$. For such mean flows with a jump in the ambient potential vorticity at $y = 0$, the conditions of Lemmas 2 and 3 are not satisfied and eigenvalues c of the spectral problem (7) below the wave continuum may exist with $c < -\beta/F$.
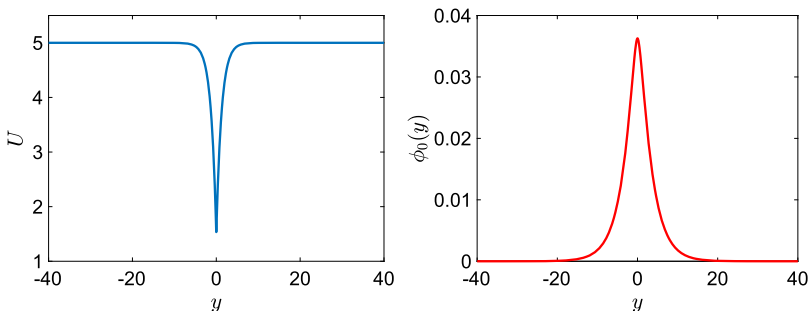

FIG. 1. Symmetric mean flow profile U in (12) with $\bar{U} = 5$, $a = 0.7$, and $b = 0.8$ for $\beta = 1$ and $F = 1$ (left) and the associated localized eigenfunction ϕ_0 (right) for the smallest eigenvalue $c_0 \approx -1.73$.
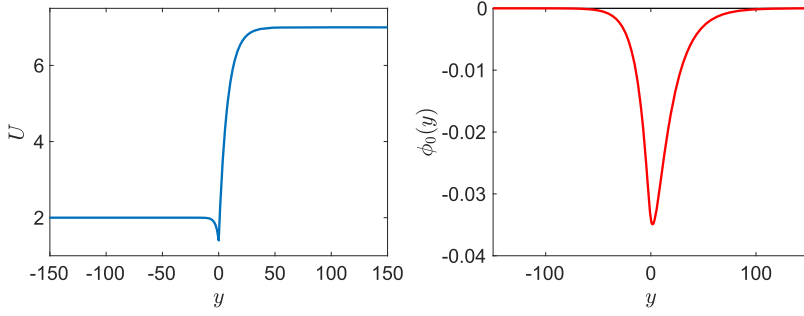
FIG. 2. Asymmetric mean flow profile U in (13) with $\bar{U}_- = 2$, $a_- = 0.3$, $b_- = 0.4$ and $\bar{U}_+ = 7$, $a_+ = 0.8$, $b_+ = 0.13$ for $\beta = 1$ and $F = 1$ (left) and the associated localized eigenfunction ϕ_0 (right) for the smallest eigenvalue $c_0 \approx -1.03$.

An example for a symmetric mean flow of the form (12) with $\bar{U} = 5$, $a = 0.7$, and $b = 0.8$ is shown in the left panel of Fig. 1. There exists a simple isolated eigenvalue $c_0 \approx -1.73$ for $\beta = 1$ and $F = 1$ with the eigenfunction ϕ_0 shown in the right panel. The left panel of Fig. 2 presents an example for an asymmetric mean flow of the form (13) with $\bar{U}_- = 2$, $a_- = 0.3$, $b_- = 0.4$, and $a_+ = 0.8$, implying $\bar{U}_+ = 7$ and $b_+ = 0.13$. The simple isolated eigenvalue is located at $c_0 \approx -1.03$ for $\beta = 1$ and $F = 1$ with the eigenfunction ϕ_0 shown in the right panel.

III. DERIVATION OF SOLITARY ROSSBY WAVE EQUATIONS

Here we derive an evolution equation describing the dynamics of localized pulses with amplitude $A(X, Y, T)$ on long spatial scales X, Y and a slow time scale T in the weakly nonlinear and slowly varying reduction of the barotropic equation (2) in an unbounded domain. We show in Sec. III A that the classical KdV and ZK equations cannot be derived and the dynamics of the amplitude A are governed by the linear wave equation

$$A_T = \lambda A_{XXX}, \tag{14}$$

where λ is a numerical constant. We then show in Sec. III B that the dynamics remain linear according to (14) even when including higher-order cubic nonlinear terms.

A. KdV and ZK equations

In addition to the fast meridional variable y over which the mean flow U changes, we introduce the long spatial and slow temporal scales

$$X = \epsilon(x - c_0 t), \quad Y = \epsilon y, \quad T = \epsilon^3 t, \tag{15}$$

where the limiting speed c_0 coincides with a simple isolated eigenvalue of the spectral problem (7). Derivative terms in the quasi-geostrophic potential vorticity equation (2) are expanded as follows:

$$\partial_t + U\partial_x = \epsilon(U - c_0)\partial_X + \epsilon^3 \partial_T,$$

$$\nabla^2 = \partial_y^2 + 2\epsilon\partial_y\partial_Y + \epsilon^2(\partial_X^2 + \partial_Y^2),$$

and

$$J(a, b) = \epsilon(a_X b_y - a_y b_X) + \epsilon^2(a_X b_Y - a_Y b_X).$$

We seek an asymptotic solution of the form

$$\tilde{\psi} = \epsilon^2 \psi^{(2)} + \epsilon^3 \psi^{(3)} + \epsilon^4 \psi^{(4)} + \cdots, \tag{16}$$

with the leading-order perturbation stream function

$$\psi^{(2)} = A(\epsilon(x - c_0 t), \epsilon y, \epsilon^3 t)\, \phi_0(y), \tag{17}$$

where $\phi_0 \in H^2(\mathbb{R})$ is the eigenfunction of the spectral problem (7) for a simple isolated eigenvalue c_0 such that $\mathcal{L}(c_0)\phi_0 = 0$. For the asymptotic expansion (16) to be a solution of the quasi-geostrophic potential vorticity equation (2), each term in the asymptotic expansion (16) must decay to zero as $|y| \to \infty$. By substituting (16) into (2) and using the expansions in terms of slow variables (15), we obtain a sequence of equations at orders of $\mathcal{O}(\epsilon^k)$ with $k \geq 3$. The choice of $\psi^{(2)}$ in (17) satisfies the equation at $\mathcal{O}(\epsilon^3)$. At the next order, $\mathcal{O}(\epsilon^4)$, we obtain the linear inhomogeneous equation

$$\mathcal{L}(c_0)\partial_X \psi^{(3)} = -2(U - c_0)A_{XY}\phi_0', \tag{18}$$

which can be solved explicitly to yield

$$\psi^{(3)} = -y\phi_0(y)A_Y(X, Y, T). \tag{19}$$

At the order $\mathcal{O}(\epsilon^5)$, we obtain the linear inhomogeneous equation

$$\mathcal{L}(c_0)\partial_X \psi^{(4)} = \partial_T A(F - \partial_y^2)\phi_0 + 2(U - c_0)(y\phi_0)'A_{XYY}$$
$$- (A_{XXX} + A_{XYY})(U - c_0)\phi_0$$
$$- AA_X(\phi_0\phi_0''' - \phi_0'\phi_0''). \tag{20}$$

Let $\theta_0 \in L^2(\mathbb{R})$ be the eigenvector of the adjoint problem (10) for the same eigenvalue c_0 such that $(U - c_0)\theta_0 \in H^2(\mathbb{R})$. By Lemma 1, if c_0 is a simple eigenvalue, then $\langle\theta_0, (F - \partial_y^2)\phi_0\rangle$ is nonzero and we normalize θ_0 such that $\langle\theta_0, (F - \partial_y^2)\phi_0\rangle = 1$. By Fredholm theory, there exists a solution $\psi^{(4)}$ to the linear inhomogeneous equation (20) decaying to zero as $|y| \to \infty$ if and only if the right-hand side of this equation is orthogonal to θ_0 in $L^2(\mathbb{R})$. This solvability condition yields the evolution equation for the amplitude $A(X, Y, T)$ which is given by the following ZK equation

$$A_T = \mu AA_X + \lambda A_{XXX} + \zeta A_{XYY}, \tag{21}$$

with the numerical coefficients given by the following inner products:

$$\mu = \left\langle\theta_0, (\phi_0\phi_0''' - \phi_0'\phi_0'')\right\rangle, \tag{22}$$

$$\lambda = \langle\theta_0, (U - c_0)\phi_0\rangle, \tag{23}$$

$$\zeta = \langle\theta_0, (U - c_0)(\phi_0 - 2(y\phi_0)')\rangle. \tag{24}$$
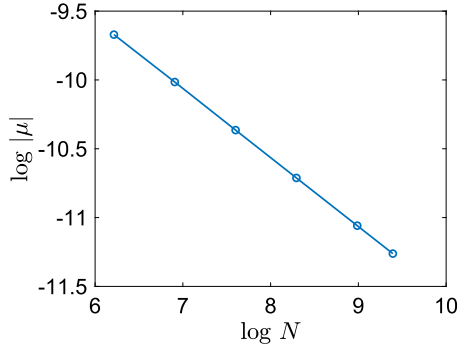
FIG. 3. Log-log plot of $|\mu|$ for increasing resolution with number of spatial grid points N for the asymmetric mean flow depicted in Fig. 2. The slope is estimated as -0.5 from linear regression.

We now show that $\zeta = 0$, so that the ZK equation cannot be derived as a two-dimensional extension to the KdV equation. By using $\mathcal{L}(c_0)(y\phi_0) = 2(U - c_0)\phi_0'$, we write (24) in the form

$$\zeta = -\langle\theta_0, (U - c_0)(\phi_0 + 2y\phi_0')\rangle$$
$$= -\langle\theta_0, (U - c_0)\phi_0\rangle - \langle y\theta_0, \mathcal{L}(c_0)(y\phi_0)\rangle.$$

Using $\mathcal{L}^\star(c_0)(y\theta_0) = 2\partial_y((U - c_0)\theta_0)$, we obtain after partial integration that $\zeta = -\zeta$ implying $\zeta = 0$.

We next show that $\mu = 0$ which precludes the role of the KdV equation to describe localized large scale perturbations. For symmetric mean flow profiles U, the linear problems (7) and (10) support even eigenfunctions ϕ_0 and θ_0. This implies that the integrand in (22) is odd so that $\mu = 0$ for symmetric flow profiles. For asymmetric mean flow profiles, we only have numerical evidence for $\mu = 0$. In Fig. 3, we plot $|\mu|$ versus the number of spatial grid points N and show that $\mu \to 0$ as $N \to \infty$ for an asymmetric flow profile on Fig. 2 with $|\mu| \sim 1/\sqrt{N}$.

Hence the dynamics of the amplitude $A(X, Y, T)$ are entirely linear with zonal dispersion only and are described by the linear dispersive wave equation (14) with λ given by (23). For the mean flow depicted in Fig. 1, we obtain $\lambda \approx -4.69$. For the asymmetric mean flow depicted in Fig. 2, we obtain $\lambda = -4.16$.

B. Modified KdV and ZK equations

Since $\mu = 0$ in the classical ZK equation (21), we should redefine the scaling of the asymptotic expansion (16) and derive a modified ZK equation with cubic nonlinear terms, analogously to Ref. 12 for the case of the KdV and modified KdV equations. We consider the same scaling (15) of the slow variables and redefine the asymptotic expansion in the form

$$\tilde{\psi} = \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \epsilon^3\psi^{(3)} + \cdots, \quad (25)$$

with the leading-order perturbation stream function

$$\psi^{(1)} = A(\epsilon(x - c_0 t), \epsilon y, \epsilon^3 t)\,\phi_0(y), \quad (26)$$

where (c_0, ϕ_0) is the same as in (17) and the corrections of the asymptotic expansion (25) are still supposed to decay to zero as $|y| \to \infty$. By substituting (25) into the quasi-geostrophic potential vorticity equation (2) and using the expansions in terms of

the slow variables (15), we obtain a sequence of equations at orders of $\mathcal{O}(\epsilon^k)$ with $k \geq 2$. The choice of $\psi^{(1)}$ in (26) satisfies the equation at $\mathcal{O}(\epsilon^2)$. At the next order, $\mathcal{O}(\epsilon^3)$, we obtain the linear inhomogeneous equation

$$\mathcal{L}(c_0)\partial_X\psi^{(2)} = -2(U - c_0)A_{XY}\phi_0' - AA_X(\phi_0\phi_0''' - \phi_0'\phi_0''). \quad (27)$$

The explicit solution in (19) would satisfy (27) if only the first term in the right-hand side were present. However, we are not able to find an explicit solution for the full linear equation (27). Therefore, we represent the solution formally as

$$\psi^{(2)} = -y\phi_0(y)A_Y(X, Y, T) - \frac{1}{2}\phi_2(y)A^2(X, Y, T), \quad (28)$$

where ϕ_2 is a solution of the inhomogeneous equation

$$\mathcal{L}(c_0)\phi_2 = \phi_0\phi_0''' - \phi_0'\phi_0''. \quad (29)$$

A solution $\phi_2 \in H^2(\mathbb{R})$ to this equation exists according to the Fredholm theory thanks to the constraint $\mu = 0$. To make the solution unique, we introduce the orthogonality condition $\langle\phi_0, \phi_2\rangle = 0$ on the correction ϕ_2.

At the order $\mathcal{O}(\epsilon^4)$, we obtain the linear inhomogeneous equation

$$\mathcal{L}(c_0)\partial_X\psi^{(3)} = \partial_T A(F - \partial_y^2)\phi_0 + 2(U - c_0)(y\phi_0)'A_{XYY}$$
$$- (A_{XXX} + A_{XYY})(U - c_0)\phi_0$$
$$+ \partial_X(AA_Y)\left[2(U - c_0)\phi_2' + y\left(\phi_0\phi_0''' - \phi_0'\phi_0''\right)\right]$$
$$+ \frac{1}{2}A^2A_X\left[\phi_0\phi_2''' + 2\phi_0'''\phi_2 - 2\phi_0'\phi_2'' - \phi_0''\phi_2'\right]. \quad (30)$$

As in Sec. III A, the solvability condition for $\psi^{(3)}$ in (30) yields the evolution equation for the amplitude $A(X, Y, T)$ which is given by the following modified ZK equation:

$$A_T = \kappa(AA_Y)_X + \nu A^2 A_X + \lambda A_{XXX} + \zeta A_{XYY}, \quad (31)$$

where λ and ζ are the same as in (23) and (24) and κ and ν are defined by

$$\kappa = -\left\langle\theta_0, \left[2(U - c_0)\phi_2' + y\left(\phi_0\phi_0''' - \phi_0'\phi_0''\right)\right]\right\rangle, \quad (32)$$

$$\nu = -\frac{1}{2}\left\langle\theta_0, \left[\phi_0\phi_2''' + 2\phi_0'''\phi_2 - 2\phi_0'\phi_2'' - \phi_0''\phi_2'\right]\right\rangle. \quad (33)$$

As we now show, $\kappa = \nu = 0$, and hence the amplitude equation is given again by the linear dispersive wave equation (14) (recall from Sec. III A that $\zeta = 0$). It is readily seen that $\kappa = 0$ since it follows from (29) that

$$\kappa = -2\left\langle\theta_0, (U - c_0)\phi_2'\right\rangle - \left\langle\mathcal{L}^\star(c_0)(y\theta_0), \phi_2\right\rangle$$
$$= -2\left\langle\theta_0, (U - c_0)\phi_2'\right\rangle - 2\left\langle\partial_y((U - c_0)\theta_0), \phi_2\right\rangle = 0.$$

We show next that $\nu = 0$ as well. Recall that since ϕ_0''' is discontinuous at $y = 0$ so is ϕ_2'', and hence ϕ_2''' involves a δ-function singularity. We therefore perform partial integration to allow for a computationally feasible expression of ν which requires splitting the integration over the regions with $y < 0$ and $y > 0$. We obtain
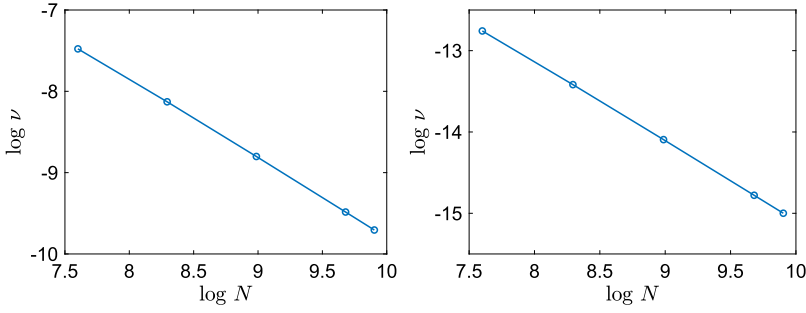
FIG. 4. Log-log plot of ν, numerically calculated from (34), for increasing resolution with number of spatial grid points N. Left: symmetric mean flow depicted in Fig. 1. Right: Asymmetric mean flow depicted in Fig. 2. The slope is estimated as -0.97 from linear regression in both cases.
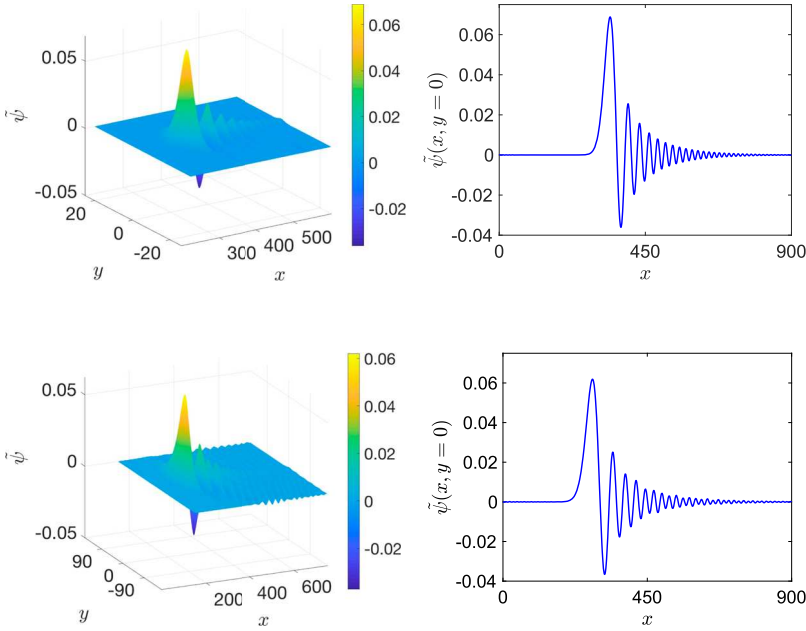

FIG. 5. Stream function $\tilde{\psi}(x, y, t = 200)$ for the localized initial condition (35) with $A_0 = 0.11$ and $w = 0.1$, for the symmetric mean flow depicted in Fig. 1. Left: surface plot of the stream function. Right: meridional cross-section at $y = 0$.
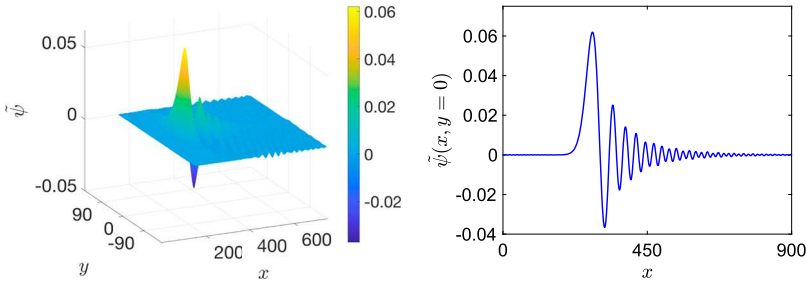

FIG. 6. Stream function $\tilde{\psi}(x, y, t = 400)$ for the localized initial condition (35) with $A_0 = 0.11$ and $w = 0.1$, for the asymmetric mean flow depicted in Fig. 2. Left: surface plot of the stream function. Right: Meridional cross-section at $y = 0$.

$$\nu = -\frac{1}{2}\left\langle \theta_0, \left[2\phi_0'''\phi_2 - 2\phi_0'\phi_2'' - \phi_0''\phi_2'\right]\right\rangle + \frac{1}{2}\left\langle \theta_0', \phi_0\phi_2''\right\rangle$$
$$+ \frac{1}{2}\left\langle \theta_0, \phi_0'\phi_2''\right\rangle - \frac{1}{2}\left[\theta_0\phi_0\phi_2''\right]_{0-\epsilon}^{0+\epsilon} \tag{34}$$

for $\epsilon \to 0$. We were not able to show analytically that $\nu = 0$ but performed careful numerical experiments for several mean flow configurations confirming $\nu = 0$. In Fig. 4, we show ν versus the number of spatial grid points N, suggesting the convergence $\nu \to 0$ as $N \to \infty$ with $\nu \sim 1/N$.

IV. NUMERICAL SIMULATION OF THE QUASI-GEOSTROPHIC EQUATIONS

Here we numerically integrate the quasi-geostrophic potential vorticity equation (2) for localized initial conditions of the form

$$\tilde{\psi}(x, y, t = 0) = A_0 \, \text{sech}^2(wx)\phi_0(y), \tag{35}$$

where ϕ_0 is given as the normalized eigenfunction of the linear problem (7) corresponding to the isolated eigenvalue c_0. We shall use the symmetric mean flow profile (12) with parameters as in Fig. 1 as well as the asymmetric mean flow profile (13) with parameters as in Fig. 2. We choose $A_0 = 0.11$ and $w = 0.1$ for both mean flow profiles.

Numerical integration is based on a finite-difference scheme where the evolution problem is split into firstly determining $\tilde{\psi}$ by solving the Helmholtz problem $(\nabla^2 - F)\tilde{\psi} = q$ for given potential vorticity q in spectral space and then, in a second step, advecting the potential vorticity in time using a second-order leapfrog scheme.[25] The discretization of the nonlinearity is performed with the Arakawa scheme[26] which conserves energy and enstrophy. We choose periodic boundary conditions on a large domain to mimic an unbounded domain. We choose a time step of $\Delta t = 0.01$ and a spatial discretization of $\Delta x = 0.3$ with a square domain of length $L = 900$ throughout. As before, we choose $\beta = F = 1$.

Figures 5 and 6 show snapshots of the stream function $\tilde{\psi}$ and its cross-section at the latitude of the discontinuity in the mean flow $y = 0$. It is clearly seen that the initially localized perturbation (35) linearly disperses along the zonal direction centered at $y = 0$. We have tested this behavior for several initial conditions as well as for several mean flow profiles and observed similar results.

V. DISCUSSION

We have shown with a combination of rigorous analytical calculations and careful computational simulations that the dynamics of small localized large-scale perturbations in the

quasi-geostrophic potential vorticity equation on unbounded domains are entirely linear. Moreover, we have shown that the dispersion is confined to the zonal direction and does not spread meridionally. This renders the typical weakly nonlinear wave equations such as the KdV and ZK equations and their higher-order modification obsolete in describing coherent structures such as atmospheric blocking events, long lived eddies in the ocean or coherent structures in the Jovian atmosphere such as the Great Red Spot. We remark, though, that the KdV and modified KdV equations can still be derived in meridionally bounded channels.[12,19,20] Since the size of these meridionally bounded channels is small compared to the typical long wave length scale of the solitary wave, the ZK or the modified ZK equations, although suggested by the linear dispersion relation (6) in the long-wave limit, are excluded as valid two-dimensional nonlinear wave models for large-scale slow localized structures. This also applies to long waves in the equatorial wave guide (see Ref. 27 and references therein); the equatorial wave guide has a meridional extent of the order of the equatorial Rossby radius of deformation. Since long waves (and solitary Rossby waves, in particular) are typically of scale larger than the equatorial Rossby radius of deformation, two dimensional solitary Rossby waves cannot be supported within the equatorial wave guide. It is an interesting question to study whether two-dimensional solitary Rossby waves are possible with a single latitudinal boundary, modelling, for example, the Antarctic circumpolar current.

We considered here the situation of the quasi-geostrophic equations on a β-plane in meridionally unconfined domain. In the β-plane approximation, the sinusoidal dependency of the Coriolis parameter, which denotes the locally vertical component of the angular momentum of the planet under consideration, is replaced by its linear approximation while the geometry is assumed to be fully planar; in particular, curvature effects in the momentum equations are neglected. This can be justified for quasi-geostrophic dynamics for small Rossby numbers and meridional extents smaller than the radius of the planet; see Ref. 28 for a discussion on the validity of the β-plane approximation. Hence our results pertain to realistic geophysical situations with a limited meridional extent. In particular, our set-up includes localized long waves in the midlatitudes of planet Earth with meridional extent of a few Rossby radii of deformation such as oceanic eddies with a typical size of 50 km and atmospheric synoptic systems with a typical size of 500 km.

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[1]G. K. Vallis, *Atmospheric and Oceanic Fluid Dynamics* (Cambridge University Press, Cambridge, UK, 2006).

[2]R. Salmon, *Lectures on Geophysical Fluid Dynamics* (Oxford University Press, New York, 1998).

[3]J. Pedlosky, *Geophysical Fluid Dynamics*, Springer study edition (Springer Verlag, 1979).

[4]J. G. Charney, "On the scale of atmospheric motions," Geofys. Publ. **17**, 251–265 (1948).

[5]A. M. Obukhov, "On the question on a geostrophic wind," Izv. Akad. Nauk SSSR, Ser. Geograf. Geofiz. **13**, 281–306 (1949).

[6]A. Hasegawa and K. Mima, "Pseudo-three dimensional turbulence in magnetized nonuniform plasmas," Geophys. Pub. **17**, 1–17 (1978).

[7]Z. Wang, C. Wu, L. Zou, Q. Wang, and Q. Ding, "Nonlinear internal wave at the interface of two-layer liquid due to a moving hydrofoil," Phys. Fluids **29**, 072107 (2017).

[8]L. Bordois, F. Auclair, A. Paci, Y. Dossmann, and C. Nguyen, "Nonlinear processes generated by supercritical tidal flow in shallow straits," Phys. Fluids **29**, 066603 (2017).

[9]S. L. Wade, B. J. Binder, T. W. Mattner, and J. P. Denier, "Steep waves in free-surface flow past narrow topography," Phys. Fluids **29**, 062107 (2017).

[10]P. Caillol, "A singular vortex Rossby wave packet within a rapidly rotating vortex," Phys. Fluids **29**, 046601 (2017).

[11]T. Maxworthy and L. Redekopp, "A solitary wave theory of the Great Red Spot and other observed features in the Jovian atmosphere," Icarus **29**, 261–271 (1976).

[12]L. G. Redekopp, "On the theory of solitary Rossby waves," J. Fluid Mech. **82**, 725–745 (1977).

[13]A. Patoine and T. Warn, "The interaction of long, quasi-stationary baroclinic waves with topography," J. Atmos. Sci. **39**, 1018–1025 (1982).

[14]T. Warn and B. Brasnett, "The amplification and capture of atmospheric solitons by topography: A theory of the onset of regional blocking," J. Atmos. Sci. **40**, 28–38 (1983).

[15]K. Haines and P. Malanotte-Rizzoli, "Isolated anomalies in westerly jet streams: A unified approach," J. Atmos. Sci. **48**, 510–526 (1991).

[16]P. Malguzzi and P. Malanotte-Rizzoli, "Nonlinear stationary Rossby waves on nonuniform zonal winds and atmospheric blocking. Part I: The analytical theory," J. Atmos. Sci. **41**, 2620–2628 (1984).

[17]P. Malguzzi and P. Malanotte-Rizzoli, "Coherent structures in a baroclinic atmosphere. Part II: A truncated model approach," J. Atmos. Sci. **42**, 2463–2477 (1985).

[18]H. Mitsudera, "Eady solitary waves: A theory of type B cyclogenesis," J. Atmos. Sci. **51**, 3137–3154 (1994).

[19]G. Gottwald and R. Grimshaw, "The formation of coherent structures in the context of blocking," J. Atmos. Sci. **56**, 3640–3662 (1999).

[20]G. Gottwald and R. Grimshaw, "The effect of topography on the dynamics of interacting solitary waves in the context of atmospheric blocking," J. Atmos. Sci. **56**, 3663–3678 (1999).

[21]K. R. Helfrich and J. Pedlosky, "Time-dependent isolated anomalies in zonal flows," J. Fluid Mech. **251**, 377–409 (1993).

[22]K. R. Helfrich and J. Pedlosky, "Large-amplitude coherent anomalies in baroclinic zonal flows," J. Atmos. Sci. **52**, 1615–1629 (1995).

[23]V. E. Zakharov and E. A. Kuznetsov, "On the stability of solitary waves in weakly dispersing media," J. Exp. Theor. Phys. **39**, 285–286 (1974).

[24]O. Bühler, *Waves and Mean Flows*, 2nd ed., Cambridge Monographs on Mechanics (Cambridge University Press, Cambridge, 2014).

[25]W. R. Holland, "The role of mesoscale eddies in the general circulation of the ocean: Numerical experiments using a wind-driven quasi-geostrophic model," J. Phys. Oceanogr. **8**, 363–392 (1978).

[26]A. Arakawa, "Computational design for long-term numerical integration of the equations of fluid motion: Two-dimensional incompressible flow. Part I," J. Comput. Phys. **1**, 119–143 (1966).

[27]J. Boyd, in *Solitary Waves in Fluids*, edited by R. Grimshaw (WIT Transactions on State-of-the-Art in Science and Engineering, 2007), Vol. 9, p. 34.

[28]W. T. M. Verkley, "On the beta plane approximation," J. Atmos. Sci. **47**, 2453–2460 (1990).