Bound states in Gross-Pitaevskii theory

Szymon Sobieszek
Supervisor: D. E. Pelinovsky

McMaster University

April 21, 2023

Setting

- We consider the focusing NLS with a harmonic potential

$$i\partial_t w = -\Delta w + |x|^2 w - |w|^{2p} w,$$

where $w(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ and $p > 0$, known as the Gross-Pitaevskii equation.

- Two conserved quantities:

$$M(w) := \int_{\mathbb{R}} |w|^2 dx \quad \text{(mass)}$$

$$E(w) := \int_{\mathbb{R}^d} \left(|\nabla w|^2 + |x|^2 |w|^2 - \frac{1}{p+1} |w|^{2p+2} \right) dx \quad \text{(energy)}$$

- We study energy-critical: $(d-2)p = 2$, and energy-supercritical: $(d-2)p > 2$ cases.

Ground states

- Standing wave solutions $w(t,x) = e^{i\lambda t}u(x)$ are found from the stationary equation

$$-\Delta u + |x|^2 u - |u|^{2p} u = \lambda u.$$

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- $\mathfrak{u}$-BVP for the ground states:

$$\begin{cases} \mathfrak{u}''(r) + \frac{d-1}{r}\mathfrak{u}'(r) - r^2\mathfrak{u}(r) + \lambda\mathfrak{u}(r) + |\mathfrak{u}(r)|^{2p}\mathfrak{u}(r) = 0, & r > 0, \\ \mathfrak{u}(r) > 0, & \mathfrak{u}'(r) < 0, \\ \lim_{r\to 0}\mathfrak{u}(r) < \infty, & \lim_{r\to\infty}\mathfrak{u}(r) = 0. \end{cases}$$

- Crandall & Rabinowitz theory $\Longrightarrow$ a family $\{\mathfrak{u}_b\}_{b\approx 0}$ of ground states parameterized by $b := \|\mathfrak{u}_b\|_\infty = \mathfrak{u}_b(0)$ bifurcates locally from $\mathfrak{u}_0 = e^{-\frac{r^2}{2}}$ when $\lambda = d$.
- Ground states for any $b > 0$ can be found from the shooting method (Joseph, Lundgren, 1973) from the IVP

$$\begin{cases} f''(r) + \frac{d-1}{r} f'(r) - r^2 f(r) + \lambda f(r) + |f(r)|^{2p} f(r) = 0, & r > 0, \\ f(0) = b, \quad f'(0) = 0. \end{cases}$$

Theorem 1 (Bizon, Ficek, Pelinovsky, Sobieszek, 2021).

Let $p = 1$, $d \geq 4$. For every $b > 0$, there exists $\lambda = \lambda(b) \in (d-4, d)$, such that the solution f to the IVP with $\lambda = \lambda(b)$ is a ground state $\mathfrak{u}_b$.

Snaking and monotone behavior

- We are interested in the global behaviour of the bifurcation curve in the (λ, b)-plane.

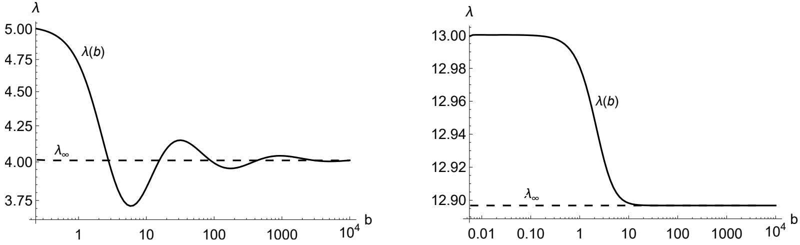

Figure 1: Graph of λ as a function of b for the ground state of the $\mathfrak{u}$-boundary-value problem for $d = 5$ (left) and $d = 13$ (right).

Energy-supercritical case

- There exists a limiting singular solution $\mathfrak{u}_\infty$ with finite energy, and $\lambda_\infty \in (d-4, d)$ s.t. $\mathfrak{u}_b \to \mathfrak{u}_\infty$ and $\lambda(b) \to \lambda_\infty$ as $b \to \infty$ (Selem et al., 2013), satisfying

$$\mathfrak{u}_\infty(r) = \frac{\sqrt{d-3}}{r}\left[1 + \mathcal{O}(r^2)\right], \quad \text{as } r \to 0.$$

- Such phenomenon has been previously observed in the NLS without the harmonic potential in a ball:

$$\begin{cases} \Delta u + \nu u + |u|^{2p} u = 0, & x \in B_1 \\ u > 0, & x \in B_1 \\ u = 0, & x \in B_1, \end{cases}$$

where $\nu > 0$, $(d-2)p > 2$.

Theorem 2 (Bizon, Ficek, Pelinovsky, Sobieszek, 2021).

Fix $d \geq 5$. Under some nondegeneracy assumptions, $\lambda(b)$ is uniquely defined near λ_∞ for $b \gg 1$, and

$$\lambda(b) - \lambda_\infty \sim A_\infty b^{-\beta} \sin(\alpha \ln b + \delta_\infty), \quad \text{if } 5 \leq d \leq 12,$$

for some $A_\infty \neq 0$ and $\delta_\infty, \alpha, \beta > 0$, and

$$\lambda(b) - \lambda_\infty \sim B_\infty b^{-\kappa}, \quad \text{if } d \geq 13,$$

for some $B_\infty \neq 0$ and $\kappa > 0$.

Ideas behind proofs

- Emden-Fowler transformation $r = e^t$, $\Psi(t) = e^t f(e^t)$.
- Second-order nonautonomous equation

$$\Psi''(t)+(d-4)\Psi'(t)-(d-3)\Psi(t)+\Psi(t)^3 = -\lambda e^{2t}\Psi(t)+e^{4t}\Psi(t)$$

- $\{\Psi_b\}_{b>0}$ family of solutions satisfying

$$\Psi_b(t) = be^t\left[1 - \frac{\lambda+b^2}{2d}e^{2t} + \mathcal{O}(e^{4t})\right], \quad \text{as } t \to -\infty.$$

- $\{\Psi_c\}_{c\in\mathbb{R}}$ family of solutions with

$$\Psi_c(t) \sim ce^{\frac{\lambda-d+2}{2}t}e^{-\frac{1}{2}e^{2t}}, \quad \text{as } t \to +\infty.$$

- Ψ-equation truncated for $t \to -\infty$:

$$\Theta''(t) + (d-4)\Theta'(t) - (d-3)\Theta(t) + \Theta(t)^3 = 0.$$

We get that $\Psi_b(t) \sim b\Theta_h(t)$ as $t \to -\infty$, where $\Theta_h(t)$ is the unique (up to translation) heteroclinic orbit, and

$$\sup_{t\in[0,T+a\log b]} |\Psi_b(t-\log b) - \Theta_h(t)| \leq Cb^{-2(1-a)}.$$

- $\Psi_c(t)$ is defined near $\Psi_\infty := e^t \mathfrak{u}_\infty(e^t)$ for λ close to λ_∞.

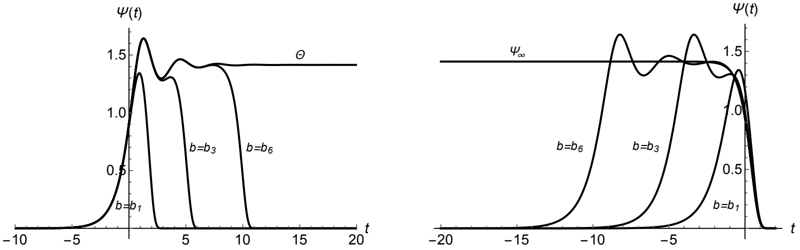

Figure 2: Plots of the solutions Ψ_b for $d = 5$ and $b = 4, 10^2, 1.3 \times 10^4$ in comparison with Θ after translation of t by $\log b$ (left) and with Ψ_∞ without translation of t (right).

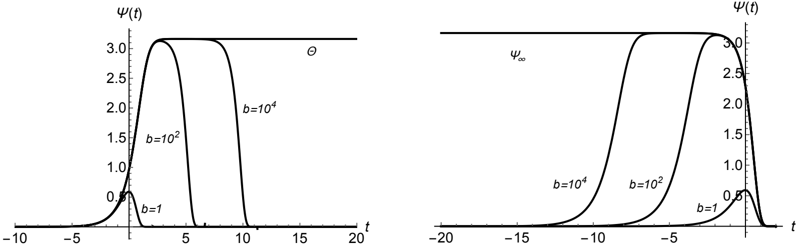

Figure 3: Plots of the solutions Ψ_b for $d = 13$ and $b = 1, 10^2, 10^4$ in comparison with Θ after translation of t by $\log b$ (left) and with Ψ_∞ without translation of t (right).

Morse index in the energy-supercritical case

- Linearization around the ground state:

$$\mathcal{L}_b := -\frac{d^2}{dr^2} - \frac{d-1}{r}\frac{d}{dr} + r^2 - \lambda(b) - 3\mathfrak{u}_b(r)^2.$$

- Linearization around the limiting singular solution:

$$\mathcal{L}_\infty := -\frac{d^2}{dr^2} - \frac{d-1}{r}\frac{d}{dr} + r^2 - \lambda_\infty - 3\mathfrak{u}_\infty(r)^2.$$

- Both operators defined in $\mathcal{E}_r = H^1_r \cap L^{2,1}_r$.
- Morse index is defined as the number of negative eigenvalues of the linearized operators.

- Note that $\mathcal{L}_b \partial_b \mathfrak{u}_b = \lambda'(b)\mathfrak{u}_b$ for $\partial_b \mathfrak{u}_b \in \mathcal{E}_r$.

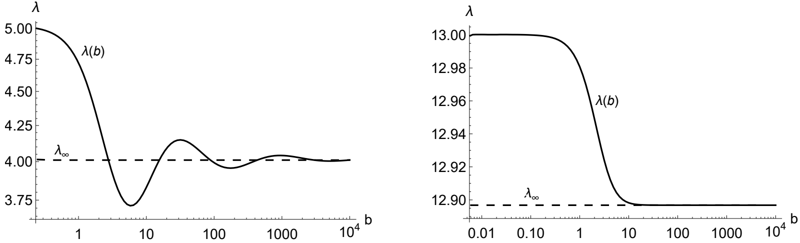

Figure 4: Graph of λ as a function of b for the ground state of the $\mathfrak{u}$-boundary-value problem for $d = 5$ (left) and $d = 13$ (right).

Theorem 3 (Pelinovsky, Sobieszek, 2022).

For every $d \geq 13$, there exists $b_0 > 0$ such that the Morse index of $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$ is finite and is independent of b for every $b \in (b_0, \infty)$. Moreover, it concides with the Morse index of $\mathcal{L}_\infty : \mathcal{E} \mapsto \mathcal{E}^*$.

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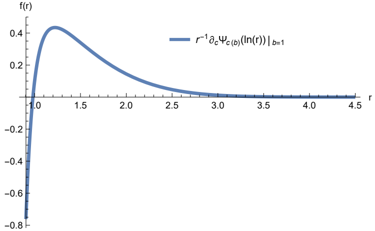
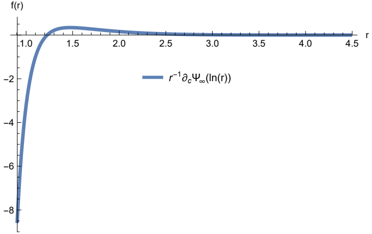
Numerical evidence below suggest that the Morse index equals one.





Energy-critical case

- General case $p > 0 \implies d = 2 + \frac{2}{p}$, $d \geq 2$.
- Existence of a family of ground states $\{\mathfrak{u}_b\}_{b>0}$, but lack of the limiting singular solution $\mathfrak{u}_\infty$.
- The b-solution is defined in the local neighborhood of $r = 0$ near the algebraic soliton

$$U_b(r) = \frac{b}{(1 + \alpha_p b^{2p} r^2)^{\frac{1}{p}}},$$

and the c-solution in the local neighborhood as $r \to \infty$ near the Tricomi function $\mathfrak{U}(z; \alpha, \beta)$

$$V_c(r) = c e^{-\frac{1}{2}r^2} \mathfrak{U}(r^2; \alpha, \beta).$$

Theorem 4 (Pelinovsky, Sobieszek, 2023).

Let $d > 4$, $p = \frac{2}{d-2}$, and $\lambda = \lambda(b)$ be the solution curve for the ground state $\mathfrak{u} = \mathfrak{u}_b$ of the stationary GP equation satisfying $\mathfrak{u}_b(0) = b$, $\mathfrak{u}_b'(r) < 0$ for $r \in (0, \infty)$, and $\mathfrak{u}_b(r) \to 0$ as $r \to \infty$. There exists C_p, such that

$$\lambda(b) \sim C_p \begin{cases} b^{-2p}, & 0 < p < \frac{1}{2}, \\ b^{-1} \log b, & p = \frac{1}{2}, \\ b^{-2(1-p)}, & \frac{1}{2} < p < 1. \end{cases} \qquad \text{as} \quad b \to \infty.$$

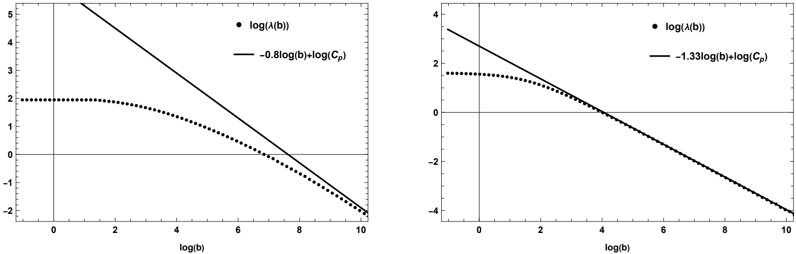


Figure 5: Graph of λ as a function of b for the ground state of the $\mathfrak{u}$-boundary-value problem for $d = 7$ (left) and $d = 5$ (right).

- Energy-critical results correspond to recent results (Wei et al., 2023), where it was shown that

$$\lambda(\varepsilon) \sim \begin{cases} 1+\varepsilon & d=3, \\ |\log \varepsilon|^{-1} & d=4, \\ \varepsilon & d=5, \\ \varepsilon^2 |\log \varepsilon| & d=6, \\ \varepsilon^2 & d \geq 7, \end{cases}$$

for $\varepsilon := b^{-p}$, which extends the previous results to $d=3$ and $d=4$.

Published work

- Bizon, P., Ficek, F., Pelinovsky, D. E., and Sobieszek, S. (2021).
  Ground state in the energy super-critical Gross–Pitaevskii equation with a harmonic potential.
  Nonlinear Analysis, 210:112358.

- Pelinovsky, D. E. and Sobieszek, S. (2022).
  Morse index for the ground state in the energy supercritical Gross–Pitaevskii equation.
  Journal of Differential Equations, 341:380–401.

- Pelinovsky, D. E. and Sobieszek, S. (2023).
  Ground state of the Gross-Pitaevskii equation with a harmonic potential in the energy-critical case.
  arXiv preprint arXiv:2302.03865.