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Bifurcations of Multi-Vortex Configurations in Rotating Bose–Einstein Condensates

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Abstract. We analyze global bifurcations along the family of radially symmetric vortices in the Gross–Pitaevskii equation with a symmetric harmonic potential and a chemical potential μ under the steady rotation with frequency Ω. The families are constructed in the small-amplitude limit when the chemical potential μ is close to an eigenvalue of the Schrödinger operator for a quantum harmonic oscillator. We show that for Ω near 0, the Hessian operator at the radially symmetric vortex of charge $m_0 \in \mathbb{N}$ has $m_0(m_0+1)/2$ pairs of negative eigenvalues. When the parameter Ω is increased, $1+m_0(m_0-1)/2$ global bifurcations happen. Each bifurcation results in the disappearance of a pair of negative eigenvalues in the Hessian operator at the radially symmetric vortex. The distributions of vortices in the bifurcating families are analyzed by using symmetries of the Gross–Pitaevskii equation and the zeros of Hermite–Gauss eigenfunctions. The vortex configurations that can be found in the bifurcating families are the asymmetric vortex ($m_0 = 1$), the asymmetric vortex pair ($m_0 = 2$), and the vortex polygons ($m_0 \geq 2$).
Keywords: Gross–Pitaevskii equation, rotating vortices, harmonic potentials, Lyapunov–Schmidt reductions, bifurcations and symmetries.

1. Introduction

This work addresses the Gross-Pitaevskii equation describing rotating Bose-Einstein condensates (BEC) placed in a symmetric harmonic trap. It is now well established from the energy minimization methods that vortex configurations become energetically favorable for larger rotating frequencies (see review [7] for physics arguments). Ignat and Millot [13, 14] confirmed that the vortex of charge one near the center of symmetry is a global minimizer of energy for a frequency above the first critical value. Seiringer [30] proved that a vortex configuration with charge m_0 becomes energetically favorable to a vortex configuration with charge $(m_0 - 1)$ for a frequency above the m_0-th critical value and that radially symmetrically vortices of charge

$m_0 \geq 2$ cannot be global minimizers of energy. The questions on how the m_0 individual vortices of charge one are placed near the center of symmetry to form an energy minimizer remain open since the time of [13, 14, 30].

For the vortex of charge one, it is shown by using variational approximations [5] and bifurcation methods [29] that the construction of energy minimizers is not trivial past the threshold value for the rotation frequency, where the radially symmetric vortex becomes a local minimizer of energy[1]. Namely, in addition to the radially symmetric vortex, which exists for all rotation frequencies, there exists another branch of the asymmetric vortex solutions above the threshold value, which are represented by a vortex of charge one displaced from the center of rotating symmetric trap. The distance from the center of the harmonic trap increases with respect to the detuning rotation frequency above the threshold value, whereas the angle is a free parameter of the asymmetric vortex solutions. Although the asymmetric vortex is not a local energy minimizer, it is nevertheless a constrained energy minimizer, for which the constraint eliminates the rotational degree of freedom and defines the angle of the solution family uniquely. Consequently, both radially symmetric and asymmetric vortices are orbitally stable in the time evolution of the Gross–Pitaevskii equation for the rotating frequency slightly above the threshold value [29].

Further results on the stability of equilibrium configurations of several vortices of charge one in rotating harmonic traps were found numerically, from the predictions given by the finite-dimensional system for dynamics of individual vortices [2, 21, 26]. The two-vortex equilibrium configuration arises again above the threshold value for the rotation frequency with the two vortices of charge one being located symmetrically with respect to the center of the harmonic trap. However, the symmetric vortex pair is stable only for small distances from the center and it losses stability for larger distances. Once it becomes unstable, another asymmetric pair of two vortices bifurcate, where one vortex has a smaller-than-critical distance from the center and the other vortex has a larger-than-critical distance from the center. The asymmetric pair is stable in numerical simulations and coexist for rotating frequencies above the threshold value with the stable symmetric vortex pair located at the smaller-than-critical distances [26]. The symmetric pair is a local minimizer of energy above the threshold value, whereas the asymmetric pair is a local constrained minimizer of energy, where the constraint again eliminates the rotational degree of freedom [18].

This work continues analysis of local bifurcations of vortex configurations in the Gross–Pitaevskii (GP) equation with a cubic repulsive interaction and a symmetric harmonic trap. In a steadily rotating frame with the rotation frequency Ω, the main model can be written in the normalized form

$$iu_t = -(\partial_x^2 + \partial_y^2)u + (x^2 + y^2)u + |u|^2 u + i\Omega(x\partial_y - y\partial_x)u, \quad (x, y) \in \mathbb{R}^2. \tag{1.1}$$

[1] The threshold value of the rotation frequency for the bifurcation of local minimizers in [5, 29] is smaller than the first critical value in [13, 14], at which the charge-one vortex becomes the global minimizer of energy.

The associated energy of the GP equation is given by

$$E(u) = \iint_{\mathbb{R}^2} \left[|\nabla u|^2 + |x|^2 |u|^2 + \frac{1}{2}|u|^4 + \frac{i}{2}\Omega \bar{u}(x\partial_y - y\partial_x)u - \frac{i}{2}\Omega u(x\partial_y - y\partial_x)\bar{u} \right] dxdy. \tag{1.2}$$

Compared to work in [29], we do not use the scaling for the semi-classical limit of the GP equation and parameterize the vortex solutions in terms of the chemical potential μ arising in the separation of variables $u(t,x,y) = e^{-i\mu t}U(x,y)$. The profile U satisfies the stationary GP equation in the form

$$\mu U = -(\partial_x^2 + \partial_y^2)U + (x^2+y^2)U + |U|^2 U + i\Omega(x\partial_y - y\partial_x)U, \quad (x,y) \in \mathbb{R}^2. \tag{1.3}$$

Local bifurcations of small-amplitude vortex solutions in the GP equation (1.1) have been addressed recently in many publications. We refer to these small-amplitude vortex solutions as the *primary branches.* Classification of localized (soliton and vortex) solutions from the triple eigenvalue was constructed by Kapitula *et al.* [16] with the Lyapunov–Schmidt reduction method. Existence, stability, and bifurcations of radially symmetric vortices with charge $m_0 \in \mathbb{N}$ were studied by Kollar and Pego [20] with shooting methods and Evans function computations. Symmetries of nonlinear terms were used to continue families of general vortex and dipole solutions from the linear limit by Contreras and García-Azpeitia [4] by using equivariant degree theory [15] and bifurcation methods [8]. Existence and stability of stationary states were analyzed in [9, 10] with the amplitude equations for the Hermite function decompositions and their truncation at the continuous resonant equation. Vortex dipoles were studied with normal form equations and numerical approximations in [12]. Numerical evidences of existence, bifurcations, and stability of such vortex and dipole solutions can be found in a vast literature [22, 23, 25, 28, 31].

Compared to the previous literature, our results will explore the recent discovery of [29] of how bifurcations of unconstrained and constrained minimizers of energy are related to the spectral stability problem of radially symmetric vortices in the small-amplitude limit, in particular, with the eigenvalues of negative Krein signature which are known to destabilize dynamics of vortices [20]. Therefore, we consider bifurcations of *secondary branches* of multi-vortex solutions from the primary branch of the radially symmetric vortex of charge $m_0 \in \mathbb{N}$. The primary branch is parameterized by only one parameter $\omega := \mu + m_0\Omega$ in the small-amplitude limit, whereas the secondary branches of multi-vortex configurations are parameterized by two parameters ω and Ω.

As a particular example with $m_0 = 2$, we show that the asymmetric pair of two vortices of charge one bifurcates from the radially symmetric vortex of charge two for Ω below but near $\Omega_0 = 2$. Similarly to the symmetric charge-two vortex [16, 20], the asymmetric pair of two charge-one vortices is born unstable but it is more energetically favorable near the bifurcation threshold compared to the charge-two vortex in the case of no rotation ($\Omega = 0$). If the charge-two vortex is a saddle point of the energy E in (1.2) with three pairs of negative eigenvalues for $\Omega = 0$, it has only one pair of negative eigenvalues for Ω below but near $\Omega_0 = 2$.

We note that the bifurcation technique developed here is not feasible by the methods developed in [16] because of the infinitely many resonances at $\Omega_0 = 2$. Nevertheless, we show that these resonances are avoided for Ω below but near $\Omega_0 = 2$.

For a charge-one vortex, a similar bifurcation happens for Ω below but near $\Omega_0 = 2$, which has been already described in [29] in other notations and with somewhat formal analysis. The results developed here allows us to give a full justification of the results of [29] for a charge-one vortex, but also to extend the analysis to the charge-two vortex, as well as to a radially symmetric vortex of a general charge $m_0 \in \mathbb{N}$.

We also consider all other secondary bifurcations of the radially symmetric vortices of charge $m_0 \in \mathbb{N}$ when the frequency parameter Ω is increased from zero in the interval $(0, 2)$. We show that each bifurcation results in the disappearance of a single pair of negative eigenvalues in the characterization of radially symmetric vortices as saddle points of the energy E in (1.2).

As a particular example, we show that the symmetric charge-two vortex has a bifurcation at Ω near $\Omega_* = 2/3$, where another secondary branch bifurcates. The new branch contains three charge-one vortices at the vertices of an equilateral triangle and a vortex of anti-charge one at the center of symmetry. Again, the secondary branch inherits instability of the radially symmetric vortex along the primary branch in the small-amplitude limit. Past the bifurcation point, the radially symmetric vortex of charge two has two pairs of negative eigenvalues. The bifurcation result near $\Omega_* = 2/3$ was not obtained in the previous work [16].

In the case of the multi-vortex configurations of the total charge two, we can conjecture that the local minimizers of energy given by the symmetric pair of two charge-one vortices as in [26] can be found from a *tertiary bifurcation* along the secondary branch given by the asymmetric pair of charge-one vortices. However, it becomes technically involved to approximate the secondary branch near the bifurcation point and to find the tertiary bifurcation point.

The following theorem represents the main result of our paper. A schematic illustration is given on Figure 1.

Theorem 1. *Fix an integer $m_0 \in \mathbb{N}$ and denote $\omega := \mu + m_0\Omega$.*

(i) *There exists a smooth family of radially symmetric vortices of charge m_0 with a positive profile U satisfying (1.3) with $\omega = \omega(a)$ given by*

$$\omega(a) = 2(m_0 + 1) + \frac{(2m_0)!}{4^{m_0}(m_0!)^2} a^2 + \mathcal{O}(a^4),$$

where the "amplitude" a parameterizes the family.

(ii) *For $\Omega = 0$ and small a, the vortices are degenerate saddle points of the energy E in (1.2) with $2N(m_0)$ negative eigenvalues, a simple zero eigenvalue, and $2Z(m_0)$ small eigenvalues of order $\mathcal{O}(a^2)$, where*

$$N(m_0) = \frac{1}{2} m_0(m_0 + 1) \quad \text{and} \quad Z(m_0) = m_0.$$

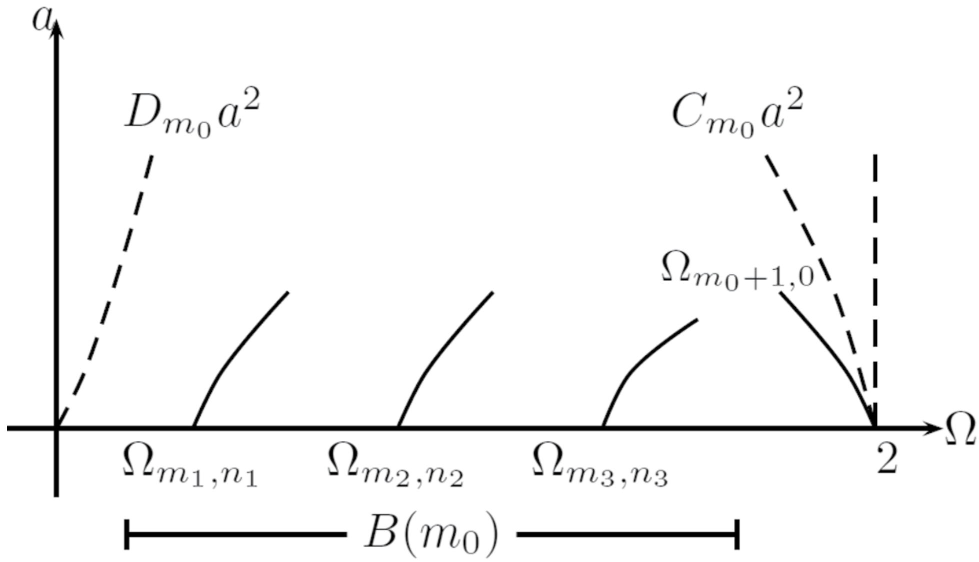

FIGURE 1. A schematic illustration of the bifurcation curves in the parameter plane (Ω, a), where a defines ω. The bifurcating solutions form surfaces parameterized by (Ω, a) close to the curves $\Omega_{m,n}$.

(iii) *There exist $C_{m_0} > 0$ and $D_{m_0} \geq 0$ such that for small a, $1 + B(m_0)$ global bifurcations occur when the parameter Ω is increased in the interval $[a^2 D_{m_0}, 2 - a^2 C_{m_0}]$, where*

$$B(m_0) = \frac{1}{2} m_0(m_0 - 1).$$

For $\Omega \gtrsim a^2 D_{m_0}$, the family of radially symmetric vortices has only $2N(m_0)$ negative eigenvalues and a simple zero eigenvalue, and it losses two of these negative eigenvalues past each non-resonant bifurcation point. If $1 \leq m_0 \leq 16$, the family has $2(m_0 - 1)$ negative eigenvalues for $\Omega \gtrsim 2 - a^2 C_{m_0}$.

(iv) *A new smooth family of multi-vortex configurations is connected to the family of radially symmetric vortices on one side of each non-resonant bifurcation point (of the pitchfork type). Furthermore, on the right (respectively, left) side of the bifurcation point, the new family has one more (respectively, one less) negative eigenvalue compared to the family of radially symmetric vortices.*

(v) *For a non-resonant bifurcation point $\Omega_{m,n} \in (0, 2)$ with $m > m_0$ and $n \geq 0$, the new family has a polygon configuration of $(m - m_0)$ charge-one vortices surrounding a center with total charge $2m_0 - m$. For the "last" bifurcation point $\Omega_{m_0+1,0} = 2 + \mathcal{O}(a^2)$, the new family consists of the charge-one asymmetric vortex ($m_0 = 1$), the asymmetric pair of charge-one vortices ($m_0 = 2$), and a configuration of vortices near the center of total charge m_0 ($m_0 \geq 3$).*

Remark 1. By global bifurcation, we mean that the bifurcating family that originates from the family of radially symmetric vortices of charge m_0 either reaches the boundaries $\Omega = 0$ or $\Omega = 2$, diverges to infinity for a value of $\Omega \in (0, 2)$, or returns to another bifurcation point along the family of radially symmetric vortices of charge m_0.

Remark 2. For $1 \leq m_0 \leq 3$, we have $D_{m_0} = 0$ in item (iii), therefore, the $1 + B(m_0)$ global bifurcations arise when Ω is increased from $\Omega = 0$ to $\Omega = 2 - a^2 C_{m_0}$. However, we do not know if $D_{m_0} = 0$ in a general case. If $D_{m_0} \neq 0$, up to $Z(m_0)$ additional bifurcations may appear if Ω is increased from $\Omega = 0$ to $\Omega = a^2 D_{m_0}$.

Remark 3. For $1 \leq m_0 \leq 3$, all bifurcation points are non-resonant in items (iii)–(v). Resonant bifurcation points may exist in a general case for $m_0 \geq 4$. In this case, the statements (i)-(iii) remain valid, but for each bifurcation point of multiplicity k, the family of radially symmetric vortices losses $2k$ negative eigenvalues past the bifurcation point. In the resonant case, the statements (iv)–(v) require further estimates. However, these resonances are unlikely to be present as the more likely scenario is that the multiple eigenvalues at $a = 0$ split into simple nonzero eigenvalues of order $\mathcal{O}(a^2)$.

Remark 4. For $m_0 \geq 4$, there are $R(m_0)$ additional bifurcations near $\Omega_0 = 2$. For $4 \leq m_0 \leq 16$, the additional bifurcation arise past the last bifurcation point at $\Omega_{m_0+1,0}$. For $m_0 \geq 17$, some of the $R(m_0)$ bifurcations arise before the "last" bifurcation point. We have found numerically that $R(4) = R(5) = 1$, $R(6) = 2$, $R(7) = R(8) = 3$, etc.

From a technical point of view, the proof of Theorem 1 is developed by using the equivariance of the bifurcation problem under the action of the group $O(2) \times O(2)$. The global bifurcation result is proven by using the restriction of the bifurcation problem to the fixed-point space of a dihedral group. This restriction leads to a simple eigenvalue in the fixed-point space, which allows us to apply the global Crandall–Rabinowitz result, see Theorem 3.4.1 of [27]. This method is also helpful to get additional information on the symmetries of the bifurcating solutions which is essential to localize the distributions of zeros for the individual vortices in the multi-vortex configurations.

The paper is structured as follows. In Section 2, we review eigenvalues of the Schrödinger operator for quantum harmonic oscillator and give definitions for the primary and secondary branches of multi-vortex solutions. In Section 3, we analyze distribution of eigenvalues of the Hessian operators along the primary branches at the secondary bifurcation points. In Section 4, we justify bifurcations of the secondary branches at the non-resonant bifurcation points by using bifurcation theorems. In Section 5, we study distribution of individual vortices in the multi-vortex configurations along the secondary branches.

2. Preliminaries

We denote the space of square integrable functions on the plane by $L^2(\mathbb{R}^2)$ and the space of radially symmetric squared integrable functions integrated with the weight r by $L^2_r(\mathbb{R}^+)$. We also use the same notations for the L^2-based Sobolev spaces such as $H^2(\mathbb{R}^2)$ and $H^2_r(\mathbb{R}^+)$. The weighted subspaces of L^2 with $\||\cdot|^2 u\|_{L^2} < \infty$ are denoted by $L^{2,2}(\mathbb{R}^2)$ and $L^{2,2}_r(\mathbb{R}^+)$.

We distinguish notations for the two sets: $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$. Notation $b \lesssim a$ means that there is an a-independent constant C such that $b \leq Ca$ for all $a > 0$ sufficiently small. If X is a Banach space, notation $u = \mathcal{O}_X(a)$ means that $\|u\|_X \lesssim a$ for all $a > 0$ sufficiently small. Similarly, $\omega = \mathcal{O}(a)$ means that $|\omega| \lesssim a$ for all $a > 0$ sufficiently small.

2.1. Schrödinger operator for quantum harmonic oscillator

Recall the quantum harmonic oscillator with equal frequencies in the space of two dimensions [3, 24]. In polar coordinates on $\mathbb{R}^2$, the energy levels of the quantum harmonic oscillator are given by eigenvalues of the Schrödinger operator L written as

$$L := -\Delta_{(r,\theta)} + r^2 : \quad H^2(\mathbb{R}^2) \cap L^{2,2}(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \tag{2.1}$$

where $\Delta_{(r,\theta)} = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2$. As is well-known [3, 24], the eigenvalues of L are distributed equidistantly and can be enumerated by two indices $m \in \mathbb{Z}$ for the angular dependence and $n \in \mathbb{N}_0$ for the number of zeros of the eigenfunctions in the radial direction. To be more precise, the eigenfunction $f_{m,n}$ for the eigenvalue $\lambda_{m,n}$ can be written in the form

$$f_{m,n}(r,\theta) = e_{m,n}(r)e^{im\theta}, \quad m \in \mathbb{Z}, \quad n \in \mathbb{N}_0,$$

where $e_{m,n}$ is an $L^2_r(\mathbb{R}^+)$-normalized solution of the differential equation

$$\left(-\Delta_m + r^2\right) e_{m,n}(r) = \lambda_{m,n} e_{m,n}(r), \quad \Delta_m := \partial_r^2 + r^{-1}\partial_r - r^{-2}m^2 \tag{2.2}$$

with n zeros on $\mathbb{R}^+$ and the eigenvalue $\lambda_{m,n}$ is given explicitly as

$$\lambda_{m,n} = 2(|m| + 2n + 1), \quad m \in \mathbb{Z}, \quad n \in \mathbb{N}_0. \tag{2.3}$$

In particular, $\lambda_{0,0} = 2$ is simple, $\lambda_{1,0} = \lambda_{-1,0} = 4$ is double, $\lambda_{2,0} = \lambda_{-2,0} = \lambda_{0,1} = 6$ is triple, and so on. For fixed $m \in \mathbb{Z}$, the spacing between the eigenvalues is 4. Multiplicity of an eigenvalue $\lambda = 2\ell$ for $\ell \in \mathbb{N}$ is ℓ.

2.2. Primary branches of radially symmetric vortices

Stationary solutions of the GP equation (1.1) are given in the form $u(t,x,y) = e^{-i\mu t}U(x,y)$, where U satisfies (1.3) and $\mu \in \mathbb{R}$ is a free parameter which has the physical meaning of the chemical potential. In polar coordinates (r,θ), U satisfies the stationary GP equation in the form

$$\mu U = -\Delta_{(r,\theta)} U + r^2 U + |U|^2 U + i\Omega\partial_\theta U. \tag{2.4}$$

Radially symmetric vortices of a fixed charge $m_0 \in \mathbb{N}$ are given in the form

$$U(r,\theta) = e^{im_0\theta}\psi_{m_0}(r), \quad \omega = \mu + m_0\Omega, \tag{2.5}$$

where (ψ_{m_0}, ω) is a root of the nonlinear operator

$$f(u,\omega) : \quad H^2_r(\mathbb{R}^+) \cap L^{2,2}_r(\mathbb{R}^+) \times \mathbb{R} \to L^2_r(\mathbb{R}^+), \tag{2.6}$$

given by $f(u,\omega) := -\Delta_{m_0} u + r^2 u + u^3 - \omega u$.

By Theorem 1 in [4], for every $m_0 \in \mathbb{N}$, there exists a unique smooth family of radially symmetric vortices of charge m_0 parameterized locally by amplitude a such that

$$\psi_{m_0}(r) \equiv \psi_{m_0}(r;a) = a e_{m_0,0}(r) + \mathcal{O}_{H^1_r}(a^3) \tag{2.7}$$

and

$$\omega \equiv \omega_{m_0}(a) = \lambda_{m_0,0} + a^2 \omega_{m_0,0} + \mathcal{O}(a^4), \tag{2.8}$$

where $\lambda_{m_0,0} = 2(m_0+1)$, $\omega_{m_0,0} = \|e_{m_0,0}\|^4_{L^4_r}$, and the normalization $\|e_{m_0,0}\|_{L^2_r} = 1$ has been used. By using the explicit expression for the $L^2_r(\mathbb{R}^+)$-normalized Hermite–Gauss solutions of the Schrödinger equation (2.2) with $\lambda_{m_0,0} = 2(m_0+1)$ given by

$$e_{m_0,0}(r) = \frac{\sqrt{2}}{\sqrt{m_0!}} r^{m_0} e^{-\frac{r^2}{2}}, \quad m_0 \in \mathbb{N}_0, \tag{2.9}$$

we compute explicitly

$$\omega_{m_0,0} = \|e_{m_0,0}\|^4_{L^4_r} = \frac{(2m_0)!}{4^{m_0}(m_0!)^2}. \tag{2.10}$$

Since $e_{m_0,0}(r) > 0$ for all $r > 0$, the property $\psi_{m_0}(r;a) > 0$, $r > 0$ holds[2] at least for sufficiently small a. The family of radially symmetric vortices approximated by (2.7) and (2.8) in the small-amplitude limit is referred to as *the primary branch*.

Remark 5. Item (i) in Theorem 1 is just a reformulation of the result of Theorem 1 in [4].

Every solution U of the stationary GP equation (2.4) is a critical point of the energy functional

$$E_\mu(u) = E(u) - \mu Q(u), \tag{2.11}$$

where $E(u)$ is given by (1.2) and $Q(u) = \|u\|^2_{L^2}$. Expanding $E_\mu(u)$ near the critical point U given by (2.5) with $u = U + v$, where v is a perturbation term in $H^1(\mathbb{R}^2) \cap L^{2,1}(\mathbb{R}^2)$, we obtain the quadratic form at the leading order

$$E_\mu(U+v) - E_\mu(U) = \langle \mathcal{H}\mathbf{v}, \mathbf{v}\rangle_{L^2} + \mathcal{O}(\|\mathbf{v}\|^3_{H^1\cap L^{2,1}}),$$

where the bold notation $\mathbf{v}$ is used for an augmented vector with components v and $\bar{v}$ and the Hessian operator $\mathcal{H}$ can be defined in the stronger sense as the linear operator

$$\mathcal{H}: \quad H^2(\mathbb{R}^2) \cap L^{2,2}(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \tag{2.12}$$

with

$$\mathcal{H} = \left[\begin{array}{cc} -\Delta_{(r,\theta)} + r^2 + i\Omega\partial_\theta - \mu + 2\psi^2_{m_0} & \psi^2_{m_0} e^{2im_0\theta} \\ \psi^2_{m_0} e^{-2im_0\theta} & -\Delta_{(r,\theta)} + r^2 - i\Omega\partial_\theta - \mu + 2\psi^2_{m_0} \end{array}\right]. \tag{2.13}$$

By using the Fourier series

$$v = \sum_{m\in\mathbb{Z}} V_m e^{im\theta}, \quad \bar{v} = \sum_{m\in\mathbb{Z}} W_m e^{im\theta}, \tag{2.14}$$

[2]More general vortex families with n_0 zeros on $\mathbb{R}^+$ have also been constructed in [4], but our work will focus on the case $n_0 = 0$.

the operator $\mathcal{H}$ is block diagonalized into blocks H_m that acts on V_m and W_{m-2m_0} for $m \in \mathbb{Z}$. We recall that $\psi_{m_0} = \psi_{m_0}(\cdot; a)$, $\omega = \mu + m_0\Omega = \omega_{m_0}(a)$, and write the blocks H_m as linear operators

$$H_m : \quad H^2_r(\mathbb{R}^+) \cap L^{2,2}_r(\mathbb{R}^+) \to L^2_r(\mathbb{R}^+), \tag{2.15}$$

with explicit dependence on the parameters (a, Ω) as follows:

$$H_m(a, \Omega) = K_m(a) - \Omega(m - m_0)R, \tag{2.16}$$

where

$$K_m(a) = \begin{bmatrix} -\Delta_m + r^2 - \omega_{m_0}(a) + 2\psi^2_{m_0}(r; a) & \psi^2_{m_0}(r; a) \\ \psi^2_{m_0}(r; a) & -\Delta_{m-2m_0} + r^2 - \omega_{m_0}(a) + 2\psi^2_{m_0}(r; a) \end{bmatrix}$$

and

$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

A secondary bifurcation along the primary branch of radially symmetric vortices given by (2.7) corresponds to the nonzero solutions in $H^2_r(\mathbb{R}^+) \cap L^{2,2}_r(\mathbb{R}^+)$ of the spectral problem

$$K_m(a) \begin{bmatrix} V_m \\ W_{m-2m_0} \end{bmatrix} = \Omega(m - m_0)R \begin{bmatrix} V_m \\ W_{m-2m_0} \end{bmatrix}. \tag{2.17}$$

This spectral problem (2.17) coincides with the stability problem for the primary branch (2.7) in the absence of rotation. The spectral parameter λ of the stability problem[3] is given for each $m \in \mathbb{Z}$ by $\lambda := \Omega(m - m_0)$. The parameter m for the angular mode satisfying the eigenvalue problem (2.17) corresponds to the bifurcating mode superposed on the primary branch of vortex solutions.

2.3. Secondary branches of multi-vortex solutions

We can look for the secondary branches bifurcating along the primary branch of radially symmetric vortices given by (2.7) and (2.8). Consequently, we write

$$U(r, \theta) = e^{im_0\theta}\psi_{m_0}(r; a) + v(r, \theta), \tag{2.18}$$

where v is a root of the nonlinear operator

$$g(v; a, \Omega) : \quad H^2(\mathbb{R}^2) \cap L^{2,2}(\mathbb{R}^2) \times \mathbb{R} \times \mathbb{R} \to L^2(\mathbb{R}^2), \tag{2.19}$$

given by

$$\begin{aligned} g(v; a, \Omega) = {} & -\Delta_{(r,\theta)}v + r^2 v + i\Omega\left(\partial_\theta v - im_0 v\right) + 2\psi^2_{m_0}(r; a)v + e^{2im_0\theta}\psi^2_{m_0}(r; a)\bar{v} \\ & + e^{-im_0\theta}\psi_{m_0}(r; a)v^2 + 2e^{im_0\theta}\psi_{m_0}(r; a)|v|^2 + |v|^2 v - \omega_{m_0}(a)v. \end{aligned} \tag{2.20}$$

The Jacobian operator of $g(v; a, \Omega)$ at $v = 0$ is given by the Hessian operator (2.13), which is block-diagonalized by the Fourier series (2.14) into blocks (2.15)–(2.16).

In the next two lemmas, we analyze symmetries of the individual blocks of the spectral problem (2.17).

[3] When the vortex is unstable, a complex eigenvalue λ of the stability problem does not correspond to the secondary bifurcation associated with the eigenvalue problem (2.17).

Lemma 1. *There exists a_0 such that for every $0<a<a_0$, the spectrum of $H_{m_0}(a,\Omega) = K_{m_0}(a)$ is strictly positive except for a simple zero eigenvalue, which is related to the gauge symmetry spanned by the eigenvector*

$$K_{m_0}(a)\left[\begin{array}{c}\psi_{m_0}\\ -\psi_{m_0}\end{array}\right] = \left[\begin{array}{c}0\\ 0\end{array}\right]. \tag{2.21}$$

Consequently, no bifurcations arise in the Ω continuation from the block $H_{m_0}(a,\Omega) = K_{m_0}(a)$.

Proof. For $m = m_0$, $H_{m_0}(a,\Omega) = K_{m_0}(a)$ is independent of the rotation frequency Ω. If the primary branch (2.7) describes vortices with $\psi_{m_0}(r;a) > 0$ for all $r > 0$, that is, if $n_0 = 0$, then the assertion on the spectrum of $K_{m_0}(a)$ for small a follows from the previous works [6, 20]. □

Lemma 2. *Eigenvalues of the spectral problem* (2.17) *with $m < m_0$ are identical to eigenvalues of the spectral problem* (2.17) *for $m > m_0$.*

Proof. We observe the symmetry $\Delta_m = \Delta_{m_0+(m-m_0)}$ and $\Delta_{m-2m_0} = \Delta_{m_0-(m-m_0)}$ with respect to the symmetry point at $m = m_0$. As a result, for each $k \in \mathbb{N}$, if $\lambda = \Omega k$ is an eigenvalue of the spectral problem (2.17) with $m = m_0 + k$ for the eigenvector $[V_{m_0+k}, W_{-m_0+k}]$, then $\lambda = \Omega k$ is the same eigenvalue of the spectral problem (2.17) with $m = m_0 - k$ for the eigenvector $[V_{m_0-k}, W_{-m_0-k}] = [W_{-m_0+k}, V_{m_0+k}]$. □

It follows from Lemmas 1 and 2 that it is sufficient to consider the spectrum of $H_m(a,\Omega)$ for $m > m_0$ and to count negative and zero eigenvalues of $H_m(a,\Omega)$ in pairs. If $a = 0$ and $\Omega = 0$, we have $H_m(0,0) = K_m(0)$, where

$$K_m(0) = \left[\begin{array}{cc} -\Delta_m + r^2 - \lambda_{m_0,0} & 0 \\ 0 & -\Delta_{m-2m_0} + r^2 - \lambda_{m_0,0}\end{array}\right]. \tag{2.22}$$

The spectrum of $K_m(0)$ is obtained from eigenvalues of the Schrödinger equation (2.2). The first diagonal entry of $K_m(0)$ has strictly positive eigenvalues

$$\mu^+_{m,n}(0) := 2(m + 2n - m_0) > 0, \quad m > m_0, \quad n \in \mathbb{N}_0.$$

The second diagonal entry of $K_m(0)$ has eigenvalues

$$\mu^-_{m,n}(0) := 2(|m - 2m_0| + 2n - m_0), \quad m > m_0, \quad n \in \mathbb{N}_0.$$

Let $N(m_0)$ and $Z(m_0)$ be the cardinality of the sets

$$\mathcal{N}(m_0) = \left\{m > m_0, \quad n \in \mathbb{N}_0 : \quad \mu^-_{m,n}(0) < 0\right\}$$

and

$$\mathcal{Z}(m_0) = \left\{m > m_0, \quad n \in \mathbb{N}_0 : \quad \mu^-_{m,n}(0) = 0\right\}.$$

The following lemma gives the count of $N(m_0)$ and $Z(m_0)$.

Lemma 3. *For every $m_0 \in \mathbb{N}$, we have*

$$N(m_0) = \frac{m_0(m_0+1)}{2}, \quad Z(m_0) = m_0. \tag{2.23}$$

Proof. To count $Z(m_0)$, we note that $\mu^-_{m,n}(0) = 0$ if and only if $|m-2m_0|+2n = m_0$. The cardinality of the set $\{(\ell, n) \in \mathbb{Z} \times \mathbb{N}_0 : \quad |\ell| + 2n = m_0\}$ coincides with the multiplicity of the eigenvalue $\lambda_{m_0,0} = 2(m_0 + 1)$ of the Schrödinger equation (2.2), which is $m_0 + 1$. Since $|\ell| \leq m_0$ translates to $m_0 \leq m \leq 3m_0$ and since $m = m_0$ contains one zero eigenvalue with $n = 0$, we obtain $Z(m_0) = m_0 + 1 - 1 = m_0$.

To count $N(m_0)$, we follow the same idea. The largest negative eigenvalue $\mu^-_{m,n}(0) = -2$ corresponds to $|m - 2m_0| + 2n = m_0 - 1$, which coincides with the multiplicity of the eigenvalue $\lambda_{m_0-1,0} = 2m_0$, which is m_0. The next negative eigenvalue $\mu^-_{m,n}(0) = -4$ corresponds to $|m - 2m_0| + 2n = m_0 - 2$, which coincides with the multiplicity of the eigenvalue $\lambda_{m_0-2,0} = 2(m_0 - 1)$, which is $m_0 - 1$. The count continues until we reach the smallest negative eigenvalue $\mu^-_{m,n}(0) = -2m_0$, which corresponds to $|m - 2m_0| + 2n = 0$ and which is simple for $m = 2m_0$ and $n = 0$. Summing integers from 1 to m_0, we obtain $N(m_0) = 1 + 2 + \cdots + m_0 = m_0(m_0 + 1)/2$. □

Remark 6. Lemma 3 yields the proof of item (ii) of Theorem 1.

Let us give some explicit examples. If $m_0 = 1$, then $\lambda_{1,0} = 4$ and

$$\begin{cases} \sigma(K_2) = \{-2, 2, 2, 6, 6, \cdots\}, \\ \sigma(K_3) = \{0, 4, 4, 8, 8, \cdots\}, \\ \sigma(K_4) = \{2, 6, 6, 10, 10, \cdots\}, \\ \vdots \end{cases} \tag{2.24}$$

so that $N(1) = 1$ and $Z(1) = 1$.

If $m_0 = 2$, then $\lambda_{2,0} = 6$ and

$$\begin{cases} \sigma(K_3) = \{-2, 2, 2, 6, 6, \cdots\}, \\ \sigma(K_4) = \{-4, 0, 4, 4, 8, 8, \cdots\}, \\ \sigma(K_5) = \{-2, 2, 6, 6, 10, 10, \cdots\}, \\ \sigma(K_6) = \{0, 4, 8, 8, 12, 12, \cdots\}, \\ \sigma(K_7) = \{2, 6, 10, 10, 14, 14, \cdots\}, \\ \vdots \end{cases} \tag{2.25}$$

so that $N(2) = 3$ and $Z(2) = 2$.

If $m_0 = 3$, then $\lambda_{3,0} = 8$ and

$$\begin{cases} \sigma(K_4) = \{-2, 2, 2, 6, 6, \cdots\}, \\ \sigma(K_5) = \{-4, 0, 4, 4, 8, 8, \cdots\}, \\ \sigma(K_6) = \{-6, -2, 2, 6, 6, 10, 10, \cdots\}, \\ \sigma(K_7) = \{-4, 0, 4, 8, 8, 12, 12, \cdots\}, \\ \sigma(K_8) = \{-2, 2, 6, 10, 10, 14, 14, \cdots\}, \\ \sigma(K_9) = \{0, 4, 8, 12, 12, 16, 16, \cdots\}, \\ \sigma(K_{10}) = \{2, 6, 10, 14, 14, 18, 18, \cdots\}, \\ \vdots \end{cases} \tag{2.26}$$

so that $N(3) = 6$ and $Z(3) = 3$.

In what follows, we fix $a > 0$ small enough and consider a continuation of eigenvalues of $H_m(a,\Omega)$ given by (2.16) with respect to the parameter Ω in the interval $(0,2)$. When one of the eigenvalues of $H_m(a,\Omega)$ reaches zero, we say that a secondary bifurcation occurs along the primary branch of radially symmetric vortices given by (2.7) and (2.8).

We will show that for every $m = m_0+2\ell$, $1 \le \ell \le m_0$, there is an a-independent constant $D_{m,m_0} \ge 0$ such that the zero eigenvalue of $K_m(0)$ becomes a positive eigenvalue of $H_m(a,\Omega)$ for small a and for $\Omega \gtrsim D_{m,m_0} a^2$. The maximum of $D_{m_0+2\ell,m_0}$ for $1 \le \ell \le m_0$ is denoted by D_{m_0}.

We further show that there is another a-independent constant $C_{m_0} > 0$ such that when Ω is increased in the interval $(D_{m_0}a^2, 2-C_{m_0}a^2)$, then $1+B(m_0)$ secondary bifurcations occur, where $B(m_0) = m_0(m_0-1)/2$, at which a negative eigenvalue of $H_m(a,\Omega)$ for some m and for Ω below the bifurcation point becomes a positive eigenvalue of $H_m(a,\Omega)$ for the same m and for Ω above the bifurcation point. The first $B(m_0)$ secondary bifurcations occur for values of Ω sufficiently distant from the value $\Omega_0 = 2$, whereas the last secondary bifurcation occurs for the value of Ω near but below the value $\Omega_0 = 2$. The latter case has to be handled in the presence of infinitely many resonances in the limit $a \to 0$. The aforementioned claims proved in Section 3 will provide proofs of item (iii) in Theorem 1.

At each non-resonant bifurcation point, a new secondary branch of vortex solutions is born for Ω on one side of the bifurcation point among the roots of the nonlinear operator g given by (2.19) and (2.20). The secondary branch represents a multi-vortex configuration near the origin of the total charge m_0, where the radial symmetry is now broken. The aforementioned claims proved in Section 4 and 5 will provide respectively proofs of items (iv) and (v) in Theorem 1.

3. Secondary bifurcations as Ω increases

Let the primary branch of radially symmetric vortices be defined by (2.7) and (2.8) in the small-amplitude limit. Expanding the family of operators $K_m(a)$ in powers of a, we obtain

$$K_m(a) = \begin{bmatrix} -\Delta_m + r^2 - \lambda_{m_0,0} & 0 \\ 0 & -\Delta_{m-2m_0} + r^2 - \lambda_{m_0,0} \end{bmatrix} + a^2 \begin{bmatrix} -\omega_{m_0,0} + 2e_{m_0,0}^2(r) & e_{m_0,0}^2(r) \\ e_{m_0,0}^2(r) & -\omega_{m_0,0} + 2e_{m_0,0}^2(r) \end{bmatrix} + \mathcal{O}(a^4),$$

where the correction term is given by a bounded potential on $\mathbb{R}^+$.

Also recall from (2.16) that the operator $H_m(a,\Omega)$ is expanded as $a \to 0$ with the leading-order term given by the diagonal operator $H_m(0,\Omega)$ with the entries given by two linear operators:

$$\left\{ \begin{array}{l} L_+ := -\Delta_m + r^2 - \lambda_{m_0,0} - \Omega(m-m_0), \\ L_- := -\Delta_{m-2m_0} + r^2 - \lambda_{m_0,0} + \Omega(m-m_0). \end{array} \right. \tag{3.1}$$

We shall now analyze how eigenvalues of $H_m(a,\Omega)$ cross zero when Ω is increased in the interval $(0,2)$.

3.1. Zero eigenvalues of $K_m(0)$

When $a=0$ and $\Omega=0$, each operator block $H_m(0,0)=K_m(0)$ has a simple zero eigenvalue for $m=m_0+2\ell$, $1\le\ell\le m_0$. See examples in (2.24), (2.25), and (2.26). The following lemma tells us that the zero eigenvalue of such $H_m(0,0)$ becomes a positive eigenvalue of $H_m(a,\Omega)$ for every sufficiently small a, provided the values of Ω are sufficiently large and positive.

Lemma 4. *For every $m_0\in\mathbb{N}$, there exists $a_0>0$ and $D_{m_0}\ge 0$ such that for every $0<a<a_0$, $\Omega>D_{m_0}a^2$, and $1\le\ell\le m_0$, there is a small positive eigenvalue of $H_{m_0+2\ell}(a,\Omega)$ which is continuous in (a,Ω) and converges to the zero eigenvalue of $K_{m_0+2\ell}(0)$ as $a\to 0$ and $\Omega\to 0$.*

Proof. The zero eigenvalue of $K_m(0)$ for $m=m_0+2\ell$, $1\le\ell\le m_0$ corresponds to the second diagonal operator in $K_m(0)$. Let us show by the perturbation theory argument that the zero eigenvalue is continued as a small $\mathcal{O}(a^2)$ eigenvalue of $K_m(a)$ for all a sufficiently small.

The eigenfunction of $K_m(0)$ with $m=m_0+2\ell$ for the zero eigenvalue is obtained from the balance

$$\lambda_{m-2m_0,n}=\lambda_{m_0,0}\quad\Rightarrow\quad n(\ell)=\frac{m_0-|2\ell-m_0|}{2}.$$

Since the zero eigenvalue of $K_{m_0+2\ell}(0)$ is simple, the regular perturbation theory in [17] implies the existence of a small eigenvalue $\mu_\ell(a)$ of the linear operator $K_{m_0+2\ell}(a)$ and the corresponding eigenvector $(V_{m_0+2\ell},W_{-m_0+2\ell})$, which are analytic functions of a. Their Taylor expansions are given by

$$\left\{\begin{array}{lcl} V_{m_0+2\ell} &=& a^2\tilde{V}_{m_0+2\ell}+\mathcal{O}_{L^2_r}(a^4),\\ W_{-m_0+2\ell} &=& c_{-m_0+2\ell}e_{|m_0-2\ell|,n(\ell)}+a^2\tilde{W}_{-m_0+2\ell}+\mathcal{O}_{L^2_r}(a^4),\\ \mu_\ell &=& a^2\tilde{\mu}_\ell+\mathcal{O}(a^4),\end{array}\right. \tag{3.2}$$

where $c_{-m_0+2\ell}\neq 0$ is arbitrary, $\tilde{V}_{m_0+2\ell}$, $\tilde{W}_{-m_0+2\ell}$, and $\tilde{\mu}_\ell$ are obtained by the standard projection algorithm, and the correction terms are defined uniquely by the method of Lyapunov–Schmidt reductions. In particular, $\tilde{\mu}_\ell$ is obtained from

$$\tilde{\mu}_\ell=-\omega_{m_0,0}+2\langle e^2_{m_0,0},e^2_{|m_0-2\ell|,n(\ell)}\rangle_{L^2_r}. \tag{3.3}$$

If $\tilde{\mu}_\ell\neq 0$, the eigenvalue $\mu_\ell(a)$ is generally nonzero but $\mathcal{O}(a^2)$ small.

It follows from (3.1) that the Ω-term in L_- is a positive perturbation to $K_m(0)$ for $m>m_0$. Therefore, there exists an a-independent constant $D_{\ell,m_0}\ge 0$ such that the eigenvalue $\mu_\ell(a)$ continued with respect to the parameter Ω is strictly positive for $\Omega>D_{\ell,m_0}a^2$. The assertion of the lemma is proved by taking the largest of D_{ℓ,m_0} for all admissible $1\le\ell\le m_0$ as D_{m_0}. □

Remark 7. Lemma 4 yields the existence of constant $D_m\ge 0$ in item (iii) of Theorem 1.

Remark 8. If $\tilde{\mu}_\ell > 0$ in the perturbation result (3.3), then $\mu_\ell > 0$ for every small $a > 0$ and $\Omega > 0$. If this is true for every $1 \leq \ell \leq m_0$, then $D_{m_0} = 0$ in Lemma 4. In particular, this is true for $1 \leq m_0 \leq 3$. Indeed, for $\ell = m_0$ and $n(\ell) = 0$, we obtain from (2.10) and (3.3):

$$\tilde{\mu}_{m_0} = \|e_{m_0,0}\|^4_{L^4_r} > 0.$$

For $\ell = m_0 - 1$ and $n(\ell) = 1$, we use the following formula for the $L^2_r(\mathbb{R}^+)$-normalized Hermite–Gauss solutions of the Schrödinger equation (2.2) with $\lambda_{m,1} = 2(m+3)$:

$$e_{m,1}(r) = \frac{\sqrt{2}}{\sqrt{(m+1)!}} r^m (m+1-r^2) e^{-\frac{r^2}{2}}. \tag{3.4}$$

Then, we obtain from (3.3) for $m_0 \geq 2$:

$$\tilde{\mu}_{m_0-1} = 2\langle e^2_{m_0,0}, e^2_{m_0-2,1}\rangle_{L^2_r} - \|e_{m_0,0}\|^4_{L^4_r} = \frac{(2m_0)!(m_0^2 + m_0 - 1)}{4^{m_0}(m_0!)^2} > 0.$$

By the symmetry, we also have $\mu_1 = \mu_{m_0-1} > 0$. Thus, for $1 \leq m_0 \leq 3$, we have $\tilde{\mu}_\ell > 0$ for all admissible $1 \leq \ell \leq m_0$.

Remark 9. It remains unclear if $\tilde{\mu}_\ell > 0$ for the other values in $2 \leq \ell \leq m_0 - 2$ for $m_0 \geq 4$.

3.2. Zero eigenvalues of $H_m(a, \Omega)$ for $\Omega \in (0, 2)$

For $m > m_0$, the leading-order diagonal operator $H_m(0, \Omega)$ given by the operators L_+ and L_- in (3.1) has an eigenbasis

$$\{(e_{m,n}, 0); (0, e_{|m-2m_0|,n})\}_{n\in\mathbb{N}_0}. \tag{3.5}$$

Since $H_m(a, \Omega)$ is self-adjoint, by regular perturbation theory in [17], the operator $H_m(a, \Omega)$ has a set of eigenvalues counted by $n \in \mathbb{N}_0$:

$$\begin{cases} \mu^+_{m,n}(a, \Omega) & := \lambda_{m,n} - \lambda_{m_0,0} - \Omega(m - m_0) + \mathcal{O}(a^2), \\ \mu^-_{m,n}(a, \Omega) & := \lambda_{m-2m_0,n} - \lambda_{m_0,0} + \Omega(m - m_0) + \mathcal{O}(a^2). \end{cases} \tag{3.6}$$

For $m > m_0$, $n \in \mathbb{N}_0$, and $\Omega < 2$, we have

$$\lambda_{m,n} - \lambda_{m_0,0} - \Omega(m - m_0) = (2 - \Omega)(m - m_0) + 4n > 0.$$

Therefore, the eigenvalues $\mu^+_{m,n}(a, \Omega)$ never become zero for small a and $\Omega < 2$. On the other hand, the eigenvalues $\mu^-_{m,n}(a, \Omega)$ become zero when $\Omega = \Omega_{m,n}(a)$ given by

$$\Omega_{m,n}(a) = 2\frac{m_0 - |m - 2m_0|}{m - m_0} - \frac{4n}{m - m_0} + \mathcal{O}(a^2). \tag{3.7}$$

Let $B(m_0)$ denote the number of eigenvalues $\mu^-_{m,n}$ crossing zero at $\Omega = \Omega_{m,n}(a)$ with $\Omega_{m,n}(0) \in (0, 2)$. The following lemma gives the count of $B(m_0)$.

Lemma 5. *For every $m_0 \in \mathbb{N}$, we have*

$$B(m_0) = \frac{m_0(m_0 - 1)}{2}. \tag{3.8}$$

Proof. To count $B(m_0)$, we count the values of $m > m_0$ for the first values of $n \in \mathbb{N}_0$, when $\Omega_{m,n}(0) \in (0, 2)$:

- For $n = 0$, the inequality $0 < m_0 - |m - 2m_0| < m - m_0$ is true for $2m_0 + 1 \leq m \leq 3m_0 - 1$.
- For $n = 1$, the inequality $0 < m_0 - |m - 2m_0| - 2 < m - m_0$ is true for $m_0 + 3 \leq m \leq 3m_0 - 3$.
- For $n = 2$, the inequality $0 < m_0 - |m - 2m_0| - 4 < m - m_0$ is true for $m_0 + 5 \leq m \leq 3m_0 - 5$.

For a general $n \in \mathbb{N}$, we have $\Omega_{m,n}(0) \in (0, 2)$ for $m_0 + 2n + 1 \leq m \leq 3m_0 - 2n - 1$ provided that the range for m is nonempty. Summing up all cases, we have

$$B(m_0) = m_0 - 1 + \sum_{n=1}^{\infty} [2m_0 - 4n - 1]_+$$

where $[a]_+$ is a when $a \geq 0$ and 0 if $a < 0$. The sum is finite as n terminates at the last entry for which $2m_0 - 4n - 1 > 0$. If m_0 is odd, then the last entry corresponds to $N = (m_0 - 1)/2$ and we obtain

$$\sum_{n=1}^{\infty} [2m_0 - 4n - 1]_+ = \sum_{n=1}^{N} (2m_0 - 4n - 1) = \frac{m_0^2 - 3m_0 + 2}{2}.$$

If m_0 is even, then the last entry corresponds to $N = m_0/2 - 1$ and we obtain

$$\sum_{n=1}^{\infty} [2m_0 - 4n - 1]_+ = \sum_{n=1}^{N} (2m_0 - 4n - 1) = \frac{m_0^2 - 3m_0 + 2}{2}.$$

Adding $m_0 - 1$ to this number, we obtain (3.8) in both cases. □

In particular, we have $B(1) = 0$, $B(2) = 1$, $B(3) = 3$, and $B(4) = 6$. See examples in (2.24), (2.25), and (2.26). Let us list the bifurcation values of Ω for these examples:

- For $m_0 = 1$, no bifurcations occur.
- For $m_0 = 2$, the only bifurcation occurs at $\Omega_{5,0}(0) = 2/3$.
- For $m_0 = 3$, three bifurcations occur at $\Omega_{7,0}(0) = 1$, $\Omega_{8,0}(0) = 2/5$, and $\Omega_{6,1}(0) = 2/3$.
- For $m_0 = 4$, six bifurcations occur at $\Omega_{9,0}(0) = 6/5$, $\Omega_{10,0}(0) = 2/3$, $\Omega_{11,0}(0) = 2/7$, $\Omega_{7,1}(0) = 2/3$, $\Omega_{8,1}(0) = 1$, and $\Omega_{9,1}(0) = 2/5$.

Remark 10. Lemma 5 yields the number $B(m_0)$ in item (iii) of Theorem 1. Note that the bifurcation points of Ω are simple for $1 \leq m_0 \leq 3$. Multiple bifurcation points exist in a general case for $m_0 \geq 4$, e.g. $\Omega_{10,0}(0) = \Omega_{7,1}(0) = 2/3$ for $m_0 = 4$.

The following proposition summarizes properties of $H_m(a, \Omega)$ near each bifurcation point. These properties are needed for the bifurcation analysis in Section 4.

Proposition 1. *For every $m_0 \in \mathbb{N}$, let $\Omega_*(a)$ be one of the bifurcation points defined by (3.7). Assume it has multiplicity k and corresponds to $m_1, \dots, m_k > m_0$. There exists $a_0 > 0$, $C_{m_0} > 0$, and $E_{m_0} > 0$ such that for every $0 < a < a_0$, $|\Omega - \Omega_*(a)| < C_{m_0} a^2$,* *and every $m > m_0$ such that $m \notin \{m_1, \dots, m_k\}$, the operator $H_m(a, \Omega)$ is invertible in $L^2_r(\mathbb{R}^+)$ with the bound*

$$\|H_m(a,\Omega)^{-1}\|_{L^2_r \to H^2_r \cap L^{2,2}_r} \leq E_{m_0}, \quad m > m_0, \quad m \notin \{m_1, \dots, m_k\}. \tag{3.9}$$

Moreover, the number of negative eigenvalues of $H_m(a, \Omega)$, $m \notin \{m_1, \dots, m_k\}$ remains the same for every Ω in $|\Omega - \Omega_*(a)| < C_{m_0} a^2$. On the other hand, the number of negative eigenvalues for $H_m(a, \Omega)$, $m \in \{m_1, \dots, m_k\}$ is reduced by one when Ω crosses $\Omega_*(a)$ in $|\Omega - \Omega_*(a)| < C_{m_0} a^2$.

Proof. First, we note that for each $m > m_0$, there may be at most one eigenvalue of $H_m(a, \Omega)$ which becomes zero at $\Omega = \Omega_*(a)$. Bound (3.9) follows from the fact that $H_m(a, \Omega_*(a))$ with $m \notin \{m_1, \dots, m_k\}$ has no eigenvalues in the neighborhood of zero. On the other hand, each simple eigenvalue of $H_m(a, \Omega_*(a))$ with $m \in \{m_1, \dots, m_k\}$ is continued in Ω according to the derivative

$$\frac{\partial H_m}{\partial \Omega}(a, \Omega) = -(m - m_0) R. \tag{3.10}$$

Let (V_m, W_{m-2m_0}) be the corresponding eigenvector for the zero eigenvalue of $H_m(a, \Omega_*(a))$. Since $m > m_0$, the eigenvalue is positive for $\Omega \gtrsim \Omega_*(a)$ and negative for $\Omega \lesssim \Omega_*(a)$ if $S_m < 0$, where

$$S_m := \|V_m\|^2_{L^2_r} - \|W_{m-2m_0}\|^2_{L^2_r}. \tag{3.11}$$

Since $V_m \to 0$ as $a \to 0$, we have $S_m < 0$ for each $m \in \{m_1, \dots, m_k\}$, provided a is small enough. □

Remark 11. The quantity S_m defined by (3.11) is referred to as the Krein quantity. The sign of S_m gives the Krein signature of the neutrally stable eigenvalues of the spectral stability problem associated with the radially symmetric vortices in the case of no rotation [20].

Definition 1. If $k = 1$ in Proposition 1, we say that the bifurcation point $\Omega_*(a)$ is non-resonant.

3.3. Zero eigenvalues of $H_m(a, \Omega)$ for Ω near 2

Consider the rotation frequency $\Omega = 2 + \mathcal{O}(a^2)$. According to (2.17) and (2.22), see examples in (2.24), (2.25), and (2.26), there are infinitely many resonances for $a = 0$.

We will show that if Ω is defined at a particular value denoted by $\Omega_{m_0+1,0}(a) = 2 + \mathcal{O}(a^2)$, for which the spectral stability problem (2.17) with $m = m_0 + 1$ admits a nontrivial solution, then the blocks $H_m(a, \Omega)$ of the Hessian operator for every $m \geq m_0 + 2$ are invertible in $L^2_r(\mathbb{R}^+)$ near $\Omega = \Omega_{m_0+1,0}(a)$ and the smallest eigenvalue of $H_m(a, \Omega_{m_0+1,0}(a))$ is proportional to $\mathcal{O}(a^2)$. At the same time, the block $H_{m_0+1}(a, \Omega_{m_0+1,0}(a))$ has a simple zero eigenvalue and a simple positive eigenvalue proportional to $\mathcal{O}(a^2)$. We also show for $1 \leq m_0 \leq 16$ that the blocks $H_m(a, \Omega_{m_0+1,0}(a))$ for $m_0 + 2 \leq m \leq 2m_0$ have exactly one small negative eigenvalue proportional to $\mathcal{O}(a^2)$, whereas all other eigenvalues are strictly positive.

The following lemma gives the precise location of $\Omega_{m_0+1,0}(a) = 2 + \mathcal{O}(a^2)$.

Lemma 6. *There exists* $a_0 > 0$ *such that for every* $0 < a < a_0$*, there exists* $\Omega_{m_0+1,0}(a) < 2$ *given asymptotically by*

$$\Omega_{m_0+1,0}(a) := 2 - \frac{(2m_0)!}{4^{m_0} m_0!(m_0+1)!} a^2 + \mathcal{O}(a^4), \tag{3.12}$$

such that $H_{m_0+1}(a, \Omega_{m_0+1,0}(a))$ *has a simple zero eigenvalue.*

Proof. We solve the bifurcation equation (2.17) for $m = m_0 + 1$ near $\Omega = 2$ in powers of a. Since $\Omega = 2$ is a double (semi-simple) eigenvalue of the bifurcation equation (2.17) at $a = 0$, we use the two-parameter perturbation theory with the Taylor expansion

$$\left\{ \begin{array}{lcl} V_{m_0+1} & = & c_{m_0+1} e_{m_0+1,0} + a^2 \tilde{V}_{m_0+1} + \mathcal{O}_{L^2_r}(a^4), \\ W_{-m_0+1} & = & c_{-m_0+1} e_{m_0-1,0} + a^2 \tilde{W}_{-m_0+1} + \mathcal{O}_{L^2_r}(a^4), \\ \Omega & = & 2 + a^2 \tilde{\Omega} + \mathcal{O}(a^4), \end{array} \right. \tag{3.13}$$

where $(c_{m_0+1}, c_{-m_0+1}) \neq (0,0)$ are to be determined, the correction terms $\tilde{V}_{m_0+1}$, $\tilde{W}_{-m_0+1}$, and $\tilde{\Omega}$ are a-independent, and the reminder terms are uniquely defined by the Lyapunov–Schmidt reductions. The admissible values of $(c_{m_0+1}, c_{-m_0+1}) \neq (0,0)$ and $\tilde{\Omega}$ are found from the matrix eigenvalue problem

$$\tilde{A} \left[\begin{array}{c} c_{m_0+1} \\ c_{-m_0+1} \end{array} \right] = \tilde{\Omega} \left[\begin{array}{c} c_{m_0+1} \\ c_{-m_0+1} \end{array} \right],$$

where

$$\tilde{A} = \left[\begin{array}{cc} \langle (-\omega_{m_0,0} + 2e^2_{m_0,0}) e_{m_0+1,0}, e_{m_0+1,0} \rangle_{L^2} & \langle e^2_{m_0,0} e_{m_0-1,0}, e_{m_0+1,0} \rangle_{L^2} \\ -\langle e^2_{m_0,0} e_{m_0+1,0}, e_{m_0-1,0} \rangle_{L^2} & -\langle (-\omega_{m_0,0} + 2e^2_{m_0,0}) e_{m_0-1,0}, e_{m_0-1,0} \rangle_{L^2} \end{array} \right]$$
$$= \left[\begin{array}{cc} \frac{(2m_0)!}{4^{m_0}(m_0-1)!(m_0+1)!} & \frac{(2m_0)!}{4^{m_0} m_0! \sqrt{(m_0-1)!(m_0+1)!}} \\ -\frac{(2m_0)!}{4^{m_0} m_0! \sqrt{(m_0-1)!(m_0+1)!}} & -\frac{(2m_0)!}{4^{m_0}(m_0!)^2} \end{array} \right],$$

and we have used the explicit formula (2.9). Eigenvalues of $\tilde{A}$ and their normalized eigenvectors are given by

$$\tilde{\Omega} = 0: \quad \left[\begin{array}{c} c_{m_0+1} \\ c_{-m_0+1} \end{array} \right] = \frac{1}{\sqrt{2m_0+1}} \left[\begin{array}{c} \sqrt{m_0+1} \\ -\sqrt{m_0} \end{array} \right] \tag{3.14}$$

and

$$\tilde{\Omega} = \tilde{\Omega}_{m_0+1,0} := -\frac{(2m_0)!}{4^{m_0} m_0!(m_0+1)!}:$$
$$\left[\begin{array}{c} c_{m_0+1} \\ c_{-m_0+1} \end{array} \right] = \frac{1}{\sqrt{2m_0+1}} \left[\begin{array}{c} \sqrt{m_0} \\ -\sqrt{m_0+1} \end{array} \right]. \tag{3.15}$$

Substituting $\tilde{\Omega} = \tilde{\Omega}_{m_0+1,0}$ from (3.15) to (3.13), we obtain the asymptotic expansion (3.12). Since $\tilde{\Omega}_{m_0+1,0} < 0$ in (3.15), we have $\Omega_{m_0+1,0}(a) < 2$ for small a. □

Remark 12. Lemma 6 yields the existence of constant $C_m > 0$ in item (iii) of Theorem 1.

In order to compute eigenvalues of the blocks $H_m(a, \Omega_{m_0+1,0}(a))$ for small a, we write explicitly the following expansion in powers of a:

$$H_m(a, \Omega_{m_0+1,0}(a)) = \begin{bmatrix} -\Delta_m + r^2 - 2(m+1) & 0 \\ 0 & -\Delta_{m-2m_0} + r^2 + 2(m - 2m_0 - 1) \end{bmatrix}$$
$$+ a^2 \begin{bmatrix} -\omega_{m_0,0} - (m - m_0)\tilde{\Omega}_{m_0+1,0} + 2e^2_{m_0,0}(r) & e^2_{m_0,0}(r) \\ e^2_{m_0,0}(r) & -\omega_{m_0,0} + (m - m_0)\tilde{\Omega}_{m_0+1,0} + 2e^2_{m_0,0}(r) \end{bmatrix}$$
$$+ \mathcal{O}(a^4).$$

We consider now eigenvalues of $H_m(a, \Omega_{m_0+1,0}(a))$ denoted by λ near zero as $a \to 0$. The following three lemmas summarize the results of computations of the perturbation theory.

Lemma 7. *There exists $a_0 > 0$ such that for every $0 < a < a_0$, the block*

$$H_{m_0+1}(a, \Omega_{m_0+1,0}(a))$$

has a simple zero eigenvalue and a simple positive eigenvalue of the order $\mathcal{O}(a^2)$, whereas all other eigenvalues are strictly positive.

Proof. For $m = m_0 + 1$, computations of the perturbation theory similar to the expansion (3.13) are repeated as follows:

$$\begin{cases} V_{m_0+1} = c_{m_0+1} e_{m_0+1,0} + a^2 \tilde{V}_{m_0+1} + \mathcal{O}_{L^2_r}(a^4), \\ W_{-m_0+1} = c_{-m_0+1} e_{m_0-1,0} + a^2 \tilde{W}_{-m_0+1} + \mathcal{O}_{L^2_r}(a^4), \\ \lambda = a^2 \tilde{\lambda} + \mathcal{O}(a^4). \end{cases} \tag{3.16}$$

The Lyapunov–Schmidt reduction method results now in the matrix eigenvalue problem

$$\tilde{A} \begin{bmatrix} c_{m_0+1} \\ c_{-m_0+1} \end{bmatrix} = \tilde{\lambda} \begin{bmatrix} c_{m_0+1} \\ c_{-m_0+1} \end{bmatrix},$$

where

$$\tilde{A} = \begin{bmatrix} \frac{(2m_0)!}{4^{m_0}(m_0!)^2} & \frac{(2m_0)!}{4^{m_0} m_0! \sqrt{(m_0-1)!(m_0+1)!}} \\ \frac{(2m_0)!}{4^{m_0} m_0! \sqrt{(m_0-1)!(m_0+1)!}} & \frac{(2m_0)!}{4^{m_0}(m_0-1)!(m_0+1)!} \end{bmatrix}.$$

Eigenvalues of $\mathcal{A}$ and their normalized eigenvectors are given by

$$\tilde{\lambda} = 0 : \quad \begin{bmatrix} c_{m_0+1} \\ c_{-m_0+1} \end{bmatrix} = \frac{1}{\sqrt{2m_0+1}} \begin{bmatrix} \sqrt{m_0} \\ -\sqrt{m_0+1} \end{bmatrix} \tag{3.17}$$

and

$$\tilde{\lambda} = \frac{(2m_0+1)!}{4^{m_0} m_0!(m_0+1)!} : \quad \begin{bmatrix} c_{m_0+1} \\ c_{-m_0+1} \end{bmatrix} = \frac{1}{\sqrt{2m_0+1}} \begin{bmatrix} \sqrt{m_0+1} \\ \sqrt{m_0} \end{bmatrix}. \tag{3.18}$$

The zero eigenvalue in (3.17) corresponds to the choice $\Omega = \Omega_{m_0+1,0}(a)$ at the bifurcation point. The positive eigenvalue in (3.18) gives the positive eigenvalue of the order $\mathcal{O}(a^2)$ in (3.16). The other eigenvalues of $H_{m_0}(0, 2)$ are strictly positive and they remain so in $H_{m_0+1}(a, \Omega_{m_0+1,0}(a))$ for small a. □

Lemma 8. *There exists $a_0 > 0$ such that for every $0 < a < a_0$, the block*

$$H_m(a, \Omega_{m_0+1,0}(a))$$

with $m \geq 2m_0 + 1$ has a simple positive eigenvalue of the order $\mathcal{O}(a^2)$, whereas all other eigenvalues are strictly positive.

Proof. For $m \geq 2m_0 + 1$, the zero eigenvalue of $H_m(0, 2)$ is simple and all other eigenvalues are strictly positive. The one-parameter perturbation expansion for the small eigenvalue is developed as follows:

$$\left\{ \begin{array}{lcl} V_m & = & c_m e_{m,0} + a^2 \tilde{V}_m + \mathcal{O}_{L^2_r}(a^4), \\ W_{m-2m_0} & = & a^2 \tilde{W}_{m-2m_0} + \mathcal{O}_{L^2_r}(a^4), \\ \lambda & = & a^2 \tilde{\lambda} + \mathcal{O}(a^4). \end{array} \right. \tag{3.19}$$

The projection condition yields the only eigenvalue given by

$$\begin{aligned} \tilde{\lambda} &= 2\langle e^2_{m_0,0} e_{m,0}, e_{m,0} \rangle_{L^2_r} - \omega_{m_0,0} - (m - m_0)\tilde{\Omega}_{m_0,0} \\ &= \frac{2(m_0 + m)!}{2^{m_0+m} m_0! m!} + \frac{(2m_0)!(m - 2m_0 - 1)}{4^{m_0} m_0!(m_0 + 1)!} > 0. \end{aligned} \tag{3.20}$$

Since $\tilde{\lambda} > 0$, the expansion (3.19) yields the positive eigenvalue of the order $\mathcal{O}(a^2)$ in the block $H_m(a, \Omega_{m_0+1,0}(a))$ for small a. □

It remains to consider the blocks $H_m(a, \Omega_{m_0+1,0}(a))$ for $m_0 + 2 \leq m \leq 2m_0$. Before continuing with the technical details, we note the example of $m_0 = 2$. The results of [16, 20] imply that no real eigenvalues exist in the neighborhood of $\Omega = 2$ and $a = 0$ among eigenvalues of the bifurcation equation (2.17) for $m = 4 = 2m_0$. This is due to oscillatory instability of the radially symmetric vortex of charge two ($m_0 = 2$), which arises in the small-amplitude limit of the primary branch. See Remark 6.9 in [16]. More general results were obtained in [10], see Proposition 8.3, where all vortices with $m_0 \geq 2$ were found unstable but the number of unstable modes is smaller than $m_0 - 1$ if m_0 is sufficiently large. The following result is in agreement with the outcomes of the stability computations in [10, 16].

Lemma 9. *Let $2 \leq m_0 \leq 16$. There exists $a_0 > 0$ such that for every $0 < a < a_0$, the block $H_m(a, \Omega_{m_0+1,0}(a))$ with $m_0 + 2 \leq m \leq 2m_0$ has two small eigenvalues of the order $\mathcal{O}(a^2)$ (one is positive and the other one is negative), whereas all other eigenvalues are strictly positive.*

Proof. For $m_0 + 2 \leq m \leq 2m_0$, the zero eigenvalue of $H_m(0, 2)$ is double and all other eigenvalues are strictly positive. The two-parameter perturbation expansion for the small eigenvalue is developed as follows:

$$\left\{ \begin{array}{lcl} V_n & = & c_m e_{m,0} + a^2 \tilde{V}_m + \mathcal{O}_{L^2_r}(a^4), \\ W_{m-2m_0} & = & c_{m-2m_0} e_{2m_0-m,0} + a^2 \tilde{W}_{m-2m_0} + \mathcal{O}_{L^2_r}(a^4), \\ \lambda & = & a^2 \tilde{\lambda} + \mathcal{O}(a^4). \end{array} \right. \tag{3.21}$$

The Lyapunov–Schmidt reduction method results in the matrix eigenvalue problem

$$\tilde{A}\left[\begin{array}{c} c_m \\ c_{m-2m_0} \end{array}\right] = \tilde{\lambda}\left[\begin{array}{c} c_m \\ c_{m-2m_0} \end{array}\right], \tag{3.22}$$

where

$$\tilde{A} = \left[\begin{array}{cc} \frac{2(m_0+m)!}{2^{m_0+m}m_0!m!} + \frac{(2m_0)!(m-2m_0-1)}{4^{m_0}m_0!(m_0+1)!} & \frac{(2m_0)!}{4^{m_0}m_0!\sqrt{m!(2m_0-m)!}} \\ \frac{(2m_0)!}{4^{m_0}m_0!\sqrt{m!(2m_0-m)!}} & \frac{2(3m_0-m)!}{2^{3m_0-m}m_0!(2m_0-m)!} - \frac{(2m_0)!(m+1)}{4^{m_0}m_0!(m_0+1)!} \end{array}\right].$$

For $m_0 = 2$ (with $m = 4$) and $m_0 = 3$ (with $m = 5, 6$), the entries of $\tilde{A}$ are computed explicitly. Since the first diagonal entry is positive and the second diagonal entry is negative, $\tilde{A}$ has one positive and one negative eigenvalue $\tilde{\lambda}$. We have checked numerically that this property remains true for every $2 \le m_0 \le 16$. The expansion (3.21) yields one positive and one negative eigenvalue of the order $\mathcal{O}(a^2)$ in the block $H_m(a, \Omega_{m_0+1,0}(a))$ for small a. □

Remark 13. For $m_0 \ge 17$, the matrix $\tilde{A}$ for some m in the range $m_0 + 2 \le m \le 2m_0$ has two negative eigenvalues and the number of such m-values grows with the number m_0. No zero eigenvalues of $\tilde{A}$ are found numerically for at least $m_0 \le 100$.

The following proposition summarizes the previous computations of the perturbation theory. The corresponding result is needed for the bifurcation analysis in Section 4.

Proposition 2. *For every integer $1 \le m_0 \le 16$, there exist $a_0 > 0$, $C_{m_0} \in (0, |\tilde{\Omega}_{m_0+1,0}|)$, and $E_{m_0} > 0$ such that for every $0 < a < a_0$, $|\Omega - \Omega_{m_0+1,0}(a)| < C_{m_0}a^2$, and every $m \ge m_0 + 2$, the operator $H_m(a, \Omega)$ is invertible in $L^2_r(\mathbb{R}^+)$ with the bound*

$$\|H_m(a,\Omega)^{-1}\|_{L^2_r \to H^2_r \cap L^{2,2}_r} \le E_{m_0} a^{-2}, \quad m \ge m_0 + 2. \tag{3.23}$$

Moreover, all eigenvalues of $H_m(a, \Omega)$ are strictly positive, except for $m_0 - 1$ simple negative eigenvalues, which correspond to $m_0 + 2 \le m \le 2m_0$. On the other hand, all eigenvalues of $H_{m_0+1}(a, \Omega)$ are strictly positive except one simple eigenvalue, which is negative for $\Omega \lesssim \Omega_{m_0+1,0}(a)$ and positive for $\Omega \gtrsim \Omega_{m_0+1,0}(a)$.

Proof. Eigenvalues and invertibility of $H_m(a, \Omega_{m_0+1,0}(a))$ for $m \ge m_0 + 2$ with the bound (3.23) follows from the outcomes of the perturbation theory in Lemmas 7, 8, and 9, where the m-independent constant E_{m_0} exists thanks to the fact that the $\mathcal{O}(a^2)$ positive eigenvalue in (3.20) is bounded away from zero.

It remains to prove that the zero eigenvalue of $H_{m_0+1}(a, \Omega_{m_0+1,0}(a))$ becomes a small positive eigenvalue of $H_{m_0+1}(a, \Omega)$ for $\Omega \gtrsim \Omega_{m_0+1,0}(a)$ and a small negative eigenvalue of $H_{m_0+1}(a, \Omega)$ for $\Omega \lesssim \Omega_{m_0+1,0}(a)$. This follows from the derivative (3.10) and the Krein signature of the zero eigenvalue of $H_{m_0+1}(a, \Omega_{m_0+1,0}(a))$ defined by (3.11). We obtain from the expansion (3.16)

$$S_{m_0+1} = \|V_{m_0+1}\|^2_{L^2_r} - \|W_{-m_0+1}\|^2_{L^2_r} = c^2_{m_0+1} - c^2_{-m_0+1} + \mathcal{O}(a^2), \tag{3.24}$$

where (c_{m_0+1}, c_{-m_0+1}) is given by the eigenvector of $\tilde{A}$ that corresponds to $\tilde{\lambda} = 0$. From (3.17), we obtain $S_{m_0+1} < 0$, hence the corresponding eigenvalue of $H_{m_0+1}(a, \Omega)$ is an increasing[4] function of Ω. $\square$

Remark 14. Propositions 1 and 2 complete the proof of item (iii) in Theorem 1.

Remark 15. Eigenvalues $\lambda := \Omega(m - m_0)$ of the bifurcation problem (2.17) near $\lambda_{m,m_0} = 2(m - m_0)$ are either complex or real for $m_0 + 2 \leq m \leq 2m_0$, depending on whether the m-th mode of the m_0-th vortex is spectrally unstable or stable. When all such eigenvalues are complex, which happens for $1 \leq m_0 \leq 3$, no other bifurcation curve is connected to the point $\Omega_0 = 2$ from below, besides the curve $\Omega_{m_0+1,0}$. When $m_0 \geq 4$, we have found that there are $R(m_0)$ pairs of real eigenvalues λ of the bifurcation problem (2.17) near λ_{m,m_0}, e.g.

- $R(4) = 1$ with $m = 8$;
- $R(5) = 1$ with $m = 10$;
- $R(6) = 2$ with $m = 11, 12$;
- $R(7) = 3$ with $m = 12, 13, 14$;
- $R(8) = 3$ with $m = 14, 15, 16$;

and so on. This finding corresponds to the result of Proposition 8.3 in [10] where the number of complex eigenvalues is found to be smaller than $m_0 - 1$ if m_0 is sufficiently large. If $R(m_0) \neq 0$, then there exist $R(m_0)$ bifurcation curves connected to the point $\Omega_0 = 2$ from below. As follows from the count on negative eigenvalues in

$$N(m_0) - B(m_0) - 1 = m_0 - 1,$$

which coincides with the number of negative eigenvalues in Lemma 9, these additional bifurcation curves for $4 \leq m_0 \leq 16$ are located above the curve $\Omega_{m_0+1,0}$. However, for $m_0 \geq 17$, thanks to the computations in Remark 13, some of the positive eigenvalues of $H_m(a, \Omega)$ for $m_0 + 2 \leq m \leq 2m_0$ become negative eigenvalues for $\Omega \lesssim \Omega_{m_0+1,0}(a)$ and the total number of negative eigenvalues at $\Omega \gtrsim \Omega_{m_0+1,0}(a)$ exceeds $m_0 - 1$. Therefore, some of the $R(m_0)$ bifurcation curves are located below the curve $\Omega_{m_0+1,0}$ for $m_0 \geq 17$.

4. Secondary branches of multi-vortex solutions

Recall that the solution U to the stationary GP equation (2.4) is a critical point of the energy functional $E_\mu(u)$ in (2.11), therefore, the bifurcation problem for $g(v; a, \Omega)$ in (2.19)–(2.20) has a variational structure. The number of negative eigenvalues of the Jacobian operator $\mathcal{H}(a, \Omega)$ in (2.12)–(2.13) (which is known as *the Morse index*) changes at every bifurcation curve as Ω crosses $\Omega_{m,n}(a)$, according to Propositions

[4] If (c_{m_0+1}, c_{-m_0+1}) in (3.24) is given by the other eigenvector of $\tilde{A}$ that corresponds to $\tilde{\lambda} > 0$, then it follows from (3.18) that $S_{m_0+1} > 0$. Hence, the corresponding small positive eigenvalue of $H_{m_0+1}(a, \Omega)$ is a decreasing function of Ω. Nevertheless, for $\Omega \gtrsim \Omega_{m_0+1,0}(a)$, these two small eigenvalues of $H_{m_0+1}(a, \Omega)$ are ranged in the same order of $\mathcal{O}(a^2)$ as at $\Omega = \Omega_{m_0+1,0}(a)$.

1 and 2, where the values of $\Omega_{m,n}(a)$ are given by Lemmas 5 and 6, see equations (3.7) and (3.12).

Here we prove that for each fixed a and for each non-resonant bifurcation point, there is a continuous branch of solutions of $g(v;a,\Omega)$ bifurcating from $(0;a,\Omega_{m,n}(a))$ on one side of the bifurcation point $\Omega=\Omega_{m,n}(a)$. The new family of multi-vortex solutions is parameterized by two parameters (a,Ω).

Besides proving the local bifurcation result, we discuss symmetries of the bifurcating branches and their global continuation with respect to parameter Ω. For definitions and methods used to prove the equivariant bifurcation we refer to [1, 11, 15].

In section 4.1, symmetries of $g(v;a,\Omega)$, in particular, its equivariant properties are analyzed. In section 4.2, we prove the local bifurcation result for a non-resonant bifurcation point $\Omega_{m,n}(a)$, with a simple zero eigenvalue of $\mathcal{H}(a,\Omega)$. We also discuss symmetries and asymptotic estimates of the bifurcating branches, which are needed to study the location of the individual vortices in the multi-vortex configurations. In section 4.3, we prove the global continuation of the solution branches.

4.1. Symmetries and equivariance of $g(v;a,\Omega)$

We define the action of the group $O(2)=S^1\cup\kappa S^1$ by

$$\rho(\varphi)v(r,\theta)=e^{-im_0\varphi}v(r,\theta+\varphi),\qquad \rho(\kappa)v(r,\theta)=\bar{v}(r,-\theta). \tag{4.1}$$

The operator $g(v;a,\Omega)$ given by (2.19)–(2.20) is $O(2)$-equivariant by the action of the group given by (4.1). That is, we have $g(\rho(\varphi)v)=\rho(\varphi)g(v)$ since

$$\begin{aligned}&e^{im_0\varphi}g(\rho(\varphi)v)(r,\theta-\varphi)\\&=\left(-\omega_m(a)-\Delta_{(r,\theta)}+r^2+\Omega i(\partial_\theta-im_0)\right)v(r,\theta)-e^{im_0\theta}\psi_{m_0}(r;a)^3\\&\quad+\left|e^{im_0\theta}\psi_{m_0}(r;a)+v(r,\theta)\right|^2\left(e^{im_0\theta}\psi_{m_0}(r;a)+v(r,\theta)\right)=g(v)(r,\theta)\end{aligned}$$

Similarly, we have $g(\rho(\kappa)v)=\rho(\kappa)g(v)$.

As is explained in Section 2.2, the component v is extended to the vector $\mathbf{v}=(v,w)$ with the constraint $w=\bar{v}$, so that the root finding problem is formulated for the analytic nonlinear operator $\mathbf{g}(\mathbf{v})=(g(v,w),\bar{g}(v,w))$. The natural extension of the action of the group $O(2)$ to the second component of $\mathbf{v}=(v,w)$ is

$$\rho(\varphi)w(r,\theta)=e^{im_0\varphi}w(r,\theta+\varphi),\qquad \rho(\kappa)w(r,\theta)=\bar{w}(r,-\theta). \tag{4.2}$$

In the Fourier basis

$$v=\sum_{m\in\mathbb{Z}}V_m(r)e^{im\theta},\qquad w=\sum_{m\in\mathbb{Z}}W_m(r)e^{im\theta},$$

the action of the group $O(2)=S^1\cup\kappa S^1$ is given by

$$\begin{aligned}\rho(\varphi)V_m&=e^{i(m-m_0)\varphi}V_m,&\qquad \rho(\kappa)V_m&=\bar{V}_m,\\ \rho(\varphi)W_m&=e^{i(m+m_0)\varphi}W_m,&\qquad \rho(\kappa)W_m&=\bar{W}_m.\end{aligned}$$

so that

$$\rho(\varphi)(V_m, W_{m-2m_0}) = e^{i(m+m_0)\varphi}(V_m, W_{m-2m_0}), \qquad (4.3)$$
$$\rho(\kappa)(V_m, W_{m-2m_0}) = (\bar{V}_m, \bar{W}_{m-2m_0}).$$

Therefore, the subspaces of functions (V_m, W_{m-2m_0}) are composed of similar irreducible representations under the action of the group $O(2)$.

The subspace (V_m, W_{m-2m_0}) has as isotropy group, the dihedral group D_{m-m_0} generated by the elements κ and $\zeta = 2\pi/(m-m_0)$. The dihedral group D_{m-m_0} will be used to find the symmetry-breaking bifurcations of the primary branch into the multi-vortex solutions along the secondary branches. Due to the symmetries of D_{m-m_0}, the multi-vortex solution is represented by a $(m-m_0)$-polygon of individual vortices.

For a fixed value of $m \in \mathbb{Z}$, the action of $\rho(\zeta)$ is given by

$$\rho(\zeta)(V_j, W_{j-2m_0}) = \exp\left(2\pi i \frac{j-m_0}{m-m_0}\right)(V_j, W_{j-2m_0}), \quad j \in \mathbb{Z}.$$

The fixed point space

$$\mathrm{Fix}(D_{m-m_0}) = \{(v,w) \in L^2(\mathbb{R}^2): \quad \rho(\gamma)(v,w) = (v,w) \text{ for } \gamma \in D_{m-m_0}\}$$

is composed of functions with real components (V_j, W_{j-2m_0}) such that $j - m_0$ is a multiple of $m - m_0$. If $(v, \bar{v}) \in \mathrm{Fix}(D_{m-m_0})$, then v can be characterized by

$$v(r,\theta) = \sum_{j \in m_0 + (m-m_0)\mathbb{Z}} V_j(r) e^{ij\theta} = e^{im_0\theta} \sum_{j \in (m-m_0)\mathbb{Z}} V_{m_0+j}(r) e^{ij\theta},$$

where all functions $\{V_j(r)\}_{j \in m_0 + (m-m_0)\mathbb{Z}}$ are real-valued. Writing $v(r,\theta) = e^{im_0\theta}\phi(r,\theta)$, we deduce that ϕ satisfies the symmetry constraints:

$$\phi(r,\theta) = \bar{\phi}(r,-\theta) = \phi(r,\theta+\zeta). \qquad (4.4)$$

Since $\mathbf{g}$ is $O(2)$-equivariant, the operator $\mathbf{g}(\mathbf{v})$ restricted to $\mathrm{Fix}(D_{m-m_0})$ is well defined. Therefore, we can consider the bifurcation problem

$$\mathbf{g}^{D_{m-m_0}}(\mathbf{v}; a, \Omega): \ X \cap \mathrm{Fix}(D_{m-m_0}) \times \mathbb{R} \times \mathbb{R} \to \mathrm{Fix}(D_{m-m_0}), \qquad (4.5)$$

where $X := H^2(\mathbb{R}^2) \cap L^{2,2}(\mathbb{R}^2)$ is the graph norm of the Jacobian operator $\mathcal{H}$. A schematic illustration of the local bifurcations of the primary and secondary branches is given on Figure 2.

Trivial solution	$O(2) \times O(2)$	$u = 0$
$\downarrow$	$\downarrow$	$\downarrow$
Vortex of charge m_0	$O(2)$	$u = e^{im_0\theta}\psi_{m_0}(r;a)$
$\downarrow$	$\downarrow$	$\downarrow$
Multi-vortex configuration	D_{m-m_0}	$u = e^{im_0\theta}\left(\psi_{m_0}(r;a) + \phi(r,\theta)\right)$

Figure 2. The isotropy lattice for the symmetry-breaking bifurcations.

By Schur's lemma, the Jacobian operator $\mathcal{H}$ for $\mathbf{g}$ has a diagonal decomposition in the subspaces of similar irreducible representations given by the components (V_m, W_{m-2m_0}). Indeed, this has been done in (2.13) and (2.16), where the operator $\mathcal{H}$ in the subspace (V_m, W_{m-2m_0}) is represented by the block H_m. Consequently, for a fixed $m \in \mathbb{Z}$, the Jacobian operator of $\mathbf{g}^{D_{m-m_0}}$ consists of the blocks H_j corresponding to $j - m_0 \in (m - m_0)\mathbb{Z}$. Moreover, in the subspace $\mathrm{Fix}(D_{m-m_0})$ we have $w = \bar{v}$, so that $W_{j-m_0} = V_{-(j-m_0)}$ and the blocks H_j with negative $j - m_0$ are determined by those with $j - m_0 \in \mathbb{N}$. Hence, we denote

$$\mathcal{H}^{D_{m-m_0}} = \mathrm{diag}\{H_j\}_{j\in m_0+(m-m_0)\mathbb{N}}.$$

By Lemma 1, the operator $\mathcal{H}$ has a zero eigenvalue in the block $j = m_0$ due to the gauge invariance of the original problem. This zero eigenvalue is not present for the operator $\mathcal{H}^{D_{m-m_0}}$ in $\mathrm{Fix}(D_{m-m_0})$ because the reflection $\kappa \in D_{m-m_0}$ excludes the gauge invariance. Furthermore, the double eigenvalues of $\mathcal{H}$ in the blocks with positive and negative $j - m_0$ become the simple eigenvalues of $\mathcal{H}^{D_{m-m_0}}$ in $\mathrm{Fix}(D_{m-m_0})$ again due to the reflection κ.

4.2. Local bifurcation results

Here we prove a local bifurcation from a simple eigenvalue of $\mathcal{H}^{D_{m-m_0}}$ that exists at $\Omega = \Omega_{m,n}(a)$ for small a, according to (3.7) and (3.12) in Lemmas 5 and 6. The restriction of the space X to the fixed-point space $\mathrm{Fix}(D_{m-m_0})$ is useful in two aspects. First, it allows us to prove the local bifurcation from a simple eigenvalue by avoiding resonances from the components that are not contained in $\mathrm{Fix}(D_{m-m_0})$. Second, it gives additional information on symmetries of the bifurcating solutions $\mathbf{v}$. The symmetries are useful to understand the distributions of individual vortices in the $(m - m_0)$-polygons.

The local bifurcation results are obtained for the non-resonance bifurcation points, according to the following definition. This definition extends Definition 1.

Definition 2. For a fixed $a > 0$, we say that $\Omega_{m,n}(a) \in (0, 2)$ is a non-resonant bifurcation point if the kernel of $\mathcal{H}^{D_{m-m_0}}(\Omega_{m,n}(a))$ has dimension one. We say that $\Omega_{m,n} \in (0, 2)$ is a non-resonant curve if this condition holds for each small a.

For each curve $\Omega_{m,n}$, the non-resonant condition is given by the following equivalent conditions:

(i) $H_j(a, \Omega_{m,n}(a))$ is invertible;
(ii) $\Omega_{m,n}(a) \neq \Omega_{j,k}(a)$;
(iii) $\mu^-_{j,k}(a, \Omega_{m,n}(a)) \neq 0$;

where j takes values in $m_0 + (m - m_0)\ell$, $\ell \in \mathbb{N}\backslash\{1\}$, $k \in \mathbb{N}_0$, and $a > 0$ is arbitrary but sufficiently small.

As we discussed in Remark 10, the bifurcation curves are all non-resonant for $1 \leq m_0 \leq 3$ and the first resonance happens for $m_0 = 4$ because $\Omega_{10,0}(0) = \Omega_{7,1}(0) = 2/3$. In view of the restriction on the range of j in the space $\mathrm{Fix}(D_{m-m_0})$, however, the bifurcation curve $\Omega_{10,0}$ is non-resonant because $\mathcal{H}^{D_6}$ is composed of blocks H_j with $j = 4, 10, 16, ...$ and the zero eigenvalue $\mu^-_{7,1}(0, \Omega_{10,0}(0))$ is not included in the spectrum of $\mathcal{H}^{D_6}$. On the other hand, the bifurcation curve $\Omega_{7,1}$ may be resonant because $\mathcal{H}^{D_3}$ is composed of blocks H_j with $j = 4, 7, 10, \ldots$ and the zero eigenvalue $\mu^-_{10,0}(0, \Omega_{7,1}(0))$ is included in the spectrum of $\mathcal{H}^{D_3}$. To know exactly if $\Omega_{7,1}(a)$ is resonant with $\Omega_{10,0}(a)$ one needs to compute the normal form in a, which is out of the scope of our presentation.

Remark 16. The curves $\Omega_{m,0}$ for $2m_0 + 1 \le m \le 3m_0 - 1$ are non-resonant. Even if the resonance occurs in $\mathcal{H}$, e.g. for $m_0 = 4$, it does not show up in $\mathcal{H}^{D_{m-m_0}}$. Indeed, if $\Omega_{m,0}(0) = \Omega_{j,k}(0)$ for $j - m_0 = (m - m_0)\ell$ with $\ell \in \mathbb{N}\backslash\{1\}$, then

$$f(m) = f(j) - \frac{2k}{j - m_0},$$

where

$$f(j) = \frac{m_0 - |j - 2m_0|}{j - m_0} = \begin{cases} 1 & m_0 < j \le 2m_0 \\ \frac{2m_0}{j-m_0} - 1 & 2m_0 < j \end{cases}$$

Note that f is a strictly decreasing function on $[2m_0, \infty)$. If $k = 0$, then $j > 2m_0$, hence $f(m) = f(j)$ is true only if $j = m$ ($\ell = 1$), which is excluded. If $k \ge 1$, then $f(m) < f(j)$, which implies that $m > j$ or $j - m_0 < m - m_0$. Therefore, the possible resonant block H_j with $m_0 < j < m$ is not in $\mathcal{H}^{D_{m-m_0}} = \mathrm{diag}(H_{m_0}, H_m, H_{2m-m_0}, \ldots)$.

Remark 17. The curve $\Omega_{m_0+1,0}(a) = 2 + \mathcal{O}(a^2)$ is non-resonant as long as the matrices $\tilde{A}$ arising in the matrix eigenvalue problem (3.22) are invertible. We have checked this condition numerically for $1 \le m_0 \le 100$.

The following proposition follows from the Crandall-Rabinowitz theorem, see Theorem I.5.1 in [19]. It covers the non-resonant bifurcation curve $\Omega_{m,n}$, for which Proposition 1 applies. It does not cover the curve $\Omega_{m_0+1,0}$ in Remark 17.

Proposition 3. *For each non-resonant curve $\Omega_{m,n} \in (0, 2)$ parameterized by $a > 0$ sufficiently small, the operator $\mathbf{g}^{D_{m-m_0}}(\mathbf{v}; a, \Omega)$ in (4.5) admits a new family of roots $\mathbf{v} \in \mathrm{Fix}(D_{m-m_0})$ and $\Omega \in (0, 2)$ parameterized by real b such that*

$$\Omega(a, b) = \Omega_{m,n}(a) + \mathcal{O}(b^2) \tag{4.6}$$

and

$$v(r, \theta; a, \Omega(a, b)) = b f_{m,n}(r, \theta; a) + \mathcal{O}_X(ab^2, b^3), \tag{4.7}$$

where

$$f_{m,n}(r, \theta; a) = e_{|m-2m_0|,n}(r) e^{i(2m_0-m)\theta} + \mathcal{O}_X(a^2) \tag{4.8}$$

is the eigenvector of $H_m(a, \Omega_{m,n}(a))$ associated with the zero eigenvalue $\mu^-_{m,n}(a, \Omega_{m,n}(a))$.

Proof. The local bifurcation problem (4.5) is well-defined for the operator $\mathbf{g}^{D_{m-m_0}}$. The operator $\mathbf{g}^{D_{m-m_0}}$ has a linearization given by $\mathcal{H}^{D_{m-m_0}}$ and its kernel is spanned by the eigenvector $f_{m,n}$ associated to the simple zero eigenvalue $\mu^-_{m,n}(a; \Omega_{m,n}(a))$ under the assumption of the proposition. Since $\mathcal{H}^{D_{m-m_0}}$ has a uniformly bounded inverse operator in the complement of the kernel, according to Proposition 1, we are in the position to define the bifurcation equation as in Theorem I.5.1 in [19].

The only condition to be verify is that

$$\partial_\Omega \mathcal{H}^{D_{m-m_0}} f_{m,n} = i\partial_\theta f_{m,n}$$

is not in the range of $\mathcal{H}^{D_{m-m_0}}$. Thanks to the basis in (3.5) and the fact that the zero eigenvalue corresponds to $\mu^-_{m,n}$, the leading-order approximation of the eigenvector $f_{m,n}$ is given by (4.8) for $a > 0$ sufficiently small. Then, $i\partial_\theta f_{m,n} \notin \mathrm{Ran}(\mathcal{H}^{D_{m-m_0}})$ because

$$\langle i\partial_\theta f_{m,n}, f_{m,n}\rangle_{L^2} = m - 2m_0 + \mathcal{O}(a^2) \neq 0.$$

The existence of the new root of $\mathbf{g}^{D_{m-m_0}}(v; a, \Omega)$ and the estimate (4.7) for $a > 0$ sufficiently small follow from the Crandall-Rabinowitz theorem, where the scaling $\mathcal{O}_X(ab^2)$ is due to the cubic terms in the expressions for g in (2.20). This theorem gives also the estimate $\Omega(a,b) = \Omega_{m,n}(a) + \mathcal{O}(b)$. Furthermore, the S^1-action (4.3) of the element $\varphi = \pi/(m+m_0)$ in the kernel generated by $f_{m,n}$ is given by $\rho(\varphi) = -1$. Therefore, the bifurcation equation is odd and $\partial_{vv}\mathbf{g}^{D_{m-m_0}}(0)(f_{m,n}, f_{m,n}) = 0$. The estimate (4.6) is obtained from formula (I.6.3) in [19]. $\square$

Remark 18. The new family (4.6) and (4.7) exists on one side of the bifurcation curve $\Omega_{m,n}$, that is,

$$\Omega(a,b) = \Omega_{m,n}(a) + cb^2 + \mathcal{O}(b^4),$$

where c can be computed from (I.6.11) in [19]. If $c > 0$ the bifurcation is supercritical pitchfork (to the right of the bifurcation curve) and the Jacobian operator at the new (secondary) branch of solutions has one more negative eigenvalue compared to that at the primary branch. If $c < 0$ the bifurcation is subcritical pitchfork (to the left of the bifurcation curve) and the Jacobian operator at the new branch of solutions has one less negative eigenvalue compared to that at the primary branch. Because the new family can be rotated in the (x,y) plane, the Jacobian operator at the new branch has an additional zero eigenvalue related to this rotation symmetry.

The following proposition covers the non-resonant bifurcation curve $\Omega_{m+1,0}$, for which Proposition 2 applies.

Proposition 4. *For the non-resonant curve $\Omega_{m_0+1,0}$ parameterized by $a > 0$ sufficiently small, the operator $\mathbf{g}^{D_1}(\mathbf{v}; a, \Omega)$ in (4.5) admits a new family of roots $\mathbf{v} \in \mathrm{Fix}(D_1)$ and $\Omega \in (0,2)$ parameterized by real b such that*

$$\Omega(a,b) = \Omega_{m_0+1,0}(a) + \mathcal{O}(a^2 b^2) \tag{4.9}$$

and

$$v(r,\theta; a, \Omega(a,b)) = a\left[b f_{m_0+1,0}(r,\theta;a) + \mathcal{O}_X(b^2)\right], \tag{4.10}$$

where $f_{m_0+1,0}$ is the eigenvector of $H_{m_0+1}(a, \Omega_{m_0+1,0}(a))$ associated with the zero eigenvalue.

Proof. The scaling of a in (4.10) is needed due to the loss of $\mathcal{O}(a^{-2})$ in the bound (3.23) on the inverse operator $(\mathcal{H}^{D_1})^{-1}$, according to Proposition 2. Since $\psi_{m_0} = \mathcal{O}(a)$, the nonlinear terms in the operator $\mathbf{g}^{D_1}(\mathbf{v}; a, \Omega)$ are now scaled by $\mathcal{O}(a^3)$, hence the loss of $\mathcal{O}(a^{-2})$ produces the terms of the expansion (4.10) at the order $\mathcal{O}(a)$ and higher. Hence the bifurcation problem is closed at the order $\mathcal{O}(a)$ and the proof follows the one in Proposition 3 with a new parameter b, which is scaled independently of a. □

Remark 19. The local bifurcation in Proposition 4 is also of the pitchfork type. Thanks to the computations in Lemma 7, the leading-order approximation of the eigenvector $f_{m_0+1,0}$ is given by

$$f_{m_0+1,0}(r,\theta;a) = c_{m_0+1}e_{m_0+1,0}(r)e^{i(m_0+1)\theta} + c_{-m_0+1}e_{m_0-1,0}(r)e^{i(m_0-1)\theta} + \mathcal{O}_X(a^2), \tag{4.11}$$

for $a > 0$ sufficiently small, where (c_{m_0+1}, c_{1-m_0}) is an eigenvector of the matrix $\tilde{A}$ computed in (3.17).

Remark 20. Propositions 3 and 4 yield the proof of item (iv) in Theorem 1.

4.3. Global bifurcation

We obtain the global bifurcation result in the fixed-point space $\mathrm{Fix}(D_{m-m_0})$ by using the topological degree theory in the case of simple eigenvalues. It is usually referred to as the global Rabinowitz result, see Theorem 3.4.1 of [27]. The global bifurcation result means that the solution branch (v, Ω) that originates at the non-resonant bifurcation curve $(0, \Omega_{m,n})$ either reaches the boundaries $\Omega = 0$ or $\Omega = 2$, returns to another bifurcation point $(0, \Omega_*)$, or diverges to infinite values of v for a finite value of $\Omega \in [0, 2)$.

The following result holds because the Jacobian operator $\mathcal{H}$ given by (2.12) and (2.13) is bounded and has closed range for $|\Omega| < 2$.

Lemma 10. *Let $X = H^2(\mathbb{R}^2) \cap L^{2,2}(\mathbb{R}^2)$ be the domain space for the Jacobian operator $\mathcal{H}$. For every $|\Omega| < 2$ there is a positive constant c such that the operator $(\mathcal{H} + cI) : X \to L^2(\mathbb{R}^2)$ is positive definite and $(\mathcal{H} + cI)^{-1} : X \to X$ is compact.*

Proof. The eigenvalues of $\mathcal{H}$ are given by $\mu^{\pm}_{m,n}(a, \Omega)$ expanded as in (3.6). For $|\Omega| < 2$, the eigenvalues $\mu^{\pm}_{m,n}$ are bounded from below and do not accumulate at a finite value. Therefore, there is a positive constant c such that the bounded operator $\mathcal{H}+cI : X \to L^2(\mathbb{R}^2)$ is positive definite and invertible. Since X is compactly included in $L^2(\mathbb{R}^2)$, then the inverse operator $(\mathcal{H} + cI)^{-1} : X \hookrightarrow L^2 \to X$ is compact. □

Remark 21. Observe in (3.6) that for $|\Omega| = 2$ the eigenvalues $\mu^{\pm}_{m,n}(a, \Omega)$ can accumulate at a finite value as $a \to 0$, while for $|\Omega| > 2$ the eigenvalues $\mu^{\pm}_{m,n}(a, \Omega)$ are unbounded both from above and from below. As a result, the operator $\mathcal{H}+cI : X \to L^2(\mathbb{R}^2)$ does not have a closed range for $|\Omega| = 2$, its inverse $(\mathcal{H} + cI)^{-1} : L^2(\mathbb{R}^2) \to X$ is not bounded, and the inverse operator $(\mathcal{H} + cI)^{-1} : X \to X$ is not compact.

By Lemma 10, the Jacobian operator $\mathcal{H}$ is Fredholm of the degree zero for $\Omega \in [0, 2)$. Also the restricted operator $\mathcal{H}^{D_{m-m_0}}(\Omega)$ is a self-adjoint Fredholm operator for every $\Omega \in [0, 2)$. Since $\mathcal{H}^{D_{m-m_0}}(\Omega)$ is invertible for Ω close but different from the non-resonant bifurcation curve $\Omega_{m,n}$, then the Morse index $n^{D_{m-m_0}}(\Omega)$ of $\mathcal{H}^{D_{m-m_0}}(\Omega)$

restricted to $\ker \mathcal{H}^{D_{m-m_0}}(\Omega_{m,n})$ for Ω close to $\Omega_{m,n}$ is well defined. Let $\eta^{D_{m-m_0}}(\Omega_{m,n})$ be the net crossing number of eigenvalues of $\mathcal{H}^{D_{m-m_0}}(\Omega)$ defined by

$$\eta^{D_{m-m_0}}(\Omega_{m,n}) := \lim_{\varepsilon\to 0}\left|n^{D_{m-m_0}}(\Omega_{m,n}+\varepsilon) - n^{D_{m-m_0}}(\Omega_{m,n}-\varepsilon)\right|. \tag{4.12}$$

If $\Omega_{m,n}$ is a non-resonant bifurcation curve, then it is obvious that $\eta^{D_{m,n}}(\Omega_{m,n}) = 1$. The following proposition gives the global bifurcation result for each non-resonant bifurcation curve.

Proposition 5. *Fix $a > 0$ sufficiently small, if $\eta^{D_{m-m_0}}(\Omega_{m,n})$ is odd for $\Omega_{m,n} \in [0,2)$, the nonlinear operator $g(v;a,\Omega)$ has a global bifurcation of solutions (v,Ω) in $\mathrm{Fix}(D_{m-m_0}) \times [0,2)$ arising from $(v,\Omega) = (0,\Omega_{m,n})$.*

Proof. Since X is a Banach algebra with respect to pointwise multiplication and $g(0;a,\Omega) = 0$, we obtain the expansion

$$g(v;a,\Omega) = \mathcal{H}(\Omega)v + \mathcal{O}_X(v^2).$$

We can apply the global Rabinowtz theorem to the nonlinear operator

$$f(v,\Omega) = (\mathcal{H}+cI)^{-1}\, g(v;a,\Omega)v = Iv - c\,(\mathcal{H}+cI)^{-1}\, v + \mathcal{O}_X(v^2),$$

where $c > 0$ is defined in Lemma 10. The operator f is also equivariant and can be restricted to $\mathrm{Fix}(D_{m-m_0})$ denoted by $f^{D_{m-m_0}}$. The index for bifurcation of f in $\mathrm{Fix}(D_{m-m_0})$ is up to an orientation factor, the jump on the local indices as Ω crosses $\Omega_{m,n}$. That is, since $\eta^{D_{m-m_0}}(\Omega_{m,n})$ is odd, then

$$\begin{aligned}
&\deg\left(\|x\| - \varepsilon, f^{D_{m-m_0}}(x,\Omega); B_{2\varepsilon}\times B_{2\rho}\right)\\
&= \deg(f^{D_{m-m_0}}(x,\Omega-\rho); B_{2\varepsilon}) - \deg(f^{D_{m-m_0}}(x,\Omega+\rho); B_{2\varepsilon})\\
&= \pm\left(1-(-1)^{\eta^{D_{m-m_0}}}\right) = \pm 2,
\end{aligned} \tag{4.13}$$

where $B_{2\varepsilon}$ and $B_{2\rho}$ are ball of radius 2ε and 2ρ around $0 \in X \cap \mathrm{Fix}(D_{m-m_0})$ and $\Omega_{m,n} \in [0,2)$, respectively. □

Remark 22. If the branch from $(0,\Omega_{m,n})$ returns to another bifurcation point $(0,\Omega_{m',n'})$, then the sum of all the bifurcation indices (4.13) at the bifurcation points has to be equal zero. Therefore, the knowledge of the exact factor $\pm$ in (4.13) is helpful to obtain information of where the branches can return. The exact factor $\pm$ in (4.13) can be computed for all the bifurcation curves using the fact that

$$\deg(f^D(x,\Omega); B_{2\varepsilon}) = (-1)^{n^{D_{m-m_0}}(\Omega)},$$

since $\mathcal{H}+cI$ is positive definite. For example, for the last bifurcation from $\Omega_{m_0+1,0}$ with $1 \le m_0 \le 16$, the exact index is

$$\deg\left(\|x\| - \varepsilon, f^{D_1}(x,\Omega); B_{2\varepsilon}\times B_{2\rho}\right) = (-1)^{m_0} - (-1)^{m_0-1} = (-1)^{m_0}2.$$

Therefore, this branch can return to a single bifurcation point Ω_0 only if the latter point has index $-2(-1)^{m_0}$.

Remark 23. Proposition 5 provides a proof of the claim in item (iii) of Theorem 1 that the bifurcations in the interval $[a^2D_{m_0}, 2 - a^2C_{m_0}]$ are global.

5. Individual vortices in the multi-vortex configurations

We can assume $a > 0$ in the expansions (2.7) and (2.8) for the primary branch after a change of phase. Also, we can choose the sign of b by a shift of θ, i.e. we can assume $b > 0$ in the expansions (4.7) and (4.10) for the secondary branch. Here we analyze the location of individual vortices in the multi-vortex configurations bifurcating along the secondary branch.

First, we prove that the total vortex charge is preserved near the origin when the secondary branch bifurcates off from the primary branch.

Lemma 11. *Fix $R > 0$. There exists $b_0 > 0$ such that the degree of the bifurcating solution U along the secondary branch on the circle of the radius R is m_0 for every $b \in [0, b_0)$.*

Proof. We recall that $a > 0$ and $\psi_{m_0}(r) > 0$ for every $r \in (0, \infty)$. For every fixed $R > 0$, there exists a sufficiently small $b_0 > 0$ such that the bifurcating solution $U(r, \theta)$ given by (2.18) is nonzero at $r = R$ for every $b \in [0, b_0)$. This follows from the smallness of the error terms in the expansions (4.7) and (4.10) in the norm of $X = H^2(\mathbb{R}^2) \cap L^{2,2}(\mathbb{R}^2)$, which is embedded in $C^0(\mathbb{R}^2)$. Since $U(r, \theta)$ is nonzero at $r = R$, the degree of U on the disk B_R of radius R is well defined and does not change for every $b \in [0, b_0)$. Since the degree is m_0 at $b = 0$, it remains m_0 for every $b \in [0, b_0)$. □

Remark 24. Because $\psi_{m_0}(r) \to 0$ as $r \to \infty$, we are not able to claim that additional zeros of $U(r, \theta)$ cannot come from infinity as $b \neq 0$. If such zeros exist, additional individual vortices come from infinity on a very small background $U(r, \theta)$.

Next, we rewrite the eigenfunctions $e_{m,n}(r)$ of the linear eigenvalue problem (2.2) in the form

$$e_{m,n}(r) = p_{m,n}(r) e^{-r^2/2}, \tag{5.1}$$

where $p_{m,n}(r)$ is a polynomial of degree $|m| + 2n$, which is chosen to be positive for r near zero. The first eigenfunctions $e_{m,0}(r)$ and $e_{m,1}(r)$ in (2.9) and (3.4) are given by (5.1) with

$$p_{m,0}(r) = \frac{\sqrt{2}}{\sqrt{m!}} r^m, \quad p_{m,1}(r) = \frac{\sqrt{2}}{\sqrt{(m+1)!}} r^m (m + 1 - r^2). \tag{5.2}$$

The following proposition deals with the secondary bifurcations described in Proposition 3.

Proposition 6. *Let $0 < b \lesssim a$ and consider the bifurcating solution to the stationary GP equation (2.4) in the form (2.18) given by the expansions (2.7) and (4.7). Let r_0 be the first positive zero of the function*

$$z(r) := a p_{m_0,0}(r) - b p_{|m-2m_0|,n}(r) \tag{5.3}$$

and assume that it is a simple zero[5]. Then, the bifurcating solution has simple zeros arranged in the $(m - m_0)$-polygon on a circle of radius ρ with $\rho = r_0 + \mathcal{O}(a^2)$.

[5]The assumption is always satisfied if $0 < b \ll a$ since $p_{|m-2m_0|,n}(r)$ is positive for small r and $|m - 2m_0| < m_0$.

Proof. By combining (2.7), (2.18), (4.7), and (4.8), we obtain an asymptotic representation of the bifurcating solutions U in the form

$$U(r,\theta) = ae_{m_0,0}(r)e^{im_0\theta} + be_{|m-2m_0|,n}(r)e^{i(2m_0-m)\theta} + \mathcal{O}_X(a^3, a^2b, ab^2, b^3).$$

Zeros of $U(r,\theta)$ are equivalent to the zeros of

$$e^{-im_0\theta}U(r,\theta) = ae_{m_0,0}(r) + be_{|m-2m_0|,n}(r)e^{-i(m-m_0)\theta} + \phi(r,\theta), \quad (5.4)$$

where $\phi = \mathcal{O}_X(a^3, a^2b, ab^2, b^3)$ satisfies the symmetry constraints (4.4).

The function $e^{-i(m-m_0)\theta}$ is real only if $\theta = k\zeta$ and $\theta = (k+1/2)\zeta$, where $\zeta = 2\pi/(m-m_0)$ and $k \in \mathbb{Z}$. For these angles, the function $\phi(\theta, r)$ is real by the symmetries (4.4). Therefore, the function (5.4) is real if and only if $\theta = k\zeta$ and $\theta = (k+1/2)\zeta$. These two choices of angles give two choices of the $(m-m_0)$-polygons of zeros along a circle of radius ρ.

To determine the small radius ρ in the limit $b \to 0$, we factorize the factor $e^{-r^2/2}$ in the eigenfunctions (5.1) and truncate the error term ϕ. Since we assume that $p_{m,n}(r)$ is positive for r near zero, then the right-hand side of (5.4) is strictly positive for $\theta = k\zeta$ and $r \gtrsim 0$. For $\theta = (k+1/2)\zeta$, we have $e^{-i(m-m_0)\theta} = -1$, hence the right-hand side of (5.4) has a zero only if $z(r)$ in (5.3) has a positive root.

Let r_0 be the first positive root of z in (5.3) and assume that it is simple. Since the function $\phi(r,\theta)$ is small in the norm of $X = H^2(\mathbb{R}^2) \cap L^{2,2}(\mathbb{R}^2)$ by Proposition 3, an application of the implicit function theorem proves that the representation (5.4) with $\theta = \zeta/2 = \pi/(m-m_0)$ has the $(m-m_0)$ polygon of simple zeros at the circle of radius ρ, where $\rho = r_0 + \mathcal{O}(a^2)$. □

Remark 25. For the last bifurcation with $m = m_0 + 1$ described in Proposition 4, a similar result cannot be proven because the small parameter a is scaled out from the expansion (4.10). The remainder term $\phi = \mathcal{O}_X(ab^2)$ in the representation for $U(r,\theta)$ may give a contribution to the distribution of individual vortices, which is comparable with the leading-order term $a\psi_{m_0}(r)e^{im_0\theta}$ and the bifurcating mode $abf_{m_0+1,0}(r,\theta;a)$.

In the rest of this section, we study individual vortices in the bifurcating multi-vortex configurations.

5.1. $(m-m_0)$-polygons of vortices

Polygons made of vortices rotating at a constant speed have been studied for many models: fluids, BECs and superconductors. It has been found that these relative equilibria of m vortices are stable for $m \leq 7$, see, e.g., [8, 21] and references therein. We have found that similar multi-vortex configurations appear along the secondary branches bifurcating from the primary branch of the radially symmetric vortex of charge $m_0 \geq 2$. As an example, we give precise information about the vortex polygons in the particular cases $n = 0$ and $n = 1$. For $n = 1$, the bifurcation is similar to the bifurcation of complex multi-vortex solutions described in Lemma 3.3 of [16] for $m_0 = 6$.

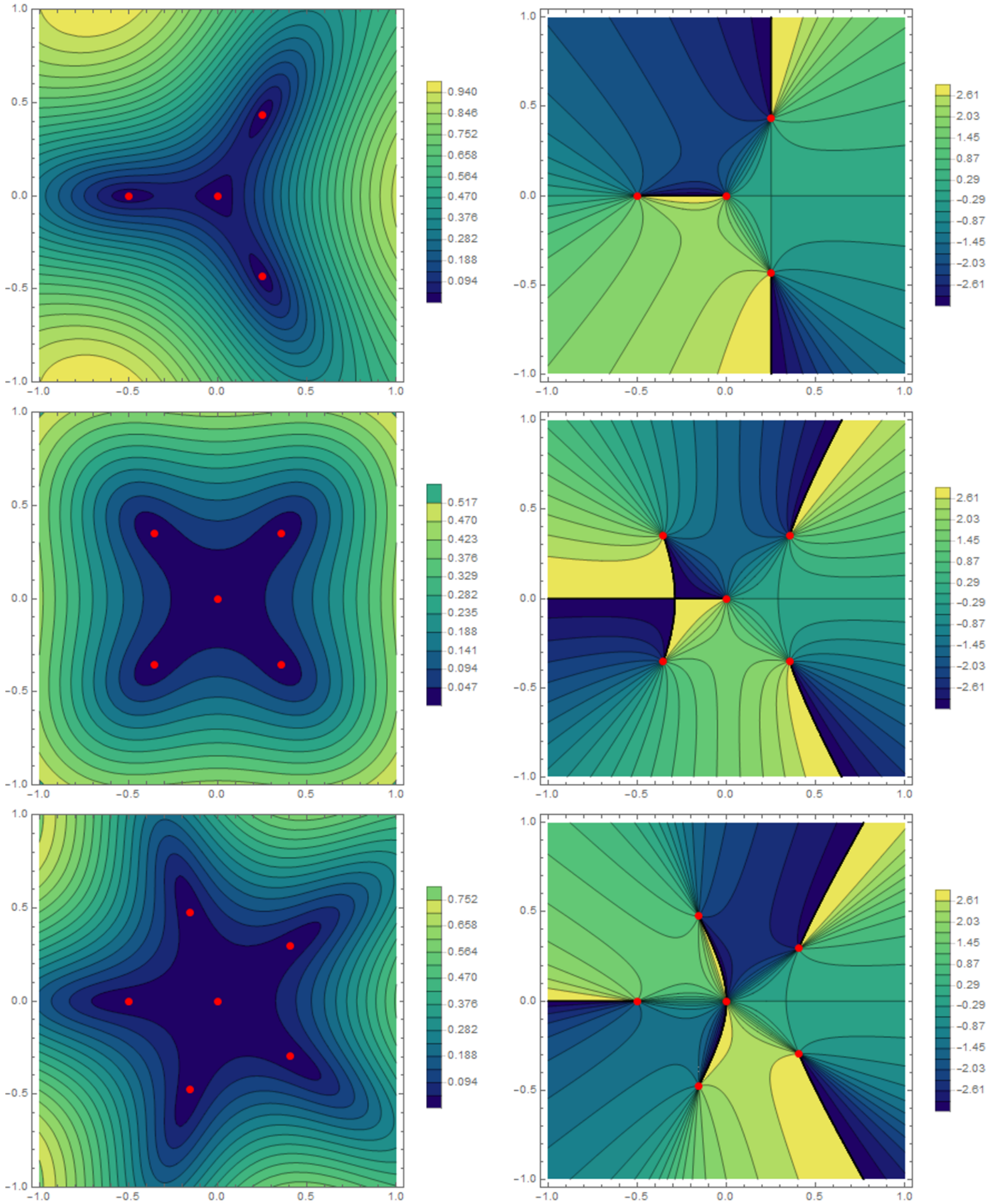

FIGURE 3. The left and right columns illustrate the norm and phase of the truncated solution U in (5.6). Top: $m_0 = 2$ near $\Omega_{5,0}$. Middle: $m_0 = 3$ near $\Omega_{7,0}$. Bottom: $m_0 = 3$ near $\Omega_{8,0}$.

5.1.1. Case $n = 0$: Vortex polygons with a central vortex. Bifurcation occurs at the bifurcation curve $\Omega_{m,n} \in (0,2)$ with $n = 0$ and $2m_0 + 1 \leq m \leq 3m_0 - 1$ (when $m_0 \geq 2$) in accordance with Lemma 5, Propositions 1, 3, and 6. By using (5.2) and (5.3), we write explicitly

$$\begin{aligned} z(r) &= ap_{m_0,0}(r) - bp_{m-2m_0,0}(r) \\ &= \frac{\sqrt{2}}{\sqrt{m_0!}} r^{m-2m_0} \left(ar^{3m_0-m} - b\frac{\sqrt{m_0!}}{\sqrt{(m-2m_0)!}} \right), \end{aligned}$$

where we recall that $2m_0 < m < 3m_0$. If $0 < b \lesssim a$, the first positive zero of $z(r)$ is located at

$$r_0 = \left(\frac{b}{a} \frac{\sqrt{m_0!}}{\sqrt{(m-2m_0)!}} \right)^{1/(3m_0-m)}. \tag{5.5}$$

By Proposition 6, we have a $(m - m_0)$-polygon of simple zeros of the function

$$U(r,\theta) = \left[ap_{m_0,0}(r) + bp_{m-2m_0,0}(r)e^{-i(m-m_0)\theta} \right] e^{-r^2/2} e^{im_0\theta} + \phi(r,\theta)e^{im_0\theta}, \tag{5.6}$$

at points $(r,\theta) = (\rho, \zeta/2 + k\zeta)$, where $\zeta = 2\pi/(m - m_0)$, $k \in \{0, 1, ..., m - m_0 - 1\}$, $\rho = r_0 + \mathcal{O}(a^2)$, and r_0 is given by (5.5).

We claim that the degree of each simple zero of $U(r,\theta)$ is $+1$, which means that each zero of U on the $(m - m_0)$-polygon represents a vortex of charge one. By symmetry of D_{m-m_0}, each zero in the $(m - m_0)$-polygon has equal degree, hence it is sufficient to compute the degree at the simple zero $(r,\theta) = (\rho, \zeta/2)$. Using Taylor expansion of U in (5.6) for $b \lesssim a$, we obtain

$$cU(r,\theta) = z'(\rho)(r - \rho) + ib(m - m_0)p_{m-2m_0,0}(\rho)(\theta - \zeta/2) + \mathcal{O}(2),$$

where $c \in \mathbb{C}$ is constant and $\mathcal{O}(2)$ denotes quadratic remainder terms of the Taylor expansion. Because $m - 2m_0 < m_0$, we have $z(r) < 0$ for $r > 0$ sufficiently small, therefore, $z'(\rho) > 0$ for $b > 0$ sufficiently small. On the other hand, $m > 2m_0$ and $p_{m-2m_0,0}(\rho) > 0$ in the same limit. Therefore, the degree of $U(r,\theta)$ at $(r,\theta) = (\rho, \zeta/2)$ is $+1$.

In addition, $U(r,\theta)$ in (5.6) has a zero at $r = 0$ if the remainder term $e^{im_0\theta}\phi(r,\theta)$ is truncated. Let d be the degree of U in a neighborhood of $r = 0$. The degree in the disk B_R of a sufficiently large radius R is equal to sum of the local degrees in the disk. By Lemma 11, we have $d + m - m_0 = m_0$, hence $d = 2m_0 - m < 0$.

When the remainder term $e^{im_0\theta}\phi(r,\theta)$ is taken into account in (5.6), the multiple zero of U at $r = 0$ may split from the origin. However, by the symmetry in D_{m-m_0}, if the central vortex splits, then it breaks into $m-m_0$ vortices of equal charge $|d|/(m - m_0)$. Since $m - 2m_0 \leq m_0 - 1 < m_0 + 1 \leq m - m_0$, then $|d|/(m - m_0) < 1$ and the central vortex never splits.

Remark 26. For the case $m_0 = 2$, we have the bifurcation point $\Omega_{5,0} = 2/3 + \mathcal{O}(a^2)$. Since $m = 5$ and $n = 0$, we have a configuration of three vortices of charge one that form an equilateral triangle and a central vortex of charge $d = 2m_0 - m = -1$ (top panel of Figure 3).

Remark 27. For the case $m_0 = 3$, we have two bifurcation points $\Omega_{7,0} = 1 + \mathcal{O}(a^2)$ and $\Omega_{8,0} = 2/5 + \mathcal{O}(a^2)$. At the former bifurcation, the bifurcating branch has four vortices of charge one that form a square and the central vortex of charge $d = 2m_0 - m = -1$ (middle panel of Figure 3). At the latter bifurcation, the bifurcating branch has five vortices of charge one that form an equilateral pentagon and the central vortex of charge $d = 2m_0 - m = -2$ (bottom panel of Figure 3).

5.1.2. Case $n = 1$: Vortex polygons without a central vortex. Bifurcation occurs at the bifurcation curve $\Omega_{m,n}$ with $n = 1$ and $m_0 + 3 \leq m \leq 3m_0 - 3$ (when $m_0 \geq 3$) in accordance with Lemma 5, Propositions 1, 3, and 6. By using (5.2) and (5.3), we write explicitly

$$\begin{aligned} z(r) &= a p_{m_0,0}(r) - b p_{|m-2m_0|,1}(r) \\ &= \frac{\sqrt{2}}{\sqrt{m_0!}} r^{|m-2m_0|} \left(a r^{m_0 - |m-2m_0|} - b C_{m,m_0} (|m - 2m_0| + 1 - r^2) \right), \end{aligned}$$

where

$$C_{m,m_0} = \frac{\sqrt{m_0!}}{\sqrt{(|m-2m_0|+1)!}}$$

and we recall that $|m - 2m_0| < m_0$. If $0 < b \ll a$, the first positive zero of $z(r)$ is located at

$$r_0 = \left(\frac{b}{a} C_{m,m_0} (|m - 2m_0| + 1) \right)^{1/(m_0 - |m-2m_0|)} \left[1 + \mathcal{O}\left(\left(\frac{b}{a} \right)^{2/(m_0 - |m-2m_0|)} \right) \right].$$

By Proposition 6, we have the $(m - m_0)$ polygon of vortices on the circle of radius $\rho = r_0 + \mathcal{O}(a^2)$. Each vortex has charge one by using the same arguments as in the case $n = 0$.

Remark 28. If $m = 2m_0$, the polygon of m_0 charge-one vortices surrounds the origin with no central vortex. For $m_0 = 3$, the bifurcation point is $\Omega_{6,1} = 2/3 + \mathcal{O}(a^2)$ and the secondary branch has three charge-one vortices located at the equilateral triangle. For $m_0 = 6$ studied in [16], the bifurcation point is $\Omega_{12,1} = 4/3 + \mathcal{O}(a^2)$ and the secondary branch has six charge-one vortices at a hexagon (top panel of Figure 4).

Remark 29. If $m \neq 2m_0$, $U(r, \theta)$ has zero at $r = 0$ if the remainder term $\phi(r, \theta)$ is truncated. By Lemma 11, the central zero of U corresponds to the vortex of charge $d = 2m_0 - m$, where $-m_0 < d < m_0$. The central vortex may split into $m - m_0$ vortices of equal charge only if $|d|$ is divisible by $m - m_0$.

5.2. Asymmetric vortex and asymmetric vortex pair

Bifurcation occurs at the bifurcation curve $\Omega_{m_0+1,0} = 2 + \mathcal{O}(a^2)$ (when $m_0 \geq 1$) in accordance with Lemma 6, Propositions 2 and 4. By using (4.10) and (4.11), we write explicitly

$$\begin{aligned} U(r,\theta) = a &\left[p_{m_0,0}(r) + b c_{m_0+1} p_{m_0+1,0}(r) e^{i\theta} + b c_{-m_0+1} p_{m_0-1,0}(r) e^{-i\theta} \right] e^{-r^2/2} e^{i m_0 \theta} \\ &+ \phi(r,\theta) e^{i m_0 \theta}, \end{aligned}$$

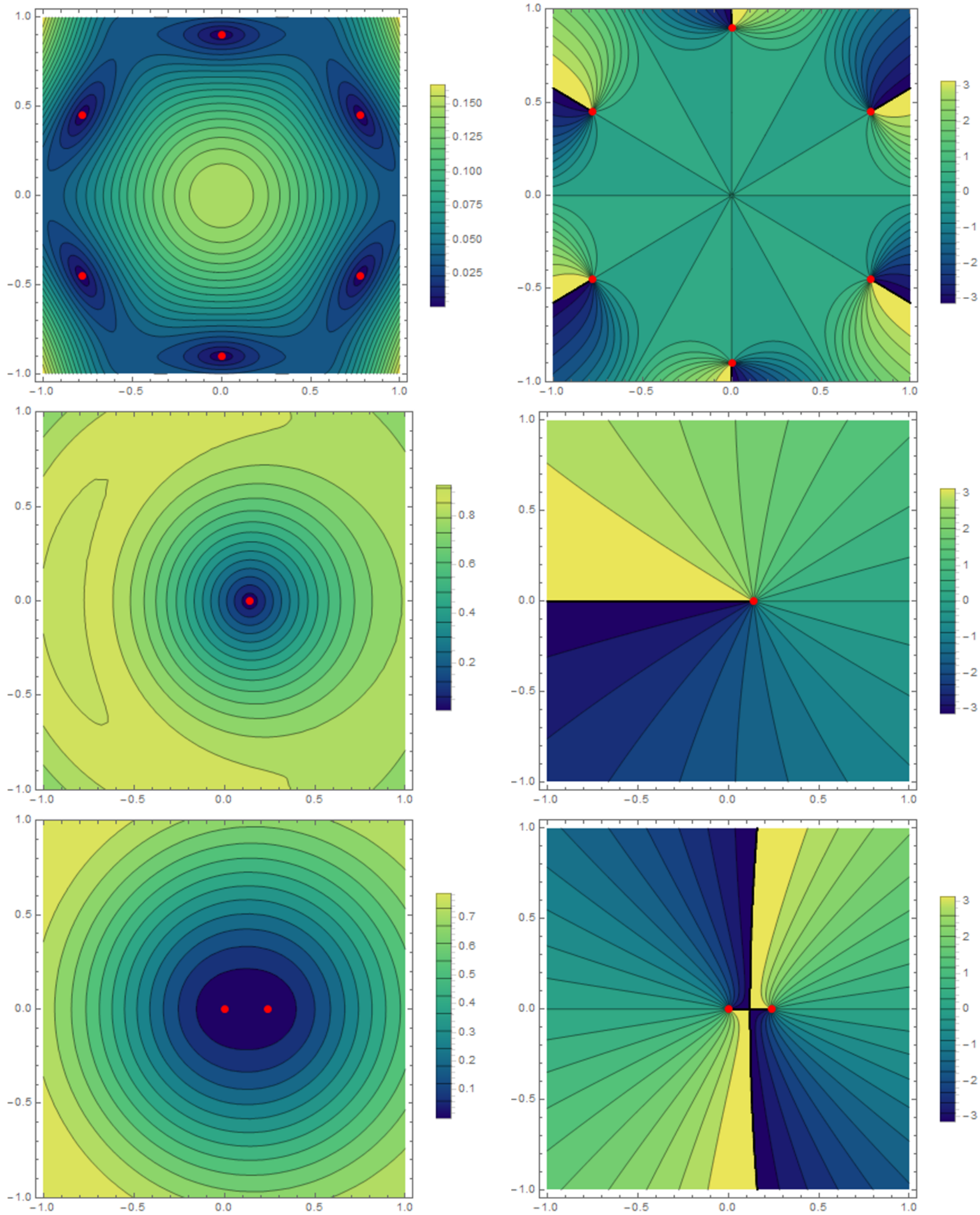

Figure 4. The left and right columns illustrate the norm and phase of the truncated solution U. Top: $m_0 = 6$ near $\Omega_{12,1}$. Center: $m_0 = 1$ near $\Omega_{2,0}$. Bottom: $m_0 = 2$ near $\Omega_{3,0}$.

where (c_{m_0+1}, c_{-m_0+1}) is obtained from the eigenvector in (3.17) and $\phi = \mathcal{O}_X(ab^2)$, see Remark 25. In particular, we have $c_{m_0+1} > 0$ and $c_{-m_0+1} < 0$.

Remark 30. If $m_0 = 1$ and $0 < b \lesssim a$, the simple zero of $U(r,\theta)$ near the origin is located at $\rho = b|c_0| + \mathcal{O}(b^2)$. The degree of U near the simple zero at $(r,\theta) = (\rho,\pi)$ is again $+1$, so that the corresponding vortex has charge one. Since no other zeros of $U(r,\theta)$ are located near the origin, the bifurcating solution at the secondary branch corresponds to the asymmetric vortex obtained in [29] (center panel of Figure 4).

Remark 31. If $m_0 = 2$ and $0 < b \lesssim a$, the double zero of $U(r,\theta)$ at the origin for $b = 0$ split to the distances $\rho_\pm = \mathcal{O}(b)$ according to the roots of the quadratic equation

$$r^2 \pm b|c_{-1}|\sqrt{2} + b^2\beta = 0, \tag{5.7}$$

where β is a numerical constant obtained from the remainder term $\phi(r,\theta)$, whereas the plus and minus signs correspond to the choice $\theta = 0$ and $\theta = \pi$ respectively. Only positive roots of the quadratic equations (5.7) are counted, and according to Lemma 11, we should have the total of two positive roots at both sign combinations. Indeed, if $\beta > 0$, the two positive roots $\rho_\pm = \mathcal{O}(b)$ exist for $\theta = \pi$ and no positive roots for $\theta = 0$, while if $\beta < 0$, one positive root ρ_+ exists for $\theta = 0$ and one positive root exists for $\theta = \pi$. In both cases, $\rho_+ \neq \rho_-$, so that the bifurcating solution at the secondary branch corresponds to the asymmetric pair of two charge-one vortices obtained in [26] (bottom panel of Figure 4 in the case $\beta < 0$).

Remark 32. If $m_0 \geq 3$ and $0 < b \lesssim a$, the multiple root of $U(r,\theta)$ at the origin for $b = 0$ split to the distances $\rho_\pm = \mathcal{O}(b)$ according to the roots of the n-th order polynomial equation, which is obtained from computations of the remainder term $\phi(r,\theta)$ up to the order of b^n. By Lemma 11, there must exist exactly n roots to the two polynomial equations for $\theta = 0$ and $\theta = \pi$ but the precise characterization of these roots depend on the coefficients of the polynomial equation.

Remark 33. Proposition 6 and computations in Sections 5.1 and 5.2 yield the proof of item (v) of Theorem 1. All items of Theorem 1 have been proved.

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