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AVERAGING OF DISPERSION-MANAGED SOLITONS: EXISTENCE AND STABILITY*

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Abstract. We consider existence and stability of dispersion-managed solitons in the two approximations of the periodic nonlinear Schrödinger (NLS) equation: (i) a dynamical system for a Gaussian pulse and (ii) an average integral NLS equation. We apply normal form transformations for finite-dimensional and infinite-dimensional Hamiltonian systems with periodic coefficients. First-order corrections to the leading-order averaged Hamiltonian are derived explicitly for both approximations. Bifurcations of soliton solutions and their stability are studied by analysis of critical points of the first-order averaged Hamiltonians. The validity of the averaging procedure is verified and the presence of ground states corresponding to dispersion-managed solitons in the averaged Hamiltonian is established.

Key words. existence and stability of pulses, optical solitons, dispersion management, averaging theory, normal form transformations, errors and convergence of asymptotic series, periodic NLS equation, integral NLS equation, Gaussian approximation

AMS subject classifications. 35Q55, 78M30, 78M35

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1. Introduction.

1.1. Motivations. Ultrafast high–bit-rate optical communication networks are enhanced by the dispersion management technology when two optical fibers of opposite dispersion are periodically concatenated into a line [1]. If the communication network has low path-averaged dispersion and high local dispersion, the data signals are optimally transmitted from the input to output ends through a periodic sequence of compression and expansion cycles. The long-haul dispersion management is technologically combined with standard loss management when a periodic chain of amplifiers compensates distributive fiber losses.

Many recent experimental groups reported revolutionary performance of dispersion-managed (DM) pulses in optical communication networks [2, 3]. Two regimes were studied in detail: DM solitons and chirped return-to-zero pulses. DM solitons are time bits transmitted stationary on the average through the long-haul communication network [2]. The chirped return-to-zero pulses are weakly broadened on the average due to transmission, and some post-transmission compression may be required at the output of the network [3].

This paper addresses the stationary DM solitons and resolves yet open problems of existence and stability of stationary DM solitons described by a periodic nonlinear Schrödinger (NLS) equation. Theoretical studies of DM solitons are based on one of the three averaging methods: (i) variational Gaussian approximation, (ii) asymptotic reduction to an integral NLS equation, and (iii) numerical split-step averaging algorithm (see the latest reviews [4, 5, 6]).

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The variational Gaussian approximation truncates the periodic NLS equation at a finite-dimensional Hamiltonian system with periodic coefficients. The truncation is performed by integrating the Lagrangian density of the NLS equation over the Gaussian pulse and varying the resulting function with respect to parameters of the Gaussian pulse [7, 8]. Periodic orbits of the nonautonomous Hamiltonian system correspond to stationary DM solitons [9, 10]. In an optimal design of the dispersion map, the evolution length for dispersion variations is much shorter than the lengths of average dispersion and fiber nonlinearity. Within this limit, the nonautonomous Hamiltonian system can be averaged over the map period. The averaging procedure results, at the leading-order approximation, in a planar dynamical system [11, 12]. Existence and stability of DM solitons can be studied by analyzing trajectories on a phase plane near the critical points of the Hamiltonian system [13]. One of the drawbacks of the Gaussian approximation is the lack of information about the error of the averaging procedure.

The asymptotic reduction to an integral NLS equation is also based on averaging of the periodic NLS equation over a short period of the dispersion map [14, 15]. The method is, however, much more general, since the kernel for the averaging transformation is the most general Fourier solution of the linear periodic Schrödinger equation, which includes the Gaussian pulse as a particular case. Stationary DM solitons are approximated by the time pulse solutions of the nonlinear eigenvalue problem associated with the integral NLS equation. The DM soliton solutions have constantly rotating complex phase along the fiber [15, 16]. Only when the integral NLS equation is approximated at the Gaussian pulse [17], the resulting dynamical system reproduces the same planar Hamiltonian system as in [13]. In the asymptotic reduction method, the integral NLS equation can be viewed as the leading-order term in a set of canonical transformations applied to the periodic NLS equation [18].

At last, the numerical split-step averaging algorithm is applied to separate the pulse resolution in time and the almost periodic evolution of the pulse along the fiber by averaging the output of the split-step method over many time periods [5, 6]. A single pulse with a preserved value of energy was found to converge to a stationary DM soliton unless various resonances and temporal instabilities resulted in an unpredictable loss of convergence of the numerical algorithm [6, 19].

We are motivated by a number of averaging methods applied to the periodic NLS equation and by contradictory results on existence and stability of DM solitons found within these methods. In order to justify and clarify these methods, we develop a systematic asymptotic procedure for averaging of the periodic NLS equation, based on normal form transformations. We extend the perturbation expansions to the next order, where the first-order corrections to the leading-order equations are derived. The validity of averaging methods and the errors (accuracy) of the leading-order and first-order approximations are proved rigorously for a two-step dispersion map. Branches of stationary DM solitons and their stability are analyzed within the averaged equations.

This paper is structured as follows. In section 1.2 we describe the physical model, parameters, and normalizations. In section 1.3 we discuss two approximations of DM solitons and summarize the previously known results, together with our main propositions. In sections 2.1–2.4 we study the Gaussian approximation, in combination with the leading and first orders of the averaging method. We find explicitly analytical curves for existence and stability of the DM solitons in this lower-dimensional approximation. In sections 3.1–3.3 we analyze the full PDE problem and prove convergence of the leading and first orders of the averaging method. The existence of ground states is proved in the averaged equation but the analytical curves are implicit in this higher-dimensional approximation. Section 4 describes open problems of analysis beyond the first-order averaging theory.

1.2. Model and parameterizations. The NLS equation for optical pulses in dispersion-compensated fibers is

$$i\frac{\partial U}{\partial Z} - \frac{1}{2}\beta_2(Z)\frac{\partial^2 U}{\partial T^2} + \gamma_2(Z)|U|^2 U = 0, \tag{1.1}$$

where $U(Z,T)$ is the electric field envelope of the carrier wave at the operating wavelength λ_0, while $\beta_2(Z)$ and $\gamma_2(Z)$ are the fiber dispersion and nonlinearity [1]:

$$\beta_2 = -\frac{\lambda_0^2}{2\pi c}D(Z), \quad \gamma_2 = \gamma \exp\left[\int_0^Z g(Z')dZ' - \alpha Z\right], \quad \gamma = \frac{2\pi n_2}{\lambda_0 A_{\rm eff}} \ (>0). \tag{1.2}$$

All units in (1.1)–(1.2) have dimensional form, such that $D(Z)$ is the dispersion coefficient measured in ps/(nm $\times$ km), c is the speed of light in km/sec, $|u|^2$ is the light intensity in mW, n_2 is the nonlinear refractive index in $(\mu\text{m})^2$/mW, $A_{\rm eff}$ is the effective fiber area in $(\mu\text{m})^2$, α is the distributive loss coefficient in km^{-1}, and $g(Z)$ is the periodic amplification. For example, if $A_{\rm eff} = 50(\mu\text{m})^2$, $c = 3\ 10^5$km/sec, $\lambda_0 = 1.5\mu$m, $n_2 = 2.5\ 10^{-11}(\mu\text{m})^2$/mW, and $D = 0.12$ps/(nm $\times$ km), then $\beta_2 \approx -0.1\text{ps}^2$/km and $\gamma \approx 2\ 10^{-3}(\text{mW} \times \text{km})^{-1}$, which are reasonable values for these coefficients.

The dispersion map $D(Z)$ consists of two piecewise-constant fibers of lengths L_1 and L_2 in km, such that $L_1 + L_2 = L_{\rm DM}$, which have dispersion values D_1 and D_2. The total number of fiber segments is $N_{\rm DM}$. The amplification map $g(Z)$ is periodic with period $L_{\rm AM}$, where the ratio $L_{\rm DM}/L_{\rm AM}$ is integer. A typical loss compensation due to erbium-doped fiber amplifiers is

$$g = \alpha L_{\rm AM} \sum_{n=1}^{N_{\rm AM}} \delta(Z - nL_{AM}), \tag{1.3}$$

where $N_{\rm AM}$ is the number of amplifiers over the transmission line: $Z \in [0, N_{\rm DM} L_{\rm DM}]$ and the amplifiers compensate the losses exactly. As a result, the fiber nonlinearity $\gamma_2(Z)$ is a periodic function with period $L_{\rm AM}$. We will use throughout the paper the lossless approximation when $\gamma_2(Z) = \gamma$ is constant. The lossless approximation occurs in the limit $L_{\rm AM} \ll L_{\rm DM}$, when $\lim_{L_{\rm AM}\to 0} \int_0^Z g(Z')dZ' = \alpha Z$. This approximation is sufficiently accurate for modeling fibers with distributed (e.g., Raman) amplification or fibers with several amplifiers at the dispersion compensation period L_{DM} [1]. In other cases, our results are still expected to hold qualitatively.

We can rescale variables (Z,T,U) by introducing characteristic pulse power P_0 in mW, characteristic pulse width T_0 in ps, and characteristic (nonlinear) length $L_{\rm NL} = (\gamma P_0)^{-1}$ in km:

$$Z = L_{\rm NL} z, \quad T = T_0 t, \quad U = \sqrt{P_0} u. \tag{1.4}$$

The periodic NLS equation (1.1) in new variables reduces to the dimensionless form [1],

$$i\frac{\partial u}{\partial z} + \frac{m}{2\epsilon} d\left(\frac{z}{\epsilon}\right)\frac{\partial^2 u}{\partial t^2} + \frac{1}{2}d_0\frac{\partial^2 u}{\partial t^2} + |u|^2 u = 0, \tag{1.5}$$

with the dimensionless parameters

$$m = \frac{\lambda_0^2 L_1 L_2 (D_1 - D_2)}{4\pi c L_{\rm DM} T_0^2}, \quad d_0 = \frac{\lambda_0^2 (D_1 L_1 + D_2 L_2)}{2\pi c \epsilon T_0^2}, \quad \epsilon = \gamma L_{\rm DM} P_0. \tag{1.6}$$

The normalized periodic function $d(\zeta)$ has the unit period for $\zeta = z/\epsilon$ and zero average: $d(\zeta+1) = d(\zeta)$ and $\int_0^1 d(\zeta) d\zeta = 0$. It is defined explicitly as

$$\begin{aligned} d &= \frac{2}{l} \quad \text{for} \quad \zeta \in [0, l), \\ d &= \frac{2}{l-1} \quad \text{for} \quad \zeta \in [l, 1), \end{aligned} \tag{1.7}$$

where $0 < l < 1$ is the ratio of the first fiber leg to the total map period, i.e., $l = L_1/L_{\rm DM}$. We assume that the first leg is for the focusing fiber, i.e., $D_1 > 0$, and the second leg is for the defocusing fiber, i.e., $D_2 < 0$. As a result, the parameter m is positive, $m > 0$. The general problem (1.5) has four parameters:

- m (> 0)—the strength of the local (varying) dispersion,
- d_0—the strength of the average dispersion,
- ϵ (> 0)—the period of the dispersion map,
- l ($0 < l < 1$)—the relative length of the focusing fiber leg to the total map period.

Parameters m and ϵ can be normalized to unity by applying the transformation to the periodic NLS equation (1.5):

$$\zeta = \frac{z}{\epsilon}, \qquad \tau = \frac{t}{\sqrt{m}}, \qquad w(\zeta, \tau) = \sqrt{\epsilon} u(z, t), \tag{1.8}$$

where $w(\zeta, \tau)$ solves the standardized periodic NLS equation:

$$i \frac{\partial w}{\partial \zeta} + \frac{1}{2} d(\zeta) \frac{\partial^2 w}{\partial \tau^2} + \frac{1}{2} D_0 \frac{\partial^2 w}{\partial \tau^2} + |w|^2 w = 0, \qquad D_0 = \frac{\epsilon d_0}{m}. \tag{1.9}$$

Thus, the periodic NLS equation (1.9) depends only on two parameters: l (through $d(\zeta)$) and D_0.

In this paper, we study a formal asymptotic limit $\epsilon \to 0$ of solutions $u(z,t)$ of the periodic NLS equation in the form (1.5). This asymptotic limit corresponds to the limit of small solutions $w(\zeta, \tau)$ (in a $L^2(\mathbb{R})$ norm) of the periodic NLS equation in the form (1.9).

1.3. DM solitons and main results on existence and stability. DM solitons can be defined as special solutions of the periodic NLS equation (1.5) in two conventional approximations: (i) Gaussian pulse [7, 8] and (ii) an averaged integral NLS equation [14, 15].

DEFINITION 1.1. *A DM soliton is an approximate quasi-periodic solution of the NLS equation* (1.5) *in the form of the Gaussian pulse with variable coefficients:*

$$u(z, t) = \sqrt{c} \, \exp\left(-\frac{t^2}{2(a + ib)} + i\phi \right), \tag{1.10}$$

where $a(z + \epsilon) = a(z)$, $b(z + \epsilon) = b(z)$, $\phi(z + \epsilon) = \phi(z) + \epsilon\mu$, and

$$c = \frac{e a^{1/2}}{\sqrt{2}(a + ib)}, \qquad e = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} |u|^2(z, t) dt. \tag{1.11}$$

The three varying parameters $a(z)$, $b(z)$, and $\phi(z)$ are the pulse width, chirp, and the gauge rotation phase, respectively. The constant parameters μ and e are the phase propagation constant and the pulse energy, respectively.

The variational equations are derived by minimizing the Lagrangian density of the periodic NLS equation (1.5) at the Gaussian pulse (1.10) (see, e.g., [22]). It is then found that the varying parameters $a(z)$ and $b(z)$ satisfy the nonautonomous dynamical system:

$$\frac{da}{dz} = \frac{ea^{5/2}b}{(a^2+b^2)^{3/2}}, \tag{1.12}$$

$$\frac{db}{dz} = \frac{m}{\epsilon} d\left(\frac{z}{\epsilon}\right) + d_0 - \frac{ea^{3/2}(a^2-b^2)}{2(a^2+b^2)^{3/2}}. \tag{1.13}$$

The phase parameter $\phi(z)$ is coupled with $a(z)$ and $b(z)$ by the nonhomogeneous equation:

$$\frac{d\phi}{dz} = \frac{ea^{1/2}(3a^2+5b^2)}{8(a^2+b^2)^{3/2}}. \tag{1.14}$$

The dynamical system (1.12)–(1.13) has been studied numerically under different parameterizations (see reviews [4, 5, 6]). The system was found to be Hamiltonian [9], where the phase plane was used for matching trajectories of two autonomous systems derived for the piecewise-constant function $d(z)$. Existence of periodic solutions of (1.12)–(1.13) was recently proved by Kunze [10]. The leading-order averaging system was derived from (1.12)–(1.13) by Turitsyn et al. [11, 12]. A single branch of periodic solutions of the system was found for $d_0 \geq 0$, while two branches coexist for $d_{\min} < d_0 < 0$ at any given e [13].

Definition 1.2. *DM soliton is a stationary pulse solution of the averaged integral NLS equation:*

$$\begin{aligned}\mu \hat{W}(\omega) = &-\frac{1}{2} d_0 \omega^2 \hat{W}(\omega) \\ &+ \iint_{-\infty}^{\infty} \frac{\sin\left[m(\omega-\omega_1)(\omega-\omega_2)\right]}{m(\omega-\omega_1)(\omega-\omega_2)} \hat{W}(\omega_1)\hat{W}(\omega_2)\hat{W}(\omega_1+\omega_2-\omega) d\omega_1 d\omega_2,\end{aligned} \tag{1.15}$$

where $\hat{W}(\omega) \in H^s(\mathbb{R})$ with $s \geq 1$ and $d_0 > 0$.

The integral NLS equation (1.15) is derived from the periodic NLS equation (1.5) in the limit $\epsilon \to 0$ by using the asymptotic averaging method explained in section 3. The integral NLS equation (1.15) follows from (3.15) for stationary pulse solutions: $\hat{V}(z,\omega) = \hat{W}(\omega)e^{i\mu z}$, where $\hat{W}(\omega)$ is real function.

Existence of stationary pulse solutions of (1.15) for $d_0 > 0$ and $\mu > 0$ was proved by Zharnitsky et al. [18]. Recently Kunze proved existence of ground state solutions $\hat{W}(\omega) \in L^2(\mathbb{R})$ for $d_0 = 0$ and $\mu > 0$ [20], which was a considerably more difficult problem due to the absence of the gradient term in the Hamiltonian. Numerical results suggest nonexistence of ground state solutions for $d_0 < 0$ due to resonance of stationary pulses with linear spectrum of the averaged integral NLS equation [13, 21]. Iterations of a numerical method for finding stationary pulse solutions of (1.15) diverge for both branches of the Gaussian pulse solutions, which exist for (1.12)–(1.13) with $d_{\min} < d_0 < 0$ (see details in [21]). No rigorous results on nonexistence of ground states of (1.15) for $d_0 < 0$ are yet available.

Definitions 1.1 and 1.2 above are commuting in the sense that (i) the system (1.12)–(1.13) can be averaged in the limit $\epsilon \to 0$ [13] and (ii) the variational Gaussian approximation can be applied to the integral NLS equation (1.15) [17]. Both the reductions result in the same set of equations for an averaged Gaussian pulse. In order to analyze the parameter dependence of DM solitons, we consider the following two equivalent parameterizations.

Suppose there exist periodic solutions of (1.12)–(1.13) or stationary pulse solutions of (1.15). The DM solitons are parameterized as $e = f_\mu(\mu; d_0, l, m, \epsilon)$, where $e = f_\mu(\mu)$ is a continuous (possibly multibranched) function of μ. Indeed, solutions $\hat{W}(\omega)$ of (1.15) smoothly depend on parameter μ in the domain of their existence, where $\hat{W}(\omega) \in H^s(\mathbb{R})$ with $s \geq 1$. Then, the function $e = f_\mu(\mu)$ is defined by (1.11) as a continuous function of μ. If there are several solutions of (1.15) for the same value of μ, the function $e = f_\mu(\mu)$ has several branches for a fixed value of μ. Alternatively, solutions $(a(z), b(z))$ of (1.12)–(1.13) smoothly depend on e in the domain of their existence. Then, μ is defined by $\mu = (\phi(z+\epsilon) - \phi(z))/\epsilon = f_\mu^{-1}(e)$. The function $e = f_\mu(\mu)$ is invertible for each branch of solution, where $f'_\mu(\mu) \neq 0$.

For an alternative parameterization, we define the effective pulse width as

$$\tau^2(z) = \frac{2}{e}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} t^2 |u|^2(z,t)dt \tag{1.16}$$

and the minimal pulse width as

$$\tau^2_{\min} = \min_{z \in [0,\epsilon]} \tau^2(z). \tag{1.17}$$

The DM solitons are parameterized as $e = f_s(s; d_0, l, m, \epsilon)$, where $s = 1/\tau^2_{\min}$ is the square inverse of the minimal pulse width. The function $e = f_s(s)$ is a continuous (possibly multibranched) function of s. For each branch of stationary pulse solutions of (1.15), there exists a continuous map $s = h_\mu(\mu)$ defined by (1.16). Then, the function $e = f_s(s)$ is parameterized by μ. Also, for each branch of periodic solutions of (1.12)–(1.13), there exists a continuous function $s = f_s^{-1}(e)$ defined by

$$\tau^2_{\min} = \min_{z \in [0,\epsilon]} \frac{(a^2+b^2)}{a} = \min_{z \in [0,\epsilon]} a(z). \tag{1.18}$$

Here we have used (1.10), (1.11), and (1.17) for the first equality and (1.12) for the second equality. The function $e = f_s(s)$ is inverted for each branch of solution, where $f'_s(s) \neq 0$.

LEMMA 1.3. *DM solitons are parameterized by three parameters: $E = \epsilon e/\sqrt{m}$ (energy), $M = \epsilon\mu$ (propagation constant), and $S = ms$ (map strength).*

Proof. The statement is proved by applying transformation (1.8) to the integral quantities (1.11) and (1.16) and to the phase $\phi(z)$ of the Gaussian pulse (1.10). ∎

The DM solitons can be analyzed in the Gaussian approximation for several alternative representations in variables E, M, and S: (i) on the plane (D_0, E) for different values of S; (ii) on the plane (S, E) for different values of S_0; (iii) on the plane (S, M) for different values of D_0; and (iv) on the plane (M, E) for different values of D_0. The four equivalent representations are shown on Figure 1.1(a)–(d), where we reproduce our main results on computations of the first-order averaging theory for the Gaussian approximation. The parameter l is fixed at $l = 0.1$. The solid curves show the result of the first-order averaging theory for $\epsilon > 0$. The dotted curves show the result of the leading-order averaging theory in the limit $\epsilon = 0$.

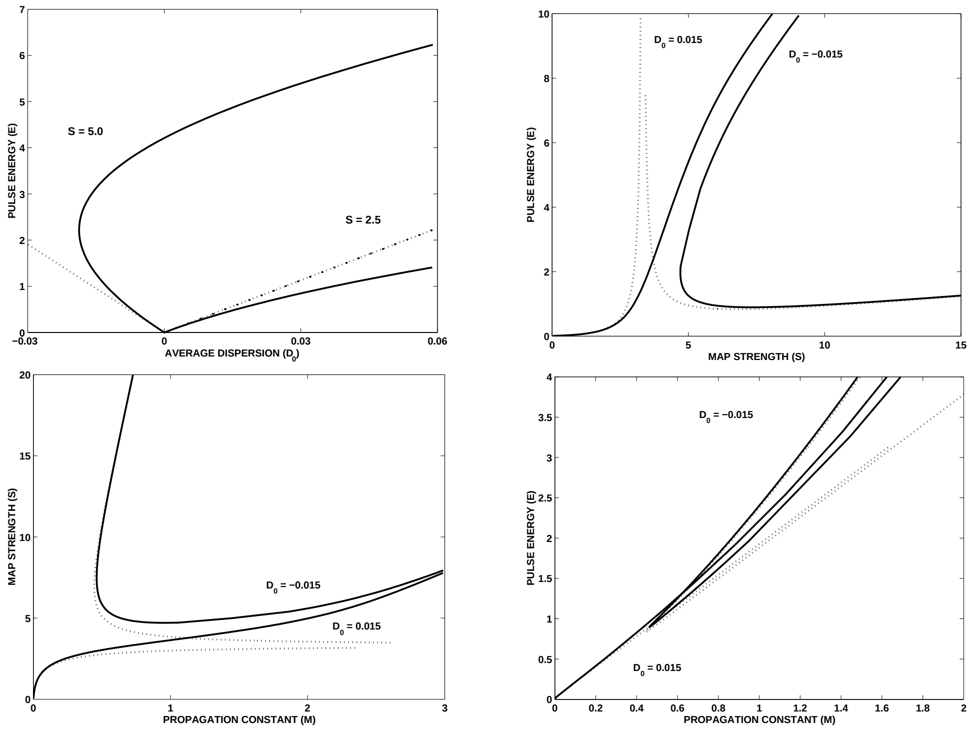

FIG. 1.1. *Parameterizations of DM solitons in the first-order averaging theory for Gaussian approximation:* (a) *plane (D_0, E) for $S = 2.5$ and $S = 5$;* (b) *plane (S, E) for $D_0 = 0.015$ and $D_0 = -0.015$;* (c) *plane (S, M) for $D_0 = 0.015$ and $D_0 = -0.015$; and* (d) *plane (M, E) for $D_0 = 0.015$ and $D_0 = -0.015$. The dotted curves display the leading-order averaging theory.*

There exists only one branch of periodic solutions of (1.12)–(1.13) for $0 < S \leq S_{\text{thr}}$, where $S_{\text{thr}} \approx 3.32$. This branch extends for $D_0 \geq 0$ (see Figure 1.1(a)). When $S > S_{\text{thr}}$, the dependence of E versus D_0 becomes two-folded: two branches of periodic solutions exist for $D_{\min} < D_0 < 0$ and one branch exists for $D_0 \geq 0$. When $S \to \infty$, all branches of periodic solutions diverge to infinitely large values of E (see Figure 1.1(b)).

The role of planes (D_0, E), (S, E), and (S, M) is different from that of the plane (M, E) in the leading-order averaging theory. Indeed, the leading-order averaged system (dotted curves on Figures 1.1(a)–(d)) describes only the lower branch of periodic solutions for $D_0 < 0$ (see Figure 1.1(a)). The functions $E = f_S(S)$ and $M = h_S(S)$ are single-branched for any S (see Figures 1.1(b), (c)). However, the function $E = f_M(M)$ has two branches at the leading-order approximation (see Figure 1.1(d)), i.e., two solutions for E correspond to the same value of M, and vice verse.

For $D_0 \geq 0$, there is only one branch of periodic solutions. This branch is bounded by the threshold value $S < S_{\text{thr}}$ in the leading-order approximation (see Figures 1.1(a)–(c)), while the function $E = f_M(M)$ is unbounded on the plane (M, E) (see Figure 1.1(d)). Thus, again the planes (D_0, E), (S, E), and (S, M) give bad leading-order approximations of periodic solutions compared to the plane (M, E).

The discrepancy between the four alternative parameterizations of periodic solutions of (1.12)–(1.13) disappears in the first-order averaging approximation, shown on Figures 1.1(a)–(d) by solid curves. Indeed, all curves are unbounded for $D_0 \geq 0$ and all curves become two-valued functions of S, M, and E for any $D_0 < 0$. The upper branch of periodic solutions on the planes (D_0, E), (S, E), and (S, M) is captured in the first-order averaging theory (see Figure 1.1(a)–(c)). On the other hand, the single branch for $D_0 \geq 0$ and the two branches for $D_0 < 0$ are weakly affected on the plane (M, E) in the first-order approximation, compared to the leading-order theory (see Figure 1.1(d)). Thus, the plane (M, E) is the most appropriate tool for analysis of periodic solutions both in the leading-order and first-order averaging theory.

We can now formulate algebraically the main results of the paper for existence and stability of DM solitons. The results are written in terms of parameters ϵ, m, d_0, e, μ, and s, while the transformation to parameters D_0, E, M, and S is prescribed by Lemma 1.3. The results are proved for the Gaussian pulse approximation by explicit computations (section 2) and for the averaged integral NLS equation by standard PDE analysis (section 3).

PROPOSITION 1.4. *Parameters of DM solitons have the following expansions in powers of ϵ:*

$$d_0 = em^{1/2} g_d^{(0)}(my) + \epsilon e^2 (1-2l) g_d^{(1)}(my) + \mathrm{O}(\epsilon^2), \tag{1.19}$$

$$\mu = \frac{e}{m^{1/2}} g_\mu^{(0)}(my) + \epsilon \frac{e^2}{m}(1-2l) g_\mu^{(1)}(my) + \mathrm{O}(\epsilon^2), \tag{1.20}$$

$$s = y + \epsilon \frac{e}{m^{3/2}} g_s^{(1)}(my, l) + \mathrm{O}(\epsilon^2), \tag{1.21}$$

where y is a parameter and the functions $g_d^{(0,1)}$, $g_\mu^{(0,1)}$, and $g_s^{(1)}$ are continuous for $y > 0$.

Proof. Here we prove the result only in the limit $\epsilon \to 0$, when the leading-order averaging theory results in the integral NLS equation (1.15). The kernel of (1.15) is independent of l. Consider the scaling transformation $\hat{W}(\omega) = \lambda \hat{W}'(\omega)$. This transformation leaves (1.15) invariant if parameters μ, d_0, e, s, and m in (1.11), (1.15), and (1.16) transform as follows:

$$\mu = \lambda^2 \mu', \quad d_0 = \lambda^2 d_0', \quad e = \lambda^2 s', \quad s = s', \quad m = m'. \tag{1.22}$$

It is clear from (1.22) that the ratios d_0/e and μ/e are invariant under this transformation and are, therefore, functions of s and m. The particular form used in (1.19)–(1.20) has been chosen to match with explicit computations of functions $g_d^{(0)}(my)$ and $g_\mu^{(0)}(my)$. ∎

COROLLARY 1.5. *In the limit $\epsilon \to 0$, the functions $e = f_s(s)$, $\mu = h_s(s)$, and $e = f_\mu(\mu)$ have the form*

$$e = \frac{d_0}{\sqrt{m} g_d^{(0)}(ms)}, \quad \mu = \frac{d_0}{m} \frac{g_\mu^{(0)}(ms)}{g_d^{(0)}(ms)}, \quad \frac{\mu\sqrt{m}}{e} = g_\mu^{(0)} (g_d^{(0)})^{-1} \left(\frac{d_0}{e\sqrt{m}} \right). \tag{1.23}$$

COROLLARY 1.6. *In the limit $\epsilon \to 0$ and $d_0 = 0$, the function $e = f_\mu(\mu)$ has the form*

$$e = \alpha\mu, \qquad \alpha = \frac{\sqrt{m}}{g_\mu^{(0)}(ms_*)},$$

where $s = s_*$ is the root of the equation $g_d^{(0)}(ms_*) = 0$.

PROPOSITION 1.7. *In the limit $\epsilon \to 0$ and $m \to 0$, the functions $e = f_\mu(\mu)$ and $s = f_\mu(\mu)$ have the form*

$$e^2 = d_0 \hat{f}_\mu(\mu), \qquad s = \frac{1}{d_0}\hat{h}_\mu(\mu). \tag{1.24}$$

Proof. In the limit $m \to 0$, the integral NLS equation (1.15) becomes the Fourier form of the NLS equation. Consider the scaling transformation for the NLS equation: $\hat{W}(\omega) = \lambda \hat{W}'(\omega')$ and $\omega = \lambda^{-1}\omega'$. This transformation leaves the NLS equation invariant if parameters μ, d_0, e, and s in (1.11), (1.15), and (1.16) transform as follows:

$$\mu = \mu', \qquad d_0 = \lambda^2 d_0', \qquad e = \lambda e', \qquad s = \lambda^{-2}s'. \tag{1.25}$$

It is clear from (1.25) that the quantities e^2/d_0 and sd_0 are invariant under this transformation and are, therefore, functions of μ. ∎

PROPOSITION 1.8. *A single branch of DM solitons exists and is linearly stable for $d_0 \geq 0$ in both Gaussian and integral NLS approximations. Two branches of DM solitons exist for $d_0 < 0$ in the Gaussian approximation. For a fixed μ, the branch with larger e is linearly unstable and the branch with smaller e is linearly stable.*

Linearized stability of DM solitons in the Gaussian approximation (1.12)–(1.13) was studied by Pelinovsky in the limit $\epsilon \to 0$ [13]. We extend this analysis in the first-order averaging theory in section 2 of this paper. Zharnitsky et al. [18] proved that the DM solitons are ground states of the averaged integral NLS equation (1.15) for $d_0 > 0$. The ground states realize a stable minimum of the Hamiltonian functional. We extend this result for the first-order averaged Hamiltonian in section 3. Open problems for nonexistence of ground states for $d_0 < 0$ and nonexistence of quasi-periodic solutions of the periodic NLS equation (1.5) are discussed in section 4.

2. Variational Gaussian approximation.

Here we analyze the dynamical system (1.12)–(1.13) derived in the variational Gaussian approximation. We construct the Hamiltonian structure for the system and develop a systematic averaging procedure based on the theory of canonical transformations in section 2.1. The first-order corrections to the leading-order averaged theory are computed in section 2.2. Existence and stability of critical points of the first-order averaged Hamiltonian are studied in sections 2.3 and 2.4.

The dynamical system (1.12)–(1.13) can be written as a Hamiltonian system in canonical variables (ξ, η):

$$\xi = b - md_{-1}\left(\frac{z}{\epsilon}\right), \qquad \eta = \frac{1}{a}, \tag{2.1}$$

where $d_{-1}(\zeta)$ is the antiderivative of $d(\zeta)$ for $\zeta = z/\epsilon$ with unit period and mean zero:

$$d_{-1}(\zeta) = \int_0^\zeta d(\zeta')d\zeta' - \int_0^1 d\zeta \int_0^\zeta d(\zeta')d\zeta'. \tag{2.2}$$

For the piecewise-constant approximation $d(\zeta)$ in (1.7), the mean-zero antiderivative $d_{-1}(\zeta)$ is

$$\begin{aligned} d_{-1} &= \frac{2\zeta}{l} - 1 \quad \text{for} \quad \zeta \in [0, l), \\ d_{-1} &= \frac{2(\zeta - 1)}{(l-1)} - 1 \quad \text{for} \quad \zeta \in [l, 1). \end{aligned} \tag{2.3}$$

The system (1.12)–(1.13) in canonical variables (ξ,η) has a classical Hamiltonian structure:

$$\frac{d\xi}{dz} = \frac{\partial H}{\partial \eta}, \qquad \frac{d\eta}{dz} = -\frac{\partial H}{\partial \xi}, \tag{2.4}$$

where the Hamiltonian $H = H(\xi,\eta,z/\epsilon)$ is

$$H = d_0\eta - e\left(\frac{\eta}{1+\eta^2(\xi + m d_{-1}(z/\epsilon))^2}\right)^{1/2}. \tag{2.5}$$

The decoupled equation (1.14) for the phase parameter $\phi(z)$ can be expressed through $H(\xi,\eta,z/\epsilon)$ as

$$\frac{d\phi}{dz} = \frac{1}{4}\left(d_0\eta + \eta\frac{\partial H}{\partial \eta} - 2H\right). \tag{2.6}$$

There exists a canonical transformation from the Hamiltonian structure (2.4)–(2.5) to the one studied in [9]. The canonical transformation (2.1) and the Hamiltonian (2.5) were first reported by Turitsyn et al. [12]. The Hamiltonian structure (2.4)–(2.5) is more convenient for developing a systematic averaging procedure based on series of canonical transformations in powers of ϵ. We will study solutions of the system (2.4)–(2.5) in the domain $\mathcal{D}_+$:

$$\mathcal{D}_+ = \{(\xi,\eta):\ \xi\in\mathbb{R},\ \eta>0\}. \tag{2.7}$$

LEMMA 2.1. *Suppose the initial point (ξ_0,η_0) belongs to $\mathcal{D}_+$. Then, a solution $(\xi(z),\eta(z))$ stays in $\mathcal{D}_+$ for any finite z: $0 \le z \le z_0 < \infty$.*

Proof. Integrating (1.12) in the canonical variables (2.1), one can find

$$\frac{1}{\eta^{1/2}} = \frac{1}{\eta_0^{1/2}} + \frac{e}{2}\int_0^z \frac{\eta(\xi + m d_{-1})dz}{(1+\eta^2(\xi+md_{-1})^2)^{3/2}}.$$

Since the integrand is never singular, the triangular inequality implies for $0 \le z \le z_0$ that

$$\left|\frac{1}{\eta^{1/2}}\right| \le \frac{1}{\eta_0^{1/2}} + \frac{eM}{2}z_0,$$

where

$$M = \max_{0\le z\le z_0} \frac{\eta|\xi+md_{-1}|}{(1+\eta^2(\xi+md_{-1})^2)^{3/2}}.$$

Therefore, the point (η,ξ) never crosses the left boundary of $\mathcal{D}_+$ at $\eta = 0$. Direct integration of (1.12)–(1.13) with the variables (2.1) and similar estimates of the resulting integrals show that $|\xi|$ and η remain bounded in the domain $\mathcal{D}_+$ for any finite z: $0 \le z \le z_0$. □

2.1. Averaging of the periodic Hamiltonian system (2.4)–(2.5). The periodic Hamiltonian system (2.4)–(2.5) is averaged according to the formalism of normal form transformations [23]. We denote $\zeta = z/\epsilon$ such that $H = H(\xi,\eta,\zeta)$. In the domain $\mathcal{D}_+$ defined by (2.7), there exists a near-identity generating function:

$$F(\xi,y,\zeta) = \xi y + \sum_{n=1}^{N+1}\epsilon^n F_n(\xi,y,\zeta) + \mathrm{O}(\epsilon^{N+2}), \tag{2.8}$$

where the correction terms $F_n(x,y,\zeta)$ for $1 \le n \le (N+1)$ are periodic mean-zero functions of ζ:

$$F_n(x,y,\zeta+1) = F_n(x,y,\zeta), \qquad \int_0^1 F_n(x,y,\zeta)d\zeta = 0. \tag{2.9}$$

The generating function $F(\xi,y,\zeta)$ defines the near-identical canonical transformation

$$x = \frac{\partial F}{\partial y}(\xi,y,\zeta), \qquad \eta = \frac{\partial F}{\partial \xi}(\xi,y,\zeta) \tag{2.10}$$

and takes the Hamiltonian $H(\xi,\eta,\zeta)$ to the form

$$\begin{aligned} H_{\text{new}}(x,y,\zeta) &= H(\xi(x,y,\zeta),\eta(x,y,\zeta),\zeta) + \frac{1}{\epsilon}\frac{\partial F}{\partial \zeta}(\xi(x,y,\zeta),y,\zeta) \\ &= H_N(x,y) + \mathrm{O}(\epsilon^{N+1}), \end{aligned} \tag{2.11}$$

where $H_N(x,y)$ is the Nth-order averaged Hamiltonian:

$$H_N(x,y) = \sum_{n=0}^{N} \epsilon^n h_n(x,y). \tag{2.12}$$

When the remainder term of order of $\mathrm{O}(\epsilon^{N+1})$ is neglected, the new canonical variables (x,y) solve the averaged Hamiltonian dynamical system:

$$\frac{dx}{dz} = \frac{\partial H_N}{\partial y}, \qquad \frac{dy}{dz} = -\frac{\partial H_N}{\partial x}. \tag{2.13}$$

The difference between solutions of the full system (2.4) and the averaged system (2.13) is controlled with the accuracy of $\mathrm{O}(\epsilon^{N+1})$ on the interval $0 \le z \le z_0$. Convergence and bounds of the normal-form transformations in Hamiltonian systems with fast dependence on time was proved by Neishtadt [24].

The canonical transformation (2.10)–(2.11) follows from the invariance of the Lagrangian of the system (2.4) [23]:

$$\mathcal{L} = \eta\frac{d\xi}{dz} - H(\xi,\eta,\zeta) = -x\frac{dy}{dz} - H_{\text{new}}(x,y,\zeta) + \frac{dF}{dz}(\xi,y,\zeta).$$

In the domain $\mathcal{D}_+$, the Hamiltonian $H(\xi,\eta,\zeta)$ is a C^∞ function of ξ and η. Then, the generating functions $F_n(\xi,y,\zeta)$ are C^∞ functions of ξ and y. Provided the asymptotic series (2.8) converges uniformly, there exists ϵ_0 such that for $0 \le \epsilon \le \epsilon_0$

$$\frac{\partial^2 F}{\partial y \partial \xi} = 1 + \sum_{n=1}^{N+1} \epsilon^n \frac{\partial^2 F}{\partial y \partial \xi} + \mathrm{O}(\epsilon^{N+2}) > 0.$$

According to the inverse function theorem, the near-identity transformations (2.10) define classical perturbation series for $\xi(x,y,\zeta)$ and $\eta(x,y,\zeta)$ in powers of ϵ (see Chapter 2.2(a) in [23]). Here, ζ is the fast "time" for periodic oscillations of $H(\xi,\eta,z/\epsilon)$ and z is the slow "time" for averaged dynamics of the new canonical variables (x,y).

For $N=0$, the leading-order averaged dynamical system is

$$\frac{dx}{dz} = \frac{\partial h_0}{\partial y}, \qquad \frac{dy}{dz} = -\frac{\partial h_0}{\partial x}, \tag{2.14}$$

where $h_0(x,y)$ is the leading-order averaged Hamiltonian $H_0(x,y)$:

$$H_0(x,y) = h_0(x,y) = \int_0^1 H(x,y,\zeta)d\zeta. \tag{2.15}$$

The leading-order averaged Hamiltonian can be computed explicitly from (2.3), (2.5), and (2.15) as

$$h_0(x,y) = d_0 y - \frac{e}{2my^{1/2}} \log\left[f_0(x,y)\right], \tag{2.16}$$

where

$$f_0(x,y) = \frac{y(x+m) + (1+y^2(x+m)^2)^{1/2}}{y(x-m) + (1+y^2(x-m)^2)^{1/2}}. \tag{2.17}$$

The leading-order Hamiltonian $h_0(x,y)$ does not depend on parameter l. However, the first-order correction term $h_1(x,y)$ does depend on l in the first-order averaged Hamiltonian $H_1(x,y)$.

2.2. First-order averaged Hamiltonian $\boldsymbol{H_1(x,y)}$. The first-order averaged Hamiltonian can be easily derived from the formalism of the normal form transformations. It follows from (2.8), (2.10), and (2.11) that the first-order correction term $F_1(x,y,\zeta)$ is the periodic mean-zero function of ζ:

$$F_1(x,y,\zeta) = \left\{h_0(x,y) - H(x,y,\zeta)\right\}_{-1}, \tag{2.18}$$

where $\{H(x,y,\zeta)\}_{-1}$ is the mean-zero antiderivative of $H(x,y,\zeta)$ in ζ; see (2.2). Expanding the near-identity canonical transformations (2.8) and (2.10) in powers of ϵ, we define the perturbation series for $\xi(x,y,\zeta)$ and $\eta(x,y,\zeta)$:

$$\xi = x + \epsilon\xi_1(x,y,\zeta) + \mathrm{O}(\epsilon^2), \qquad \xi_1 = -\frac{\partial F_1}{\partial y}(x,y,\zeta), \tag{2.19}$$

$$\eta = y + \epsilon\eta_1(x,y,\zeta) + \mathrm{O}(\epsilon^2), \qquad \eta_1 = \frac{\partial F_1}{\partial x}(x,y,\zeta). \tag{2.20}$$

Similarly, the first-order correction term $h_1(x,y)$ is found in the form

$$h_1(x,y) = \int_0^1 \left(-\frac{\partial H}{\partial x}\frac{\partial F_1}{\partial y} + \frac{\partial H}{\partial y}\frac{\partial F_1}{\partial x} - \frac{\partial^2 F_1}{\partial\zeta\partial x}\frac{\partial F_1}{\partial y}\right)(x,y,\zeta)d\zeta \tag{2.21}$$

$$= \int_0^1 \left(\frac{\partial H}{\partial y}\frac{\partial F_1}{\partial x}\right)(x,y,\zeta)d\zeta, \tag{2.22}$$

where we have used (2.9) and (2.18) for the second equality in (2.22). The first-order averaged dynamical system (2.13) then takes the form

$$\frac{dx}{dz} = \frac{\partial h_0}{\partial y} + \epsilon\frac{\partial h_1}{\partial y}, \qquad \frac{dy}{dz} = -\frac{\partial h_0}{\partial x} - \epsilon\frac{\partial h_1}{\partial x}. \tag{2.23}$$

A remarkable result is that the first-order correction term $h_1(x,y)$ vanishes in the case of symmetric maps, when $l = 1/2$.

LEMMA 2.2. *When the dispersion map is symmetric, i.e., $l = 1/2$, then $h_1(x,y) = 0$ and $F_1(x,y,0) = F_1(x,y,l) = 0$.*

Proof. When $d(\zeta) = 4$ for $z \in [0, 1/2)$ and $d(\zeta) = -4$ for $z \in [-1/2, 0)$ (see (1.7)), the mean-zero antiderivative function $d_{-1}(\zeta)$ is even in ζ, i.e., $d_{-1}(-\zeta) = d_{-1}(\zeta)$ (see (2.3)). As a result, the Hamiltonian $H(x, y, \zeta)$ in (2.5) can be represented by the Fourier cosine-series:

$$H(x, y, \zeta) = h_0(x, y) + \sum_{n=1}^{\infty} c_n(x, y) \cos(2\pi n\zeta), \tag{2.24}$$

where $c_n(x, y)$ are some Fourier coefficients. As a result, the first-order correction term $F_1(x, y, \zeta)$ given by (2.18) is computed as the Fourier sine-series:

$$F_1(x, y, \zeta) = -\sum_{n=1}^{\infty} \frac{1}{2\pi n} c_n(x, y) \sin(2\pi n\zeta). \tag{2.25}$$

It is clear that $F_1(x, y, 0) = F_1(x, y, 1/2) = 0$. The first-order correction $h_1(x, y)$ given by (2.22) is the average of the product of the Fourier cosine- and sine-series, which is zero. ∎

In general case, when $l \neq 1/2$, the first-order averaging theory is equivalent to the following result. If (x, y) solve the averaged equations (2.23) and (ξ, η) solve the full equations (2.4)–(2.5) with close initial values—$|\xi_0 - x_0 - \epsilon\xi_1(x_0, y_0, 0)| \leq c_x\epsilon^2$ and $|\eta_0 - y_0 - \epsilon\eta_1(x_0, y_0, 0)| \leq c_y\epsilon^2$, where c_x and c_y are some constants, then the solutions (x, y) and (ξ, η) remain within the linear accuracy in ϵ at the distances $0 \leq z \leq z_0$:

$$\sup_{z\in[0,z_0]} |\xi(z) - x(z) - \epsilon\xi_1(x, y, \zeta)| \leq C_x\epsilon^2, \qquad \sup_{z\in[0,z_0]} |\eta(z) - y(z) - \epsilon\eta_1(x, y, \zeta)| \leq C_y\epsilon^2, \tag{2.26}$$

where C_x and C_y are some constants. The standard proof of this statement is based on convergence of the perturbation series (2.19)–(2.20) [23].

When the dispersion map is symmetric with equal legs, i.e., $l = 1/2$, the corrections $\xi_1(x, y, \zeta)$ and $\eta_1(x, y, \zeta)$ vanish at the points $\zeta = 0$ and $\zeta = \frac{1}{2}$. As a result, the distance between solutions (x, y) and (ξ, η) remains within the quadratic accuracy in ϵ at the ends of the dispersion map, i.e., at $z = k\epsilon$ and $z = (k - \frac{1}{2})\epsilon$, where $k \in \mathbb{Z}_+$:

$$\sup_{z\in[0,z_0]} |\xi(z = k\epsilon) - x(z = k\epsilon)| \leq C_x\epsilon^2, \qquad \sup_{z\in[0,z_0]} |\eta(z = k\epsilon) - y(z = k\epsilon)| \leq C_y\epsilon^2. \tag{2.27}$$

This result is related to the Strang's work [25] on symmetrization of the split-step methods for solving PDEs. The quadratic convergence occurs only at the ends of the dispersion map, while it is linear in the interior of the dispersion map.

Remark 2.1. The symmetric dispersion map with $l = 1/2$ can be translated for any ζ_0 such that $d(\zeta + \zeta_0) = -d(\zeta_0 - \zeta)$. The first-order correction $h_1(x, y)$ vanishes for all such symmetric maps. In particular, the symmetric dispersion map used in numerical modeling of the NLS equation by the split-step method is $d(\zeta) = 4$ for $\zeta \in [0, 1/4) \cup [3/4, 1)$ and $d(\zeta) = -4$ for $\zeta \in [1/4, 3/4)$. This map is equivalent to our symmetric map with $l = 1/2$ by the translation with $\zeta_0 = 1/4$.

The first-order correction term $h_1(x, y)$ can be found explicitly by direct computations from (2.3), (2.5), (2.16), (2.18), and (2.22). The explicit formula is

$$
\begin{aligned}
h_1(x,y) &= \frac{e^2(1-2l)}{8m^2}\left[\frac{(x+m)}{1+y^2(x+m)^2} - \frac{(x-m)}{1+y^2(x-m)^2}\right] \\
&+ \frac{e^2(1-2l)}{16m^3y^2}\left[\log\left[f_0(x,y)\right] + \frac{2y(x-m)}{(1+y^2(x-m)^2)^{1/2}} - \frac{2y(x+m)}{(1+y^2(x+m)^2)^{1/2}}\right] \\
&\times \left[\log\left[f_0(x,y)\right] + \frac{xy}{(1+y^2(x-m)^2)^{1/2}} - \frac{xy}{(1+y^2(x+m)^2)^{1/2}}\right] \\
&+ \frac{e^2(1-2l)}{16m^3y^2}\left[\frac{3+y^2(x+m)^2}{(1+y^2(x+m)^2)^{1/2}} - \frac{3+y^2(x-m)^2}{(1+y^2(x-m)^2)^{1/2}}\right] \\
&\times \left[\frac{1}{(1+y^2(x+m)^2)^{1/2}} - \frac{1}{(1+y^2(x-m)^2)^{1/2}}\right], \qquad (2.28)
\end{aligned}
$$

where $f_0(x,y)$ is defined by (2.17). We confirm from (2.28) that $h_1(x,y) = 0$ for $l = 1/2$. The first-order averaged Hamiltonian $H_1(x,y)$ is analyzed next for existence and stability of critical points. The critical points of the averaged Hamiltonian correspond to the Gaussian approximation of the DM solitons.

2.3. Existence of critical points of the first-order averaged Hamiltonian. The first-order averaged Hamiltonian is $H_1(x,y) = h_0(x,y) + \epsilon h_1(x,y)$, where $h_0(x,y)$ and $h_1(x,y)$ are given explicitly in (2.16) and (2.28).

LEMMA 2.3. *The points $(0,y_*)$ are critical points of the first-order averaged Hamiltonian $H_1(x,y)$ if y_* is an extremum of the function $H_1(0,y)$:*

$$
\begin{aligned}
H_1(0,y) &= d_0 y - \frac{e}{2my^{1/2}}\log\left[\hat{f}_0(my)\right] \\
&+ \epsilon\frac{e^2(1-2l)}{16m^3y^2}\left[\log^2\left[\hat{f}_0(my)\right] - \frac{4my}{(1+m^2y^2)^{1/2}}\log\left[\hat{f}_0(my)\right] + \frac{4m^2y^2}{1+m^2y^2}\right], \qquad (2.29)
\end{aligned}
$$

where $\hat{f}_0(my) = f_0(0,y)$, i.e.,

$$
\hat{f}_0(my) = \left[\frac{(1+m^2y^2)^{1/2}+my}{(1+m^2y^2)^{1/2}-my}\right].
$$

Proof. The variation of $h_0(x,y)$ in x leads to the only solution $x = 0$. The same solution gives also an extremum of $h_1(x,y)$ in x. The variation of $H_1(0,y)$ in y defines the critical point $y = y_*$. □

Proof of Proposition 1.4. The first equation (1.19) follows from the condition $H_1'(0,y) = 0$. The continuous functions $g_d^{(0,1)}$ are computed explicitly:

$$
g_d^{(0)} = -\frac{1}{4m^{3/2}y^{3/2}}\left[\log\left[\hat{f}_0(my)\right] - \frac{4my}{(1+m^2y^2)^{1/2}}\right], \qquad (2.30)
$$

$$
\begin{aligned}
g_d^{(1)} &= \frac{1}{8m^3y^3}\left[\log^2\left[\hat{f}_0(my)\right]\right. \\
&\left. - \frac{2my(2+3m^2y^2)}{(1+m^2y^2)^{3/2}}\log\left[\hat{f}_0(my)\right] + \frac{4m^2y^2(1+2m^2y^2)}{(1+m^2y^2)^2}\right]. \qquad (2.31)
\end{aligned}
$$

The second equation (1.20) follows from the nonhomogeneous equation (2.6) reduced for the perturbation expansion (2.19)–(2.20):

$$(2.32)\quad \frac{d\phi}{dz} = \frac{1}{4}\left[d_0 y + y\frac{dx}{dz} + y\frac{\partial \xi_1}{\partial \zeta} - 2H(x,y,\zeta) + \epsilon\left(d_0\eta_1 + \eta_1\frac{\partial H}{\partial y}(x,y,\zeta)\right.\right.$$
$$\left.\left. +\ y\left(\frac{\partial \xi_1}{\partial x}\frac{dx}{dz} + \frac{\partial \xi_1}{\partial y}\frac{dy}{dz}\right) + y\frac{\partial \xi_2}{\partial \zeta} - 2\left(\frac{\partial H}{\partial x}\xi_1 + \frac{\partial H}{\partial y}\eta_1\right)(x,y,\zeta)\right) + \mathrm{O}(\epsilon^2)\right].$$

Integrating (2.33) over $\zeta \in [0,1]$ at the critical point $(0, y_*)$, we define μ as

$$(2.33)\qquad \mu = \frac{1}{\epsilon}\left(\phi(z+\epsilon) - \phi(z)\right) = \frac{1}{4}\left[d_0 y_* - 2h_0(0,y_*) - 3\epsilon h_1(0,y_*) + \mathrm{O}(\epsilon^2)\right],$$

where we have utilized (2.15) and (2.22). The continuous functions $g_d^{(0,1)}$ in (1.20) are computed explicitly:

$$(2.34)\qquad g_\mu^{(0)} = \frac{1}{16 m^{1/2} y^{1/2}}\left[5\log\left[\hat{f}_0(my)\right] - \frac{4my}{(1+m^2y^2)^{1/2}}\right],$$

$$(2.35)\qquad g_\mu^{(1)} = -\frac{1}{64 m^2 y^2}\left[5\log^2\left[\hat{f}_0(my)\right]\right.$$
$$\left. -\ \frac{2my(8+9m^2y^2)}{(1+m^2y^2)^{3/2}}\log\left[\hat{f}_0(my)\right] + \frac{4m^2y^2(4+5m^2y^2)}{(1+m^2y^2)^2}\right].$$

The third equation (1.21) follows from (1.18):

$$(2.36)\qquad s = \frac{1}{\tau_{\min}^2} = \max_{0\le z\le\epsilon}\eta(z) = y + \epsilon\max_{0\le\zeta\le 1}\eta_1(0,y_*,\zeta) + \mathrm{O}(\epsilon^2).$$

If (a,b) is a nonconstant periodic solution of z, then $a(z)$ has at least two extremal points on the interval $z \in [0,\epsilon)$. The extremal values for $a(z)$ occur for $z = z_*$, where $b(z_*) = 0$; see (1.12). It follows from (2.3) and (2.18) that $d_{-1}(\zeta_*) = 0$ and $F_1(0,y_*,\zeta_*) = 0$ for $\zeta_* = l/2$ and $\zeta_* = (1+l)/2$. As a result, $b(z_*) = \mathrm{O}(\epsilon^2)$ and $\xi(\zeta_*) = \mathrm{O}(\epsilon^2)$, see (2.1) and (2.19), i.e., $a(z)$ and $\eta(z)$ have extrema for $z_* = \epsilon\zeta_*$. Computing the derivative of $F_1(x,y,\zeta)$ in x for $(0,y_*,\zeta)$, we find that the maximal value of $\eta_1(0,y_*,\zeta_*)$ occurs at $\zeta_* = l/2$ and the continuous functions $g_s^{(1)}$ in (1.21) are computed explicitly:

$$(2.37)\qquad g_s^{(1)} = \frac{m^{1/2}y^{1/2}}{2}\left(l + \frac{l-1}{(1+m^2y^2)^{1/2}}\right) + \frac{(1-2l)}{4m^{1/2}y^{1/2}}\log\left[\hat{f}_0(my)\right]. \qquad \square$$

Figures 1.1(a)–(d) are constructed with the help of explicit formulas (1.19)–(1.21), (2.30)–(2.31), (2.34)–(2.35), and (2.37). The dotted curves show the limit $\epsilon = 0$, the solid curves show the results of the first-order averaging theory for $\epsilon > 0$, with $l = 0.1$ fixed. The first-order averaging theory corresponds well to numerical analysis of the full equations (1.12)–(1.13); see [4, 7, 8].

2.4. Stability of critical points of the first-order averaged Hamiltonian. Linear stability of the critical points $(0, y_*)$ in the first-order averaged system (2.23) is defined by concavity of the quadratic form:

$$(2.38)\qquad H_1(x,y) - H_1(0,y_*) = \frac{1}{2}\frac{\partial^2 H_1}{\partial x^2}\bigg|_{(0,y_*)} x^2 + \frac{1}{2}\frac{\partial^2 H_1}{\partial y^2}\bigg|_{(0,y_*)}(y-y_*)^2,$$

since the derivative of $H_1(x,y)$ in x is zero for any $(0,y)$. The critical point $(0,y_*)$ is linearly stable if it corresponds to an extremum of the quadratic form (2.38); the stable critical points are centers on the phase plane (x,y). The critical point is linearly unstable if it corresponds to a saddle point of the quadratic form (2.38). The unstable critical points appear as saddle points on the phase plane (x,y). It is easy to analyze the linear stability of the critical point $(0,y_*)$ with the help of the function $e = f_s(s)$ shown on Figure 1.1(b).

LEMMA 2.4. *The critical point* $(0,y_*)$ *of the first-order averaged Hamiltonian* $H_1(x,y)$ *is stable for* $d_0 \geq 0$. *For* $d_0 < 0$, *define* $s_{\rm thr}$ *and* $s_{\rm stab}$ *as the turning and minimal points of the curve* $e = f_s(s)$, *i.e.,* $f_s'(s_{\rm thr}) = \infty$ *and* $f_s'(s_{\rm stab}) = 0$. *The critical point* $(0,y_*)$ *is stable for* $d_0 < 0$ *in the following cases:* (i) *for the upper branch of the curve* $e = f_s(s)$, *when* $s \geq s_{\rm thr}$ *and* (ii) *for the lower branch of the curve* $e = f_s(s)$, *when* $s_{\rm thr} < s < s_{\rm stab}$.

Proof. It follows from (2.16) that

$$\left.\frac{\partial^2 h_0}{\partial x^2}\right|_{x=0} = \frac{e y^{5/2}}{(1+m^2y^2)^{3/2}} > 0$$

for any $y > 0$. Therefore, there exists ϵ_0 such that the curvature of $H_1(x,y)$ is positive in x for $0 \leq \epsilon \leq \epsilon_0$. Then, the stability criterion is $H_1''(0,y_*) > 0$, where

$$H_1''(0,y) = -em^{3/2} g_d^{(0)\prime}(my) - \epsilon e^2 m(1-2l) g_d^{(1)\prime}(my) = f_s'(s) W(s),$$

where

$$W(s) = \left[g_d^{(0)}(my) + 2\epsilon e(1-2l) g_d^{(1)}(my) + \mathrm{O}(\epsilon^2)\right]\left[1 + \epsilon \frac{e}{m^{1/2}} g_s^{(1)\prime}(my,l) + \mathrm{O}(\epsilon^2)\right].$$

For $d_0 > 0$, it follows from Figure 1.1(b) and (1.19) that $f_s'(s) > 0$ and $g_d^{(0)}(my) > 0$ for any $y > 0$. As a result, there exists ϵ_0 such that the curvature of $H_1(0,y)$ is positive in y for $0 \leq \epsilon \leq \epsilon_0$. Therefore, the critical point $(0,y_*)$ is stable for $d_0 \geq 0$ and $0 \leq \epsilon \leq \epsilon_0$.

For $d_0 < 0$, the upper and lower branches of the function $e = f_s(s)$ are separated by the point $s = s_{\rm thr}$, where $f_s'(s_{\rm thr}) = \infty$ (see Figure 1.1(b)). Since $H_1''(0,y)$ may not be singular in the domain $y > 0$, the function $W(s)$ changes sign at $s = s_{\rm thr}$. It follows from (1.19) that $g_d^{(0)}(my) < 0$ for $d_0 < 0$. Therefore, it is clear that $W(s) > 0$ for the upper branch of $e = f_s(s)$ and $W(s) < 0$ for the lower branch of $e = f_s(s)$ on Figure 1.1(b). On the other hand, $f_s'(s) > 0$ for the upper branch of $e = f_s(s)$ and $f_s'(s) < 0$ for the lower branch of $e = f_s(s)$ between $s_{\rm thr} < s < s_{\rm stab}$, where $f_s'(s_{\rm stab}) = 0$; see Figure 1.1(b). As a result, the curvature of $H_1(0,y)$ in y is positive for the two cases (i) and (ii). For the lower branch of $e = f_s(s)$ at $s > s_{\rm stab}$, the curvature of $H_1(0,y)$ in y is negative, since $f_s'(s) > 0$ and $W(s) < 0$. As a result, the critical point $(0,y_*)$ is linearly unstable for the lower branch of $e = f_s(s)$ at $s > s_{\rm stab}$. ∎

Lemma 2.4 corresponds to Proposition 1.8 for Gaussian pulses in the first-order averaging theory. At the plane (μ, e), the point $s = s_{\rm stab}$ is the point of minimal e, i.e., it is a branching point of the function $e = f_\mu(\mu)$. As a result, for $d_0 < 0$, the upper branch of the function $e = f_\mu(\mu)$ is linearly unstable, while the lower branch of the function $e = f_\mu(\mu)$ is linearly stable [13].

We compute the phase plane $H_1(x,y) = \text{const}$ of the first-order averaged Hamiltonian from (2.16) and (2.28). The phase plane is shown on Figure 2.1(a)–(b) in standardized variables $X = x/m$ and $Y = my$ for $D_0 = 0.015$ and $D_0 = -0.015$,

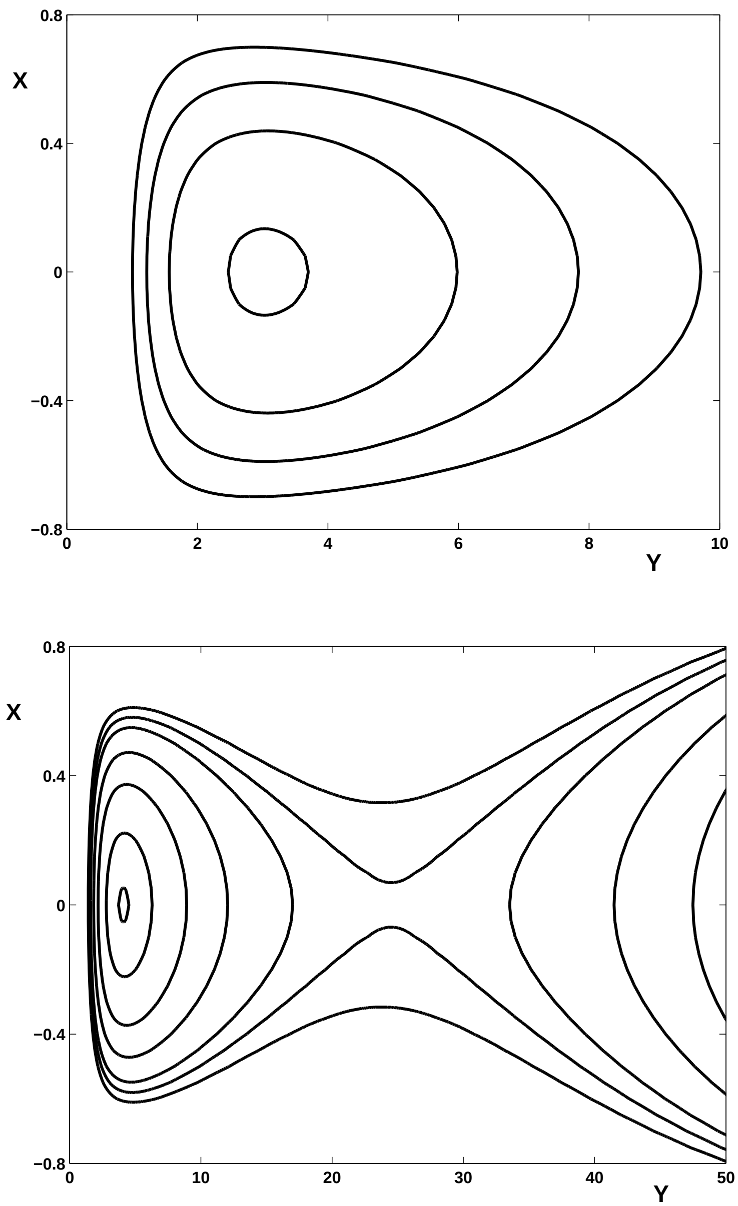

FIG. 2.1. *The contour levels of the first-order averaged Hamiltonian* $H_1(X, Y)$ *for* $D_0 = 0.015$ (a) *and* $D_0 = -0.015$ (b). *The other parameters are* $E = 2$ *and* $l = 0.1$.

with $l = 0.1$. In the initial-value problem, the energy parameter is constant, taken as $E = 2$.

For $D_0 \geq 0$ there is only one critical point, which is a center (see Figure 2.1(a)). The trajectories of the dynamical system (2.23) are all closed around the stable center point. This dynamics corresponds to small oscillations of the DM soliton, perturbed by an initial condition.

For $D_0 < 0$, two critical points coexist for the same value of E. The critical point with a larger value of Y_* is unstable (saddle point), while that with smaller value of Y_* is stable (center) (see Figure 2.1(b)). The critical point with a larger value of Y_* corresponds to a shorter DM soliton. If the shorter soliton is shortened by an initial perturbation, i.e., $Y(0) > Y_*$, it is destroyed, since the trajectory (X, Y) is unbounded on the phase plane of the first-order averaged system. We speculate that the shorter soliton transforms into chirped quasi-linear waves but this process is beyond the variational Gaussian approximation. Since solutions of (1.12)–(1.13) are bounded in the domain $\mathcal{D}_+$ for any finite z, the transformation happens over infinite propagation distances z.

In the other case, when the shorter DM soliton is broadened by the initial perturbation, i.e., $Y(0) < Y_*$, the trajectory (X, Y) is trapped inside the separatrix loop of the center point. In this case, the pulse undertakes large-amplitude oscillations around the stable longer DM soliton, similarly to the case $D_0 \geq 0$. Instability of short DM solitons along the lower branch of the (E, S) curve is confirmed in numerical computations [7].

3. Reduction to an averaged integral NLS equation. Here we analyze and extend the integral NLS equation (1.15), derived by means of averaging of the periodic NLS equation (1.5). We develop a formal method of canonical transformations for PDEs in section 3.1. In the leading and first orders in powers of ϵ, we prove convergence of the periodic NLS equation (1.5) to an averaged integral NLS equation in section 3.2. Existence of ground states of the first-order averaged Hamiltonian is proved for the case $d_0 > 0$ in section 3.3.

The periodic NLS equation (1.5) has the standard Hamiltonian structure:

$$i\frac{\partial u}{\partial z} = \frac{\delta H}{\delta \bar{u}}, \qquad -i\frac{\partial \bar{u}}{\partial z} = \frac{\delta H}{\delta u}, \tag{3.1}$$

where the Hamiltonian $H = H(u, \bar{u}, z/\epsilon)$ is

$$H = \frac{1}{2}\int_{-\infty}^{\infty} \left[\frac{m}{\epsilon} d\left(\frac{z}{\epsilon}\right) \left|\frac{\partial u}{\partial t}\right|^2 + d_0 \left|\frac{\partial u}{\partial t}\right|^2 - |u|^4 \right] dt. \tag{3.2}$$

LEMMA 3.1. *Define a fundamental solution of the linear periodic equation*

$$i\frac{\partial u}{\partial z} + \frac{m}{2\epsilon} d\left(\frac{z}{\epsilon}\right) \frac{\partial^2 u}{\partial t^2} = 0 \tag{3.3}$$

in the operator form:

$$u(z, t) = T\left(\frac{z}{\epsilon}\right) u(0, t). \tag{3.4}$$

The operator $T(\zeta)$ for $\zeta = z/\epsilon$ is a unitary operator with unit period:

$$T^{-1}(\zeta) = \overline{T}(\zeta), \qquad T(\zeta + 1) = T(\zeta), \tag{3.5}$$

where $\bar{T}$ is complex conjugate.

Proof. In the Fourier space of t, the operator $T(\zeta)$ is a multiplication operator:

$$\hat{u}(z, \omega) = T_\omega(\zeta)\hat{u}(0, \omega), \qquad T_\omega(\zeta) = e^{-\frac{im}{2} d_{-1}(\zeta)\omega^2}, \tag{3.6}$$

where $d_{-1}(\zeta)$ is given by (2.2)–(2.3) and $\hat{u}(\zeta,\omega)$ is the Fourier transform of $u(\zeta,t)$ in t. The two properties (3.5) follow from the Fourier form (3.6). $\square$

Using a linear canonical transformation

$$u(z,t) = T(\zeta)v(z,t), \qquad \bar{u}(z,t) = T^{-1}(\zeta)\bar{v}(z,t), \qquad \zeta = \frac{z}{\epsilon}, \tag{3.7}$$

we eliminate the fast periodic term from (1.5) and rewrite the Hamiltonian system (3.1) in new canonical variables $(v,\bar{v})$,

$$i\frac{\partial v}{\partial z} = \frac{\delta H}{\delta \bar{v}}, \qquad -i\frac{\partial \bar{v}}{\partial z} = \frac{\delta H}{\delta v}, \tag{3.8}$$

with the new Hamiltonian $H = H(v,\bar{v},\zeta)$:

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\left[d_0\left|\frac{\partial v}{\partial t}\right|^2 - |T(\zeta)v|^4\right]dt. \tag{3.9}$$

The periodic NLS equation in new variables $(v,\bar{v})$ can be written in the operator form:

$$i\frac{\partial v}{\partial z} + \frac{1}{2}d_0\frac{\partial^2 v}{\partial t^2} + T^{-1}(\zeta)\left(|T(\zeta)v|^2\, T(\zeta)v\right) = 0. \tag{3.10}$$

In the Fourier space of t, the operator equation (3.10) takes the form of a periodic integral NLS equation:

$$i\frac{\partial \hat{v}}{\partial z}(\omega) - \frac{1}{2}d_0\omega^2\hat{v}(\omega) + \iint_{-\infty}^{\infty} K_\omega(\zeta)\hat{v}(\omega_1)\hat{v}(\omega_2)\bar{\hat{v}}(\omega_1+\omega_2-\omega)d\omega_1 d\omega_2 = 0, \tag{3.11}$$

where $K_\omega(\zeta)$ is defined by

$$K_\omega(\zeta) = e^{-\frac{im}{2}d_{-1}(\zeta)(\omega_1^2+\omega_2^2-(\omega_1+\omega_2-\omega)^2-\omega^2)} = e^{imd_{-1}(\zeta)(\omega-\omega_1)(\omega-\omega_2)}. \tag{3.12}$$

The asymptotic reduction of (3.11) to an integral NLS equation is based on averaging of the Hamiltonian (3.9) in ζ [18]. A direct averaging method produces the following leading-order averaged Hamiltonian $H_0(V,\bar{V})$:

(3.13)
$$H_0(V,\bar{V}) = h_0(V,\bar{V}) = \int_0^1 H(V,\bar{V},\zeta)d\zeta = \frac{1}{2}\int_{-\infty}^{\infty}\left[d_0\left|\frac{\partial V}{\partial t}\right|^2 - \int_0^1 |T(\zeta)V|^4 d\zeta\right]dt.$$

The leading-order averaged Hamiltonian $H_0(V,\bar{V})$ generates the averaged integral NLS equation in the operator form:

$$i\frac{\partial V}{\partial z} + \frac{1}{2}d_0\frac{\partial^2 V}{\partial t^2} + \int_0^1 T^{-1}(\zeta)\left(|T(\zeta)v|^2\, T(\zeta)v\right)d\zeta = 0. \tag{3.14}$$

In the Fourier space of t, the integral NLS equation (3.14) takes an explicit form:

(3.15)
$$i\frac{\partial \hat{V}}{\partial z}(\omega) - \frac{1}{2}d_0\omega^2\hat{V}(\omega) + \iint_{-\infty}^{\infty}\langle K_\omega\rangle\hat{V}(\omega_1)\hat{V}(\omega_2)\bar{\hat{V}}(\omega_1+\omega_2-\omega)d\omega_1 d\omega_2 = 0,$$

where $\langle K_\omega \rangle$ is the average of $K_\omega(\zeta)$ over $\zeta \in [0,1]$. When the antiderivative function $d_{-1}(\zeta)$ is given by (2.3), the kernel $\langle K_\omega \rangle$ is computed explicitly as

$$\langle K_\omega \rangle = \frac{\sin m(\omega - \omega_1)(\omega - \omega_2)}{m(\omega - \omega_1)(\omega - \omega_2)}. \tag{3.16}$$

The integral equation (3.15) with the kernel (3.16) becomes (1.15) for stationary pulse solutions: $\hat{V}(z,\omega) = \hat{W}(\omega)e^{i\mu z}$.

The asymptotic reduction of the periodic NLS equation (1.5) to the averaged integral NLS equation (3.15) was first derived in [14, 15]. Higher-order corrections to the averaged integral NLS equation were considered in [26, 27] with the help of formal Lie transformations. We develop a method of formal canonical transformations for the Hamiltonian $H(v,\bar{v},\zeta)$ and, in addition, we prove convergence of the averaging procedure at the leading and first orders in powers of ϵ.

3.1. Averaging of the periodic integral NLS equation (3.11)–(3.12). The periodic integral NLS equation (3.11)–(3.12) can be averaged with the help of the normal form transformations, formally generalized for infinite-dimensional Hamiltonian systems. In this generalization, the generating functional $F(v,\bar{V},\zeta)$ replaces the generating function $F(\xi,y,\zeta)$ (see (2.8)):

$$F(v,\bar{V},\zeta) = \int_{-\infty}^{\infty} dt \left[v\bar{V} + \sum_{n=1}^{N+1} \epsilon^n F_n(v,\bar{V},\zeta) + \mathrm{O}(\epsilon^{N+2}) \right], \tag{3.17}$$

where the correction terms $F_n(V,\bar{V},\zeta)$ for $1 \le n \le (N+1)$ are periodic mean-zero functions of ζ:

$$F_n(V,\bar{V},\zeta+1) = F_n(V,\bar{V},\zeta), \qquad \int_0^1 F_n(V,\bar{V},\zeta)d\zeta = 0. \tag{3.18}$$

The generating functional $F(v,\bar{V},\zeta)$ defines the near-identical canonical transformation

$$\bar{v} = \frac{\delta F}{\delta v}, \qquad V = \frac{\delta F}{\delta \bar{V}}, \tag{3.19}$$

and takes the Hamiltonian $H(v,\bar{v},\zeta)$ to the form

$$\begin{aligned} H_{\mathrm{new}}(V,\bar{V},\zeta) &= H(v(V,\bar{V},\zeta),\bar{v}(V,\bar{V},\zeta),\zeta) + \frac{i}{\epsilon}\frac{\partial F}{\partial \zeta}(v(V,\bar{V},\zeta),\bar{V},\zeta) \\ &= H_N(V,\bar{V}) + \mathrm{O}(\epsilon^{N+1}), \end{aligned} \tag{3.20}$$

where $H_N(V,\bar{V})$ is the Nth-order averaged Hamiltonian

$$H_N(V,\bar{V}) = \sum_{n=0}^{N} \epsilon^n h_n(V,\bar{V}). \tag{3.21}$$

When the remainder term of order of $\mathrm{O}(\epsilon^{N+1})$ is neglected, the new canonical variables $(V,\bar{V})$ solve the averaged Hamiltonian dynamical system:

$$i\frac{\partial V}{\partial z} = \frac{\delta H_N}{\delta \bar{V}}, \qquad -i\frac{\partial \bar{V}}{\partial z} = \frac{\delta H_N}{\delta V}, \tag{3.22}$$

The difference between solutions of the full system (3.8) and the averaged system (3.22) is controlled with the accuracy of $O(\epsilon^{N+1})$ on the interval $0 \leq z \leq z_0$.

The Lagrangian functional for the system (3.8) is transformed as follows:

(3.23)
$$L = i\int_{-\infty}^{\infty} \bar{v}\frac{\partial v}{\partial z}dt - H(v,\bar{v},\zeta) = -i\int_{-\infty}^{\infty} V\frac{\partial \bar{V}}{\partial z}dt - H_{\rm new}(V,\bar{V},\zeta) + i\frac{dF}{dz}(v,\bar{V},\zeta),$$

where

$$\frac{dF}{dz} = \frac{1}{\epsilon}\frac{\partial F}{\partial \zeta} + \int_{-\infty}^{\infty} dt\left(\frac{\partial v}{\partial z}\frac{\delta F}{\delta v} + \frac{\partial \bar{V}}{\partial z}\frac{\delta F}{\delta \bar{V}}\right).$$

If $F(V,\bar{V},\zeta)$ generates $\bar{v}$ and V according to (3.19), then the Hamiltonian $H(v,\bar{v},\zeta)$ transforms according to (3.20). The method of normal form transformations in (3.17)–(3.23) is a formal algorithmic procedure. Still we are able to prove convergence of the first-order averaged theory in a suitable function space; see section 3.2.

The difference between solutions of the averaged integral NLS equation (3.15) and the periodic integral NLS equation (3.11) is defined with the help of the first-order correction $F_1(V,\bar{V},\zeta)$ in (3.17). The first-order correction can be found from (3.9), (3.13), and (3.20):

$$F_1(V,\bar{V},\zeta) = \frac{-i}{2}\left\{|T(\zeta)V|^4 - \int_0^1 |T(\zeta)V|^4\,d\zeta\right\}_{-1}, \tag{3.24}$$

where $\{*\}_{-1}$ stands for the mean-zero antiderivative in ζ defined by (2.2). Expanding the near-canonical transformations (3.17) and (3.19) in powers of ϵ, we define the perturbation series for $v(V,\bar{V},\zeta)$:

$$v = V + \epsilon i\Phi(V,\bar{V},\zeta) + O(\epsilon^2), \qquad \bar{v} = \bar{V} - \epsilon i\overline{\Phi(V,\bar{V},\zeta)} + O(\epsilon^2), \tag{3.25}$$

where $\Phi(V,\bar{V},\zeta)$ is formally computed as

$$\Phi = \left\{T^{-1}(\zeta)\left(|T(\zeta)V|^2\,T(\zeta)V\right) - \int_0^1 T^{-1}(\zeta)\left(|T(\zeta)V|^2\,T(\zeta)V\right)d\zeta\right\}_{-1}. \tag{3.26}$$

In the Fourier form, $\Phi(V,\bar{V},\zeta)$ is expressed explicitly as $\hat{\Phi}_\omega(V,\bar{V},\zeta)$:

$$\hat{\Phi}_\omega = \iint_{-\infty}^{\infty} \{K_\omega(\zeta) - \langle K_\omega\rangle\}_{-1}\,\hat{V}(\omega_1)\hat{V}(\omega_2)\hat{\bar{V}}(\omega_1+\omega_2-\omega)d\omega_1 d\omega_2. \tag{3.27}$$

With the use of correction $\Phi(V,\bar{V},\zeta)$, the first-order correction term $h_1(V,\bar{V})$ of the new averaged Hamiltonian is found in the form

$$h_1(V,\bar{V}) = -i\int_{-\infty}^{\infty} dt\int_0^1 \left(|T(\zeta)V|^2\left(T^{-1}(\zeta)\bar{V}\right)T(\zeta)\Phi(V,\bar{V},\zeta)\; - \text{c.c.}\right)d\zeta, \tag{3.28}$$

where c.c. stands for complex conjugation and we have used the periodicity of $\Phi(V,\bar{V},\zeta)$ in ζ. The first-order correction $h_1(V,\bar{V})$ vanishes in the case of symmetric maps, when $l = 1/2$.

LEMMA 3.2. *When the dispersion map is symmetric, i.e., $l = 1/2$, then $h_1(V,\bar{V}) = 0$ and $\Phi(V,\bar{V},0) = \Phi(V,\bar{V},l) = 0$.*

Proof. If $l = 1/2$, then the mean-zero antiderivative function $d_{-1}(\zeta)$ is even: $d_{-1}(-\zeta) = d_{-1}(\zeta)$. The operator $\hat{T}_\omega(\zeta)$ and the kernel $K_\omega(\zeta)$ in (3.6) and (3.12) are even functions of ζ and can be expanded into the Fourier cosine-series, e.g.,

$$K_\omega(\zeta) = \langle K_\omega \rangle + \sum_{n=1}^{\infty} k_{\omega n} \cos(2\pi n \zeta). \tag{3.29}$$

It follows from (3.27) that the first-order correction $\hat{\Phi}_\omega(V, \bar{V}, \zeta)$ is expanded into the Fourier sine-series:

$$\hat{\Phi}_\omega(V, \bar{V}, \zeta) = \sum_{n=1}^{\infty} \phi_{\omega n}(V, \bar{V}) \sin(2\pi n \zeta). \tag{3.30}$$

The integrand of (3.28) contains only the product of the Fourier cosine- and sine-series, which has zero mean. ◻

The first-order averaged Hamiltonian is finally defined as $H_1(V, \bar{V}) = h_0(V, \bar{V}) + \epsilon h_1(V, \bar{V})$, where $h_0(V, \bar{V})$ and $h_1(V, \bar{V})$ are given by (3.13) and (3.28).

3.2. Averaging theorems for the first-order averaged integral NLS equation. Here we justify the first-order averaging theory for the periodic integral NLS equation (3.10). In order to shorten notation, we introduce the operator $Q(v, \zeta)$ for the cubic nonlinear term in (3.10):

$$Q(v, \zeta) = Q(v, v, v, \zeta), \qquad Q(u, v, w, \zeta) = T^{-1}(\zeta) \left(T(\zeta) u \, T(\zeta) v \, \overline{T(\zeta) w} \right). \tag{3.31}$$

In the operator form,

$$\Phi(V, \bar{V}, \zeta) = \{ Q(V, \zeta) - \langle Q \rangle (V) \}_{-1},$$

and the first-order averaged integral NLS equation can be written in the form

$$i \frac{\partial V}{\partial z} + \frac{1}{2} d_0 \frac{\partial^2 V}{\partial t^2} + \langle Q \rangle (V) + \epsilon \langle Q_1 \rangle (V) = 0, \tag{3.32}$$

where

$$\langle Q \rangle (V) = \int_0^1 Q(V, \zeta) d\zeta, \tag{3.33}$$

and

$$\langle Q_1 \rangle (V) = \frac{\delta h_1}{\delta \overline{V}} = i \int_0^1 \left[Q(V, V, \Phi, \zeta) - 2Q(V, \Phi, V, \zeta) \right] d\zeta. \tag{3.34}$$

First, we list some properties of Q and Φ and formulate a local existence result for the first-order averaged integral NLS equation (3.32).

PROPOSITION 3.3. *Let u, v, w be in $H^s(\mathbb{R})$ $(s \geq 0)$; then the following inequalities hold:*

$$||Q(u, v, w, \zeta)||_{H^s} \leq C_s ||u||_{H^s} ||v||_{H^s} ||w||_{H^s}, \tag{3.35}$$

$$||Q(u, \zeta)||_{H^s} \leq C_s ||u||^3_{H^s}, \tag{3.36}$$

$$||\langle Q(u) \rangle||_{H^s} \leq C_s ||u||^3_{H^s}, \tag{3.37}$$

$$||\Phi(u, \bar{u}, \zeta)|| \leq C_s ||u||^3_{H^s}, \tag{3.38}$$

$$||Q(u, u, \Phi(u, \bar{u}, \zeta), \zeta)|| \leq C_s ||u||^5_{H^s}, \tag{3.39}$$

$$||\langle Q(u, u, \Phi(u, \bar{u}, *), *) \rangle|| \leq C_s ||u||^5_{H^s}. \tag{3.40}$$

Proof. The first inequality is proven using the well-known property of $H^s(\mathbb{R})$ with, e.g., $s \geq 1$,

$$||uv||_{H^s} \leq C_s ||u||_{H^s} ||v||_{H^s},$$

the isometric properties of $T(\zeta)$,

$$||T(\zeta)u||_{H^s} = ||u||_{H^s},$$

and

$$T(\zeta)Q(u,v,w,\zeta) = T(\zeta)uT(\zeta)v\overline{T(\zeta)w} \Rightarrow ||T(\zeta)Q(u,v,w,\zeta)||_{H^s}$$
$$\leq C||T(\zeta)u||_{H^s}||T(\zeta)v||_{H^s}||T(\zeta)w||_{H^s} \Rightarrow ||Q(u,v,w,\zeta)||_{H^s} \leq C||u||_{H^s}||v||_{H^s}||w||_{H^s}.$$

The remaining inequalities (3.36)–(3.40) are obtained by direct application of the first inequality (3.35). ∎

PROPOSITION 3.4. *Let $V(0) \in H^s(\mathbb{R})$ with $s \geq 1$. Then there exists $z_0 > 0$ such that the first-order averaged equation* (3.32) *has a unique solution $V(z) \in L^\infty([0, z_0], H^s(\mathbb{R}))$.*

Proof. The local existence for (3.32) with $\epsilon = 0$ has been proven in [28] by using the standard application of contraction mapping. In the general case, when $\epsilon \neq 0$, the proof of local existence is similar. First, we rewrite (3.32) in the integral form:

$$V(z) = T_0(z)V(0) + \int_0^Z T_0(z - z') \left(\langle Q\rangle(V(z')) + \epsilon\langle Q_1\rangle(V(z'))\right) dz',$$

where $T_0(z)$ is the operator associated with the fundamental solution of the linear Schrödinger equation:

$$i\frac{\partial V}{\partial z} + \frac{1}{2}d_0\frac{\partial^2 V}{\partial t^2} = 0.$$

Estimating the difference between two solutions, we obtain

$$||V_1(z) - V_2(z)||_{H^s} \leq z_0 C_s(||V_1(0)||_{H^s}, ||V_2(0)||_{H^s}, \epsilon)||V_1(0) - V_2(0)||_{H^s},$$

which is a contraction if z_0 is sufficiently small (uniformly in ϵ). Using the standard energy estimates, we also obtain

$$\frac{\partial}{\partial z}||V(z)||^2_{H^s} \leq C_s(||V(z)||^2_{H^s}, \epsilon)||V(z)||^2_{H^s},$$

where C_s is a smooth function in both variables, thus implying uniqueness. ∎

PROPOSITION 3.5. *Let $V(0) \in H^1(\mathbb{R})$ and $d_0 \neq 0$. Then there exists a global solution $V \in L^\infty([0,\infty), H^1(\mathbb{R}))$ with initial data $V(0)$.*

Proof. For the proof we use first-order averaged Hamiltonian $H_1(V, \bar{V})$, conserved in z. It is shown in section 3.3 that the Hamiltonian $H_1(V, \bar{V})$ is bounded uniformly in $\epsilon \in [0, \epsilon_0]$, provided $||V||_{L^2}$ is fixed. Therefore, since the Hamiltonian is conserved in z, the gradient term must be bounded:

$$\int_{-\infty}^{\infty} |\partial_t V(z)|^2 dt \leq C(||V(0)||_{L^2}, ||\partial_t V(0)||_{L^2}, d_0),$$

which implies that $||V(z)||_{H^1}$ is uniformly bounded, thus proving global existence of solutions. ∎

Remark 3.1. If $d_0 = 0$, then a global solution $V(z)$ still exists in $H^1(\mathbb{R})$, although it is not uniformly bounded.

Before proving convergence of the first-order averaging theory, we reproduce the leading-order averaging theory from [28].

THEOREM 3.6 (see [28]). *Let $V(z) \in L^\infty([0,z_0], H^s(\mathbb{R}))$, where $s \geq 2$, be a solution of the averaged NLS equation* (3.32) *with $\epsilon = 0$ and $v(z)$ be a solution of the full equation* (3.10) *such that $||v(0) - V(0)||_{H^{s-2}} \leq C\epsilon$. Then, for sufficiently small positive $\epsilon < \epsilon_0$ we have $v(z) \in L^\infty([0,z_0], H^{s-2}(\mathbb{R}))$ and the solutions stay close at the distances $0 \leq z \leq z_0$:*

$$\sup_{z\in[0,z_0]} ||v(z) - V(z)||_{H^{s-2}} \leq C\epsilon. \tag{3.41}$$

We prove the analogous theorem for the first-order averaged integral NLS equation (3.32).

THEOREM 3.7. *Let $V(z) \in L^\infty([0,z_0], H^s(\mathbb{R}))$, where $s \geq 4$, be a solution of the first-order averaged integral NLS equation* (3.32) *and $v(z)$ be a solution of the full equation* (3.10) *such that $||v(0) - V(0) - i\epsilon\Phi(V(0),\bar{V}(0),0)||_{H^{s-4}} \leq C\epsilon$. Then, for sufficiently small positive $\epsilon < \epsilon_0$ we have $v(z) \in L^\infty([0,z_0], H^{s-4}(\mathbb{R}))$ and the solutions are ϵ-close on $0 \leq z \leq z_0$:*

$$\sup_{z\in[0,z_0]} ||v(z) - V(z) - i\epsilon\Phi(V,\bar{V},\zeta)||_{H^{s-4}} \leq C\epsilon^2. \tag{3.42}$$

Proof. We start with the averaged integral NLS equation (3.32) and use near-identical transformations to transform it to the periodic integral NLS equation (3.10). In the last step we compare the solutions of the transformed and the reduced equations by using Gronwall's inequality. This approach has a technical advantage over the "direct" approach, which starts from the original equation (3.10) and transforms it to the averaged equation (3.32). Indeed, for the periodic equation (3.10), there is no a priori ϵ-independent estimate on the existence interval.

Let us make a transformation $V = v_1 - w_1$ in (3.32), where v_1 is a new variable and w_1 is a small correction. We formally obtain

$$\begin{aligned} & i\frac{\partial v_1}{\partial z} + \frac{1}{2}d_0\frac{\partial^2 v_1}{\partial t^2} + Q(v_1,\zeta) \\ &\quad = i\frac{\partial w_1}{\partial z} + \frac{1}{2}d_0\frac{\partial^2 w_1}{\partial t^2} + Q(v_1,\zeta) - \langle Q\rangle(V) + \epsilon\langle Q_1\rangle(V). \end{aligned} \tag{3.43}$$

Choosing $w_1 = i\epsilon\Phi(V,\bar{V},\zeta)$, we obtain

$$i\frac{\partial v_1}{\partial z} + \frac{1}{2}d_0\frac{\partial^2 v_1}{\partial t^2} + Q(v_1,\zeta) = R_1(V,\zeta), \tag{3.44}$$

where

$$\begin{aligned} R_1(V,\zeta) = & -\epsilon\frac{\partial}{\partial z}\Phi(V,\bar{V},\zeta) + \epsilon i\frac{1}{2}d_0\frac{\partial^2}{\partial t^2}\Phi(V,\bar{V},\zeta) \\ & + Q(v_1,\zeta) - Q(V,\zeta) + \epsilon\langle Q_1\rangle(V). \end{aligned} \tag{3.45}$$

We expand the right-hand side of (3.46) as

$$\begin{aligned} Q(v_1,\zeta) - Q(V,\zeta) &= Q(V + i\epsilon\Phi,\zeta) - Q(V,\zeta) \\ &= -i\epsilon Q(V,V,\Phi,\zeta) + 2i\epsilon Q(V,\Phi,V,\zeta) - \epsilon^2 Q(\Phi,\Phi,V,\zeta) \\ &\quad + 2\epsilon^2 Q(\Phi,v_1,\Phi,\zeta) + i\epsilon^3 Q(\Phi,\zeta). \end{aligned}$$

If $\langle Q_1 \rangle(V)$ is defined by (3.34), then (3.46) transforms to the periodic NLS equation with a mean-zero error mismatch of order $O(\epsilon)$:

$$\begin{aligned} R_1 = & -\epsilon \frac{\partial}{\partial z}\Phi(V,\bar{V},\zeta) + \epsilon i \frac{1}{2} d_0 \frac{\partial^2}{\partial t^2}\Phi(V,\bar{V},\zeta) \\ & - i\epsilon\{Q(V,V,\Phi,\zeta)\} + 2i\epsilon\{Q(V,\Phi,V,\zeta)\} \\ & - \epsilon^2 Q(\Phi,\Phi,V,\zeta) + 2\epsilon^2 Q(\Phi,v_1,\Phi,\zeta) + i\epsilon^3 Q(\Phi,\zeta), \end{aligned}$$

where $\{Q\}$ stands for the mean-zero periodic part of Q in ζ. By using properties of Q and Φ from Proposition 3.3 and taking into account the loss of two derivatives, we find the estimate $||R_1(V,\zeta)||_{H^{s-2}} \leq C\epsilon$.

Now, we carry out another transformation $v_1 = v_2 - w_2$, where $w_2 = -i\epsilon\,\{R_1(V,\zeta)\}_{-1}$. The mean-zero antiderivative of $R_1(V,\zeta)$ in ζ satisfies the estimate $||w_2||_{H^{s-2}} \leq C\epsilon^2$. After rearranging the terms we recover the equation

$$i\frac{\partial v_2}{\partial z} + \frac{1}{2} d_0 \frac{\partial^2 v_2}{\partial t^2} + Q(v_2,\zeta) = R_2(V,\zeta), \tag{3.46}$$

where $R_2(V,\zeta)$ has a long expression in powers of ϵ^2 and higher. With the help of Proposition 3.3, we can estimate all terms of R_2 as $||R_2(V,\zeta)||_{H^{s-4}} \leq C\epsilon^2$. Comparing solutions of (3.46) and (3.10), we obtain an equation for the difference $f := v_2 - v$:

$$i\frac{\partial f}{\partial z} + \frac{1}{2} d_0 \frac{\partial^2 f}{\partial t^2} + Q(v_2,\zeta) - Q(v,\zeta) = R_2(V,\zeta). \tag{3.47}$$

The difference in the left-hand side of (3.47) can be estimated as

$$\begin{aligned} ||Q(v_2,\zeta) - Q(v,\zeta)||_{H^{s-2}} &= ||Q(v_2,\zeta) - Q(f - v_2,\zeta)||_{H^{s-2}} \\ &\leq C_s(||f||_{H^{s-2}}, ||v_2||_{H^{s-2}})||f||_{H^{s-2}}. \end{aligned}$$

The growth of f can be estimated by using the standard energy estimates. We differentiate the equation for f $1,2,\ldots,n$ times, multiply each of them with $\partial_k \overline{f}$ $(k = 1,2,\ldots,n)$, subtract complex conjugates, and finally take the sum to obtain

$$\frac{\partial}{\partial z}||f||^2_{H^n} \leq C_s(||f||_{H^n}, ||v_2||_{H^n})||f||^2_{H^n} + C||R_2||_{H^n}||f||_{H^n}.$$

In the last inequality, we can take n: $0 \leq n \leq s-4$ (thus, we have to assume $s \geq 4$) and using Gronwall's inequality, we obtain

$$||f(z)||_{H^{s-4}} \leq C_1(e^{C_2 z}\epsilon^2 + ||f(0)||_{H^{s-4}}),$$

which proves (3.42). ∎

COROLLARY 3.8. *Suppose the dispersion map $d(\zeta)$ is symmetric with equal legs, i.e., $l = 1/2$. If the solutions $V(z)$ and $v(z)$ are close in the sense of $||v(0) - V(0)||_{H^{s-4}} \leq C\epsilon^2$, then for sufficiently small positive $\epsilon < \epsilon_0$ the solutions remain within the quadratic accuracy at the distances $0 \leq z \leq z_0$ at the points $z = k\epsilon$ and $z = (k - \frac{1}{2})\epsilon$, where $k \in \mathbb{Z}_+$:*

$$\sup_{z\in[0,z_0]} ||v(z = k\epsilon) - V(z = k\epsilon)||_{H^{s-4}} \leq C\epsilon^2. \tag{3.48}$$

The quadratic convergence is based on the fact that $h_1(V,\bar{V}) = 0$ for $l = 1/2$ and $\Phi(V,\bar{V},0) = \Phi(V,\bar{V},l) = 0$; see Lemma 3.2. As a result, we have an improved (quadratic) convergence between solutions of the periodic integral NLS equation (3.11) and the integral NLS equation (3.15). It is only the linear convergence between solutions of the full and averaged equations valid at any point z of the dispersion map in the general case $l \neq 1/2$.

3.3. Existence and stability of ground states of the first-order averaged Hamiltonian. The first-order averaged Hamiltonian functional $H_1(V,\bar{V})$ is a constant of motion in the averaged system; therefore its extrema are expected to be stable solutions. Unfortunately, Hamiltonians in such problems are not bounded from either above or below. The way out is to consider a constrained variational problem, since there exists another conserved quantity e defined by (1.11). We show that the obtained Hamiltonian possesses a constrained minimum for the case $d_0 > 0$. The constrained minimum implies stability of a stationary pulse in this case.

Let us consider the following minimization problem:

$$P_E = \inf\left\{H_1(V,\bar{V}), V \in H^1(\mathbb{R}), \int_{-\infty}^{+\infty} |V|^2 dt = E\right\}. \tag{3.49}$$

First, we show that the Hamiltonian is bounded from below, $P_E > -\infty$, which is a necessary condition for the presence of a smooth minimizer. Note that the Hamiltonian is unbounded from above for $d_0 > 0$ because of the gradient term in (3.13).

Proposition 3.9. *The Hamiltonian functional $H_1(V,\bar{V})$ is uniformly bounded from below if $d_0 \geq 0$ and E is fixed.*

Proof. Since the gradient term is positive, we need only to establish the boundedness of the other two terms. The leading-order term $h_0(V,\bar{V})$ can be bounded by applying Hölder and Strichartz estimates [28]:

$$\int_0^1 \int_{-\infty}^{+\infty} |T(\zeta)V|^4 dt d\zeta = \int_0^1 \int_{-\infty}^{+\infty} |T(\zeta)V||T(\zeta)V|^3 dt d\zeta$$

$$\leq \left(\int_0^1 \int_{-\infty}^{+\infty} |T(\zeta)V|^2 dt d\zeta\right)^{\frac{1}{2}} \left(\int_0^1 \int_{-\infty}^{+\infty} |T(\zeta)V|^6 dt d\zeta\right)^{\frac{1}{2}} \leq E^{\frac{1}{2}} C_S E^{\frac{3}{2}} = C_S E^2,$$

where we have used the isometry of $T(\zeta)$ in $L^2(\mathbb{R})$ as well as the Strichartz inequality:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |T(\zeta)V|^6 dt dz \leq C_s^2 E^3.$$

Now we estimate the first-order term $h_1(V,\bar{V})$ as

$$\begin{aligned}
&\left|\int_0^1 \int_{-\infty}^{+\infty} \overline{T(\zeta)V}^2\, T(\zeta)V\, T(\zeta)\Phi(V,\bar{V},\zeta)\ dt d\zeta\right| \\
&\leq \left(\int_0^1 \int_{-\infty}^{+\infty} |T(\zeta)V|^6 dt d\zeta\right)^{\frac{1}{2}} \left(\int_0^1 \int_{-\infty}^{+\infty} |T(\zeta)\Phi(V,\bar{V},\zeta)|^2 dt d\zeta\right)^{\frac{1}{2}} \\
&\leq C_S E^{\frac{3}{2}} \left(\int_0^1 \int_{-\infty}^{+\infty} |\Phi(V,\bar{V},\zeta)|^2 dt d\zeta\right)^{\frac{1}{2}}.
\end{aligned}$$

The integral of $|\Phi(V,\bar{V},\zeta)|^2$ in ζ can be estimated from the definition (3.26), rewritten as

$$\Phi(V,\bar{V},\zeta) = \int_0^{\zeta} \Psi(\zeta_1,t)d\zeta_1 - \int_0^1 \int_0^{\zeta_2} \Psi(\zeta_1,t)d\zeta_1 d\zeta_2 - \left(\zeta - \frac{1}{2}\right)\int_0^1 \Psi(\zeta_1,t)d\zeta_1,$$

where we used the notation

$$\Psi(\zeta,t) = T^{-1}(\zeta)\left(|T(\zeta)V|^2\, T(\zeta)V\right).$$

The product $|\Phi|^2$ contains 10 terms, which can be estimated in a straightforward way using Strichartz estimate. We give an example of how to carry out one of these estimates:

$$\left|\int_0^1 d\zeta \int_{-\infty}^{\infty} dt \left[\int_0^{\zeta} \Psi(\zeta_1,t)d\zeta_1 \int_0^1 \int_0^{\zeta_3} \overline{\Psi(\zeta_2,t)}d\zeta_2 d\zeta_3\right]\right|$$
$$\leq \left|\int_0^1 d\zeta \int_{-\infty}^{\infty} dt \left[\int_0^1 |\Psi(\zeta_1,t)|d\zeta_1 \int_0^1 \int_0^1 |\Psi(\zeta_2,t)|d\zeta_2 d\zeta_3\right]\right|$$
$$= \left|\int_{-\infty}^{\infty} dt \left[\int_0^1 |\Psi(\zeta_1,t)|d\zeta_1 \int_0^1 |\Psi(\zeta_2,t)|d\zeta_2\right]\right|$$
$$= \int_0^1 d\zeta_2 \int_0^1 d\zeta_1 \int_{-\infty}^{\infty} dt|\Psi(\zeta_1,t)||\Psi(\zeta_2,t)|$$
$$\leq \left(\int_0^1 d\zeta_1 \int_0^1 d\zeta_2 \int_{-\infty}^{\infty} dt|\Psi(\zeta_1,t)|^2\right)^{\frac{1}{2}} \left(\int_0^1 d\zeta_1 \int_0^1 d\zeta_2 \int_{-\infty}^{\infty} dt|\Psi(\zeta_2,t)|^2\right)^{\frac{1}{2}}$$
$$\leq \int_0^1 \int_{\infty}^{\infty} |\Psi(\zeta,t)|^2 dt d\zeta.$$

The last integral is estimated using the definition of $\Psi(\zeta,t)$ and the Strichartz estimate:

$$\int_0^1 \int_{-\infty}^{\infty} |\Psi(\zeta,t)|^2 dt d\zeta = \int_0^1 \int_{-\infty}^{\infty} \left|T^{-1}(\zeta)\left(T(\zeta)V^2\overline{T(\zeta)V}\right)\right|^2 dt d\zeta$$
$$= \int_0^1 \int_{-\infty}^{\infty} |T(\zeta)V|^6 dt d\zeta \leq C_S^2 E^3. \tag{3.50}$$

Therefore the term $h_1(V,\bar{V})$ in the Hamiltonian $H_1(V,\bar{V})$ is bounded by $C_S E^{3/2} C_S E^{3/2} = C_S^2 E^3$. ∎

The next step is to verify the subadditivity condition which is necessary for the construction of a converging minimizing sequence [31]. The subadditivity property holds in the case $\epsilon = 0$ (see [28]). Here we show that it also holds for sufficiently small ϵ.

Lemma 3.10. *For any $E > 0$ there exist $\epsilon_0 > 0$ (which may depend on E) such that for any $0 < \epsilon < \epsilon_0$ any minimizing sequence V_n possesses a subsequence V_{n_k} satisfying the subadditivity property*

$$P_{E_1+E2} < P_{E_1} + P_{E_2} \ \textit{provided} \ E = E_1 + E_2. \tag{3.51}$$

Proof. The proof is a simple application of a scaling argument, followed by some estimates using smallness of ϵ. Consider a one-parameter family $V^\lambda = \sqrt{\lambda}V$ with $\lambda \in (0,1)$; then

$$E^\lambda = \int_{-\infty}^{\infty} |V^\lambda|^2 dt = \lambda E.$$

Introducing the notation for the Hamiltonian,

$$H_1(V,\bar{V}) = H^{(2)}(V,\bar{V}) - H^{(4)}(V,\bar{V}) + \epsilon H^{(6)}(V,\bar{V}),$$

where $H^{(2,4,6)}(V,\bar{V})$ represent quadratic gradient term, positive quartic term, and the sixth order perturbation term, respectively. The Hamiltonian then scales as follows:

$$H_1^\lambda = \lambda H^{(2)} - \lambda^2 H^{(4)} + \lambda^3 \epsilon H^{(6)}$$

and then

$$H_1^\lambda - \lambda H_1 = (\lambda - \lambda^2) H^{(4)} + (\lambda - \lambda^3)\epsilon H^{(6)} = \lambda(1-\lambda)(H^{(4)} + (1+\lambda)\epsilon H^{(6)}).$$

Note that for $\epsilon = 0$, $H_1^\lambda > \lambda H_1$, which implies the subadditivity $P_{\lambda E} > \lambda P_E$. The latter results in (3.51) for the same E. For sufficiently small ϵ, the condition (3.51) is expected to hold since $H^{(6)}$ is uniformly bounded. Indeed, if we fix $E > 0$, then for $\epsilon = 0$ the infimum is negative, $P_E^0 < 0$, as shown in [28]. For positive ϵ, the infimum cannot change by more than $\epsilon C_S^2 E^3$; therefore $P_E^\epsilon \le P_E^0 + \epsilon C_S^2 E^3$ remains negative.

By definition, for any minimizing sequence we have $H(V_n) \to P_E^\epsilon$ and therefore for sufficiently large $n \ge N$ the quartic term $H^{(4)}$ has to be bounded from below:

$$H^{(4)} \ge |P_E^\epsilon| - \epsilon C_S^2 E^3 - \delta(N) \Rightarrow H^{(4)} \ge |P_E^0| - 2\epsilon C_S^2 E^3 - \delta(N).$$

Then we prove the estimate:

$$\begin{aligned} H^{(4)} + (1+\lambda)\epsilon H^{(6)} &\ge |P_E^0| - 2\epsilon C_S^2 E^3 - \delta(N) - 2\epsilon C_S^2 E^3 \\ &= |P_E^0| - 4\epsilon C_S^2 E^3 - \delta(N) \ge |P_E^0| - 5\epsilon C_S^2 E^3, \end{aligned}$$

where the last step in the inequalities was done by taking N sufficiently large. Therefore, by requiring that

$$5\epsilon C_S^2 E^3 < \frac{1}{2}|P_E^0|$$

we achieve the subadditivity condition for the minimizing sequence. ∎

We will also use lemma on localization from [28]. The lemma says that finite energy cannot propagate too far in the linear Schrödinger equation if the initial data are sufficiently smooth.

LEMMA 3.11 (see [28]). *Let $V \in H^1(\mathbb{R})$, $T(\zeta)$ be a free Schrödinger propagator and let*

$$\epsilon(\zeta) = \sup_{\xi \in \mathbb{R}} \int_{\xi-1}^{\xi+1} |T(\zeta)V|^2 dt. \tag{3.52}$$

Then the following estimate holds:

$$\epsilon(\zeta) \le 2\epsilon(0) + \sqrt{\epsilon^2(0) + 2C\epsilon(0)\zeta}. \tag{3.53}$$

Now, we are ready to establish the convergence of a minimizing sequence. The two results above make the convergence proof straightforward and very similar to the one with $\epsilon = 0$; see [28]. Therefore, we sketch only the proof of the main result, providing details only when they are different from the case $\epsilon = 0$.

PROPOSITION 3.12. *If $d_0 > 0$ and $0 < \epsilon < \frac{|P_E^0|}{10\, C_S^2 E^3}$, then there exists a minimizer $W \in H^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$ of the constrained minimization problem* (3.49).

Proof. First we observe that any minimizing sequence $V_n \in H^1(\mathbb{R})$ must possess a bounded derivative

$$\int_{-\infty}^{\infty} \left| \frac{\partial V_n}{\partial t} \right|^2 dt < C, \tag{3.54}$$

for otherwise $\{H_1(V_n, \bar{V}_n)\}_{n=0}^{\infty}$ would have an unbounded subsequence (since $H^{(2)}(V_n, \bar{V}_n)$ would dominate over $H^{(4)}(V_n, \bar{V}_n)$ and $H^{(6)}(V_n, \bar{V}_n)$, which are uniformly bounded). Then there exists a weakly converging subsequence in $L^2(\mathbb{R})$. In order to assure the strong convergence $V_n \to W$ with W satisfying the constraint (3.49), we have to show that the sequence is tight (the energy does not escape to infinity[1]). We assume that $n > N$ with N sufficiently large so that the subadditivity condition would hold. Now, we use the concentration-compactness principle [29], which says that there exists a subsequence V_{n_k}, denoted by V_k, for which one of the following statements is true:

1. (convergence) For some sequence $\{t_k\}_{k=0}^{\infty}$ the translated sequence converges to some limit $V_k(t - t_k) \to W$ (as $k \to \infty$) satisfying the constraint (3.49).
2. (vanishing) The following identity is true:

$$\sup_{y\in\mathbb{R}} \int_{y-1}^{y+1} |V_k|^2 dt \to 0 \quad \text{as} \quad k \to \infty.$$

3. (splitting) There exist $E_1, E_2 > 0\, (E = E_1 + E_2)$ such that for any $\epsilon > 0$ one can find two sequences v_k, w_k and $K > 0$ so that for any $k > K$ we have

$$\int_{-\infty}^{\infty} |V_k - (W_k + U_k)|^2 dt < \epsilon,$$

where

$$\int_{-\infty}^{\infty} |W_k|^2 dt = E_1, \qquad \int_{-\infty}^{\infty} |U_k|^2 dt = E_2$$

such that

$$\text{dist}(\text{supp}(W_k), \text{supp}(U_k)) \to \infty.$$

Our goal is to rule out the second and the third possibilities in order to prove convergence of a minimizing sequence. It has been shown in [28] for $\epsilon = 0$ that vanishing implies that $H^{(4)}(V_k, \bar{V}_k) \to 0$. This is in contradiction with the sequence being minimizing as the infimum is negative and $H_1(V_k, \bar{V}_k) \to 0$. The proof that $H^{(4)}(V_k, \bar{V}_k) \to 0$ is based on Cazenave's estimate [30],

$$\int_{-\infty}^{\infty} |V|^4 dt \le C\, ||V||_{H^1}^2 \sup_{y\in\mathbb{R}} \int_{y-1}^{y+1} |V|^2 dt, \tag{3.55}$$

and on the lemma on localization (3.52)–(3.53). Combing these estimates in a similar way, we prove that $H^{(6)}(V_k) \to 0$.

We also show that the splitting may not occur. By contradiction, we assume that splitting occurs and show that the sequence is not minimizing by using the subadditivity condition (3.51). The proof is identical to the one in [28] and therefore is omitted here.

Since both the vanishing and splitting scenarios do not occur, the concentration-compactness principle implies that the sequence $V_k \to W$ strongly in $L^2(\mathbb{R})$ [29]. Using the standard argument (see section 3.1 in [28]), we show that $V_k \to W$ strongly in $H^1(\mathbb{R})$. The minimizer weakly satisfies the Euler–Lagrange equation

$$-\mu W + \frac{1}{2} d_0 W''(t) + \langle Q \rangle(W) + \epsilon \langle Q_1 \rangle(W) = 0. \tag{3.56}$$

[1] There is no problem with the local loss of compactness since on any finite interval $I \subset \mathbb{R}$ the space $H^1(I)$ is compactly embedded in $L^2(I)$.

Using the bootstrapping procedure, we show that the solution is smooth. If $W \in H^1(\mathbb{R})$, then $\langle Q\rangle(W) + \epsilon\langle Q_1\rangle(W) \in H^1(\mathbb{R})$. Due to the presence of the term $d_0 W''(t)$ in (3.56), the solution is extended to function space $W \in H^3(\mathbb{R})$. Continuing this, we obtain that $W \in H^s(\mathbb{R})$ for any $s \geq 1$ and $W \in C^\infty(\mathbb{R})$. ∎

The minimizer $W(t)$ obtained in Proposition 3.12 defines a stationary pulse solution $V(z,t) = W(t)e^{i\mu z}$ of the first-order averaged integral NLS equation, where $\mu = f_\mu^{-1}(e)$ and $e = f_\mu(\mu)$ is a continuous function. Thus, the existence and stability of a single branch of DM solitons is proved for $d_0 > 0$ in the first-order averaged integral NLS equation (3.32). This completes the proof of Proposition 1.8 for the integral NLS approximation. If the stationary pulse solutions $W(t)$ are computed numerically, all other parameters of DM solitons can be computed for the case $d_0 > 0$, in direct correspondence with Proposition 1.4. Otherwise, the analytical dependencies in (1.19)–(1.21) remain implicitly defined by the averaged integral equation (3.56).

4. Conclusion. We have studied existence and stability of dispersion-managed (DM) solitons for the periodic NLS equation. We defined the DM solitons either as periodic solutions of a low-dimensional system for parameters of a Gaussian pulse or as stationary pulse solutions of the averaged integral NLS equation. In both cases, we have found and analyzed the first-order averaged Hamiltonian. Some open problems appear beyond this analysis and are worth mentioning here.

First, it is a conjecture that DM solitons do not exist as quasi-periodic solutions of the periodic NLS equation (1.5), contrary to the approximating Gaussian pulses. Recent work of Yang and Kath [19] discusses parametric resonances between localized pulses and linear Bloch waves associated with the varying dispersion $d(z)$. Asymptotic and numerical analysis confirmed that the quasi-periodic pulses produce nonlocalized radiation tails, which escape the localized region to infinity [19]. The radiation tail is exponentially small in the limit $\epsilon \to 0$, i.e., it appears beyond any asymptotic expansion in powers of ϵ. In our analysis, all the resonant terms are removed from the leading and first order of the asymptotic series. As a result, the quasi-periodic pulses exist in the averaged integral NLS equation (3.32), at least for $d_0 > 0$.

Second, the first-order constrained Hamiltonian $H_1(V,\bar{V})$ was shown to possess a constrained minimum only for $d_0 > 0$. With the use of the new work by Kunze [20], the constrained minimum can be shown to exist for $d_0 = 0$. However, it is impossible to prove whether or not a local extremum of the averaged Hamiltonian exists for $d_0 < 0$ even in the limit $\epsilon \to 0$. Indeed, the operator $\mu - \frac{1}{2}d_0\partial_{tt}$ is not positive-definite for $\mu > 0$ and $d_0 < 0$, and a strong resonance occurs between spectra of a localized pulse and linear waves. As a result, the Hamiltonian functional $H_1(V,\bar{V})$ is unbounded from below even for the constrained problem (3.49).

Two branches of Gaussian pulse solutions exist for $d_0 < 0$: one is stable and the other one is unstable in the propagation in z. However, iterations of a numerical method quickly diverge for the branch of unstable Gaussian pulses [13] and slowly diverge for the branch of stable Gaussian pulses [21]. Rigorous analysis of existence or nonexistence of stationary solutions of the problem (3.56) with $d_0 < 0$ is not completed yet.

Finally, the higher-order averaged Hamiltonian can be found and analyzed for the case $d_0 > 0$ in a similar manner. However, the constrained minimization procedure fails already for the second-order Hamiltonian, which has a correction $H^{(8)}(V,\bar{V})$ that contains eight powers of V and $\bar{V}$. Because of such higher-order nonlinearity, the correction $H^{(8)}(V,\bar{V})$ is not be bounded from below by the Strichartz estimate (3.50). Therefore, higher-order averaged equations become less useful for analysis.

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