

Global solutions to the shallow water system with a method of an additional argument

Sergey N. Alekseenko^a, Marina V. Dontsova^b and Dmitry E. Pelinovsky^{a,c}

^aDepartment of Applied Mathematics, Nizhny Novgorod State Technical University, Nizhny Novgorod, Russia;

^bDepartment of Applied Mathematics, Nizhny Novgorod State Pedagogical University, Nizhny Novgorod, Russia;

^cDepartment of Mathematics, McMaster University, Hamilton, Canada

ABSTRACT

The classical system of shallow water (Saint–Venant) equations describes long surface waves in an inviscid incompressible fluid of a variable depth. Although shock waves are expected in this quasi-linear hyperbolic system for a wide class of initial data, we find a sufficient condition on the initial data that guarantee existence of a global classical solution continued from a local solution. The sufficient conditions can be easily satisfied for the fluid flow propagating in one direction with two characteristic velocities of the same sign and two monotonically increasing Riemann invariants. We prove that these properties persist in the time evolution of the classical solutions to the shallow water equations and provide no shock wave singularities formed in a finite time over a half-line or an infinite line. On a technical side, we develop a novel method of an additional argument, which allows to obtain local and global solutions to the quasi-linear hyperbolic systems in physical rather than characteristic variables.

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1. Introduction

The shallow water system arises in the dispersionless limit of Euler equations and describes long waves on the surface of an inviscid incompressible fluid (e.g. water). We assume that the surface waves are two-dimensional in the (x, z) -variables and that the fluid is located between the hard bottom of the varying depth at $z = -h(x)$ and the free surface at $z = \eta(t, x)$, where h is given and η is unknown.

In the case of surface waves free of vorticity, the velocity vector of the fluid's motion is given by the gradient of the velocity potential, which is found from the Laplace equation in variables (x, z) . In the dispersionless limit, for which the horizontal length of wave motion is much larger compared to the vertical length, the Euler equations reduce to the shallow water system

$$\begin{cases} \partial_t \eta + \partial_x [u(h(x) + \eta)] = 0, \\ \partial_t u + u \partial_x u + g \partial_x \eta = 0, \end{cases} \quad (1.1)$$

where $u(t, x)$ is the horizontal component of velocity at the free surface $z = \eta(t, x)$, and g is the gravitational constant. In what follows, we set $g = 1$ without loss of generality.

The shallow water system (1.1), which is also known as the Saint–Venant equations, is reviewed in many texts and monographs (see, e.g. Section 5.1.1 in [1]). Recently, interest in the shallow water system arises due to modeling of run-up of water waves towards the beach. [2] In particular, when the bottom topography changes like $h(x) \sim x^{4/3}$, the waves propagating towards the beach are free of reflections. [3]

Using the standard technique of Riemann invariants, one can diagonalize the quasi-linear system (1.1) in new coordinates

$$z_{\pm}(t, x) := u(t, x) \pm 2\sqrt{h(x) + \eta(t, x)}, \tag{1.2}$$

which are real if $h(x) + \eta(t, x) > 0$. This constraint corresponds to the hyperbolicity of the shallow water system (1.1) and, physically, to the nonzero depth of the fluid flow over the variable bottom. Substitution of (1.2) into (1.1) yields the system of symmetric quasi-linear equations

$$\begin{cases} \partial_t z_+ + \frac{1}{4}(3z_+ + z_-)\partial_x z_+ = h'(x), \\ \partial_t z_- + \frac{1}{4}(z_+ + 3z_-)\partial_x z_- = h'(x). \end{cases} \tag{1.3}$$

The characteristic speeds of the system (1.3) are given by

$$c_{\pm} := \frac{1}{4}(3z_{\pm} + z_{\mp}) = u \pm \sqrt{h(x) + \eta}. \tag{1.4}$$

System (1.3) in Riemann invariants is well-known, see, e.g. Sections 5.7 and 13.10 in [4]. Some explicit solutions can be obtained in the case $h'(x) = \text{const}$ using the hodograph transformation method, see, e.g. recent works [2,5] and references therein. Review of exact solutions to the shallow water system can be found in Section 16.2.1 in [6].

The Cauchy problem is posed for the system (1.3) with initial data

$$z_{\pm}(0, x) = \varphi_{\pm}(x). \tag{1.5}$$

If the initial data φ_{\pm} are defined on the infinite line in Sobolev spaces $H^s(\mathbb{R})$, then the Cauchy problem (1.3) and (1.5) is known to be locally well-posed for $s > \frac{3}{2}$. [7] The method of characteristics can be used in a local neighborhood of any point if the initial data φ_{\pm} are C^1 functions near this point and their first derivatives are Lipschitz continuous. [8] A good survey of the method of characteristics can be found in [9,10]. Using modern analytical techniques, local solutions with only Lipschitz continuous functions φ_{\pm} can be obtained from a weak formulation of the quasi-linear system (1.3).

It is easy to recover the solution (u, η) to the shallow water system (1.1) from the solution (z_+, z_-) to the system (1.3). Indeed, inverting (1.2) yields

$$u(t, x) = \frac{1}{2} [z_+(t, x) + z_-(t, x)], \quad \eta(t, x) = \frac{1}{16} [z_+(t, x) - z_-(t, x)]^2 - h(x). \tag{1.6}$$

The initial data for u and η are given by

$$u_0(x) = \frac{1}{2} [\varphi_+(x) + \varphi_-(x)], \quad \eta_0(x) = \frac{1}{16} [\varphi_+(x) - \varphi_-(x)]^2 - h(x), \tag{1.7}$$

where positivity of $h(x) + \eta_0(x) > 0$ is assumed for every x .

For most quasi-linear systems, local solutions in Sobolev spaces $H^s(\mathbb{R})$ are not continued for all times t because wave breaking occurs in a finite time, resulting in appearance of the shock waves. [11] However, depending on the initial values φ_{\pm} and the given profile h , the wave breaking may be avoided and the local solutions can be continued for all finite times. We term such solutions as global solutions and warn that these solutions are allowed to diverge in some norm as $t \rightarrow \infty$.

The alternative of the global existence is the wave breaking in a finite time, which happens when the shock waves are formed in the quasi-linear hyperbolic systems. [11] We note that the wave breaking can also occur in the presence of weak dispersion, if the initial data are sufficiently large in some norm. [12,13]

This paper is devoted to the solvability of the classical system (1.3) both locally and globally. We will consider the semi-infinite line $[0, \infty)$ for x . Generally speaking, a boundary condition is required

at the finite boundary $x = 0$ for all positive times $t > 0$. However, if we find a condition on the initial values φ_{\pm} and the given profile h which ensure that both characteristic speeds c_{\pm} in (1.4) are negative near $x = 0$ for all $t > 0$, then we can avoid setting boundary conditions at $x = 0$. Moreover, with additional constraints on φ_{\pm} and h , one can also continue classical solutions to the shallow water system (1.1) globally in time and thus avoid wave breaking in a finite time.

Both local and global solutions to the quasi-linear system (1.3) are obtained using a similar analytical technique, which we develop in this work. The technique relies on a novel method, termed as the *method of an additional argument*. This method was pioneered for scalar conservation laws in [14, 15] and for systems of conservation laws in [16,17]. This method allows us to avoid technical problems arising in other techniques such as the method of characteristics or the method of generalized solutions. [10] For instance, the solvability condition in the method of characteristics relies on invertibility of the characteristic variables, which may be difficult to prove (see discussion in Section 2.2 below). Compared to the method of characteristics, the method of an additional argument allows us to obtain the local and global solvability of the quasi-linear system directly in physical coordinates.

In what follows, for a given $T > 0$, we use notation

$$\Omega_T := \{(t, x) : t \in (0, T), x \in \mathbb{R}^+\},$$

for the domain of definition of the Cauchy problem associated with the system (1.3). We denote by $C^{1,1}(\Omega_T)$ the space of bounded functions of two variables in Ω_T , which are continuously differentiable both in t and x with bounded first derivatives. We also introduce the norm in the space of functions $C_b^n(\mathbb{R}^+)$ with bounded and continuous derivatives up to the n th order:

$$\|h\|_{C_b^n} := \sup_{x \in \mathbb{R}^+} |h(x)| + \sum_{j=1}^n \sup_{x \in \mathbb{R}^+} |h^{(j)}(x)|, \quad h \in C_b^n(\mathbb{R}^+).$$

The following two theorems present the main results obtained in this paper.

Theorem 1: Assume that $u_0, \eta_0 \in C_b^1(\mathbb{R}^+)$ and $h \in C_b^2(\mathbb{R}^+)$ satisfy the conditions

$$h(x) \geq 0, \quad h'(x) \leq 0, \quad x \in \mathbb{R}^+, \tag{1.8}$$

and

$$\eta_0(x) \geq C, \quad u_0(x) \leq -2\sqrt{h(x) + \eta_0(x)}, \quad x \in \mathbb{R}^+, \tag{1.9}$$

for a fixed positive constant C . Then, for every $T > 0$ satisfying the constraint

$$T \leq \min\left(\frac{C_\varphi}{C_h}, \frac{1}{15C_\varphi}\right), \tag{1.10}$$

where $C_h := \|h\|_{C_b^2}$ and $C_\varphi := \max\{\|\varphi_+\|_{C_b^1}, \|\varphi_-\|_{C_b^1}\}$ with the initial data $\varphi_{\pm} := u_0 \pm \sqrt{h + \eta}$, there exists a unique classical solution $u, \eta \in C^{1,1}(\Omega_T)$ to the shallow water system (1.1) such that $u|_{t=0} = u_0$ and $\eta|_{t=0} = \eta_0$.

Theorem 2: In addition to the conditions of Theorem 1, assume that $u_0, \eta_0 \in C_b^1(\mathbb{R}^+)$ and $h \in C_b^2(\mathbb{R}^+)$ satisfy the conditions

$$h''(x) \geq 0, \quad x \in \mathbb{R}^+ \tag{1.11}$$

and

$$u'_0(x) \geq \frac{|h'(x) + \eta'_0(x)|}{\sqrt{h(x) + \eta_0(x)}}, \quad x \in \mathbb{R}^+. \tag{1.12}$$

Then, for every $T > 0$, there exists a unique classical solution $u, \eta \in C^{1,1}(\Omega_T)$ to the shallow water system (1.1) such that $u|_{t=0} = u_0$ and $\eta|_{t=0} = \eta_0$.

Remark 1: It follows from the definition (1.2) for Riemann invariants that conditions (1.9) are satisfied if

$$\varphi_+(x) \leq 0, \quad \varphi_-(x) \leq 0, \quad x \in \mathbb{R}^+. \tag{1.13}$$

Similarly, condition (1.12) is satisfied if

$$\varphi'_+(x) \geq 0, \quad \varphi'_-(x) \geq 0, \quad x \in \mathbb{R}^+. \tag{1.14}$$

Remark 2: Since the quasi-linear system (1.3) is written in the symmetric form, the result of Theorem 1 agrees with the result of Kato [7] on the infinite line, since Sobolev space $H^s(\mathbb{R})$ with $s > \frac{3}{2}$ is continuously embedded into the space $C^1_b(\mathbb{R})$. However, the Cauchy problem (1.3) and (1.5) on the half-line cannot be solved by the method of Kato [7] unless a boundary condition is set at $x = 0$ in one way or another.

Remark 3: The result of Theorem 1 is stronger than the corresponding result of Courant and Lax [8], which establishes the existence of classical solutions with Lipschitz continuity for their spatial derivatives in a local neighborhood of any point x on \mathbb{R}^+ , provided the initial data are available near this point. Although the formulations of the method of characteristics in [8] and the method of an additional argument here are similar, our technique allows us to obtain the solution to the quasi-linear system (1.3) in physical rather than characteristic coordinates. Also we obtain a stronger result using the Schauder fixed point theorem (see Lemma 2 below) instead of the Arzelà–Ascoli theorem on convergence of bounded and equicontinuous sequences for spatial derivatives.

Remark 4: Using the integral reformulation of the quasi-linear system (1.3) as in Section 2.1 below, one can relax conditions on the initial data φ_{\pm} . Lipschitz continuity of φ_{\pm} is sufficient for local solvability of the integral equations, similar to the technique developed in the method of characteristics in [9]. However, for equivalence between the system of integral equations and the quasi-linear system (1.3), we need to work with classical solutions, which are only obtained from the C^1 functions φ_{\pm} . Therefore, the condition of C^1 initial data is considered to be optimal in this context.

The physical relevance of the conditions (1.8), (1.9), (1.11), and (1.12) is to provide the bottom topography h and the initial values for u and η such that both the Riemann invariants z_{\pm} and their characteristic speeds c_{\pm} given by (1.2) and (1.4) are strictly negative, whereas the Riemann invariants are monotonically increasing, see (1.13) and (1.14). Under these conditions, the surface waves do not break in a finite time, because they move convectively to the finite boundary at $x = 0$, through which they radiate away. These conditions can be satisfied, for instance, if

$$h(x) = (1 + x)^{-p}, \quad \eta_0(x) = C, \quad u_0(x) = -2\sqrt{C + h(x)}, \tag{1.15}$$

where $p > 0$ and $C > 0$ are fixed. Thus, the bottom topography becomes deeper near $x = 0$ and uniform as $x \rightarrow \infty$, whereas the initial horizontal velocity is negative everywhere and the current is stronger near $x = 0$ and becomes uniform as $x \rightarrow \infty$. Such configurations can model river waterfalls, e.g. Niagara falls in Ontario, Canada.

Theorems 1 and 2 can be extended to the infinite line without any restrictions, as long as the conditions (1.8), (1.9), (1.11), and (1.12) hold on the infinite line. The main example (1.15) does not make sense on the infinite line, but the conditions can be satisfied for the shear flow on the flat bottom with sign-definite, monotonically increasing velocity u_0 , which may vanish at one infinity but has a non-vanishing background flow at the other infinity.

In a single-wave reduction of the system (1.3) with $h'(x) \equiv 0$ and $z_+(t, x) \equiv 0$, the constraint (1.12) guarantees that

$$\varphi'_-(x) = u'_0(x) - \frac{\eta'_0(x)}{\sqrt{h + \eta_0(x)}} \geq 0, \quad x \in \mathbb{R}^+.$$

This condition is well known [11] to exclude shock waves in the Cauchy problem posed for the inviscid Burgers equation

$$\begin{cases} \partial_t z_- + \frac{3}{4} z_- \partial_x z_- = 0, \\ z_-|_{t=0} = \varphi_-. \end{cases} \tag{1.16}$$

In the same context, the constraint (1.9) ensures that $\varphi_-(x) \leq 0$ for every $x \in \mathbb{R}^+$, the latter constraint is only required to avoid the boundary condition on z_- at $x = 0$ for the evolution problem (1.16) on the semi-infinite line \mathbb{R}^+ .

The rest of this paper is organized as follows. Section 2 is devoted to the reformulation of the quasi-linear system (1.3) as a system of integral equations using the method of an additional argument. The equivalence between the quasi-linear system (1.3) and the system of integral equations is established. In Section 3, we obtain a local solution of Theorem 1. In Section 4, we show that the local solution in $C^{1,1}(\Omega_T)$ can be extended for every $T > 0$ as in Theorem 2. The additional constraints (1.11) and (1.12) allow us to control the rate of change of the spatial derivatives of the solution z_{\pm} during the time evolution of the quasi-linear system (1.3).

2. Reformulation with the method of an additional argument

Here we adopt the method of an additional argument in order to reformulate the Cauchy problem given by (1.3) and (1.5) as a boundary-value problem along characteristic coordinates. For a given point $(t, x) \in \Omega_T$, we introduce the extended characteristic coordinates $\eta_+(s; t, x)$ and $\eta_-(s; t, x)$ from solutions to the system of differential equations

$$\begin{cases} \frac{d\eta_+}{ds}(s; t, x) = \frac{1}{4} [3z_+(s, \eta_+(s; t, x)) + z_-(s, \eta_+(s; t, x))], \\ \frac{d\eta_-}{ds}(s; t, x) = \frac{1}{4} [z_+(s, \eta_-(s; t, x)) + 3z_-(s, \eta_-(s; t, x))], \end{cases} \quad 0 \leq s \leq t, \tag{2.1}$$

starting with the boundary values $\eta_{\pm}(t; t, x) = x$. In the characteristic variables, the system (1.3) can be rewritten as the system of differential equations

$$\begin{cases} \frac{dz_+}{ds}(s, \eta_+(s; t, x)) = h'(\eta_+(s; t, x)), \\ \frac{dz_-}{ds}(s, \eta_-(s; t, x)) = h'(\eta_-(s; t, x)), \end{cases} \quad 0 \leq s \leq t, \tag{2.2}$$

starting with the initial values $z_{\pm}(0, \eta_{\pm}(0; t, x)) = \varphi_{\pm}(\eta_{\pm}(0; t, x))$. The domain of definition of the systems (2.1) and (2.2) is given by

$$\Gamma_T := \{(s, t, x) : 0 \leq s \leq t \leq T, \quad x \in \mathbb{R}^+\}, \tag{2.3}$$

for a given $T > 0$. We denote by $C^{k,k,m}(\Gamma_T)$ the space of bounded functions of three variables in Γ_T , which are differentiable k -times with respect to s and t , m -times with respect to x , with bounded derivatives. We also denote the supremum norm of a function $U \in C^{0,0,0}(\Gamma_T)$ by

$$\|U\| := \sup_{(s,t,x) \in \Gamma_T} |U(s; t, x)|. \tag{2.4}$$

The variable s is referred to as the additional argument of the system (2.1) and (2.2). The main difference of the method of an additional argument from the method of characteristics is that the system (2.1) is integrated backward in s from the current time t to the initial time 0, whereas the system (2.2) is integrated forward in s from the initial time 0 to the current time t . Although the combined system (2.1) and (2.2) represents a boundary-value problem instead of the Cauchy problem, we are still able to rewrite the systems (2.1) and (2.2) as a system of integral equations and to solve it by the Picard method of successful iterations. Compared to the method of characteristics, the solutions $z_{\pm}(t, x) \equiv z_{\pm}(t, \eta_{\pm}(t; t, x))$ appear in physical rather than characteristic coordinates.

2.1. Integral equations for classical solutions of system (1.3)

Integrating (2.1) backward in s , we obtain the system of integral equations

$$\begin{cases} \eta_+(s; t, x) = x - \frac{1}{4} \int_s^t [3z_+(v, \eta_+(v; t, x)) + z_-(v, \eta_+(v; t, x))] dv, \\ \eta_-(s; t, x) = x - \frac{1}{4} \int_s^t [z_+(v, \eta_-(v; t, x)) + 3z_-(v, \eta_-(v; t, x))] dv, \end{cases} \quad 0 \leq s \leq t. \quad (2.5)$$

Integrating (2.2) forward in s , we obtain another system of integral equations

$$\begin{cases} z_+(s, \eta_+(s; t, x)) = \varphi_+(\eta_+(0; t, x)) + \int_0^s h'(\eta_+(v; t, x)) dv, \\ z_-(s, \eta_-(s; t, x)) = \varphi_-(\eta_-(0; t, x)) + \int_0^s h'(\eta_-(v; t, x)) dv, \end{cases} \quad 0 \leq s \leq t. \quad (2.6)$$

From the geometric definition of the characteristic curves in the domain Ω_T on the (t, x) plane, we have the connection formulas

$$\begin{cases} z_-(s, \eta_+(s; t, x)) = z_-(s, \eta_-(s; s, \eta_+(s; t, x))), \\ z_+(s, \eta_-(s; t, x)) = z_+(s, \eta_+(s; s, \eta_-(s; t, x))), \end{cases} \quad 0 \leq s \leq t. \quad (2.7)$$

Let us denote

$$Z_{\pm}(s; t, x) := z_{\pm}(s, \eta_{\pm}(s; t, x)) \quad \text{and} \quad Y_{\pm}(s; t, x) := z_{\mp}(s, \eta_{\pm}(s; t, x)). \quad (2.8)$$

It follows from the boundary conditions $\eta_{\pm}(t; t, x) = x$ that $Z_{\pm}(t; t, x) = z_{\pm}(t, x)$ and $Y_{\pm}(t; t, x) = z_{\mp}(t, x)$. Furthermore, the system (2.7) is equivalent to the following relations between variables Z_{\pm} and Y_{\pm} :

$$Y_+(s; t, x) = Z_-(s; s, \eta_+(s; t, x)), \quad Y_-(s; t, x) = Z_+(s; s, \eta_-(s; t, x)). \quad (2.9)$$

From now on, we will be writing systems using one equation with two subscripts. Using new notations, we rewrite system (2.5) in the following form:

$$\eta_{\pm}(s; t, x) = x - \frac{1}{4} \int_s^t [3Z_{\pm}(v; t, x) + Y_{\pm}(v; t, x)] dv, \quad 0 \leq s \leq t. \quad (2.10)$$

Therefore, the characteristic coordinates can be eliminated from the systems (2.6) and (2.9), after which we obtain the following integral equations for unknown functions Z_{\pm} and Y_{\pm} in Γ_T :

$$\begin{aligned} Z_{\pm}(s; t, x) = & \varphi_{\pm} \left(x - \frac{1}{4} \int_0^t [3Z_{\pm}(v; t, x) + Y_{\pm}(v; t, x)] dv \right) \\ & + \int_0^s h' \left(x - \frac{1}{4} \int_v^t [3Z_{\pm}(\tau; t, x) + Y_{\pm}(\tau; t, x)] d\tau \right) dv, \end{aligned} \quad (2.11)$$

and

$$Y_{\pm}(s; t, x) = Z_{\mp} \left(s; s, x - \frac{1}{4} \int_s^t [3Z_{\pm}(v; t, x) + Y_{\pm}(v; t, x)] dv \right). \quad (2.12)$$

Our first result states that the system (2.11) and (2.12) is closed in Γ_T for every $T > 0$ under conditions (1.8) and (1.13) on h and φ_{\pm} .

Proposition 1: Assume that $h \in C_b^1(\mathbb{R}^+)$ and $\varphi_{\pm} \in C_b^0(\mathbb{R}^+)$. Under the conditions

$$h'(x) \leq 0, \quad \varphi_+(x) \leq 0, \quad \varphi_-(x) \leq 0, \quad x \in \mathbb{R}^+, \quad (2.13)$$

the system (2.11) and (2.12) is closed in Γ_T for every $T > 0$ in the sense that if a unique solution (Z_{\pm}, Y_{\pm}) exists in $C^{0,0,0}(\Gamma_T)$, then

$$\begin{cases} \eta_{\pm}(s; t, x) \geq 0, \\ Z_{\pm}(s; t, x) \leq 0, \\ Y_{\pm}(s; t, x) \leq 0, \end{cases} \quad (s, t, x) \in \Gamma_T. \tag{2.14}$$

Proof: We obtain from (2.10), (2.11), and (2.12) for every $(s, t, x) \in \Gamma_T$,

$$\eta_{\pm}(s; t, x) \geq x, \quad Z_{\pm}(s; t, x) \leq \varphi_{\pm}(\eta_{\pm}(0; t, x)) \leq 0, \quad Y_{\pm}(s; t, x) = Z_{\mp}(s; s, \eta_{\pm}(s; t, x)) \leq 0,$$

by using conditions (2.13) and the continuation arguments. Then, constraints (2.14) follow. □

Next, we show how the classical solutions to the Cauchy problem (1.3) and (1.5) are obtained from suitable solutions to the integral system (2.11)–(2.12).

Proposition 2: Assume that $h \in C_b^2(\mathbb{R}^+)$ and $\varphi_{\pm} \in C_b^1(\mathbb{R}^+)$. If there exists a unique solution $(Z_{\pm}, Y_{\pm}) \in C^{1,1,1}(\Gamma_T)$ of the system (2.11) and (2.12), then $z_{\pm}(t, x) = Z_{\pm}(t; t, x)$ is a classical solution to system (1.3) in $C^{1,1}(\Omega_T)$ such that $z_{\pm}(0, x) = \varphi_{\pm}(x)$ for $x \in \mathbb{R}^+$.

Proof: Let us introduce two differential operators W_{\pm} given by

$$W_{\pm}f := \frac{\partial f}{\partial t} + \frac{1}{4} (3Z_{\pm}(t; t, x) + Z_{\mp}(t; t, x)) \frac{\partial f}{\partial x}.$$

Applying W_+ to the corresponding integral equation in the system (2.11) and using $Y_+(t; t, x) = Z_-(t; t, x)$ from (2.9), we obtain

$$\begin{aligned} (W_+Z_+)(s; t, x) &= -\frac{1}{4}\varphi'_+(\cdot) \int_0^t [3(W_+Z_+)(s; t, x) + (W_+Y_+)(s; t, x)] ds \\ &\quad -\frac{1}{4} \int_0^s h''(\cdot) \left(\int_v^t [3(W_+Z_+)(\tau; t, x) + (W_+Y_+)(\tau; t, x)] d\tau \right) dv, \end{aligned}$$

where the arguments of $\varphi'_+(\cdot)$ and $h''(\cdot)$ are the same as in (2.11). Since we have the correspondence between Y_+ and Z_- from the system (2.12), we obtain similarly

$$(W_+Y_+)(s; t, x) = -\frac{1}{4}\partial_x Z_-(\cdot) \int_s^t [3(W_+Z_+)(v; t, x) + (W_+Y_+)(v; t, x)] dv,$$

where the argument of $Z_-(\cdot)$ is the same as in (2.12). Using the norm in Γ_T defined by (2.4), we obtain the following estimate

$$3\|W_+Z_+\| + \|W_+Y_+\| \leq \frac{1}{4} \left(3\|\varphi_+\|_{C_b^1} t + \frac{3}{2}\|h\|_{C_b^2} t^2 + \|\partial_x Z_-\| t \right) (3\|W_+Z_+\| + \|W_+Y_+\|).$$

Note that $\|\partial_x Z_-\| < \infty$ due to the assumption $Z_- \in C^{1,1,1}(\Gamma_T)$, whereas $\|\varphi_+\|_{C_b^1} < \infty$ and $\|h\|_{C_b^2} < \infty$ due to the assumptions on φ_+ and h . Let T_+ be the smallest positive root of the algebraic equation

$$\frac{1}{4} \left(3\|\varphi_+\|_{C_b^1} t + \frac{3}{2}\|h\|_{C_b^2} t^2 + \|\partial_x Z_-\| t \right) = 1.$$

Then, for every $t \in [0, t_+]$ with $t_+ := \min(T_+, T)$, we obtain

$$\|W_+Z_+\| + \|W_+Y_+\| = 0,$$

which imply $W_+Z_+ = W_+Y_+ = 0$ in Γ_{t_+} .

Applying W_- to the corresponding integral equations into the system (2.11) and (2.12), we obtain similar estimates

$$3\|W_-Z_-\| + \|W_-Y_-\| \leq \frac{1}{4} \left(3\|\varphi_-\|_{C_b^1}t + \frac{3}{2}\|h\|_{C_b^2}t^2 + \|\partial_x Z_+\|t \right) (3\|W_-Z_-\| + \|W_-Y_-\|).$$

Let T_- be the smallest positive root of the algebraic equation

$$\frac{1}{4} \left(3\|\varphi_-\|_{C_b^1}t + \frac{3}{2}\|h\|_{C_b^2}t^2 + \|\partial_x Z_+\|t \right) = 1.$$

Then, for every $t \in [0, t_-]$ with $t_- := \min(T_-, T)$, we obtain

$$\|W_-Z_-\| + \|W_-Y_-\| = 0,$$

which implies $W_-Z_- = W_-Y_- = 0$ in Γ_{t_-} .

Let $z_{\pm}(t, x) := Z_{\pm}(t; t, x)$ and $T_0 := \min(T_+, T_-, T)$. Then, for every $t \in [0, T_0]$, we use $\partial_s Z_{\pm}(t; t, x) = h'(x)$ that follows from system (2.11) and obtain

$$\frac{\partial z_+}{\partial t} + \frac{1}{4}(3z_+ + z_-)\frac{\partial z_+}{\partial x} = \frac{\partial Z_+}{\partial s}(t; t, x) + (W_+Z_+)(t; t, x) = h'(x)$$

and

$$\frac{\partial z_-}{\partial t} + \frac{1}{4}(z_+ + 3z_-)\frac{\partial z_-}{\partial x} = \frac{\partial Z_-}{\partial s}(t; t, x) + (W_-Z_-)(t; t, x) = h'(x),$$

which is nothing but system (1.3). Therefore, $z_{\pm} \in C^{1,1}(\Omega_{T_0})$ is a solution to system (1.3) for $T_0 \leq T$. If $T_0 < T$, then the continuation of the solution to the entire domain Ω_T can be performed in a finite number of steps. □

2.2. Integral equations for x-derivatives of system (1.3)

Let us denote $u_{\pm}(t, x) := \partial_x z_{\pm}(t, x)$. If $z_{\pm} \in C^{1,1}(\Omega_T)$ as in Proposition 2, then $u_{\pm} \in C^{0,0}(\Omega_T)$. Differentiating (2.5) with respect to x , we obtain a system of integral equations for x -derivatives of the characteristic coordinates:

$$\xi_{\pm}(s; t, x) = 1 - \frac{1}{4} \int_s^t [3u_{\pm}(v, \eta_{\pm}(v; t, x)) + u_{\mp}(v, \eta_{\pm}(v; t, x))] \xi_{\pm}(v; t, x) dv, \quad 0 \leq s \leq t, \tag{2.15}$$

where $\xi_{\pm}(s; t, x) := \partial_x \eta_{\pm}(s; t, x)$ satisfies the initial conditions $\xi_{\pm}(t; t, x) = 1$. There exists a unique solution of the system (2.15) in the form

$$\xi_{\pm}(s; t, x) = e^{-\frac{1}{4} \int_s^t [3u_{\pm}(v, \eta_{\pm}(v; t, x)) + u_{\mp}(v, \eta_{\pm}(v; t, x))] dv}, \quad 0 \leq s \leq t. \tag{2.16}$$

The main difficulty in the method of characteristics is to control positivity of $\xi_{\pm}(s; t, x)$ in Γ_T as T increases. The explicit expression (2.16) shows that positivity of $\xi_{\pm}(s; t, x)$ in Γ_T follows from boundness of $u_{\pm}(t, x)$ in Ω_T , but this property is hard to control. On the other hand, in the method of an additional argument, we introduce

$$U_{\pm}(s; t, x) := \partial_x Z_{\pm}(s; t, x), \quad V_{\pm}(s; t, x) := \partial_x Y_{\pm}(s; t, x) \tag{2.17}$$

and define using (2.8) and the chain rule

$$U_{\pm}(s; t, x) = u_{\pm}(s, \eta_{\pm}(s; t, x))\xi_{\pm}(s; t, x), \quad V_{\pm}(s; t, x) = u_{\mp}(s, \eta_{\pm}(s; t, x))\xi_{\pm}(s; t, x). \quad (2.18)$$

It follows from the boundary conditions $\eta_{\pm}(t; t, x) = x$ and $\xi_{\pm}(t; t, x) = 1$ that $U_{\pm}(t; t, x) = u_{\pm}(t, x)$ and $V_{\pm}(t; t, x) = u_{\mp}(t, x)$. If $(Z_{\pm}, Y_{\pm}) \in C^{1,1,1}(\Gamma_T)$ as in Proposition 2, then $(U_{\pm}, V_{\pm}) \in C^{0,0,0}(\Gamma_T)$. By differentiating the system (2.11) and (2.12) with respect to x , we obtain the system of integral equations:

$$U_{\pm}(s; t, x) = \varphi'_{\pm}(\cdot) \left(1 - \frac{1}{4} \int_0^t [3U_{\pm}(v; t, x) + V_{\pm}(v; t, x)] dv \right) + \int_0^s h''(\cdot) \left(1 - \frac{1}{4} \int_v^t [3U_{\pm}(\tau; t, x) + V_{\pm}(\tau; t, x)] d\tau \right) dv, \quad (2.19)$$

and

$$V_{\pm}(s; t, x) = U_{\mp} \left(s; s, x - \frac{1}{4} \int_s^t [3Z_{\pm}(v; t, x) + Y_{\pm}(v; t, x)] dv \right) \times \left(1 - \frac{1}{4} \int_s^t [3U_{\pm}(v; t, x) + V_{\pm}(v; t, x)] dv \right), \quad (2.20)$$

where the arguments of $\varphi'_{\pm}(\cdot)$ and $h''(\cdot)$ are the same as in the system (2.11). On the other hand, differentiating (2.10) in x yields the following relation

$$\xi_{\pm}(s; t, x) = 1 - \frac{1}{4} \int_s^t [3U_{\pm}(v; t, x) + V_{\pm}(v; t, x)] dv, \quad 0 \leq s \leq t. \quad (2.21)$$

This relation is complementary to the expression (2.16).

The following proposition states that the variables U_{\pm} and V_{\pm} are sign-definite in Γ_T for every $T > 0$, for which a solution $(Z_{\pm}, Y_{\pm}) \in C^{1,1,1}(\Gamma)$ exists, under additional conditions (1.11) and (1.14) on h and φ_{\pm} .

Proposition 3: Assume that $h \in C_b^2(\mathbb{R}^+)$ and $\varphi_{\pm} \in C_b^1(\mathbb{R}^+)$ satisfy (2.13), and the additional conditions

$$h''(x) \geq 0, \quad \varphi'_+(x) \geq 0, \quad \varphi'_-(x) \geq 0, \quad x \in \mathbb{R}^+. \quad (2.22)$$

If a solution (Z_{\pm}, Y_{\pm}) to the system (2.11) and (2.12) exists in $C^{1,1,1}(\Gamma_T)$, then

$$\begin{cases} \xi_{\pm}(s; t, x) \leq 1, \\ U_{\pm}(s; t, x) \geq 0, \\ V_{\pm}(s; t, x) \geq 0, \end{cases} \quad (s, t, x) \in \Gamma_T. \quad (2.23)$$

Proof: Assuming existence of solution $(Z_{\pm}, Y_{\pm}) \in C^{1,1,1}(\Gamma_T)$ to the system (2.11) and (2.12), we have by Proposition 2 and the definition (2.17) that $(U_{\pm}, V_{\pm}) \in C^{0,0,0}(\Gamma_T)$ and $u_{\pm} \in C^{0,0}(\Omega_T)$. By (2.16), we have $\xi_{\pm}(s; t, x) > 0$ for every $(s, t, x) \in \Gamma_T$. Then, using relations (2.21), conditions (2.22), and the result of Proposition 1, we obtain from the system (2.19) and (2.20) that $U_{\pm}(s; t, x) \geq 0$ and $V_{\pm}(s; t, x) \geq 0$ for every $(s, t, x) \in \Gamma_T$. Using relations (2.21) again, we have $\xi_{\pm}(s; t, x) \leq 1$ for every $(s, t, x) \in \Gamma_T$. Thus, constraints (2.23) have been proved. \square

Generally speaking, the chain rule (2.18) and the representation (2.21) only show that if $U_{\pm}(s; t, x)$ and $V_{\pm}(s; t, x)$ remain bounded and positive for $(s, t, x) \in \Gamma_T$, then $\xi_{\pm}(s; t, x)$ may still vanish at the same points $(s, t, x) \in \Gamma_T$ for which either $u_{\pm}(s, \eta_{\pm}(s; t, x))$ or $u_{\mp}(s, \eta_{\pm}(s; t, x))$ become unbounded. However, divergence of $u_{\pm}(t, x)$ for $(t, x) \in \Omega_T$ contradicts to the result of Proposition 2, if the

solution $(Z_{\pm}, Y_{\pm}) \in C^{1,1,1}(\Gamma_T)$ to the system (2.11) and (2.12) is obtained. Therefore, the essence of the method of an additional argument is to ensure solvability of the system (2.11) and (2.12) in $C^{1,1,1}(\Gamma_T)$, which would guarantee strict positivity of $\xi_{\pm}(s; t, x)$ for every $(s, t, x) \in \Gamma_T$.

For completeness, we mention that if we substitute (2.16), (2.18) and (2.21) to the integral equation (2.19), then we obtain

$$u_{\pm}(s, \eta_{\pm}(s; t, x)) = \varphi'_{\pm}(\eta_{\pm}(0; t, x))e^{-\frac{1}{4} \int_0^s \ddagger 3u_{\pm}(v, \eta_{\pm}(v; t, x)) + u_{\mp}(v, \eta_{\pm}(v; t, x)) \ddagger dv} + \int_0^s h''(\eta_{\pm}(v; t, x))e^{-\frac{1}{4} \int_v^s \ddagger 3u_{\pm}(\tau, \eta_{\pm}(\tau; t, x)) + u_{\mp}(\tau, \eta_{\pm}(\tau; t, x)) \ddagger d\tau} dv, \tag{2.24}$$

which can be thought as a weak formulation of the system of differential equations

$$\begin{cases} \frac{du_+}{ds}(s, \eta_+(s; t, x)) + \frac{3}{4}u_+^2(s, \eta_+(s; t, x)) + \frac{1}{4}u_+(s, \eta_+(s; t, x))u_-(s, \eta_+(s; t, x)) = h''(\eta_+(s; t, x)), \\ \frac{du_-}{ds}(s, \eta_-(s; t, x)) + \frac{3}{4}u_-^2(s, \eta_-(s; t, x)) + \frac{1}{4}u_+(s, \eta_-(s; t, x))u_-(s, \eta_-(s; t, x)) = h''(\eta_-(s; t, x)), \end{cases} \tag{2.25}$$

where $0 \leq s \leq t$, subject to the initial conditions $u_{\pm}(0, \eta_{\pm}(0; t, x)) = \varphi'_{\pm}(\eta_{\pm}(0; t, x))$ and the consistency conditions $u_{\mp}(s, \eta_{\pm}(s; t, x)) = u_{\mp}(s, \eta_{\mp}(s; s, \eta_{\pm}(s; s, x)))$. The differential system (2.25) can be derived by differentiating system (1.3) with respect to x for appropriate solutions $z_{\pm} \in C^{2,2}(\Omega_T)$ and using the characteristic equation (2.1). Again, control of boundness of $u_{\pm}(t, x)$ for $(t, x) \in \Omega_T$ is very difficult within the evolution problem (2.25) or the system (2.24). However, all these difficult steps are avoided in the method of an additional argument.

3. Local solution to system (2.11)–(2.12)

Here, we use the method of Picard’s successive approximations to prove existence of a local solution to the system (2.11) and (2.12). At first, we are looking for local solutions in the space $C^{0,0,0}(\Gamma_T)$. The fixed existence time $T > 0$ is supposed to be small to ensure that the contraction method works. Then, we obtain local solutions in the space $C^{1,1,1}(\Gamma_T)$ from the Schauder fixed-point theorem. Assumptions of both Propositions 1 and 2 are satisfied for the local solutions in $C^{1,1,1}(\Gamma_T)$. Thus, by correspondence between solutions to the system (2.11) and (2.12) and the quasi-linear system (1.3), the results obtained in this section yield the proof of Theorem 1.

The main difficulty in the proof of existence of a local solution to the system (2.11) and (2.12) in $C^{0,0,0}(\Gamma_T)$ is due to the fact that the integral equation (2.12) is composed of unknown functions. As a result, the method of successive approximations consists of two levels, similar to what is described in [17]. In order to close the system (2.11) and (2.12) in Γ_T , we use the conditions (1.8) and (1.9) on the function h and initial data u_0 and η_0 , the latter conditions are rewritten for φ_{\pm} in the form (1.13).

Lemma 1: Assume $h \in C_b^2(\mathbb{R}^+)$ and $\varphi_{\pm} \in C_b^1(\mathbb{R}^+)$ satisfying the constraints (1.8) and (1.13). Define

$$T := \min\left(\frac{C_{\varphi}}{C_h}, \frac{1}{15C_{\varphi}}\right), \tag{3.1}$$

where $C_{\varphi} := \max\{\|\varphi_+\|_{C_b^1}, \|\varphi_-\|_{C_b^1}\}$ and $C_h := \|h\|_{C_b^2}$. Then, the system (2.11) and (2.12) admits a unique solution in class $(Z_{\pm}, Y_{\pm}) \in C^{0,0,0}(\Gamma_T)$ such that

$$\|Z_{\pm}\|, \|Y_{\pm}\| \leq 2C_{\varphi}. \tag{3.2}$$

Proof: By Proposition 1, the system (2.11) and (2.12) is closed in Γ_T in the sense of bounds (2.14). In order to apply the Picard method, we start with the initial approximations

$$Z_{\pm(0)}(s; t, x) = Y_{\pm(0)}(s; t, x) = \varphi_{\pm}(x) \tag{3.3}$$

and define the successive approximations $\{Z_{\pm(n)}, Y_{\pm(n)}\}_{n \in \mathbb{N}}$ from the recursive iterations based on the system (2.11) and (2.12) for $n \in \mathbb{N}$:

$$Z_{\pm(n)}(s; t, x) = \varphi_{\pm} \left(x - \frac{1}{4} \int_0^t [3Z_{\pm(n)}(v; t, x) + Y_{\pm(n)}(v; t, x)] dv \right) + \int_0^s h' \left(x - \frac{1}{4} \int_v^t [3Z_{\pm(n)}(\tau; t, x) + Y_{\pm(n)}(\tau; t, x)] d\tau \right) dv, \quad (3.4)$$

and

$$Y_{\pm(n)}(s; t, x) = Z_{\mp(n-1)} \left(s; s, x - \frac{1}{4} \int_s^t [3Z_{\pm(n)}(v; t, x) + Y_{\pm(n)}(v; t, x)] dv \right). \quad (3.5)$$

The system (3.4)–(3.5) is implicit in $(Z_{\pm(n)}, Y_{\pm(n)})$. Therefore, for each $n \in \mathbb{N}$, we obtain $Z_{\pm(n)}, Y_{\pm(n)}$ from another sequence of successive approximations $\{Z_{\pm(n)}^{(k)}, Y_{\pm(n)}^{(k)}\}_{k \in \mathbb{N}}$ starting with the initial approximations

$$Z_{\pm(n)}^{(0)}(s; t, x) = Z_{\pm(n-1)}(s; t, x) \quad \text{and} \quad Y_{\pm(n)}^{(0)}(s; t, x) = Y_{\pm(n-1)}(s; t, x), \quad n \in \mathbb{N}, \quad (3.6)$$

which is defined at least for $n = 1$. Successive approximations $\{Z_{\pm(n)}^{(k)}, Y_{\pm(n)}^{(k)}\}_{k \in \mathbb{N}}$ are defined by the explicit iteration scheme for $k \in \mathbb{N}$:

$$Z_{\pm(n)}^{(k)}(s; t, x) = \varphi_{\pm} \left(x - \frac{1}{4} \int_0^t [3Z_{\pm(n)}^{(k-1)}(v; t, x) + Y_{\pm(n)}^{(k-1)}(v; t, x)] dv \right) + \int_0^s h' \left(x - \frac{1}{4} \int_v^t [3Z_{\pm(n)}^{(k-1)}(\tau; t, x) + Y_{\pm(n)}^{(k-1)}(\tau; t, x)] d\tau \right) dv, \quad (3.7)$$

and

$$Y_{\pm(n)}^{(k)}(s; t, x) = Z_{\mp(n-1)} \left(s; s, x - \frac{1}{4} \int_s^t [3Z_{\pm(n)}^{(k-1)}(v; t, x) + Y_{\pm(n)}^{(k-1)}(v; t, x)] dv \right). \quad (3.8)$$

The construction of successive approximations to the two-level system in $C^{0,0,0}(\Gamma_T)$ is broken into three steps.

Step 1. We prove for every $n \in \mathbb{N}$ that the sequence $\{Z_{\pm(n)}^{(k)}, Y_{\pm(n)}^{(k)}\}_{k \in \mathbb{N}}$ satisfying (3.6), (3.7), and (3.8) converges in $C^{0,0,0}(\Gamma_T)$ for a fixed $T > 0$ satisfying (3.1), so that we can define

$$Z_{\pm(n)}(s; t, x) := \lim_{k \rightarrow \infty} Z_{\pm(n)}^{(k)}(s; t, x) \quad \text{and} \quad Y_{\pm(n)}(s; t, x) := \lim_{k \rightarrow \infty} Y_{\pm(n)}^{(k)}(s; t, x), \quad n \in \mathbb{N}. \quad (3.9)$$

Let us introduce $C_{\varphi} := \max\{\|\varphi_+\|_{C_b^1}, \|\varphi_-\|_{C_b^1}\}$ and $C_h := \|h\|_{C_b^2}$. It follows from (3.7) and (3.8) that

$$\|Z_{\pm(n)}^{(k)}\| \leq C_{\varphi} + C_h T \leq 2C_{\varphi}, \quad \|Y_{\pm(n)}^{(k)}\| = \|Z_{\pm(n-1)}\|, \quad k \in \mathbb{N}, \quad (3.10)$$

where we have used $C_h T \leq C_{\varphi}$ according to the constraint (3.1). Since the bounds (3.10) are independent of k , if convergence to the limits (3.9) can be proved for each $n \in \mathbb{N}$, then by the induction method, we have

$$\|Z_{\pm(n)}\|, \|Y_{\pm(n)}\| \leq 2C_{\varphi}, \quad n \in \mathbb{N}. \quad (3.11)$$

Bounds (3.11) are also satisfied for $n = 0$. Now, we establish convergence to the limits in (3.9).

Using the fundamental theorem of calculus and the estimates similar to those in the proof of Proposition 2, we derive the bounds on the distance between two successive approximations:

$$3\|Z_{\pm(n)}^{(k+1)} - Z_{\pm(n)}^{(k)}\| + \|Y_{\pm(n)}^{(k+1)} - Y_{\pm(n)}^{(k)}\| \leq K_{\pm}(T) \left(3\|Z_{\pm(n)}^{(k)} - Z_{\pm(n)}^{(k-1)}\| + \|Y_{\pm(n)}^{(k)} - Y_{\pm(n)}^{(k-1)}\| \right), \tag{3.12}$$

where we have denoted

$$K_{\pm}(T) := \frac{1}{4} \left(3C_{\varphi}T + \frac{3}{2}C_hT^2 + \|\partial_x Z_{\mp(n-1)}\|T \right). \tag{3.13}$$

Let us assume by induction that $(Z_{\pm(n-1)}, Y_{\pm(n-1)}) \in C^{0,0,1}(\Gamma_T)$ satisfying

$$\|\partial_x Z_{\pm(n-1)}\| \leq 3C_{\varphi}, \quad \|\partial_x Y_{\pm(n-1)}\| \leq 4C_{\varphi}, \quad n \in \mathbb{N}, \tag{3.14}$$

which is satisfied at least for $n = 1$. It follows from (3.11) and (3.12) that

$$\|Z_{\pm(n)}^{(1)} - Z_{\pm(n)}^{(0)}\|, \|Y_{\pm(n)}^{(1)} - Y_{\pm(n)}^{(0)}\| \leq 8K_{\pm}(T)C_{\varphi}. \tag{3.15}$$

Continuing on with (3.12) and (3.15), we obtain

$$\|Z_{\pm(n)}^{(k+1)} - Z_{\pm(n)}^{(k)}\|, \|Y_{\pm(n)}^{(k+1)} - Y_{\pm(n)}^{(k)}\| \leq (4K_{\pm}(T))^k (8K_{\pm}(T)C_{\varphi}), \quad k \in \mathbb{N}. \tag{3.16}$$

Therefore, the sequence $\{Z_{\pm(n)}^{(k)}, Y_{\pm(n)}^{(k)}\}_{k \in \mathbb{N}}$ is Cauchy in $C^{0,0,0}(\Gamma_T)$ for each $n \in \mathbb{N}$ if $4K_{\pm}(T) < 1$. From the definition (3.13), bound (3.14), and $C_hT \leq C_{\varphi}$, we have

$$4K_{\pm}(T) \leq 3C_{\varphi}T + \frac{3}{2}C_{\varphi}T + 3C_{\varphi}T = \frac{15}{2}C_{\varphi}T \leq \frac{1}{2},$$

if $T \leq \frac{1}{15C_{\varphi}}$, according to the constraint (3.1). Hence, for each $n \in \mathbb{N}$, the sequence $\{Z_{\pm(n)}^{(k)}, Y_{\pm(n)}^{(k)}\}_{k \in \mathbb{N}}$ converges as $k \rightarrow \infty$ to a limit denoted by $(Z_{\pm(n)}, Y_{\pm(n)})$ in $C^{0,0,0}(\Gamma_T)$, as in (3.9).

Taking the limit $k \rightarrow \infty$ in the recursive system (3.7)–(3.8), we obtain the recursive system (3.4)–(3.5) for $(Z_{\pm(n)}, Y_{\pm(n)})$ in $C^{0,0,0}(\Gamma_T)$. Therefore, $(Z_{\pm(n)}, Y_{\pm(n)})$ is a local solution to the system (3.4)–(3.5) for each $n \in \mathbb{N}$ that satisfies bounds (3.11). Moreover, from the contraction principle, it follows that the local solution to the system (3.4)–(3.5) is unique in $C^{0,0,0}(\Gamma_T)$ for each $n \in \mathbb{N}$.

Step 2. We prove that for each $n \in \mathbb{N}$, the solution $(Z_{\pm(n)}, Y_{\pm(n)}) \in C^{0,0,0}(\Gamma_T)$ to the system (3.4) and (3.5) constructed in Step 1 belongs actually to $C^{0,0,1}(\Gamma_T)$ and satisfies the same bounds (3.14) as the previous approximation $(Z_{\pm(n-1)}, Y_{\pm(n-1)})$. By differentiating the system (3.4)–(3.5) with respect to x , we obtain a system of linear integral equations

$$U_{\pm(n)}(s; t, x) = \varphi'_{\pm}(\cdot) \left(1 - \frac{1}{4} \int_0^t (3U_{\pm(n)}(v; t, x) + V_{\pm(n)}(v; t, x))dv \right) + \int_0^s h''(\cdot) \left(1 - \frac{1}{4} \int_v^t (3U_{\pm(n)}(\tau; t, x) + V_{\pm(n)}(\tau; t, x))d\tau \right) dv \tag{3.17}$$

and

$$V_{\pm(n)}(s; t, x) = \partial_x Z_{\mp(n-1)}(\cdot) \left(1 - \frac{1}{4} \int_0^t (3U_{\pm(n)}(v; t, x) + V_{\pm(n)}(v; t, x))dv \right), \tag{3.18}$$

where the arguments of φ'_{\pm} , h'' , and $\partial_x Z_{\mp(n-1)}$ are the same as in the system (3.4)–(3.5). We recall that φ'_{\pm} , h'' are continuous and by the method of induction, $\partial_x Z_{\mp(n-1)}$ is also taken to be continuous,

for each $n \in \mathbb{N}$. Since $(Z_{\pm(n)}, Y_{\pm(n)}) \in C^{0,0,0}(\Gamma_T)$ is substituted in the arguments of φ'_{\pm} , h'' , and $\partial_x Z_{\mp(n-1)}$, we know that the coefficients of the linear system (3.17) and (3.18) are all continuous functions in Γ_T .

We first claim that there exists a unique solution of the linear system (3.17) and (3.18) in $C^{0,0,0}(\Gamma_T)$. Indeed, let us rewrite the system in the form

$$(I + P) \begin{bmatrix} U_{\pm(n)} \\ V_{\pm(n)} \end{bmatrix} = \begin{bmatrix} \varphi'_{\pm}(\cdot) + \int_0^s h''(\cdot) d\nu \\ \partial_x Z_{\mp(n-1)}(\cdot) \end{bmatrix},$$

where P is a perturbation to the identity matrix I given by

$$P \begin{bmatrix} U_{\pm(n)} \\ V_{\pm(n)} \end{bmatrix} := \frac{1}{4} \begin{bmatrix} \varphi'_{\pm}(\cdot) \int_0^t (3U_{\pm(n)}(\nu; t, x) + V_{\pm(n)}(\nu; t, x)) d\nu \\ \quad + \int_0^s h''(\cdot) \int_{\nu}^t (3U_{\pm(n)}(\tau; t, x) + V_{\pm(n)}(\tau; t, x)) d\tau d\nu \\ \partial_x Z_{\mp(n-1)}(\cdot) \int_0^t (3U_{\pm(n)}(\nu; t, x) + V_{\pm(n)}(\nu; t, x)) d\nu \end{bmatrix}.$$

We estimate the norm of each component of the perturbation P in $C^{0,0,0}(\Gamma_T)$ as follows

$$\left\| P \begin{bmatrix} U_{\pm(n)} \\ V_{\pm(n)} \end{bmatrix} \right\| \leq \frac{1}{4} TC_{\varphi} \begin{bmatrix} 6 & 2 \\ 9 & 3 \end{bmatrix} \begin{bmatrix} \|U_{\pm(n)}\| \\ \|V_{\pm(n)}\| \end{bmatrix}, \tag{3.19}$$

where we have used $C_h T \leq C_{\varphi}$ and $\|\partial_x Z_{\mp(n-1)}\| \leq 3C_{\varphi}$. Eigenvalues of the matrix in (3.19) are 0 and 9. If $TC_{\varphi} \leq \frac{1}{15}$, the norm induced by the perturbation P is strictly smaller than one. Therefore, the matrix integral operator $I + P$ is invertible and a unique solution $(U_{\pm(n)}, V_{\pm(n)})$ to the linear system (3.17) and (3.18) exists in $C^{0,0,0}(\Gamma_T)$.

Next, for every $(s, t, x_0) \in \Gamma_T$, we claim that the quotients

$$\frac{Z_{\pm(n)}(s; t, x) - Z_{\pm(n)}(s; t, x_0)}{x - x_0} \quad \text{and} \quad \frac{Y_{\pm(n)}(s; t, x) - Y_{\pm(n)}(s; t, x_0)}{x - x_0}$$

remain bounded as $x \rightarrow x_0$ for every $x_0 \in \mathbb{R}^+$. This is shown by repeating the estimates for the system (3.4) and (3.5), where we are using the constraint on T in (3.1), and the smoothness properties on φ_{\pm} , h , and $Z_{\mp(n-1)}$. Now, by repeating the estimates for bounded functions

$$E_{\pm(n)}(s; t, x, x_0) := \frac{Z_{\pm(n)}(s; t, x) - Z_{\pm(n)}(s; t, x_0)}{x - x_0} - U_{\pm(n)}(s; t, x_0)$$

and

$$F_{\pm(n)}(s; t, x, x_0) := \frac{Y_{\pm(n)}(s; t, x) - Y_{\pm(n)}(s; t, x_0)}{x - x_0} - V_{\pm(n)}(s; t, x_0)$$

and using uniqueness of solutions to the system (3.4) and (3.5) and their first variations (3.17)–(3.18), we obtain for every $(s, t, x_0) \in \Gamma_T$ that

$$\lim_{x \rightarrow x_0} E_{\pm(n)}(s; t, x, x_0) = 0 \quad \text{and} \quad \lim_{x \rightarrow x_0} F_{\pm(n)}(s; t, x, x_0) = 0.$$

Therefore, $(Z_{\pm(n)}, Y_{\pm(n)})$ are continuously differentiable with respect to x at every $x_0 \in \mathbb{R}^+$ and

$$\partial_x Z_{\pm(n)}(s; t, x) = U_{\pm(n)}(s; t, x) \quad \text{and} \quad \partial_x Y_{\pm(n)}(s; t, x) = V_{\pm(n)}(s; t, x), \quad (s, t, x) \in \Gamma_T. \tag{3.20}$$

It remains to verify bounds (3.14) for $(Z_{\pm(n)}, Y_{\pm(n)})$. It follows from the second line of (3.19) substituted to (3.18) that

$$\begin{aligned} \|V_{\pm(n)}\| &\leq \frac{3C_\varphi}{1 - \frac{3}{4}C_\varphi T} \left(1 + \frac{3T}{4}\|U_{\pm(n)}\|\right) \\ &\leq \frac{60C_\varphi}{19} \left(1 + \frac{3T}{4}\|U_{\pm(n)}\|\right). \end{aligned} \tag{3.21}$$

where we have used $C_\varphi T \leq \frac{1}{15}$. Substituting this estimate to the first line of (3.19) and to Equation (3.17) yields

$$\begin{aligned} \|U_{\pm(n)}\| &\leq \frac{2C_\varphi}{1 - 3C_\varphi T} (1 + C_\varphi T) \\ &\leq \frac{5}{2}C_\varphi (1 + C_\varphi T) \leq \frac{8}{3}C_\varphi, \end{aligned} \tag{3.22}$$

where we have used again $C_\varphi T \leq \frac{1}{15}$. Using the correspondence (3.20), we obtain

$$\|\partial_x Z_{\pm(n)}\| \leq 3C_\varphi, \quad \|\partial_x Y_{\pm(n)}\| \leq 4C_\varphi, \quad n \in \mathbb{N}. \tag{3.23}$$

The validity of the bounds (3.14) for every $n \in \mathbb{N}$ is verified by the induction method.

Step 3. We prove under the same constraint (3.1) on T that the sequence $\{Z_{\pm(n)}, Y_{\pm(n)}\}_{n \in \mathbb{N}}$ defined by the recursive system (3.4)–(3.5) converges in $C^{0,0,0}(\Gamma_T)$ to the solution $(Z_\pm, Y_\pm) \in C^{0,0,0}(\Gamma_T)$ satisfying the system (2.11) and (2.12) and the bound (3.2).

After the convergence to the limits (3.9) is proved, the index n in the system (3.4) and (3.5) can be incremented by one using the induction method. Convergence of iterations $\{Z_{\pm(n)}, Y_{\pm(n)}\}_{n \in \mathbb{N}}$ can be considered in $C^{0,0,0}(\Gamma_T)$ with standard methods.

It follows from (3.4) and (3.5) with the fundamental theorem of calculus that

$$\|Z_{\pm(n+1)} - Z_{\pm(n)}\| \leq \frac{1}{4} \left(C_\varphi T + \frac{1}{2} C_h T^2 \right) (3 \|Z_{\pm(n+1)} - Z_{\pm(n)}\| + \|Y_{\pm(n+1)} - Y_{\pm(n)}\|)$$

and

$$\begin{aligned} \|Y_{\pm(n+1)} - Y_{\pm(n)}\| &\leq \frac{1}{4} T \|\partial_x Z_{\mp(n)}\| (3 \|Z_{\pm(n+1)} - Z_{\pm(n)}\| + \|Y_{\pm(n+1)} - Y_{\pm(n)}\|) \\ &\quad + \|Z_{\mp(n)} - Z_{\mp(n-1)}\|, \end{aligned}$$

where C_φ and C_h are the same constants as above. Under the conditions (3.1) and (3.23), we obtain

$$\|Z_{\pm(n+1)} - Z_{\pm(n)}\| \leq \frac{1}{40} (3 \|Z_{\pm(n+1)} - Z_{\pm(n)}\| + \|Y_{\pm(n+1)} - Y_{\pm(n)}\|)$$

and

$$\|Y_{\pm(n+1)} - Y_{\pm(n)}\| \leq \frac{1}{20} (3 \|Z_{\pm(n+1)} - Z_{\pm(n)}\| + \|Y_{\pm(n+1)} - Y_{\pm(n)}\|) + \|Z_{\mp(n)} - Z_{\mp(n-1)}\|.$$

From the inequalities above, we obtain

$$\|Z_{\pm(n+1)} - Z_{\pm(n)}\| \leq \frac{1}{35} \|Z_{\mp(n)} - Z_{\mp(n-1)}\|,$$

and hence

$$\|Z_{+(n+1)} - Z_{+(n)}\| + \|Z_{-(n+1)} - Z_{-(n)}\| \leq \frac{1}{35} (\|Z_{+(n)} - Z_{+(n-1)}\| + \|Z_{-(n)} - Z_{-(n-1)}\|).$$

Therefore, the iteration map defined by the system (3.4)–(3.5) is a contraction in $C^{0,0,0}(\Gamma_T)$. Hence, the sequence $\{Z_{\pm(n)}, Y_{\pm(n)}\}_{n \in \mathbb{N}}$ is Cauchy in $C^{0,0,0}(\Gamma_T)$ and it converges as $n \rightarrow \infty$ to a limit, denoted as (Z_{\pm}, Y_{\pm}) , defined in the same function space. Moreover, taking the limit $n \rightarrow \infty$ in the iterative system (3.4)–(3.5), we obtain the system (2.11) and (2.12) for the limiting functions (Z_{\pm}, Y_{\pm}) . Therefore, the limiting functions (Z_{\pm}, Y_{\pm}) are solutions of the system (2.11)–(2.12) in $C^{0,0,0}(\Gamma_T)$. Since the sequence $\{Z_{\pm(n)}, Y_{\pm(n)}\}_{n \in \mathbb{N}}$ in $C^{0,0,0}(\Gamma_T)$ satisfies the bounds (3.11) that are independent of n , the limiting functions (Z_{\pm}, Y_{\pm}) satisfy the same bounds, which become bounds (3.2). Finally, it follows from the contraction method that the local solution (Z_{\pm}, Y_{\pm}) is unique in $C^{0,0,0}(\Gamma_T)$. \square

Lemma 2: *Under conditions of Lemma 1, the unique local solution to the system (2.11) and (2.12) belongs to the class $(Z_{\pm}, Y_{\pm}) \in C^{0,0,1}(\Gamma_T)$ and satisfies*

$$\|\partial_x Z_+\| + \|\partial_x Z_-\| \leq 15C_{\varphi}, \quad \|\partial_x Y_+\| + \|\partial_x Y_-\| \leq 45C_{\varphi}. \tag{3.24}$$

Proof: First, we prove existence of a unique solution $(U_{\pm}, V_{\pm}) \in C^{0,0,0}(\Gamma_T)$ to the system (2.19) and (2.20) under the conditions of Lemma 1. Since solutions for $(Z_{\pm}, Y_{\pm}) \in C^{0,0,0}(\Gamma_T)$ are already obtained in Lemma 1, the coefficients of the integral equation (2.19) and the arguments of the unknown functions U_{\mp} in (2.20) are all continuous functions in Γ_T .

Equation (2.19) represents a linear relation between U_{\pm} and V_{\pm} . Equation (2.20) is linear with respect to (V_+, V_-) and quadratic with respect to (U_+, U_-) . Therefore, first we solve (2.20) to obtain a unique map from (U_+, U_-) to (V_+, V_-) , then we substitute the map to (2.19) and solve the system uniquely in (U_+, U_-) by using the Schauder fixed-point theorem.

Let us define a ball in $C^{0,0,0}(\Gamma_T)$ of a finite radius given by

$$\|U_+\| + \|U_-\| \leq 15C_{\varphi} =: \delta. \tag{3.25}$$

The integral equation (2.20) is rewritten in the explicit form

$$V_{\pm}(s; t, x) + \frac{1}{4}U_{\mp}(\cdot) \int_s^t V_{\pm}(v; t, x)dv = F_{\pm} := U_{\mp}(\cdot) \left(1 - \frac{3}{4} \int_s^t U_{\pm}(v; t, x)dv \right) \tag{3.26}$$

where $U_{\mp}(\cdot)$ refers to

$$U_{\mp} \left(s; s, x - \frac{1}{4} \int_s^t [3Z_{\pm}(v; t, x) + Y_{\pm}(v; t, x)] dv \right). \tag{3.27}$$

For every (U_+, U_-) in the ball given by (3.25), we have

$$\left\| \frac{1}{4}U_{\mp}(\cdot) \int_s^t V_{\pm}(v; t, x)dv \right\| \leq \frac{1}{4}T\|U_{\mp}\|\|V_{\pm}\| \leq \frac{1}{4}\|V_{\pm}\|, \tag{3.28}$$

where we have used the constraint $C_{\varphi}T \leq \frac{1}{15}$. Therefore, the second term in (3.26) is strictly smaller than the first term in (3.26). Inverting the linear operator on V_{\pm} in $C^{0,0,0}(\Gamma_T)$ implies that for every U_{\pm} in the ball given by (3.25), there exists a unique solution $V_{\pm} \in C^{0,0,0}(\Gamma_T)$ of Equation (3.26) such that

$$\|V_{\pm}\| \leq \frac{4}{3}\|F_{\pm}\| \leq \frac{4}{3} \left(1 + \frac{3}{4}T\|U_{\pm}\| \right) \|U_{\mp}\| \leq \frac{7}{3}\|U_{\mp}\| \leq 3\|U_{\mp}\|. \tag{3.29}$$

This unique solution defines a map from $(U_+, U_-) \in C^{0,0,0}(\Gamma_T)$ to $(V_+, V_-) \in C^{0,0,0}(\Gamma_T)$. Since the integral equation (3.26) is a quadratic polynomial on $(U_+, U_-) \in C^{0,0,0}(\Gamma_T)$, the map $C^{0,0,0}(\Gamma_T) \ni (U_+, U_-) \mapsto (V_+, V_-) \in C^{0,0,0}(\Gamma_T)$ is C^∞ in the ball (3.25).

Let us estimate the Lipschitz constant for the map $C^{0,0,0}(\Gamma_T) \ni (U_+, U_-) \mapsto (V_+, V_-) \in C^{0,0,0}(\Gamma_T)$. Denote the values (V'_+, V'_-) that correspond to the values (U'_+, U'_-) . Note that the arguments of (U'_+, U'_-) are the same as those of (U_+, U_-) given by (3.27). Subtracting (3.26) for (U_+, U_-) and (U'_+, U'_-) , we obtain

$$\begin{aligned} V_\pm - V'_\pm + \frac{1}{4}(U_\mp - U'_\mp) \int_s^t V_\pm dv + \frac{1}{4}U'_\mp \int_s^t (V_\pm - V'_\pm)dv \\ = (U_\mp - U'_\mp) \left(1 - \frac{3}{4} \int_s^t U_\pm dv\right) - \frac{3}{4}U'_\pm \int_s^t (U_\pm - U'_\pm)dv. \end{aligned}$$

Using estimates similar to (3.28) and (3.29), we obtain

$$\begin{aligned} \|V_\pm - V'_\pm\| &\leq \frac{4}{3} \left(1 + \frac{3}{4}T\|U_\pm\| + \frac{1}{4}T\|V_\pm\|\right) \|U_\mp - U'_\mp\| + T\|U'_\mp\| \|U_\pm - U'_\pm\| \\ &\leq \frac{7}{3} \|U_\mp - U'_\mp\| + \|U_\pm - U'_\pm\| \leq 3\|U_\mp - U'_\mp\| + \|U_\pm - U'_\pm\|. \end{aligned} \tag{3.30}$$

Next, we substitute the map $C^{0,0,0}(\Gamma_T) \ni (U_+, U_-) \mapsto (V_+, V_-) \in C^{0,0,0}(\Gamma_T)$ to the integral equation (2.19) and rewrite it in the explicit form:

$$\begin{aligned} U_\pm(s; t, x) + \frac{1}{4}\varphi'_\pm(\cdot) \int_0^t (3U_\pm(v; t, x) + V_\pm(v; t, x))dv \\ + \frac{1}{4} \int_0^s h''(\cdot) \int_v^t [3U_\pm(\tau; t, x) + V_\pm(\tau; t, x)] d\tau dv = G_\pm := \varphi'_\pm(\cdot) + \int_0^s h''(\cdot) dv, \end{aligned} \tag{3.31}$$

where the arguments for φ'_\pm and h'' are uniquely defined continuous functions in Γ_T . Since the mapping $C^{0,0,0}(\Gamma_T) \ni (U_+, U_-) \mapsto (V_+, V_-) \in C^{0,0,0}(\Gamma_T)$ is nonlinear, we solve the system (3.31) by using the Schauder fixed-point theorem in the ball (3.25). Using bounds (3.29) and the constraint $C_h T \leq C_\varphi$, we estimate the integral terms in the left-hand side of system (3.31) as follows:

$$\left\| \frac{1}{4}\varphi'_\pm(\cdot) \int_0^t (3U_\pm(v; t, x) + V_\pm(v; t, x))dv \right\| \leq \frac{1}{4}TC_\varphi(3\|U_\pm\| + \|V_\pm\|) \leq \frac{1}{20} (\|U_+\| + \|U_-\|)$$

and

$$\begin{aligned} \left\| \frac{1}{4} \int_0^s h''(\cdot) \int_v^t [3U_\pm(\tau; t, x) + V_\pm(\tau; t, x)] d\tau dv \right\| &\leq \frac{1}{4}T^2C_h(3\|U_\pm\| + \|V_\pm\|) \\ &\leq \frac{1}{20} (\|U_+\| + \|U_-\|), \end{aligned}$$

where we have used the constraint $TC_\varphi \leq \frac{1}{15}$. The integral terms in system (3.31) are strictly smaller than the identity terms in the ball (3.25). Therefore, writing the fixed-point problem in the form

$$\begin{bmatrix} U_+ \\ U_- \end{bmatrix} = \begin{bmatrix} G_+ \\ G_- \end{bmatrix} + \mathcal{T} \begin{bmatrix} U_+ \\ U_- \end{bmatrix} \tag{3.32}$$

shows that the nonlinear integral operator \mathcal{T} maps the ball (3.25) to its smaller subset. The inhomogeneous terms G_\pm given by (3.31) are bounded by $\|G_\pm\| \leq 2C_\varphi$. By the Schauder fixed-point theorem, there exists a solution $(U_+, U_-) \in C^{0,0,0}(\Gamma_T)$ to the fixed-point problem (3.32) in the ball

(3.25). The solution to the system of integral Equation (3.31) satisfies the bound

$$\|U_+\| + \|U_-\| \leq \frac{4C_\varphi}{1 - 3TC_\varphi/2} \leq \frac{40}{9}C_\varphi < \delta$$

and hence belongs to the ball (3.25). The solution is unique if the operator \mathcal{T} is a contraction in the ball (3.25).[18] This is proved directly using the Lipschitz continuity of the map $C^{0,0,0}(\Gamma_T) \ni (U_+, U_-) \mapsto (V_+, V_-) \in C^{0,0,0}(\Gamma_T)$ with the Lipschitz constant given by (3.30). Indeed, we have

$$\begin{aligned} \left\| \frac{1}{4}\varphi'_\pm \int_0^t [3(U_\pm - U'_\pm) + (V_\pm - V'_\pm)]dv \right\| &\leq \frac{1}{4}TC_\varphi(3\|U_\pm - U'_\pm\| + \|V_\pm - V'_\pm\|) \\ &\leq \frac{1}{15}\|U_\pm - U'_\pm\| + \frac{1}{20}\|U_\mp - U'_\mp\| \end{aligned}$$

and a similar estimate for the second term in T . Therefore, the operator \mathcal{T} is a contraction in the ball (3.25) so that the solution $(U_+, U_-) \in C^{0,0,0}(\Gamma_T)$ is unique.

For every $(s, t, x_0) \in \Gamma_T$, we repeat the estimates for the quotients

$$\frac{Z_\pm(s; t, x) - Z_\pm(s; t, x_0)}{x - x_0} \quad \text{and} \quad \frac{Y_\pm(s; t, x) - Y_\pm(s; t, x_0)}{x - x_0}$$

and prove that they remain bounded as $x \rightarrow x_0$ for every $x_0 \in \mathbb{R}^+$. Furthermore, by repeating the estimates for bounded functions

$$E_\pm(s; t, x, x_0) := \frac{Z_\pm(s; t, x) - Z_\pm(s; t, x_0)}{x - x_0} - U_\pm(s; t, x_0)$$

and

$$F_\pm(s; t, x, x_0) := \frac{Y_\pm(s; t, x) - Y_\pm(s; t, x_0)}{x - x_0} - V_\pm(s; t, x_0)$$

and using uniqueness of solutions to the system (2.11) and (2.12) and their first variations (2.19)–(2.20), we obtain for every $(s, t, x_0) \in \Gamma_T$ that

$$\lim_{x \rightarrow x_0} E_\pm(s; t, x, x_0) = 0 \quad \text{and} \quad \lim_{x \rightarrow x_0} F_\pm(s; t, x, x_0) = 0.$$

Therefore, (Z_\pm, Y_\pm) are continuously differentiable with respect to x at every $x_0 \in \mathbb{R}^+$ and the correspondence (2.17) is established. Bounds (3.24) follow from bounds (3.25) and (3.29). \square

Remark 5: Bounds (3.24) are bigger than the n -independent bounds (3.23). Nevertheless, the bigger bounds (3.24) are still sufficient for invertibility of the characteristic coordinates $\xi_\pm(s; t, x)$ with respect to x for every $(s, t, x) \in \Gamma_T$. Indeed, bounds (3.24) imply that

$$\left\| \frac{1}{4} \int_s^t [3U_\pm(v; t, x) + V_\pm(v; t, x)] dv \right\| \leq \frac{1}{4}T(3\|U_\pm\| + \|V_\pm\|) \leq \frac{3}{4}T(\|U_+\| + \|U_-\|) \leq \frac{3}{4},$$

where the constraint $C_\varphi T \leq \frac{1}{15}$ has been used. Therefore, it follows from (2.21) that if (U_\pm, V_\pm) are x -derivatives of the local solution (Z_\pm, Y_\pm) in Lemmas 1 and 2, then $\xi_\pm(s; t, x) > 0$ for every $(s, t, x) \in \Gamma_T$.

Lemma 3: Under conditions of Lemma 1, the unique local solution to the system of integral Equations (2.11) and (2.12) belongs to the class $(Z_\pm, Y_\pm) \in C^{1,1,1}(\Gamma_T)$.

Proof: By Lemmas 1 and 2, there exists a unique solution $(Z_\pm, Y_\pm) \in C^{0,0,1}(\Gamma_T)$ to the system (2.11) and (2.12). We show that the solution actually belongs to $C^{1,1,1}(\Gamma_T)$.

Let us compute the derivatives of the system (2.11) and (2.12) in t :

$$\begin{aligned} \partial_t Z_{\pm}(s; t, x) &= -\frac{1}{4}\varphi'_{\pm}(\cdot) (3Z_{\pm}(t; t, x) + Y_{\pm}(t; t, x)) - \frac{1}{4} \int_0^s h''(\cdot) dv (3Z_{\pm}(t; t, x) + Y_{\pm}(t; t, x)) \\ &\quad - \frac{1}{4}\varphi'_{\pm}(\cdot) \int_0^t (3\partial_t Z_{\pm}(v; t, x) + \partial_t Y_{\pm}(v; t, x)) dv \\ &\quad - \frac{1}{4} \int_0^s h''(\cdot) \left(\int_v^t (3\partial_t Z_{\pm}(\tau; t, x) + \partial_t Y_{\pm}(\tau; t, x)) d\tau \right) dv \end{aligned} \tag{3.33}$$

and

$$\partial_t Y_{\pm}(s; t, x) = -\frac{1}{4}\partial_x Z_{\mp}(\cdot) \int_0^t (3\partial_t Z_{\pm}(v; t, x) + \partial_t Y_{\pm}(v; t, x)) dv, \tag{3.34}$$

where the arguments of φ'_{\pm} , h'' , and $\partial_x Z_{\pm}$ are the same as in the system (2.11)–(2.12). They are given continuous functions of their arguments.

Using similar estimates as in Step 2 in the proof of Lemma 1, we can use invertibility of the linear integral operators and prove existence and uniqueness of solutions to the system (3.33)–(3.34) for $(\partial_t Z_{\pm}, \partial_t Y_{\pm})$ in $C^{0,0,0}(\Gamma_T)$. Moreover, the t -derivatives of (Z_{\pm}, Y_{\pm}) satisfy the following bounds:

$$\|\partial_t Z_{\pm}\| \leq \frac{1}{4} \left(C_{\varphi} T + \frac{1}{2} C_h T^2 \right) (3\|\partial_t Z_{\pm}\| + \|\partial_t Y_{\pm}\|) + \frac{1}{4} (C_{\varphi} + C_h T) (3\|Z_{\pm}\| + \|Y_{\pm}\|)$$

and

$$\|\partial_t Y_{\pm}\| \leq \frac{1}{4} \|\partial_x Z_{\mp}\| T (3\|\partial_t Z_{\pm}\| + \|\partial_t Y_{\pm}\|).$$

Using bounds (3.1), (3.2), and (3.24), we confirm that $\|\partial_t Z_{\pm}\|$ and $\|\partial_t Y_{\pm}\|$ are bounded in Γ_T . Therefore, the solution (Z_{\pm}, Y_{\pm}) to the system (2.11) and (2.12) belongs to $C^{0,1,1}(\Gamma_T)$.

Finally, we compute the derivatives of the system (2.11) and (2.12) in s :

$$\partial_s Z_{\pm}(s; t, x) = h' \left(x - \frac{1}{4} \int_s^t (3Z_{\pm}(v; t, x) + Y_{\pm}(v; t, x)) dv \right) \tag{3.35}$$

and

$$\partial_s Y_{\pm}(s; t, x) = \partial_s Z_{\mp}(\cdot) + \partial_t Z_{\mp}(\cdot) + \frac{1}{4} \partial_x Z_{\mp}(\cdot) (3Z_{\pm}(s; t, x) + Y_{\pm}(s; t, x)). \tag{3.36}$$

From (3.35), we confirm that $\|\partial_s Z_{\pm}\|$ is bounded in Γ_T . Then, from (3.36) and the bounds on $Z_{\pm} \in C^{1,1,1}(\Gamma_T)$, we confirm that $\|\partial_s Y_{\pm}\|$ is also bounded in Γ_T . Therefore, the solution (Z_{\pm}, Y_{\pm}) to the system (2.11) and (2.12) belongs to $C^{1,1,1}(\Gamma_T)$. \square

The proof of Theorem 1 follows from the results of Lemmas 1–3, as well as the correspondence result of Proposition 2. Solutions to the shallow water system (1.1) are related to the solutions to the system (1.3) using the transformation (1.6).

4. Global solution to system (2.11)–(2.12)

It follows from the correspondence $z_{\pm}(t, x) = Z_{\pm}(t; t, x)$ for $(t, x) \in \Omega_T$ and the bounds (3.2) and (3.24) that the local solution to the system (1.3) at time $t = T$ satisfies the estimates

$$\|z_{\pm}(T, \cdot)\|_{C_b^1} \leq 15C_{\varphi}. \tag{4.1}$$

If we attempt to continue this local solution beyond the time $t = T$ by a recurrent use of Lemmas 1–3, then we will run into the following obstacle.

Let us denote the solution to the system (2.11) and (2.12) given by Lemmas 1–3 extended from time T_{m-1} to T_m by $(Z_{\pm}^{(m)}, Y_{\pm}^{(m)})$ for $m \in \mathbb{N}$, where $T_0 = 0$. Then, iterating bound (4.1) with the bounds (3.2) and (3.24), we obtain

$$\|z_{\pm}^{(m)}(T_m, \cdot)\|_{C_b^1} \leq 15^m C_{\varphi}, \quad m \in \mathbb{N}. \tag{4.2}$$

Furthermore, using the constraint (3.1) on the continuation time, we have

$$T_m - T_{m-1} \leq \frac{1}{15^{m+1} C_{\varphi}}, \quad m \in \mathbb{N}. \tag{4.3}$$

Since the series $\sum_{m \in \mathbb{N}} 15^{-m}$ converges, we have $T_{\infty} := \lim_{m \rightarrow \infty} T_m < \infty$, so that the continuation technique results in a local solution to the system (1.3) over a finite time span $[0, T_{\infty})$.

In order to be able to extend the local solution to the system (2.11) and (2.12) without restriction on time T , we shall find sharper bounds on the growth of the x -derivatives of the solution (Z_{\pm}, Y_{\pm}) . This is only possible under additional conditions (1.11) and (1.12) on the function h and initial data, the latter conditions are rewritten in the form (1.14). The key result is the following lemma.

Lemma 4: *In addition to the conditions of Lemma 1, assume that conditions (1.11) and (1.14) are satisfied. Then, the unique solution $(Z_{\pm}, Y_{\pm}) \in C^{0,0,1}(\Gamma_T)$ to the system (2.11) and (2.12) constructed in Lemmas 1 and 2 satisfy the improved bounds*

$$\|\partial_x Z_{\pm}\|, \|\partial_x Y_{\pm}\| \leq 2C_{\varphi}. \tag{4.4}$$

Proof: The components (U_{\pm}, V_{\pm}) satisfy the system (2.19) and (2.20) with the correspondence (2.17). By Proposition 3 and Remark 5, we have $0 < \xi_{\pm}(s; t, x) \leq 1$, $U_{\pm}(s; t, x) \geq 0$, and $V_{\pm}(s; t, x) \geq 0$ for every $(s, t, x) \in \Gamma_T$, where ξ_{\pm} are related to U_{\pm} and V_{\pm} by (2.21). Therefore, the system (2.19) and (2.20) implies the bounds

$$\|U_{\pm}\| \leq C_{\varphi} + C_h T \leq 2C_{\varphi}, \quad \|V_{\pm}\| \leq \|U_{\mp}\| \leq 2C_{\varphi},$$

where we have used $C_h T \leq C_{\varphi}$ as in Lemma 1. Due to the correspondence (2.17), we have obtained the bounds (4.4). □

The sharper bounds (4.4) can be used to continue the local solution $z_{\pm}(t, x) = Z_{\pm}(t; t, x)$ to the system (1.3) globally in time. The next lemma establish piecewise continuation of solutions to the system (2.11) and (2.12) in $C^{1,1,1}(\Gamma_T)$ for larger values of T .

Lemma 5: *Let $(Z_{\pm}^{(m)}, Y_{\pm}^{(m)})$ for $m \in \mathbb{N}$ denote the sequence of solutions to the system (2.11) and (2.12) on the interval $[T_{m-1}, T_m]$ starting with initial data*

$$z_{\pm}(T_{m-1}, x) = Z_{\pm}^{(m-1)}(T_{m-1}; T_{m-1}, x),$$

where $T_0 = 0$ and $Z_{\pm}^{(0)}(0; 0, x) = \varphi_{\pm}(x)$. Assume $h \in C_b^2(\mathbb{R}^+)$ and $\varphi_{\pm} \in C_b^1(\mathbb{R}^+)$ satisfy the bounds (1.8), (1.11), (1.13), and (1.14). Define $C_{\varphi} := \max\{\|\varphi_+\|_{C_b^1}, \|\varphi_-\|_{C_b^1}\}$ and $C_h := \|h\|_{C_b^2}$. Assume that $(Z_{\pm}^{(m)}, Y_{\pm}^{(m)}) \in C^{1,1,1}(\Gamma_{T_m - T_{m-1}})$ for an $m \in \mathbb{N}$ satisfies the bounds

$$\|Z_{\pm}^{(m)}\|, \|Y_{\pm}^{(m)}\|, \|\partial_x Z_{\pm}^{(m)}\|, \|\partial_x Y_{\pm}^{(m)}\| \leq (m + 1)C_{\varphi}. \tag{4.5}$$

Then, the system (2.11) and (2.12) admits a unique solution in class

$$(Z_{\pm}^{(m+1)}, Y_{\pm}^{(m+1)}) \in C^{1,1,1}(\Gamma_{T_{m+1}-T_m})$$

satisfying the bounds

$$\|Z_{\pm}^{(m+1)}\|, \|Y_{\pm}^{(m+1)}\|, \|\partial_x Z_{\pm}^{(m+1)}\|, \|\partial_x Y_{\pm}^{(m+1)}\| \leq (m+2)C_{\varphi}, \tag{4.6}$$

while the time span $[T_m, T_{m+1}]$ is defined by

$$T_{m+1} - T_m := \min\left(\frac{C_{\varphi}}{C_h}, \frac{1}{15(m+1)C_{\varphi}}\right). \tag{4.7}$$

Proof: The first step of the induction method with bound (4.5) and the time constraint (4.7) is justified by Lemmas 1–4.

By Proposition 1, the system (2.11) and (2.12) remains closed in $\Gamma_{T_m-T_{m-1}}$, so that $z_{\pm}(T_m, x) \leq 0$ and $\partial_x z_{\pm}(T_m, x) \geq 0$ remain true for every $x \in \mathbb{R}^+$. Then, the system (2.11) and (2.12) remains closed in $\Gamma_{T_{m+1}-T_m}$ as long as the solution exists. Let us denote $T := T_{m+1} - T_m$.

We review bounds used in the proof of Lemma 1. Since the superscript now refer to the $(m+1)$ th local solution defined on the interval $[T_m, T_{m+1}]$, we only look at the convergence of iterations defined by the implicit system (3.4) and (3.5). It follows from these integral equations that bounds (3.11) for the successive approximations $\{Z_{\pm(n)}^{(m+1)}, Y_{\pm(n)}^{(m+1)}\}_{n \in \mathbb{N}}$ become

$$\begin{cases} \|Z_{\pm(n)}^{(m+1)}\| \leq (m+1)C_{\varphi} + C_h T \leq (m+2)C_{\varphi}, \\ \|Y_{\pm(n)}^{(m+1)}\| = \|Z_{\pm(n-1)}^{(m+1)}\| \leq (m+2)C_{\varphi}, \end{cases} \quad n \in \mathbb{N}, \tag{4.8}$$

where we have used $C_h T \leq C_{\varphi}$ according to the constraint (4.7). If convergence of successive approximations $\{Z_{\pm(n)}^{(m+1)}, Y_{\pm(n)}^{(m+1)}\}_{n \in \mathbb{N}}$ as $n \rightarrow \infty$ is proved, then bounds (4.8) yield the first bounds in (4.6). To prove the convergence, we first assume as in Step 1 that

$$\|\partial_x Z_{\pm(n-1)}^{(m+1)}\| \leq (2m+3)C_{\varphi}, \quad \|\partial_x Y_{\pm(n-1)}^{(m+1)}\| \leq (3m+4)C_{\varphi}, \quad n \in \mathbb{N}, \tag{4.9}$$

which is true for $n = 1$. From the definition (3.13), bounds (4.5), (4.9), and $C_h T \leq C_{\varphi}$, convergence of successive approximations at the second level of Picard iterations in $C^{0,0,0}(\Gamma_{T_{m+1}-T_m})$ (Step 1) is guaranteed if

$$4K_{\pm}(T) \leq 3(m+1)C_{\varphi}T + \frac{3}{2}C_h T^2 + (2m+3)C_{\varphi}T \leq \frac{5(2m+3)}{2}C_{\varphi}T \leq \frac{2m+3}{6(m+1)} < 1, \tag{4.10}$$

where we have used $C_{\varphi}T \leq \frac{1}{15(m+1)}$ as in the constraint (4.7). Thus, successive approximations at the second level of Picard iterations converge in $C^{0,0,0}(\Gamma_T)$ to the solution $\{Z_{\pm(n)}^{(m+1)}, Y_{\pm(n)}^{(m+1)}\}_{n \in \mathbb{N}}$ for every $n \in \mathbb{N}$.

We hence check that $\{Z_{\pm(n)}^{(m+1)}, Y_{\pm(n)}^{(m+1)}\}_{n \in \mathbb{N}}$ belongs to $C^{0,0,1}(\Gamma_T)$ (Step 2). Let us now rewrite bounds (3.19) in order to check consistency with the bounds (4.9). We obtain

$$\left\| P \begin{bmatrix} U_{\pm(n)} \\ V_{\pm(n)} \end{bmatrix} \right\| \leq \frac{1}{4}TC_{\varphi} \begin{bmatrix} 3(m+2) & (m+2) \\ 3(2m+3) & (2m+3) \end{bmatrix} \begin{bmatrix} \|U_{\pm(n)}\| \\ \|V_{\pm(n)}\| \end{bmatrix}, \tag{4.11}$$

where we have used $C_h T \leq C_{\varphi}$ and $\|\partial_x Z_{\pm(n-1)}^{(m+1)}\| \leq (2m+3)C_{\varphi}$. Since the upper bound in (4.11) has the norm being strictly smaller than one, under the constraint (4.7) on the time step T , we establish

existence and uniqueness of partial derivatives of $(Z_{\pm(n)}^{(m+1)}, Y_{\pm(n)}^{(m+1)})$ in x for each $n \in \mathbb{N}$. Moreover, we can estimate them by obtaining bounds similar to (3.21) and (3.22). Using (4.11), we obtain

$$\begin{aligned} \|\partial_x Y_{\pm(n)}^{(m+1)}\| &\leq \frac{(2m+3)C_\varphi}{1 - \frac{2m+3}{4}C_\varphi T} \left(1 + \frac{3T}{4}\|\partial_x Z_{\pm(n)}^{(m+1)}\|\right) \\ &\leq \frac{20(2m+3)C_\varphi}{19} \left(1 + \frac{3T}{4}\|\partial_x Z_{\pm(n)}^{(m+1)}\|\right), \end{aligned} \tag{4.12}$$

where we have used (4.7) as well as $2m+3 \leq 3(m+1)$. By using (4.7), (4.11), (4.12), $2m+3 \leq 3(m+1)$, and $m+2 \leq 2(m+1)$, we obtain

$$\begin{aligned} \|\partial_x Z_{\pm(n)}^{(m+1)}\| &\leq \frac{(m+2)C_\varphi}{1 - \frac{15(m+2)}{19}C_\varphi T} \left(1 + \frac{5(2m+3)}{19}C_\varphi T\right) \\ &\leq \frac{20}{17}(m+2)C_\varphi \leq (2m+3)C_\varphi. \end{aligned} \tag{4.13}$$

Substituting (4.13) to (4.12), we obtain

$$\begin{aligned} \|\partial_x Y_{\pm(n)}^{(m+1)}\| &\leq \frac{20(2m+3)C_\varphi}{19} \left(1 + \frac{3(2m+3)}{4}C_\varphi T\right) \\ &\leq \frac{23(2m+3)}{19}C_\varphi \leq (3m+4)C_\varphi. \end{aligned} \tag{4.14}$$

By the induction method, we obtain that bounds (4.9) are valid for every $n \in \mathbb{N}$.

Convergence of the successive approximations $\{Z_{\pm(n)}^{(m+1)}, Y_{\pm(n)}^{(m+1)}\}_{n \in \mathbb{N}}$ at the first level of Picard iterations is proved in $C^{0,0,0}(\Gamma_T)$ similarly to the proof of Lemma 1 (Step 3). Since the sequence $\{Z_{\pm(n)}^{(m+1)}, Y_{\pm(n)}^{(m+1)}\}_{n \in \mathbb{N}}$ satisfies the bounds (4.8) that are independent of n , the limiting functions $(Z_{\pm}^{(m+1)}, Y_{\pm}^{(m+1)}) \in C^{0,0,0}(\Gamma_T)$ satisfy the first two bounds in (4.6).

Although the bounds (4.9) are independent of n , we still need to prove that $(Z_{\pm}^{(m+1)}, Y_{\pm}^{(m+1)})$ belong to $C^{0,0,1}(\Gamma_T)$. We hence follow the proof of Lemma 2 and obtain $(Z_{\pm}^{(m+1)}, Y_{\pm}^{(m+1)}) \in C^{0,0,1}(\Gamma_T)$ together with the bounds

$$\|\partial_x Z_+^{(m+1)}\| + \|\partial_x Z_-^{(m+1)}\| \leq 15(m+1)C_\varphi, \quad \|\partial_x Y_+^{(m+1)}\| + \|\partial_x Y_-^{(m+1)}\| \leq 45(m+1)C_\varphi. \tag{4.15}$$

Although the bounds (4.15) are bigger than bounds (4.9), which are independent of n , they are sufficient to control the local solution $(Z_{\pm}^{(m+1)}, Y_{\pm}^{(m+1)})$ on Γ_T . In particular, the characteristic coordinates are still invertible in x , because the integral part of (2.21) is estimated as follows:

$$\frac{1}{4}T \left(3\|\partial_x Z_{\pm}^{(m+1)}\| + \|\partial_x Y_{\pm}^{(m+1)}\|\right) \leq \frac{3}{4}15(m+1)C_\varphi T \leq \frac{3}{4}.$$

As a result, for the local solution in $(Z_{\pm}^{(m+1)}, Y_{\pm}^{(m+1)}) \in C^{0,0,1}(\Gamma_T)$, we still have $\xi_{\pm}(s; t, x) > 0$ for every $(s, t, x) \in \Gamma_T$.

The proof of Lemma 3 applies verbatim, so that we actually have $(Z_{\pm}^{(m+1)}, Y_{\pm}^{(m+1)}) \in C^{1,1,1}(\Gamma_T)$.

Finally, we improve the bounds (4.15) by using the technique in Lemma 4. In particular, we have $\partial_x Z_{\pm}^{(m+1)}(s; t, x) \geq 0$ and $\partial_x Y_{\pm}^{(m+1)}(s; t, x) \geq 0$, and $\xi_{\pm}(s; t, x) \leq 1$ for every $(s, t, x) \in \Gamma_T$. As a result, the system (2.19) and (2.20) imply the bounds

$$\|\partial_x Z_{\pm}^{(m+1)}\| \leq (m+1)C_\varphi + C_h T \leq (m+2)C_\varphi, \quad \|\partial_x Y_{\pm}^{(m+1)}\| \leq \|\partial_x Z_{\mp}^{(m+1)}\| \leq (m+2)C_\varphi,$$

which yields the last two bounds in (4.6). □

With Lemma 5, we finally extend the local solution to every $T > 0$ and thus prove Theorem 2. By Lemma 5 and the induction method, we construct a sequence of local solutions $\{(Z_{\pm}^{(m)}, Y_{\pm}^{(m)})\}_{m \in \mathbb{N}} \in C^{1,1,1}(\Gamma_{T_m - T_{m-1}})$ to the system (2.11) and (2.12). The sequence is extended to the time T_m , which is obtained from (4.7) as

$$T_m = \sum_{k=1}^m T_k - T_{k-1} = \sum_{k=1}^m \frac{1}{15kC_{\varphi}}, \quad (4.16)$$

where we assumed $C_h \leq 15C_{\varphi}^2$ for simplicity. Since the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the sequence of local solutions is extended to arbitrary time $T > 0$ by incrementing the values of m .

By Proposition 2, we obtain the classical solution to system (1.3) by $z_{\pm}(t, x) = Z_{\pm}(t; t, x)$ for every $(t, x) \in \Gamma_T$ and every $T > 0$. Using the transformation formulas (1.6), we obtain the classical solution (u, η) to the shallow water system (1.1). Thus, the proof of Theorem 2 is complete.

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No potential conflict of interest was reported by the authors.

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