



# Stability of smooth solitary waves in the $b$ -Camassa–Holm equation

Stéphane Lafortune<sup>a,\*</sup>, Dmitry E. Pelinovsky<sup>b,c</sup>

<sup>a</sup> Department of Mathematics, College of Charleston, Charleston, SC 29401, USA

<sup>b</sup> Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1

<sup>c</sup> Department of Applied Mathematics, Nizhny Novgorod State Technical University n.a. R. E. Alekseev, 603950 Nizhny Novgorod, Russia

## ARTICLE INFO

### Article history:

Received 20 January 2022

Received in revised form 21 July 2022

Accepted 22 July 2022

Available online 1 August 2022

Communicated by Víctor M. Pérez-García

### Keywords:

Camassa–Holm equation

Traveling wave solutions

Stability of solitary waves

Integrability

## ABSTRACT

We derive the precise stability criterion for smooth solitary waves in the  $b$ -family of Camassa–Holm equations. The smooth solitary waves exist on the constant background. In the integrable cases  $b = 2$  and  $b = 3$ , we show analytically that the stability criterion is satisfied and smooth solitary waves are orbitally stable with respect to perturbations in  $H^3(\mathbb{R})$ . In the non-integrable cases, we show numerically and asymptotically that the stability criterion is satisfied for every  $b > 1$ . The orbital stability theory relies on a Hamiltonian formulation of the  $b$ -family which is different from the Hamiltonian formulations available for  $b = 2$  and  $b = 3$ .

© 2022 Elsevier B.V. All rights reserved.

## 1. Introduction

The  $b$ -family of Camassa–Holm equations (which we simply call  $b$ -CH) is written for the scalar velocity variable  $u = u(t, x)$  in the form

$$u_t - u_{xxx} + (b + 1)uu_x = bu_xu_{xx} + uu_{xxx}, \quad (1)$$

where  $b$  is arbitrary parameter. The  $b$ -CH model was introduced in [1,2] by using transformations of the integrable hierarchy of KdV equations. The  $b$ -CH model is not integrable in general but it has the same asymptotic accuracy as the integrable cases of  $b = 2$  called the Camassa–Holm equation [3] and  $b = 3$  called the Degasperis–Procesi equation [4]. The  $b$ -CH model describes the horizontal velocity  $u = u(t, x)$  for the unidirectional propagation of waves along the surface of a shallow water flowing over a flat bed at a certain depth [5]. The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations for modeling of shallow water waves was discussed in [6–8].

Peaked and smooth solitary waves exist in the  $b$ -CH Eq. (1), depending on the values of the parameter  $b$  and the parameter  $k$  for the constant background. Traveling waves in the  $b$ -CH equation were studied by using dynamical system methods [9] and hodograph transformations [10].

Early numerical simulations on the zero background ( $k = 0$ ) in [11,12] showed that the initial data resolves into a sequence of peaked solitary waves called *peakons* for  $b > 1$  and a sequence

of smooth solitary waves called *leftons* for  $b < -1$ . In the intermediate case of  $b \in (-1, 1)$ , the initial data generates a rarefactive wave with exponentially decaying tails. Recent numerical experiments in [13] added more examples of dynamics of peaked solitary waves which appear to be unstable for  $b < 1$  and stable for  $b > 1$ .

Stability of both peaked and smooth solitary waves on the zero background ( $k = 0$ ) has been now well understood. Orbital stability of *leftons* for  $b < -1$  was shown in [14] by using the Hamiltonian structure and conservation laws from [15–17] and analyzing perturbations in some exponentially weighted spaces. Orbital stability of *peakons* was shown for  $b = 2$  in [18,19] and for  $b = 3$  in [20] by using conservation of two energy integrals in the energy space  $H^1(\mathbb{R})$  for  $b = 2$  and  $L^2(\mathbb{R}) \cap L^3(\mathbb{R})$  for  $b = 3$ . However, the initial-value problem for the  $b$ -CH equation with  $b > 1$  is ill-posed in  $H^s(\mathbb{R})$  for  $s < \frac{3}{2}$  [21] due to the lack of continuous dependence and norm inflation, hence the orbital stability in the energy space is only conditional with respect to the existence of local solutions. It was shown in [22] that  $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$  is the largest space where the initial-value problem near the peaked solitary waves is defined. However, the  $W^{1,\infty}(\mathbb{R})$  norm of perturbations grows generally and induces the nonlinear instability of *peakons* in  $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ . This was first shown with the method of characteristics for  $b = 2$  in [23] (and in the periodic setting, in [24]) and for the cubic Novikov equation in [25]. In our previous work [26], we have proven spectral and linear instability of *peakons* in the  $b$ -CH equation for every  $b$ .

Solitary waves on the constant background  $k \neq 0$  are smooth. The smooth solitary waves can be found from the transformation  $u(t, x) = k + v(t, x - kt)$ , where  $v(t, x)$  satisfies the equivalent

\* Corresponding author.

E-mail addresses: [lafortunes@cofc.edu](mailto:lafortunes@cofc.edu) (S. Lafortune), [dmpeli@math.mcmaster.ca](mailto:dmpeli@math.mcmaster.ca) (D.E. Pelinovsky).

version of the  $b$ -CH equation:

$$v_t - v_{bx} + (b + 1)vv_x = kv_x + bv_x v_{xx} + v v_{xxx}. \tag{2}$$

If  $v(t, x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , then  $u(t, x) \rightarrow k$  as  $|x| \rightarrow \infty$ .

Orbital stability of smooth solitary waves was obtained for  $b = 2$  in [27] and for  $b = 3$  in [28] by working with the conserved energy integrals in the energy space. Since the Hamiltonian structure used in [27,28] is a special feature due to integrability of the  $b$ -CH equations for  $b = 2$  and  $b = 3$ , these results cannot be extended to other values of  $b$ .

The purpose of this work is to study orbital stability of the smooth solitary waves in the  $b$ -CH equation for any  $b > 1$ . We explore the Hamiltonian formulation of the  $b$ -CH equation [15–17] and the characterization of smooth traveling waves [29]. As a result of relevant computations, we obtain the precise criterion for orbital stability of the smooth solitary waves with respect to perturbations in  $H^3(\mathbb{R})$ . We show analytically for  $b = 2$  and  $b = 3$  and numerically for other values of  $b > 1$  that the stability criterion is satisfied. This yields orbital stability of the smooth solitary waves with respect to perturbations in  $H^3(\mathbb{R})$ .

The Hamiltonian formulation of Eq. (1) is developed by using the momentum density  $m := u - u_{xx}$ , for which Eq. (1) can be rewritten in the form:

$$m_t + um_x + bmu_x = 0. \tag{3}$$

If  $b \neq 1$ , this equation can be cast in the Hamiltonian form

$$\frac{dm}{dt} = J_m \frac{\delta E}{\delta m}, \tag{4}$$

where

$$J_m = -\frac{1}{b-1}(bm\partial_x + m_x)(1 - \partial_x^2)^{-1}\partial_x^{-1}(b\partial_x m - m_x) \tag{5}$$

is the skew-adjoint operator in  $L^2(\mathbb{R})$  such that  $J_m^* = -J_m$  and

$$E(m) = \int_{\mathbb{R}} m dx \tag{6}$$

is the conserved mass integral. When solutions of Eq. (3) are considered on the zero background, there exist two other conserved quantities [15] given by

$$F_1(m) = \int_{\mathbb{R}} m^{\frac{1}{b}} dx \tag{7}$$

and

$$F_2(m) = \int_{\mathbb{R}} \left( \frac{m_x^2}{b^2 m^2} + 1 \right) m^{-\frac{1}{b}} dx. \tag{8}$$

As is explained in Appendix, the two conserved quantities  $F_1$  and  $F_2$  are the Casimir functionals for the Hamiltonian operator  $J_m$  satisfying

$$J_m \frac{\delta F_1}{\delta m} = 0, \quad J_m \frac{\delta F_2}{\delta m} = 0.$$

The conserved quantities  $E$  and  $F_2$  were used in the study of orbital stability of *leftons* for  $b < -1$  in the exponentially decaying spaces, for which these two functionals are well-defined [14].

In the integrable cases  $b = 2, 3$ , the nonlocal Hamiltonian operator (5) defines just one set of compatible Hamiltonian structures. When  $b = 2$ , the first two Hamiltonian operators and their corresponding energy functionals are given by [3,6]

$$J_1 = -\partial_x(1 - \partial_x^2), \quad E_1(m) = \frac{1}{2} \int_{\mathbb{R}} (u^3 + uu_x^2) dx \tag{9}$$

and

$$J_2 = -(\partial_x m + m\partial_x), \quad E_2(m) = \frac{1}{2} \int_{\mathbb{R}} (u^2 + u_x^2) dx, \tag{10}$$

where  $\frac{\delta E_{1,2}}{\delta m}$  are computed with  $u = (1 - \partial_x^2)^{-1}m$ . The nonlocal operator  $J_m$  in (5) defines the third Hamiltonian structure, and is given by  $J_2 J_1^{-1} J_2$  up to scaling [15–17]. For  $b = 3$ ,  $J_m$  in (5) defines the second Hamiltonian structure, in addition to the first Hamiltonian structure [2] given by

$$J_1 = \partial_x(1 - \partial_x^2)(4 - \partial_x^2), \quad E_1(m) = \frac{1}{3} \int_{\mathbb{R}} u^3 dx. \tag{11}$$

When solutions of Eq. (3) are considered on the nonzero constant background with  $m(t, x) \rightarrow k$  decaying fast as  $|x| \rightarrow \infty$ , we have to redefine the conserved quantities  $F_1(m)$  and  $F_2(m)$  in order to eliminate divergence of the constant background. For  $k > 0$ , we consider the class of functions in the set

$$X_k = \{m - k \in H^1(\mathbb{R}) : m(x) > 0, x \in \mathbb{R}\} \tag{12}$$

and redefine the conserved quantities for  $m \in X_k$  as

$$\hat{E}(m) = \int_{\mathbb{R}} (m - k) dx, \tag{13}$$

$$\hat{F}_1(m) = \int_{\mathbb{R}} \left[ m^{\frac{1}{b}} - k^{\frac{1}{b}} \right] dx \tag{14}$$

and

$$\hat{F}_2(m) = \int_{\mathbb{R}} \left[ \left( \frac{m_x^2}{b^2 m^2} + 1 \right) m^{-\frac{1}{b}} - k^{-\frac{1}{b}} \right] dx. \tag{15}$$

Since  $m(x) \rightarrow k$  as  $|x| \rightarrow \infty$ ,  $k > 0$ , and  $m(x) > 0$ , there exists  $m_0 > 0$  such that  $m(x) \geq m_0$  for every  $x \in \mathbb{R}$ . Therefore,  $\hat{F}_1(m)$  and  $\hat{F}_2(m)$  are well-defined in  $X_k$  due to the Banach algebra property of  $H^1(\mathbb{R})$  and the conservation of  $\hat{E}(m)$ . The functionals  $\hat{E}$ ,  $\hat{F}_1$ , and  $\hat{F}_2$  are conserved on  $X_k$  because the time derivative of the integrands is still a total derivative and the asymptotic values of the functions  $m$  and  $u$  at  $x \rightarrow \pm\infty$  are the same.

Local well-posedness of Eq. (2) with  $k > 0$  can be easily shown for the initial data  $v(0, \cdot) = v_0 \in H^s(\mathbb{R})$  with  $s > \frac{3}{2}$  [30–33]. This well-posedness theory for the function  $v \in H^s(\mathbb{R})$  translates to the function  $m \in X_k$  if  $s = 3$ . Moreover, it is well-known that the local solutions exist for all times and do not break if the momentum density  $m(t, x)$  is strictly positive [31–33]. Positivity of  $m$  is preserved in the time evolution so that  $X_k$  is invariant under the flow of Eq. (3). Assuming that the local and global well-posedness of Eq. (3) is well-known, we adopt the following definition of orbital stability of traveling waves in the set  $X_k$ .

**Definition 1.** Let  $m(t, x) = \mu(x - ct)$  be the traveling wave solution of Eq. (3) with  $\mu \in X_k$ . We say that the traveling wave is orbitally stable in  $X_k$  if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that for every  $m_0 \in X_k$  satisfying  $\|m_0 - \mu\|_{H^1} < \delta$ , there exists a unique solution  $m \in C^0(\mathbb{R}, X_k)$  of Eq. (3) with the initial datum  $m(0, \cdot) = m_0$  satisfying

$$\inf_{x_0 \in \mathbb{R}} \|m(t, \cdot) - \mu(\cdot - x_0)\|_{H^1} < \varepsilon, \quad t \in \mathbb{R}.$$

The following theorem presents the main result of this work.

**Theorem 1.** For fixed  $b > 1$ ,  $c > 0$ , and  $k \in (0, (b + 1)^{-1}c)$ , there exists a unique solitary wave  $m(t, x) = \mu(x - ct)$  of Eq. (3) with profile  $\mu \in C^\infty(\mathbb{R})$  satisfying  $\mu(x) > 0$  for  $x \in \mathbb{R}$ ,  $\mu'(0) = 0$ , and  $\mu(x) \rightarrow k$  as  $|x| \rightarrow \infty$  exponentially fast. The solitary wave satisfies the condition  $0 < \phi(x) < c$  for  $x \in \mathbb{R}$ ,  $\phi'(0) = 0$ , and  $\phi(x) \rightarrow k$  as  $|x| \rightarrow \infty$  exponentially fast, where  $\phi := k + (1 - \partial_x^2)^{-1}(\mu - k)$  is uniquely defined, and it is orbitally stable in  $X_k$  if the mapping

$$k \mapsto Q(\phi) := \int_{\mathbb{R}} \left[ b \left( \frac{c - k}{c - \phi} \right) - \left( \frac{c - k}{c - \phi} \right)^b - b + 1 \right] dx \tag{16}$$

is strictly increasing.

**Remark 1.** The existence and uniqueness of the solitary wave in [Theorem 1](#) is established in [Lemma 1](#) and the relation between  $\mu$  and  $\phi$  can be written in the form

$$\mu = k \left( \frac{c - k}{c - \phi} \right)^b.$$

**Remark 2.** [Theorem 1](#) implies that if  $u(t, x) = \phi(x - ct)$  is the traveling wave solution of Eq. (1) with  $\phi \in Y_k$  given by

$$Y_k = \{u - k \in H^3(\mathbb{R}) : u(x) - u''(x) > 0, \quad x \in \mathbb{R}\}, \quad (17)$$

then it is orbitally stable in  $Y_k$  in the  $H^3(\mathbb{R})$  norm. Indeed, integration by parts yields

$$\|m\|_{H^1}^2 = \int_{\mathbb{R}} (m^2 + m_x^2) dx = \int_{\mathbb{R}} (u^2 + 3u_x^2 + 3u_{xx}^2 + u_{xxx}^2) dx,$$

which is equivalent to the squared  $H^3(\mathbb{R})$  norm on  $u$ .

**Remark 3.** The stability criterion of [Theorem 1](#) is verified for  $b = 2$  and  $b = 3$  based on analytical computations. This gives an alternative proof of the orbital stability of smooth solitary waves in the Camassa–Holm and Degasperis–Procesi equations compared to [\[27\]](#) and [\[28\]](#), respectively.

**Remark 4.** The stability criterion of [Theorem 1](#) is only verified numerically for other values of  $b > 1$  and asymptotically in the limit  $k \rightarrow 0$  and  $k \rightarrow (b + 1)^{-1}c$ . It is an open question on proving the stability criterion analytically in the general case.

**Remark 5.** Orbital stability of smooth multi-soliton solutions of the Camassa–Holm equation with  $b = 2$  was recently proven in [\[34\]](#). The proof relies on the bi-Hamiltonian structure of the Camassa–Holm equation and may not be generalized for a general case  $b > 1$ , for which the only Hamiltonian structure is given by [\(4\)](#).

The article is organized as follows. Traveling waves including the smooth solitary waves on the nonzero constant background are described in [Section 2](#). Variational characterization of the traveling solitary waves in terms of the mass and energy integrals is developed in [Section 3](#). Derivation and proof of the stability criterion in [Theorem 1](#) are given in [Section 4](#). Verification of the stability criterion is reported in [Section 5](#). We give the summary and discuss open directions in [Section 6](#).

## 2. Traveling waves

Let us consider traveling waves of Eq. (1) in the form  $u(t, x) = \phi(x - ct)$  with speed  $c$  and profile  $\phi$  found from the third-order differential equation

$$-(c - \phi)(\phi''' - \phi') + b\phi'(\phi'' - \phi) = 0. \quad (18)$$

The following lemma describes the family of solitary waves on the nonzero constant background parameterized by the arbitrary parameter  $k > 0$ .

**Lemma 1.** For fixed  $b > 1$  and  $c > 0$ , there exists a one-parameter family of smooth solitary waves with profile  $\phi \in C^\infty(\mathbb{R})$  satisfying  $\phi'(0) = 0$  and  $\phi(x) \rightarrow k$  as  $|x| \rightarrow \infty$  if and only if the arbitrary parameter  $k$  belongs to the interval  $(0, (b + 1)^{-1}c)$ . Moreover,

$$0 < \phi(x) < c, \quad \mu(x) = \phi(x) - \phi''(x) > 0, \quad x \in \mathbb{R}, \quad (19)$$

and the family is smooth with respect to parameter  $k$ .

**Proof.** Multiplying [\(18\)](#) by  $(c - \phi)^{b-1}$  and integrating in  $x$  yield the second-order equation:

$$-(c - \phi)^b(\phi'' - \phi) = a, \quad (20)$$

where  $a$  is the integration constant. The second-order equation [\(20\)](#) is conservative and admits the first-order invariant:

$$\frac{1}{2}(b - 1)(\phi'^2 - \phi^2) + \frac{a}{(c - \phi)^{b-1}} = g, \quad (21)$$

where  $g$  is another integration constant.

Smooth solitary wave solutions with profile  $\phi \in C^\infty(\mathbb{R})$  satisfying  $\phi(x) \rightarrow k$  as  $|x| \rightarrow \infty$  correspond to the homoclinic orbit from the equilibrium point  $(\phi, \phi') = (k, 0)$ . Taking the limit as  $|x| \rightarrow \infty$  in [\(20\)](#) and [\(21\)](#) yields the relations:

$$a = k(c - k)^b, \quad g = kc - \frac{1}{2}(b + 1)k^2. \quad (22)$$

For fixed  $b > 1$  and  $c > 0$ , the first-order invariant [\(21\)](#) represents the energy conservation for a Newtonian particle with the mass  $(b - 1) > 0$  and energy  $g$  under a force with the potential energy

$$U(\phi) := -\frac{1}{2}(b - 1)\phi^2 + \frac{a}{(c - \phi)^{b-1}}. \quad (23)$$

For smooth solutions  $\phi \in C^\infty(\mathbb{R})$ , we consider the restriction  $\phi \in (-\infty, c)$  since no smooth bounded solutions exist for  $\phi \in (c, \infty)$  if  $b > 1$ .

Critical points of  $U$  in  $(-\infty, c)$  are given by roots of the algebraic equation  $\phi(c - \phi)^b = a$  for  $\phi \in (-\infty, c)$ . The mapping  $M(\phi) : (-\infty, c) \mapsto \mathbb{R}$  defined by  $M(\phi) := \phi(c - \phi)^b$  has one critical point given by a global maximum that occurs at  $\phi_0 := \frac{c}{b+1} < c$ . Furthermore,  $M(\phi) \rightarrow -\infty$  as  $\phi \rightarrow -\infty$  and  $M(c) = 0$ . Therefore, there exists only one critical point of  $U$  for  $a \in (-\infty, 0]$  and  $a = a$ , two critical points of  $U$  for  $a \in (0, a)$ , and no critical points of  $U$  for  $a \in (a, \infty)$ , where

$$a := M(\phi_0) = \frac{b^b c^{b+1}}{(b + 1)^{b+1}}. \quad (24)$$

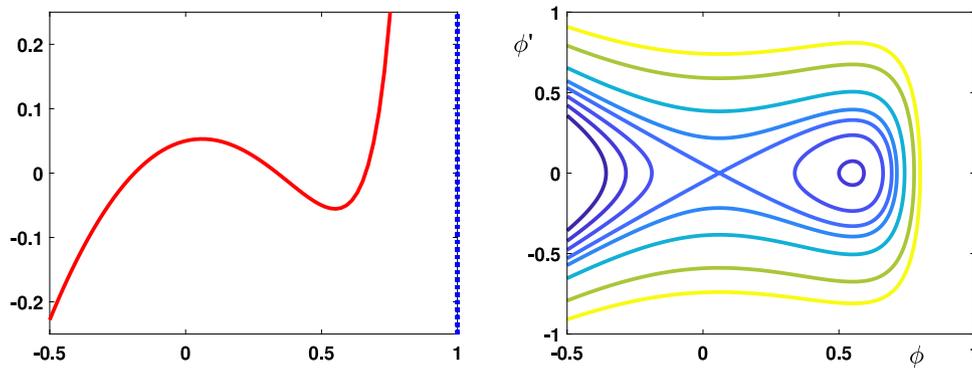
Homoclinic orbits with  $\phi \in (-\infty, c)$  exist only if at least two roots of the algebraic equation  $\phi(c - \phi)^b = a$  exist in  $(-\infty, c)$ , which happens if and only if  $a \in (0, a)$ . The two roots can be ordered as follows:

$$0 < \phi_1 < \frac{c}{b + 1} < \phi_2 < c. \quad (25)$$

We will now show that the homoclinic orbit does exist if  $a \in (0, a)$ .

For fixed  $b > 1$  and  $c > 0$ , the local maximum and minimum points of  $U$  give respectively the saddle point  $(\phi_1, 0)$  and the center point  $(\phi_2, 0)$  of the second-order equation [\(20\)](#). There exists a punctured neighborhood of the center  $(\phi_2, 0)$  enclosed by the homoclinic orbit connecting the saddle  $(\phi_1, 0)$ . Thus,  $\phi_1 \equiv k \in (0, (b + 1)^{-1}c)$  is taken as the arbitrary parameter of the homoclinic orbit, which specifies  $a$  and  $g = U(k)$  by the relation [\(22\)](#). Since  $U(\phi) \rightarrow +\infty$  as  $\phi \rightarrow c$  from the left, the homoclinic orbit belongs to the vertical stripe  $\{(\phi, \phi') : 0 < \phi < c\}$  and represents the smooth solitary wave with the profile  $\phi \in C^\infty(\mathbb{R})$  satisfying  $\phi(x) \rightarrow k$  as  $|x| \rightarrow \infty$ . By the translational invariance, the solitary wave profile satisfying  $\phi'(0) = 0$  is uniquely defined.

Finally, smoothness of the family in  $k$  is due to the fact that the differential equations depend smoothly on  $\phi, g, a$ , and  $c$  if  $\phi < c$ , whereas parameters  $a$  and  $g$  depend smoothly on  $k$  in [\(22\)](#) if  $k \in (0, (b + 1)^{-1}c)$ .



**Fig. 1.** Left:  $U$  versus  $\phi$  for  $b = 3$ ,  $c = 1$ , and  $a = 0.05 \in (0, a)$ . Right: the phase portrait of the second-order Eq. (20) constructed from the level curves of the first-order invariant (21) on the phase plane  $(\phi, \phi')$  for the same parameter values.

As for the sign of the momentum density for the solitary wave with profile  $\phi$  given by  $\mu := \phi - \phi''$ , Eq. (20) gives the relation:

$$\mu = \frac{a}{(c - \phi)^b}. \tag{26}$$

It then follows from the inequalities  $0 < \phi(x) < c$  and  $a > 0$  that  $\mu(x) > 0$ .  $\square$

For illustration of the proof of Lemma 1, Fig. 1 (left) shows the graph of  $U$  versus  $\phi$  for  $b = 3$ ,  $c = 1$ , and  $a = 0.05 \in (0, a)$ , where  $a \approx 0.11$ . The corresponding phase portrait on the phase plane  $(\phi, \phi')$  is shown on the right panel.

**Remark 6.** The standard integration of Eq. (18) in  $x$  gives a different second-order equation

$$-(c - \phi)(\phi'' - \phi) + \frac{1}{2}(b - 1)(\phi'^2 - \phi^2) = g, \tag{27}$$

which can be integrated to the same first-order invariant (21). Consequently, Eq. (27) is redundant in view of the two Eqs. (20) and (21).

**Remark 7.** The one-parameter family of smooth solitary waves exists for fixed  $b > 1$  and  $c < 0$ . It is obtained from the family of solutions to (18) presented in Lemma 1 by using the symmetry transformation:

$$c \mapsto -c, \quad \phi \mapsto -\phi, \tag{28}$$

which leaves Eq. (18) invariant. For  $b > 1$  and  $c < 0$ , the arbitrary parameter  $k$  belongs to the interval  $((b + 1)^{-1}c, 0)$ . This family satisfies  $\mu(x) < 0$  for all  $x \in \mathbb{R}$  so it does not belong to  $X_k$  in (12).

**Remark 8.** In the limiting case  $k \rightarrow 0$ , the profile  $\phi$  is no longer smooth since  $a \rightarrow 0$  for which  $\phi_1 = 0$  and  $\phi_2 = c$ . This limit recovers the peaked solitary waves with the profile  $\phi(x) = ce^{-|x|}$  considered in our previous work [26]. In the limiting case  $k \rightarrow (b + 1)^{-1}c$ , the profile  $\phi$  is constant in  $x$  since  $a \rightarrow a$  for which  $\phi_1 = \phi_2 = (b + 1)^{-1}c$ .

**Remark 9.** The relation (26) can be recovered with the traveling wave reduction  $m(t, x) = \mu(x - ct)$  of Eq. (3), from which  $\mu$  satisfies the differential equation

$$\mu'(\phi - c) + b\mu\phi' = 0. \tag{29}$$

After multiplying (29) by  $(c - \phi)^{b-1}$  and integration in  $x$ , we obtain (26).

**Remark 10.** Inequalities (19) for the smooth solitary waves of Lemma 1 are strict in the sense that for every fixed  $b > 1$ ,  $c > 0$ , and  $k \in (0, (b + 1)^{-1}c)$ , there exist positive constants  $C_{\pm}(b, c, k)$  such that

$$C_-(b, c, k) \leq c - \phi(x) \leq C_+(b, c, k), \quad x \in \mathbb{R}. \tag{30}$$

Consequently, there exist positive constants  $\hat{C}_{\pm}(b, c, k)$  such that

$$\hat{C}_-(b, c, k) \leq \mu(x) \leq \hat{C}_+(b, c, k), \quad x \in \mathbb{R}. \tag{31}$$

In view of Remark 8,  $C_-(b, c, k) \rightarrow 0$  and  $\hat{C}_+(b, c, k) \rightarrow \infty$  as  $k \rightarrow 0$  when the smooth solitary wave becomes the peaked solitary wave for every  $b > 1$  and  $c > 0$ .

**Remark 11.** By a similar phase plane analysis as in the proof of Lemma 1, we can identify additional families of smooth solitary waves in  $X_k$  for  $b < 1$ . The analysis is different when  $b < 1$  due to the fact that the potential in (23) must be multiplied by  $-1$ . Indeed, in that case, what plays the role of the mass in (21) must be taken to be  $(1 - b) > 0$ . In the case  $0 < b < 1$ , we identified a family of smooth solitary waves in  $X_k$  with  $0 < \phi < c$ . We also identified a family of solitary waves for  $\phi > c$  when  $-1 < b < 0$ . Taking  $c - \phi$  negative changes the way (18) is integrated. The stability analysis of those additional solutions can be developed similarly to the proof of Theorem 1 in the case  $b > 1$ . In the case  $b = 1$ , one would obtain a different expression than (21) due to the fact that different  $J_m$  and  $E(m)$  must be used in (4) [15].

### 3. Variational characterization

The solitary waves of Lemma 1 can be characterized variationally if the system of Eqs. (20) and (21) can be shown to arise as the Euler-Lagrange equation of the action functional

$$\Lambda_{\omega_1, \omega_2}(m) := \omega_1 \hat{F}_1(m) + \omega_2 \hat{F}_2(m) - \hat{E}(m), \tag{32}$$

where the normalized mass  $\hat{E}(m)$  and the two Casimir functionals  $\hat{F}_1(m)$  and  $\hat{F}_2(m)$  are given by (13), (14), and (15). The following lemma states that the variational characterization is possible if and only if the Lagrange multipliers  $\omega_1$  and  $\omega_2$  are uniquely related to parameters of Eqs. (20) and (21).

**Lemma 2.** For fixed  $b > 1$  and  $c > 0$ , a critical point  $\mu \in X_k$  of the action functional  $\Lambda_{\omega_1, \omega_2}$  in (32) coincides with the solitary wave solution  $\mu = \phi - \phi'' \in C^\infty(\mathbb{R})$  of Lemma 1 satisfying (20) and (21) if and only if

$$\omega_1 = \frac{1}{2} [(b - 1)c + (b + 1)k] k^{-\frac{1}{b}}, \quad \omega_2 = \frac{1}{2} (b - 1)(c - k) k^{\frac{1}{b}}, \tag{33}$$

where  $k \in (0, (b + 1)^{-1}c)$  is an arbitrary parameter.

**Proof.** Let  $\mu \in X_k$  be a critical point of  $\Lambda_{\omega_1, \omega_2}$ . After straightforward simplifications, the equation  $\Lambda'_{\omega_1, \omega_2}(\mu) = 0$  gives the following differential equation,

$$\frac{\omega_1}{b} \mu^{\frac{1}{b}-1} - \frac{\omega_2}{b\mu^{\frac{1}{b}+1}} \left[ \frac{2\mu''}{b\mu} - \frac{(2b+1)(\mu')^2}{b^2\mu^2} + 1 \right] - 1 = 0. \quad (34)$$

If  $\mu \in X_k$  is a weak solution of Eq. (34), then  $\mu \in C^\infty(\mathbb{R})$  by bootstrapping arguments, so that the set of solitary wave solutions is given by Lemma 1 if we can establish equivalence between the differential equations.

Assuming the relation (26), we introduce  $\phi$  from  $\mu(c-\phi)^b = a$  and obtain

$$\mu' = \frac{b\phi'}{c-\phi}\mu, \quad \mu'' = \frac{b\phi''}{c-\phi}\mu + \frac{b(b+1)(\phi')^2}{(c-\phi)^2}\mu.$$

Substituting these relations into (34) gives after straightforward simplifications:

$$\omega_1 a^{\frac{1}{b}} - \omega_2 a^{-\frac{1}{b}} \left[ 2(c-\phi)\phi'' - (\phi')^2 + (c-\phi)^2 \right] - ab(c-\phi)^{1-b} = 0. \quad (35)$$

Substituting (27) into (35) yields

$$\omega_1 a^{\frac{1}{b}} - \omega_2 a^{-\frac{1}{b}} \left[ b(\phi')^2 - b\phi^2 + c^2 - 2g \right] - ab(c-\phi)^{1-b} = 0. \quad (36)$$

Substituting (21) into (36) gives the unique choice for Lagrange multipliers,

$$\omega_1 = \frac{1}{2a^{\frac{1}{b}}} \left[ 2g + (b-1)c^2 \right], \quad \omega_2 = \frac{1}{2} a^{\frac{1}{b}} (b-1). \quad (37)$$

Finally, substituting (22) into (37) gives (33).  $\square$

**Remark 12.** Only one parameter  $k \in (0, (b+1)^{-1}c)$  is arbitrary for fixed  $b > 1$  and  $c > 0$ . Although the action functional (32) has two Lagrange multipliers, first variations of  $\hat{E}(m)$ ,  $\hat{F}_1(m)$ , and  $\hat{F}_2(m)$  are not defined independently of each other for  $m \in X_k$ . Indeed, due to nonzero boundary conditions at infinity, those functionals are not Fréchet differentiable on  $X_k$ , but the Lagrange multipliers in (32) are such that  $\Lambda_{\omega_1, \omega_2}$  is differentiable with

$$\Lambda'_{\omega_1, \omega_2}(\mu) = \frac{b+1}{2b} k^{1-\frac{1}{b}} \hat{F}'(\mu) + \frac{b-1}{2b} \hat{G}'(\mu) = 0,$$

where

$$\hat{F}(m) = b\hat{F}_1(m) - k^{\frac{1}{b}-1}\hat{E}(m),$$

$$\hat{G}(m) = b(c-k)k^{\frac{1}{b}}\hat{F}_2(m) + bck^{-\frac{1}{b}}\hat{F}_1(m) - \hat{E}(m),$$

where  $\hat{F}$  and  $\hat{G}$  are Fréchet differentiable on  $X_k$ .

The variational characterization of Lemma 2 implies two important properties when we add a perturbation  $\tilde{m} := m - \mu$  to the solitary wave with the profile  $\mu \in C^\infty(\mathbb{R})$ . Since  $\mu(x)$  is strictly positive and bounded by Remark 10, we have  $m \in X_k$  if  $\tilde{m} \in H^1(\mathbb{R})$  and the  $H^1(\mathbb{R})$  norm of  $\tilde{m}$  is sufficiently small.

The following two results describe the second-order variation of the action functional  $\Lambda_{\omega_1, \omega_2}$  and the first-order constraint on the perturbation  $\tilde{m} \in H^1(\mathbb{R})$ . In what follows,  $\langle \cdot, \cdot \rangle$  denotes the standard inner product in  $L^2(\mathbb{R})$ ,  $\mu := \phi - \phi'' \in C^\infty(\mathbb{R})$  is defined by Lemma 1, and  $(\omega_1, \omega_2)$  are defined by Lemma 2.

**Corollary 1.** *There exists a sufficiently small positive  $\varepsilon_0$  such that for every  $\tilde{m} \in H^1(\mathbb{R})$  satisfying  $\|\tilde{m}\|_{H^1} \leq \varepsilon_0$ , we have*

$$\Lambda_{\omega_1, \omega_2}(\mu + \tilde{m}) - \Lambda_{\omega_1, \omega_2}(\mu) = \frac{(b-1)}{2b^2} \langle \mathcal{L}\tilde{m}, \tilde{m} \rangle + R(\tilde{m}), \quad (38)$$

where

$$\begin{aligned} \mathcal{L} = & -\frac{d}{dx} \frac{c-\phi}{\mu^2} \frac{d}{dx} + \frac{(b+1)(c-\phi)}{2\mu^2} + \frac{(2b+1)(c-\phi)\mu''}{b\mu^3} \\ & - \frac{(2b+1)(3b+1)(c-\phi)(\mu')^2}{2b^2\mu^4} - \frac{(c-k)[(b-1)c + (b+1)k]}{2(c-\phi)\mu^2} \end{aligned} \quad (39)$$

and  $R(\tilde{m})$  is the remainder term satisfying  $\|R(\tilde{m})\|_{H^1} \leq C_0 \|\tilde{m}\|_{H^1}^3$  for some  $\tilde{m}$ -independent positive constant  $C_0$ .

**Proof.** The expression for  $\mathcal{L}$  is obtained by straightforward computations with the use of relations (26) and (33). Coefficients of the Sturm-Liouville operator  $\mathcal{L}$  are smooth and bounded since  $(c-\phi)$ ,  $\mu$  are strictly positive, bounded, and smooth on  $\mathbb{R}$  by Lemma 1 and Remark 10.

Similarly,  $R(\tilde{m})$  is computed by using the Taylor expansion of the energy densities in  $\hat{F}_1(m)$  and  $\hat{F}_2(m)$  at  $m = \mu$ . The leading-order term in  $R(\tilde{m})$  is cubic and the Sobolev space  $H^1(\mathbb{R})$  forms a Banach algebra with respect to multiplication so that the estimate  $\|R(\tilde{m})\|_{H^1} \leq C_0 \|\tilde{m}\|_{H^1}^3$  follows from the Taylor expansion of  $\hat{F}_1(m)$ ,  $\hat{F}_2(m)$  and the smallness of  $\|\tilde{m}\|_{H^1}$ .  $\square$

**Corollary 2.** *Let  $\tilde{m} \in H^1(\mathbb{R})$  be a small perturbation to  $\mu$  such that  $m = \mu + \tilde{m} \in X_k$  does not change the conserved quantity  $\hat{F}(m) := b\hat{F}_1(m) - k^{\frac{1}{b}-1}\hat{E}(m)$  up to the first order. Then,  $\tilde{m} \in H^1(\mathbb{R})$  satisfies the constraint*

$$\langle \mu^{\frac{1}{b}-1} - k^{\frac{1}{b}-1}, \tilde{m} \rangle = 0. \quad (40)$$

**Proof.** Since  $\hat{F}'_1(\mu) = b^{-1}\mu^{\frac{1}{b}-1}$  and  $\hat{E}'(\mu) = 1$ , the linear combination in  $\hat{F}(m)$  is chosen in such way that  $\hat{F}'(\mu) = \mu^{\frac{1}{b}-1} - k^{\frac{1}{b}-1}$  decays to zero at infinity exponentially fast. Then, the constraint (40) is well-defined for every  $\tilde{m} \in H^1(\mathbb{R})$  and expresses fixed constraint of  $\hat{F}(m)$  up to the first order.  $\square$

**Remark 13.** It follows from (22) and (26) that

$$\mu = k \left( \frac{c-k}{c-\phi} \right)^b. \quad (41)$$

As a result, the constraint (40) can be equivalently written as

$$\langle (c-\phi)^{b-1} - (c-k)^{b-1}, \tilde{m} \rangle = 0, \quad (42)$$

where  $(c-\phi)^{b-1} - (c-k)^{b-1}$  decays to zero at infinity exponentially fast.

#### 4. Stability criterion

The results of Lemma 2 suggest that the smooth solitary wave of Lemma 1 with the profile  $\mu = \phi - \phi''$  is a critical point of the action functional  $\Lambda_{\omega_1, \omega_2}$  with uniquely selected parameters  $(\omega_1, \omega_2)$ . In order to prove Theorem 1, we need to derive the criterion for the critical point to be a local non-degenerate minimizer of  $\Lambda_{\omega_1, \omega_2}$  subject to the fixed value of another conserved quantity  $\hat{F}$ . This is done by using the second derivative test, which relies on Corollaries 1 and 2. We will show that the Hessian operator  $\mathcal{L}$  of the action functional  $\Lambda_{\omega_1, \omega_2}$  defined by (38) and (39) has exactly one simple negative eigenvalue and a simple zero eigenvalue isolated from the rest of the spectrum. Then, we add the constraint (40) in order to derive a precise condition when the Hessian operator is positive under the constraint with the only degeneracy due to the translational symmetry.

The following lemma gives the spectral properties of the linear operator  $\mathcal{L}$ .

**Lemma 3.** *The linear operator  $\mathcal{L}$  defined by (39) is extended as a self-adjoint operator in  $L^2(\mathbb{R})$  with the dense domain  $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$ . There exists  $\delta > 0$  such that the spectrum of  $\mathcal{L}$  in  $(-\infty, \delta)$  consists of a simple zero eigenvalue and a simple negative eigenvalue.*

**Proof.** The linear operator  $\mathcal{L}$  defined by (39) belongs to the class of self-adjoint Sturm–Liouville operators in  $L^2(\mathbb{R})$  with the dense domain  $H^2(\mathbb{R})$ . Since coefficients of  $\mathcal{L}$  are smooth and bounded, see the proof of Corollary 1, standard properties of the Sturm–Liouville operators hold.

Since  $\mu(x) \rightarrow k$  as  $|x| \rightarrow \infty$  exponentially fast, Weyl’s Lemma states that the essential spectrum of  $\mathcal{L}$  is given by the essential spectrum of the linear operator with constant coefficients  $\mathcal{L}_\infty$  given by

$$\mathcal{L}_\infty = -\frac{c-k}{k^2} \frac{d^2}{dx^2} + \frac{c-(b+1)k}{k^2}. \tag{43}$$

Since  $c > (b+1)k$ , the spectrum of  $\mathcal{L}_\infty$  coincides with

$$\Sigma_\infty := [k^{-2}(c - (b+1)k), \infty),$$

which is strictly positive. Hence, the spectrum of the self-adjoint operator  $\mathcal{L}$  in  $\mathbb{R} \setminus \Sigma_\infty$  consists of isolated semi-simple eigenvalues of finite multiplicity. Moreover, each eigenvalue is simple because the Wronskian  $W(v_1, v_2)$  of two solutions  $v_1, v_2$  of the second-order differential equation  $\mathcal{L}v = \lambda v$  satisfies the Liouville formula,

$$W(v_1, v_2) = \frac{W_0 \mu^2}{c - \phi},$$

where  $W_0 \neq 0$  is constant so that if  $v_1 \in H^2(\mathbb{R})$  decays to zero as  $|x| \rightarrow \infty$ , then  $v_2 \notin H^2(\mathbb{R})$  diverges to infinity as  $|x| \rightarrow \infty$ .

It remains to characterize the zero and negative eigenvalues of  $\mathcal{L}$ . Due to the translation symmetry of Eq. (3),  $\mu' \in H^2(\mathbb{R})$  belongs to the kernel of  $\mathcal{L}$  so that 0 is in the spectrum of  $\mathcal{L}$ . Sturm’s Oscillation Theorem states that the  $n$ th simple eigenvalue corresponds to the eigenfunction with  $(n-1)$  simple zeros on  $\mathbb{R}$ . Since  $\mu' = \frac{b\mu}{c-\phi} \phi'$  has only one zero on  $\mathbb{R}$ , 0 is the second eigenvalue of  $\mathcal{L}$  and there exists only one simple negative eigenvalue.  $\square$

In order to prove the next lemma, we first derive formal relations on the family of solutions in Eqs. (20) and (21). Solutions depend on three parameters  $g, a,$  and  $c$ , which are considered independently for the moment. Formal differentiating of  $(c - \phi)^b \mu = a$  in  $g, a,$  and  $c$  yields

$$\partial_g \mu = \frac{b\mu \partial_g \phi}{c - \phi}, \quad \partial_a \mu = \frac{b\mu \partial_a \phi}{c - \phi} + \frac{\mu}{a}, \quad \partial_c \mu = \frac{b\mu \partial_c \phi}{c - \phi} - \frac{b\mu}{c - \phi}. \tag{44}$$

We note the following result due to the scaling transformation.

**Proposition 1.** *Let  $\mu$  be defined by (26) and assume that  $\mu$  is smooth with respect to parameters  $g, a,$  and  $c$ . Then, it satisfies*

$$c \partial_c \mu + (b+1) a \partial_a \mu + 2 g \partial_g \mu = \mu. \tag{45}$$

**Proof.** Solutions  $\phi(x; g, a, c)$  of the system of Eqs. (20) and (21) enjoy the scaling transformation:

$$\phi(x; g, a, c) = c \varphi(x; \gamma, \alpha), \quad g = c^2 \gamma, \quad a = c^{b+1} \alpha, \tag{46}$$

where  $\varphi, \gamma,$  and  $\alpha$  are independent of  $c$ . It follows from (46) for a smooth  $\phi(x; g, a, c)$  that

$$c \partial_c \phi + (b+1) a \partial_a \phi + 2 g \partial_g \phi = \phi. \tag{47}$$

The relation (45) follows from (47) by using (44).  $\square$

Long but straightforward computations based on the explicit expression (39) with

$$(c - k)[(b - 1)c + (b + 1)k] = 2g + (b - 1)c^2$$

yield

$$\mathcal{L} \partial_g \mu = \frac{b}{a(1-b)} (c - \phi)^{b-1}, \tag{48}$$

$$\mathcal{L} \partial_a \mu = \frac{2g + (b - 1)c^2}{a^2(b - 1)} (c - \phi)^{b-1} - \frac{b}{a(b - 1)}, \tag{49}$$

and

$$\mathcal{L} \partial_c \mu = -\frac{bc}{a} (c - \phi)^{b-1}, \tag{50}$$

where  $b \neq 1$  and  $a \neq 0$  are assumed. It follows from (45) with the help of (48), (49), and (50) that

$$\mathcal{L} \mu = \frac{2g + (b - 1)c^2}{a(b - 1)} (c - \phi)^{b-1} - \frac{b(b + 1)}{b - 1}. \tag{51}$$

We also acknowledge the obvious translation symmetry of Eq. (3).

**Proposition 2.** *Let  $\mu(t, x)$  be a solution of Eq. (3). Then  $\mu(t + t_0, x + x_0)$  is also a solution of Eq. (3) for every  $t_0 \in \mathbb{R}$  and  $x_0 \in \mathbb{R}$ .*

**Proof.** The proof is immediate since Eq. (3) has constant coefficients in  $(t, x)$ .  $\square$

We are now ready to prove the following lemma which gives the precise condition for the smooth solitary wave with the profile  $\mu$  to become a constrained minimizer of the action functional  $\Lambda_{\omega_1, \omega_2}$  given by (32) and (33).

**Lemma 4.** *The solitary wave with profile  $\mu$  is a local constrained minimizer of  $\Lambda_{\omega_1, \omega_2}$  given by (32) and (33) under the constraint (42) if and only if the mapping*

$$k \mapsto Q(\phi) := \int_{\mathbb{R}} \left[ b \left( \frac{c - k}{c - \phi} \right) - \left( \frac{c - k}{c - \phi} \right)^b - b + 1 \right] dx \tag{52}$$

is increasing. The minimizer is only degenerate due to the translational symmetry if and only if the mapping (52) is strictly increasing.

**Proof.** Under the spectral properties of Lemma 3, it is well-known since the pioneering work [35] that the operator  $\mathcal{L}$  is positive definite on a subspace of  $L^2(\mathbb{R})$  defined by the scalar constraint (42) if and only if

$$\langle \mathcal{L}^{-1} v_0, v_0 \rangle \leq 0, \quad \text{where } v_0 := (c - \phi)^{b-1} - (c - k)^{b-1}. \tag{53}$$

Moreover, also as a consequence of [35], the operator  $\mathcal{L}$  is strictly positive definite in

$$L_c^2 := \{ \tilde{m} \in L^2(\mathbb{R}) : \langle v_0, \tilde{m} \rangle = 0, \langle \mu', \tilde{m} \rangle = 0 \} \tag{54}$$

if and only if  $\langle \mathcal{L}^{-1} v_0, v_0 \rangle < 0$ . It remains to show that the sign of  $\langle \mathcal{L}^{-1} v_0, v_0 \rangle$  is opposite to the sign of the derivative of the mapping (52).

Let us recall that parameters  $g$  and  $a$  are expressed in terms of  $k$  for fixed  $b > 1$  and  $c > 0$  from (22) and that the family of solitary waves is smooth with respect to parameter  $k$ . Differentiating  $\mu$  with respect to  $k$  yields with the chain rule that

$$\partial_k \mu = (c - (b + 1)k) [\partial_g \mu + (c - k)^{b-1} \partial_a \mu]. \tag{55}$$

From (55), we use (48) and (49) to obtain

$$\mathcal{L} \partial_k \mu = \frac{c - k(b + 1)}{ak(b - 1)} [(k + c(b - 1))(c - \phi)^{b-1} - bk(c - k)^{b-1}]. \tag{56}$$

We also rewrite (51) as

$$\mathcal{L}\mu = \frac{c-k}{a(b-1)} [(k(b+1) + c(b-1))(c-\phi)^{b-1} - bk(b+1)(c-k)^{b-1}]. \tag{57}$$

Since  $\mu(x) \rightarrow k$  as  $|x| \rightarrow \infty$  exponentially fast, then  $\partial_k \mu(x) \rightarrow 1$  as  $|x| \rightarrow \infty$ , so that  $\partial_k \mu(x)$  does not decay to 0 at infinity. However,  $\partial_k \mu(x) - k^{-1} \mu(x)$  does decay to 0 at infinity, so we use (56) and (57) to compute

$$\mathcal{L}(\partial_k \mu - k^{-1} \mu) = -\frac{cb^2}{a(b-1)} [(c-\phi)^{b-1} - (c-k)^{b-1}].$$

Therefore,

$$\mathcal{L}^{-1}v_0 = -\frac{a(b-1)}{cb^2} (\partial_k \mu - k^{-1} \mu), \tag{58}$$

and we can compute

$$\langle \mathcal{L}^{-1}v_0, v_0 \rangle = -\frac{a(b-1)}{cb^2} \times \int_{\mathbb{R}} [(c-\phi)^{b-1} - (c-k)^{b-1}] \left[ \partial_k \mu - \frac{\mu}{k} \right] dx,$$

which becomes in view of relation (41),

$$\begin{aligned} \langle \mathcal{L}^{-1}v_0, v_0 \rangle &= -\frac{ak(b-1)}{cb} (c-k)^{b-1} \\ &\times \int_{\mathbb{R}} \left[ 1 - \left( \frac{c-k}{c-\phi} \right)^{b-1} \right] \frac{\partial}{\partial k} \left( \frac{c-k}{c-\phi} \right) dx \\ &= -\frac{ak(b-1)}{cb^2} (c-k)^{b-1} \frac{\partial}{\partial k} \\ &\times \int_{\mathbb{R}} \left[ b \left( \frac{c-k}{c-\phi} \right) - \left( \frac{c-k}{c-\phi} \right)^b - b + 1 \right] dx, \end{aligned}$$

where the integrands converge to zero at infinity. Since  $a > 0$ ,  $b > 1$ , and  $c > 0$ , the sign of  $\langle \mathcal{L}^{-1}v_0, v_0 \rangle$  is opposite to the sign of the derivative of the mapping (52).  $\square$

**Remark 14.** The result of Theorem 1 follows from the second derivative test in Lemma 4, the local well-posedness theory for the  $b$ -CH equation (3) in  $X_k$ , and the orbital stability theory pioneered in [36]. One of the assumptions made in [36] is that the operator  $J_m$  in (4) be onto. However, in the proof of stability in [36, Section 3], that condition is never used. The condition on the operator  $J_m$  is used only to prove the instability. Since we only prove the stability, we do not need to use any conditions on  $J_m$ .

**5. Verification of the stability criterion**

Here we verify the stability criterion of Theorem 1. To do so, we rewrite Eq. (27) with  $g = kc - \frac{1}{2}(b+1)k^2$  as

$$(c-\phi)(\phi-\phi'') + \frac{1}{2}(b-1)(\phi'^2 - \phi^2) = ck - \frac{1}{2}(b+1)k^2. \tag{59}$$

By using the transformation (which generalizes the one used in [27,29] for  $b = 2$ ),

$$z = \int_0^x \frac{dx}{[c-\phi(x)]^{\frac{b-1}{2}}}, \quad \phi(x) = \psi(z), \tag{60}$$

we obtain from (59) the equivalent second-order equation,

$$-\psi''(z) + (\psi-k)(c-\psi)^{b-2} \left[ c - \frac{1}{2}(b+1)(\psi+k) \right] = 0. \tag{61}$$

Using the following transformation

$$\zeta = \sqrt{c-k(b+1)(c-k)} \frac{b-2}{z}, \quad \psi(z) = k + (c-k)\varphi(\zeta), \tag{62}$$

we rewrite Eq. (61) in the normalized form

$$-\varphi''(\zeta) + \varphi(1-\varphi)^{b-2} [1 - (2\gamma)^{-1}(b+1)\varphi] = 0, \tag{63}$$

where

$$\gamma := \frac{c-k(b+1)}{c-k}. \tag{64}$$

It follows that  $\gamma \in (0, 1)$  if  $k \in (0, (b+1)^{-1}c)$ . Substituting (60) and (62) into  $Q(\phi)$  given by (16) yields

$$\begin{aligned} Q(\phi) &= \int_{\mathbb{R}} \left[ b \frac{\phi-k}{c-\phi} + 1 - \left( \frac{c-k}{c-\phi} \right)^b \right] dx \\ &= \int_{\mathbb{R}} \left[ b(\psi-k)(c-\psi)^{\frac{b-3}{2}} + (c-\psi)^{\frac{b-1}{2}} \right. \\ &\quad \left. - (c-k)^b (c-\psi)^{-\frac{b+1}{2}} \right] dz \\ &= \gamma^{-1/2} \int_{\mathbb{R}} \left[ b\varphi(1-\varphi)^{\frac{b-3}{2}} + (1-\varphi)^{\frac{b-1}{2}} - (1-\varphi)^{-\frac{b+1}{2}} \right] d\zeta. \end{aligned} \tag{65}$$

Although we do not write it explicitly, the wave profile  $\varphi$  depends on  $\gamma$  since Eq. (63) for  $\varphi$  depends on  $\gamma$ . If  $b > 1$  and  $c > 0$  are fixed, monotonicity of the mapping  $k \mapsto Q(\phi)$  given by (16) is determined with the chain rule from monotonicity of the mappings  $k \mapsto \gamma$  and  $\gamma \mapsto Q(\phi)$  given by (64) and (65). Since

$$\frac{d\gamma}{dk} = \frac{-bc}{(c-k)^2} < 0,$$

the mapping (16) is strictly increasing if and only if

$$\frac{d}{d\gamma} Q(\phi) < 0. \tag{66}$$

The following two lemmas report explicit computations of  $Q(\phi)$  for the integrable cases  $b = 2$  and  $b = 3$ , from which the stability criterion (66) can be proven analytically.

**Lemma 5.** The mapping (16) is strictly increasing for  $b = 2$ ,  $c > 0$ , and  $k \in (0, \frac{1}{3}c)$ .

**Proof.** Eq. (63) with  $b = 2$  admits the exact solution for the solitary wave centered at  $\zeta = 0$ :

$$\varphi(\zeta) = \gamma \operatorname{sech}^2\left(\frac{1}{2}\zeta\right). \tag{67}$$

Substituting (67) to  $Q(\phi)$  in (65) for  $b = 2$  yields

$$\begin{aligned} Q(\phi) &= -\gamma^{-1/2} \int_{\mathbb{R}} (1-\varphi)^{-3/2} \varphi^2 d\zeta \\ &= -\gamma^{3/2} \int_{\mathbb{R}} \frac{\operatorname{sech}^4(\frac{1}{2}\zeta)}{(1-\gamma \operatorname{sech}^2(\frac{1}{2}\zeta))^{3/2}} d\zeta, \end{aligned}$$

from which we compute

$$\begin{aligned} \frac{d}{d\gamma} Q(\phi) &= -\frac{3}{2} \gamma^{1/2} \int_{\mathbb{R}} \left[ \frac{\operatorname{sech}^4(\frac{1}{2}\zeta)}{(1-\gamma \operatorname{sech}^2(\frac{1}{2}\zeta))^{3/2}} \right. \\ &\quad \left. + \frac{\gamma \operatorname{sech}^6(\frac{1}{2}\zeta)}{(1-\gamma \operatorname{sech}^2(\frac{1}{2}\zeta))^{5/2}} \right] d\zeta \\ &= -\frac{3}{2} \gamma^{1/2} \int_{\mathbb{R}} \frac{\operatorname{sech}^4(\frac{1}{2}\zeta)}{(1-\gamma \operatorname{sech}^2(\frac{1}{2}\zeta))^{5/2}} d\zeta < 0, \end{aligned}$$

hence the stability criterion (66) is satisfied.  $\square$

**Lemma 6.** The mapping (16) is strictly increasing for  $b = 3$ ,  $c > 0$ , and  $k \in (0, \frac{1}{4}c)$ .

**Proof.** Eq. (63) with  $b = 3$  admits the exact solution for the solitary wave centered at  $\zeta = 0$ :

$$\varphi(\zeta) = \frac{3\gamma}{2 + \gamma + \sqrt{(1 - \gamma)(4 - \gamma)} \cosh(\zeta)}. \tag{68}$$

Substituting (68) to  $Q(\phi)$  in (65) for  $b = 3$  yields

$$Q(\phi) = -\gamma^{-1/2} \int_{\mathbb{R}} \frac{\varphi^2(3 - 2\varphi)}{(1 - \varphi)^2} d\zeta.$$

Writing  $\partial_\gamma \varphi = \gamma^{-1} \varphi + \hat{\varphi}$  with

$$\hat{\varphi}(\zeta) = \frac{3\gamma [(5 - 2\gamma) \cosh(\zeta) - 2\sqrt{(1 - \gamma)(4 - \gamma)}]}{2\sqrt{(1 - \gamma)(4 - \gamma)}(2 + \gamma + \sqrt{(1 - \gamma)(4 - \gamma)} \operatorname{sech}(\zeta))^2},$$

we obtain explicitly

$$\begin{aligned} \frac{d}{d\gamma} Q(\phi) &= -\frac{1}{2\gamma^{3/2}} \int_{\mathbb{R}} \frac{\varphi^2(9 - 7\varphi + 2\varphi^2)}{(1 - \varphi)^3} d\zeta \\ &\quad - \frac{2}{\gamma^{1/2}} \int_{\mathbb{R}} \frac{\varphi(3 - 3\varphi + \varphi^2)}{(1 - \varphi)^3} \hat{\varphi} d\zeta. \end{aligned}$$

Both terms are strictly negative. Indeed, the first term is negative because

$$\min_{\varphi \in [0,1]} (9 - 7\varphi + 2\varphi^2) = 4 > 0.$$

The second term is negative if  $\hat{\varphi}(\zeta) > 0$  for every  $\zeta \in \mathbb{R}$ , which is true for every  $\gamma \in (0, 1)$  because

$$\begin{aligned} (5 - 2\gamma) \cosh(\zeta) - 2\sqrt{(1 - \gamma)(4 - \gamma)} \\ \geq (5 - 2\gamma) - 2\sqrt{(1 - \gamma)(4 - \gamma)} > 0. \end{aligned}$$

Hence the stability criterion (66) is satisfied.  $\square$

The next lemma shows that the stability criterion (66) is satisfied in the asymptotic limits  $\gamma \rightarrow 0$  and  $\gamma \rightarrow 1$  for every  $b > 1$ . In view of the transformation (64), they correspond to the asymptotic limits  $k \rightarrow 0$  and  $k \rightarrow (b + 1)^{-1}c$  for fixed  $b > 1$  and  $c > 0$ .

**Lemma 7.** The mapping (16) is strictly increasing for  $b > 1$  and  $c > 0$  in the asymptotic limits  $k \rightarrow 0$  and  $k \rightarrow (b + 1)^{-1}c$ .

**Proof.** It follows from (63) that  $\varphi$  depends on  $\gamma$  analytically with  $\varphi \rightarrow 0$  as  $\gamma \rightarrow 0$ . Therefore, it satisfies the Taylor expansion

$$\varphi(\zeta) = \gamma\varphi_1(\zeta) + \gamma^2\varphi_2(\zeta) + \mathcal{O}(\gamma^3) \quad \text{as } \gamma \rightarrow 0,$$

where  $\varphi_1$  is computed similarly to (67) in the form

$$\varphi_1(\zeta) = \frac{3}{b + 1} \operatorname{sech}^2\left(\frac{1}{2}\zeta\right).$$

Similarly to the proof of Lemma 6, we write  $\partial_\gamma \varphi = \gamma^{-1} \varphi + \hat{\varphi}$ , where  $\hat{\varphi} = \gamma\varphi_2 + \mathcal{O}(\gamma^2)$ , and obtain from  $Q(\phi)$  in (65) that

$$\begin{aligned} \frac{d}{d\gamma} Q(\phi) &= -\frac{1}{2\gamma^{3/2}} \int_{\mathbb{R}} \left[ ((b + 2)\varphi - 1)(1 - \varphi)^{-\frac{b+3}{2}} \right. \\ &\quad \left. + (1 - 3\varphi + (b - 1)(b - 2)\varphi^2)(1 - \varphi)^{\frac{b-5}{2}} \right] d\zeta \\ &\quad - \frac{1}{2\gamma^{1/2}} \int_{\mathbb{R}} \left[ (b + 1)(1 - \varphi)^{-\frac{b+3}{2}} \right. \\ &\quad \left. + ((b - 1)^2\varphi - (b + 1))(1 - \varphi)^{\frac{b-5}{2}} \right] \hat{\varphi} d\zeta. \end{aligned}$$

We note that

$$\begin{aligned} f(\varphi) &:= ((b + 2)\varphi - 1) + (1 - 3\varphi + (b - 1)(b - 2)\varphi^2)(1 - \varphi)^{b-1} \\ &= \frac{3}{2}b(b - 1)\varphi^2 + \mathcal{O}(\varphi^3) \quad \text{as } \varphi \rightarrow 0 \end{aligned}$$

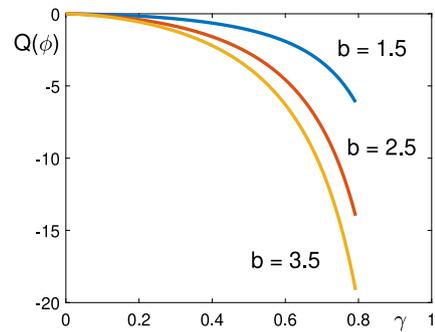


Fig. 2.  $Q(\phi)$  versus  $\gamma$  for three values of  $b$ .

and

$$\begin{aligned} g(\varphi) &:= (b + 1) + ((b - 1)^2\varphi - (b + 1))(1 - \varphi)^{b-1} \\ &= 2b(b - 1)\varphi + \mathcal{O}(\varphi^2) \quad \text{as } \varphi \rightarrow 0. \end{aligned}$$

By substituting the asymptotic expansions with  $\varphi = \mathcal{O}(\gamma)$  and  $\hat{\varphi} = \mathcal{O}(\gamma)$ , we obtain

$$\frac{d}{d\gamma} Q(\phi) = -\frac{3}{4}b(b - 1)\gamma^{1/2} \int_{\mathbb{R}} \varphi_1^2 d\zeta + \mathcal{O}(\gamma^{3/2}) \quad \text{as } \gamma \rightarrow 0,$$

so that the stability criterion (66) is satisfied as  $\gamma \rightarrow 0$  for every  $b > 1$ .

In the opposite asymptotic limit  $\gamma \rightarrow 1$ , we integrate Eq. (63) with the boundary conditions  $\varphi(\zeta) \rightarrow 0$  as  $|\zeta| \rightarrow \infty$  and obtain the first-order invariant:

$$\begin{aligned} -(\varphi')^2 + \frac{(1 - \varphi)^{b-1}}{\gamma b(b - 1)} [2(1 - \gamma) + 2(1 - \gamma)(b - 1)\varphi + b(b - 1)\varphi^2] \\ = \frac{2(1 - \gamma)}{\gamma b(b - 1)}. \end{aligned} \tag{69}$$

If the center of the solitary wave is translated to the origin, so that  $\varphi'(0) = 0$ , we can parameterize the integral for  $Q(\phi)$  in (65) and obtain the equivalent representation

$$Q(\phi) = \int_0^{\varphi_0} \frac{2\sqrt{b(b - 1)}(1 - \varphi)^{-\frac{b+1}{2}} [(1 - \varphi)^{b-1}[(b - 1)\varphi + 1] - 1] d\varphi}{\sqrt{(1 - \varphi)^{b-1} [2(1 - \gamma) + 2(1 - \gamma)(b - 1)\varphi + b(b - 1)\varphi^2] - 2(1 - \gamma)}}, \tag{70}$$

where  $\varphi_0$  is the turning point, for which the integral is weakly singular due to vanishing denominator. It follows from (70) that  $\varphi_0 \rightarrow 1$  as  $\gamma \rightarrow 1$  for  $b > 1$  and

$$\lim_{\gamma \rightarrow 1} Q(\phi) = 2 \int_0^1 \frac{(1 - \varphi)^{b-1} [(b - 1)\varphi + 1] - 1}{\varphi(1 - \varphi)^b} d\varphi = -\infty,$$

so that the stability criterion (66) is satisfied as  $\gamma \rightarrow 1$  for every  $b > 1$ .  $\square$

In the general case  $b > 1$ , we verify the stability criterion (66) numerically. Fig. 2 shows dependence of  $Q(\phi)$  computed from the numerical quadrature of the integral (70) versus  $\gamma$  in  $[0, 1]$  for different integer values of  $b$ . The turning point  $\varphi_0$  was obtained by using the Newton-Raphson iterative method. The integral in the expression (70) was computed by using the composite midpoint rule. It follows from the behavior of  $Q(\phi)$  versus  $\gamma$  that the stability criterion (66) is satisfied for every  $b > 1$ .

## 6. Conclusion

We have derived the precise criterion for orbital stability of smooth solitary waves on the nonzero constant background in

the  $b$ -CH equation with  $b > 1$ . Perturbations to the horizontal velocity  $u(t, x)$  are controlled in  $H^3(\mathbb{R})$ . Verification of this stability criterion analytically for integrable cases  $b = 2$  and  $b = 3$  give alternative proofs of orbital stability of smooth solitary waves in the Camassa–Holm and Degasperis–Procesi equations compared to the work in [27] and [28] respectively. We also verified the stability criterion analytically for every  $b > 1$  and  $c > 0$  in the asymptotic limits, where the family of smooth solitary waves terminates. The stability criterion is verified numerically in the general case. It is still open to verify the stability criterion analytically for every  $b > 1$ ,  $c > 0$ , and  $k \in (0, (b + 1)^{-1}c)$ .

This work opens roads to further studies of traveling waves in the  $b$ -CH model. Stability of smooth periodic waves for the integrable case  $b = 2$  was considered in [29] by using two alternative Hamiltonian structures (9) and (10). This approach also leads to the precise stability criterion which can only be verified numerically in the general case. It would be natural to explore the Hamiltonian structure (4) for stability of smooth periodic waves both for  $b = 2$  and generally for  $b > 1$ .

Stability of smooth traveling waves in the  $b$ -CH model with  $b \leq 1$  is also of interest from the points of physical applications. Some smooth traveling solitary waves with positive  $\phi$  and  $\mu$  exist for  $b \leq 1$  and their orbital stability can be clarified by using the same analysis as in the proof of Theorem 1. Orbital stability of periodic waves for  $b \leq 1$  is also an open problem.

### CRediT authorship contribution statement

**Stéphane Lafortune:** Conceptualization, Funding acquisition, Investigation, Methodology, Resources, Software, Validation, Visualization, Writing – original draft, Writing – review & editing.  
**Dmitry E. Pelinovsky:** Conceptualization, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Validation, Visualization, Writing – original draft, Writing – review & editing.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Acknowledgments

The research of S. Lafortune was supported by a Collaboration Grants for Mathematicians from the Simons Foundation, USA (award # 420847). D. E. Pelinovsky acknowledges the funding of this study provided by Grants No. FSWE, Russian Federation-2020-0007 and No. NSH, Russian Federation-70.2022.1.5. The authors thank R. M. Chen, B. Ehrman, D. Holm, A. Hone, M. Johnson, and T. Kapitula for useful comments.

### Appendix. Casimirs for the operator $J_m$

While it is known that  $F_1$  and  $F_2$  in (7) and (8) are Casimirs for the operator  $J_m$  in (5) [10,14], the proof of that fact has not been formally presented before. We reproduce an argument graciously provided by Andrew Hone to us.

Suppose that  $J_m f = 0$ . Then we have

$$J_m f = (1 - b)^{-1} b^2 m^{1-1/b} \partial_x (m^{1/b} \partial_x^{-1} (1 - \partial_x^2)^{-1} (m^{1/b} \partial_x (m^{1-1/b} f))) = 0,$$

where we have used the alternate expression for  $J_m$  given in [16, Equation (1.10)]. Multiplying by  $(1 - b)b^{-2}m^{1/b-1}$  and performing one integration gives

$$\partial_x^{-1} (1 - \partial_x^2)^{-1} (m^{1/b} \partial_x (m^{1-1/b} f)) = k_1 m^{-1/b},$$

where  $k_1$  is an arbitrary constant. This implies

$$\begin{aligned} \partial_x (m^{1-1/b} f) &= k_1 m^{-1/b} \partial_x (1 - \partial_x^2) m^{-1/b} \\ &= k_1 \partial_x (b^{-1} m^{-2/b-1} m_{xx} \\ &\quad - (2b)^{-1} (b^{-1} + 2) m^{-2/b-2} m_x^2 + 2^{-1} m^{-2/b}). \end{aligned}$$

Integrating once more gives

$$f = k_1 (b^{-1} m^{-1/b-2} m_{xx} - (2b)^{-1} (b^{-1} + 2) m^{-1/b-3} m_x^2 + 2^{-1} m^{-1/b-1}) + k_2 m^{1/b-1},$$

for another arbitrary constant  $k_2$ . This expression is a linear combination of  $\frac{\delta F_1}{\delta m}$  and  $\frac{\delta F_2}{\delta m}$ .

### References

- [1] H.R. Dullin, G.A. Gottwald, D.D. Holm, An integrable shallow water equation with linear and nonlinear dispersion, *Phys. Rev. Lett.* 87 (2001) 194501.
- [2] A. Degasperis, D.D. Holm, A.N.W. Hone, A new integrable equation with peakon solutions, *Theor. Math. Phys.* 133 (2002) 1461–1472.
- [3] R. Camassa, D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* 71 (1993) 1661–1664.
- [4] A. Degasperis, M. Procesi, in: A. Degasperis, G. Gaeta (Eds.), *Symmetry and Perturbation Theory*, in: *Asymptotic Integrability*, World Scientific Publishing, Singapore, 1999, pp. 23–37.
- [5] R.I. Ivanov, Water waves and integrability, *Phil. Trans. R. Soc. A* 365 (2007) 2267–2280.
- [6] R. Camassa, D. Holm, J. Hyman, A new integrable shallow water equation, *Adv. Appl. Mech.* 31 (1994) 1–33.
- [7] R.S. Johnson, Camassa–Holm, Korteweg–de Vries and related models for water waves, *J. Fluid Mech.* 455 (2002) 63–82.
- [8] A. Constantin, D. Lannes, The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations, *Arch. Ration. Mech. Anal.* 192 (2009) 165–186.
- [9] B. Guo, Z. Liu, Periodic cusp wave solutions and single-solitons for the  $b$ -equation, *Chaos Solitons Fractals* 23 (2005) 1451–1463.
- [10] L.E. Barnes, A.N.W. Hone, Similarity reductions of peakon equations: the  $b$ -family, 2022, arXiv:2201.00265.
- [11] D.D. Holm, M.F. Staley, Nonlinear balance and exchange of stability in dynamics of solitons, peakons, ramps/cliffs and leftons in a  $1+1$  nonlinear evolutionary PDE, *Phys. Lett. A* 308 (2003) 437–444.
- [12] D.D. Holm, M.F. Staley, Wave structure and nonlinear balances in a family of evolutionary PDEs, *SIAM J. Appl. Dyn. Syst.* 2 (2003) 323–380.
- [13] E.G. Charalampidis, R. Parker, P.G. Kevrekidis, S. Lafortune, The stability of the  $b$ -family of peakon equations, 2020, arXiv:2012.13019, submitted to *Nonlinearity*.
- [14] A.N.W. Hone, S. Lafortune, Stability of stationary solutions for nonintegrable peakon equations, *Physica D* 269 (2014) 28–36.
- [15] A. Degasperis, D.D. Holm, A.N.W. Hone, Integrable and non-integrable equations with peakons, in: *Proceedings of Nonlinear Physics – Theory and Experiment II*, World Scientific, Singapore, 2003, pp. 37–43.
- [16] A.N.W. Hone, J.P. Wang, Prolongation algebras and Hamiltonian operators for peakon equations, *Inverse Problems* 19 (2003) 129–145.
- [17] D.D. Holm, A.N.W. Hone, A class of equations with peakon and pulson solutions, *J. Nonl. Math. Phys.* 12 (2005) 380–394.
- [18] A. Constantin, L. Molinet, Orbital stability of solitary waves for a shallow water equation, *Physica D* 157 (2001) 75–89.
- [19] A. Constantin, W.A. Strauss, Stability of peakons, *Comm. Pure Appl. Math.* 53 (2000) 603–610.
- [20] Z. Lin, Y. Liu, Stability of peakons for the Degasperis–Procesi equation, *Comm. Pure Appl. Math.* 62 (2009) 125–146.
- [21] A. Himonas, K. Grayshan, C. Holliman, Ill-posedness for the  $b$ -family of equations, *J. Nonlinear Sci.* 26 (2016) 1175–1190.
- [22] F. Linares, G. Ponce, Th.C. Sideris, Properties of solutions to the Camassa–Holm equation on the line in a class containing the peakons, *Adv. Stud. Pure Math.* 81 (2019) 196–245.
- [23] F. Natali, D.E. Pelinovsky, Instability of  $H^1$ -stable peakons in the Camassa–Holm equation, *J. Differential Equations* 268 (2020) 7342–7363.
- [24] A. Madiyeva, D.E. Pelinovsky, Growth of perturbations to the peaked periodic waves in the Camassa–Holm equation, *SIAM J. Math. Anal.* 53 (2021) 3016–3039.
- [25] R.M. Chen, D.E. Pelinovsky,  $W^{1,\infty}$  instability of  $H^1$ -stable peakons in the Novikov equation, *Dyn. Partial Differ. Equ.* 18 (2021) 173–197.
- [26] S. Lafortune, D.E. Pelinovsky, Spectral instability of peakons in the  $b$ -family of the Camassa–Holm equations, *SIAM J. Math. Anal.* 54 (2022) 4572–4590.
- [27] A. Constantin, W.A. Strauss, Stability of the Camassa–Holm solitons, *J. Nonlinear Sci.* 12 (2002) 415–422.
- [28] J. Li, Y. Liu, Q. Wu, Spectral stability of smooth solitary waves for the Degasperis–Procesi equation, *J. Math. Pures Appl.* 142 (2020) 298–314.

- [29] A. Geyer, R.H. Martins, F. Natali, D.E. Pelinovsky, Stability of smooth periodic traveling waves in the Camassa–Holm equation, *Stud. Appl. Math.* 148 (2022) 27–61.
- [30] A. Constantin, J. Escher, On the Cauchy problem for a family of quasilinear hyperbolic equations, *Comm. Partial Differential Equations* 23 (1998) 1449–1458.
- [31] A. Constantin, J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.* 181 (1998) 229–243.
- [32] J. Escher, Z. Yin, Well-posedness, blow-up phenomena, and global solutions for the b-equation, *J. Reine Angew. Math.* 624 (2008) 51–80.
- [33] Y. Zhou, On solutions to the holm-staley b-family of equations, *Nonlinearity* 23 (2010) 369–381.
- [34] Y. Liu, Z. Wang, Stability of smooth multi-solitons for the Camassa–Holm equation, 2021, preprint.
- [35] N.G. Vakhitov, A.A. Kolokolov, Stationary solutions of the wave equation in a medium with nonlinearity saturation, *Radiophys. Quantum Electron.* 16 (1973) 783–789.
- [36] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry, I, *J. Funct. Anal.* 74 (1987) 160–197.