On the orbital stability of Gaussian solitary waves in the log-KdV equation

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Abstract

We consider the logarithmic Korteweg–de Vries (log-KdV) equation, which models solitary waves in anharmonic chains with Hertzian interaction forces. By using an approximating sequence of global solutions of the regularized generalized KdV equation in $H^1(\mathbb{R})$ with conserved L^2 norm and energy, we construct a weak global solution of the log-KdV equation in a subset of $H^1(\mathbb{R})$. This construction yields conditional orbital stability of Gaussian solitary waves of the log-KdV equation, provided that uniqueness and continuous dependence of the constructed solution holds.

Furthermore, we study the linearized log-KdV equation at the Gaussian solitary wave and prove that the associated linearized operator has a purely discrete spectrum consisting of simple purely imaginary eigenvalues in addition to the double zero eigenvalue. The eigenfunctions, however, do not decay like Gaussian functions but have algebraic decay. Using numerical approximations, we show that the Gaussian initial data do not spread out but produce visible radiation at the left slope of the Gaussian-like pulse in the time evolution of the linearized log-KdV equation.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Solitary waves in anharmonic chains with Hertzian interaction forces are modelled by the Fermi–Pasta–Ulam (FPU) lattices with non-smooth nonlinear potentials [15]. Recently, the

FPU lattice equations in the limit of small anharmonicity of the Hertzian interaction forces were reduced to the following logarithmic Korteweg–de Vries (log-KdV) equation [7, 10]:

$$v_t + v_{xxx} + (v \log |v|)_x = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}. \tag{1.1}$$

Here and in what follows, the subscripts denote the partial derivatives.

The log-KdV equation (1.1) has a two-parameter family of Gaussian solitary waves

$$v(x, t) = e^c v_G(x - ct - a), \quad a, c \in \mathbb{R}, \tag{1.2}$$

where v_G is a symmetric standing wave given by

$$v_G(x) := e^{\frac{1}{2} - \frac{x^2}{4}}, \quad x \in \mathbb{R}. \tag{1.3}$$

The ultimate goal of this work is to prove the nonlinear orbital stability of Gaussian solitary waves (1.2) in the log-KdV equation (1.1). The main problem is, of course, the limited smoothness of the log-KdV equation, where the nonlinearity $f(v) = v \log |v|$ is continuous but not differentiable at $v = 0$, whereas the energy

$$E(v) = \frac{1}{2} \int_{\mathbb{R}} \left[(v_x)^2 - v^2 \log |v| \right] dx + \frac{1}{4} \int_{\mathbb{R}} v^2 \, dx \tag{1.4}$$

is not a C^2 functional at $v = 0$.

Although $E(v)$ is not C^2 at $v = 0$, the second variation of $E(v)$ at v_G is well determined by the Schrödinger operator with a harmonic potential

$$L := -\partial_x^2 + \frac{1}{4}(x^2 - 6). \tag{1.5}$$

Note that the spectrum of L in $L^2(\mathbb{R})$ consists of equally spaced simple eigenvalues

$$\sigma(L) = \{-1, 0, 1, 2, \ldots\},$$

which include exactly one negative eigenvalue with the eigenvector v_G (defined without normalization). Therefore, $E(v)$ is not convex at v_G in a subspace of $H^1(\mathbb{R})$. Nevertheless, the second variation of $E(v)$ at v_G given by $E_c(u) = \frac{1}{2} \langle Lu, u \rangle_{L^2}$ is positive in the constrained space

$$X_c := \left\{ u \in H^1(\mathbb{R}) \cap L_1^2(\mathbb{R}) : \langle v_G, u \rangle_{L^2} = 0 \right\}, \tag{1.6}$$

where $L_1^2(\mathbb{R}) := \{u \in L^2(\mathbb{R}) : xu \in L^2(\mathbb{R})\}$. At the linearized approximation, this constraint fixes $\|v\|_{L^2}^2$ at $\|v_G\|_{L^2}^2$. Based on these facts, the linear orbital stability of the Gaussian solitary wave v_G can be deduced in the following sense.

Consider the time evolution of the linearized log-KdV equation at the Gaussian wave v_G,

$$\begin{cases} u_t = \partial_x Lu, & t > 0, \\ u|_{t=0} = u_0. \end{cases} \tag{1.7}$$

Note that the quadratic energy function $E_c(u)$ is constant in time for smooth solutions of the linearized log-KdV equation (1.7). We say that the Gaussian solitary wave v_G is linearly orbitally stable in $H^1(\mathbb{R})$, if for every $u_0 \in X_c$, there exists a unique global solution u of the linearized log-KdV equation (1.7) in a subspace of $L^\infty(\mathbb{R}, H^1(\mathbb{R}))$ which satisfies the following bound

$$\|u(\cdot, t)\|_{H^1} \leqslant C(\|u_0\|_{H^1}, \|u_0\|_{L_1^2}), \quad t \in \mathbb{R}, \tag{1.8}$$

for some t-independent positive constant C that depends on the initial norms $\|u_0\|_{H^1}$ and $\|u_0\|_{L_1^2}$.

The following theorem is based on the fact that the conserved quantity $E_c(u)$ is positive if $u \in X_c$ and controls the squared H^1 norm of the solution u in time t. Although this theorem was not formulated in [10], it can be deduced from the arguments developed in this work, which rely on symplectic projections and the energy method. Due to the scaling invariance (1.2), the same stability result holds for all values of parameters a and c.

Theorem 1.1. *[10] Gaussian solitary wave v_G of the log-KdV equation (1.1) is linearly orbitally stable in $H^1(\mathbb{R})$.*

To extend this result to the proof of nonlinear orbital stability of Gaussian solitary waves, we need local and global well-posedness of the log-KdV equation (1.1). In a similar context of the log-NLS equation, global well-posedness and orbital stability of the Gaussian solitary waves were proved by Cazenave and Haraux [6] and Cazenave [4] (these results were later summarized in the monograph [5, section 9.3]).

The idea from [4, 6] is to approximate the logarithmic nonlinearity by a smooth nonlinearity, to construct a sequence of global bounded solutions of the regularized system in $C(\mathbb{R}, H^1(\mathbb{R}))$, and then to prove convergence of a subsequence to the limit, which solves the original log-NLS equation in a weak sense. Uniqueness of the weak solution does not come in this formalism for free, but it can be proved with an additional trick involving log-nonlinearity [5, lemma 9.3.5], which is based on the bound (2.23) below. Similar ideas were recently used by Carles and Gallo [2] for analysis of the NLS equation with power-like nonlinearity and a sublinear damping term (see also [3] for various extensions).

While trying to adopt the programme above to the log-KdV equation (1.1), we come to two main difficulties. The first one is that local solutions of the generalized KdV equation with smooth nonlinearities exist in the space $C([-t_0, t_0], H^s(\mathbb{R}))$ for $s > \frac{3}{2}$ (see [12, 18] for two independent proofs of these results). Therefore, to push s to lower values, in particular, to $s = 1$, we need to adopt the formalism of Kenig *et al* [13], which was originally developed to the KdV equation with integer powers. The second difficulty is that the proof of uniqueness requires us to consider more restrictive solutions of the log-KdV equation than the ones constructed in the proof of existence. As a result, the solutions we establish in a subspace of $H^1(\mathbb{R})$ have non-decreasing L^2 norm and energy. Unlike the case of weak solutions of the NLS equation (e.g., treated in [9]), we cannot establish even the L^2 conservation in the log-KdV equation (1.1). See remark 2.4 below for further details.

To make sense of the energy (1.4) of the log-KdV equation (1.1), we shall work with functions in the class

$$X := \left\{ v \in H^1(\mathbb{R}) : v^2 \log |v| \in L^1(\mathbb{R}) \right\}. \tag{1.9}$$

The following theorem gives the main result on the existence of weak solutions of the log-KdV equation (1.1) in the energy space X.

Theorem 1.2. *For any $v_0 \in X$, there exists a global solution $v \in L^\infty(\mathbb{R}, X)$ of the log-KdV equation (1.1) such that*

$$\|v(t)\|_{L^2} \leqslant \|v_0\|_{L^2}, \quad E(v(t)) \leqslant E(v_0), \quad \text{for all } t \in \mathbb{R}. \tag{1.10}$$

Moreover, if $\partial_x \log |v| \in L^\infty((-t_0, t_0) \times \mathbb{R})$, then the solution v exists in $C((-t_0, t_0), X)$, is unique for every $t \in (-t_0, t_0)$, depends continuously on the initial data $v_0 \in X$, and satisfies conservation of $\|v(t)\|_{L^2}$ and $E(v(t))$ for all $t \in (-t_0, t_0)$.

Unfortunately, $\partial_x \log |v|$ is unbounded as $|x| \to \infty$ for the Gaussian solitary wave v_G. Therefore, it is unclear from theorem 1.2 if uniqueness and continuous dependence hold for such Gaussian solutions. Convexity of the energy functional $E(v)$ at v_G and global well-posedness of the Cauchy problem for the log-KdV equation (1.1) in X are the two main ingredients of the nonlinear orbital stability of the Gaussian solitary wave v_G in $H^1(\mathbb{R})$. Because of limitations in theorem 1.2, we can only obtain the conditional nonlinear orbital stability, where the condition is that the global solution $v \in L^\infty(\mathbb{R}, X)$ of the log-KdV equation (1.1) constructed in theorem 1.2 is unique and depends continuously on the initial data $v_0 \in X$.

To define the nonlinear orbital stability, we use the standard theory (see [1] for review of this theory). We say that the Gaussian solitary wave v_G is orbitally stable in $H^1(\mathbb{R})$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $v_0 \in X \subset H^1(\mathbb{R})$ satisfying

$$\|v_0 - v_G\|_{H^1} \leqslant \delta, \tag{1.11}$$

there exists a unique solution $v \in C(\mathbb{R}, X)$ of the log-KdV equation (1.1) satisfying

$$\inf_{a \in \mathbb{R}} \|v(\cdot, t) - v_G(\cdot + a)\|_{H^1} \leqslant \varepsilon, \tag{1.12}$$

for all $t \in \mathbb{R}$. The following conditional orbital stability result follows from theorems 1.1 and 1.2 with the standard arguments [1].

Corollary 1.3. *Gaussian solitary wave v_G of the log-KdV equation (1.1) is orbitally stable in $H^1(\mathbb{R})$ under condition that the solution $v \in L^\infty(\mathbb{R}, X)$ in Theorem 1.2 is unique and depends continuously on the initial data $v_0 \in X$.*

Since corollary 1.3 does not give a proper nonlinear orbital stability result, we shall also look at the orbital stability problem from a different point of view. We first inspect properties of the linearized operator $\partial_x L$ in the linearized log-KdV equation (1.7). The following theorem gives the spectral stability of the Gaussian solitary wave v_G with precise characterization of eigenvalues and eigenvectors of the linear operator $\partial_x L$ in $L^2(\mathbb{R})$.

Theorem 1.4. *The spectrum of $\partial_x L$ in $L^2(\mathbb{R})$ is purely discrete and consists of a double zero eigenvalue and a symmetric sequence of simple purely imaginary eigenvalues $\{\pm i\omega_n\}_{n \in \mathbb{N}}$ such that $0 < \omega_1 < \omega_2 < \ldots$ and $\omega_n \to \infty$ as $n \to \infty$. The double zero eigenvalue corresponds to the Jordan block*

$$\partial_x L \partial_x v_G = 0, \quad \partial_x L v_G = -\partial_x v_G, \tag{1.13}$$

whereas the purely imaginary eigenvalues $\lambda = \pm i\omega_n$ correspond to the eigenfunctions $u = u_{\pm n}(x)$, which are smooth in x but decay algebraically as $|x| \to \infty$.

Remark 1.5. Because the spectrum of $\partial_x L$ is purely discrete by theorem 1.4, we have no chance to assume dispersive decay estimates near the Gaussian solitary wave v_G. This indicates that no asymptotic stability result can hold for solitary waves in the log-KdV equation (1.1).

Remark 1.6. Remark 1.5 agrees with the result of [4, proposition 4.3] stating that the L^p norms at the solution v for any $p \geqslant 2$ including $p = \infty$ may not vanish as $t \to \infty$ (or in a finite time), hence the solution does not scatter to zero. Although this statement was proved for the log-NLS equation, it is based on the consideration of the same energy functional $E(v)$ as in the log-KdV equation (1.1). This non-scattering result is related to the fact that the L^p norm at the family (1.2) can be scaled to be arbitrarily small by using the scaling parameter $c \in \mathbb{R}$.

Theorem 1.4 can be used to provide an alternative proof of theorem 1.1. On the other hand, because of the algebraic decay of the eigenfunctions in theorem 1.4, any function of x that decays like the Gaussian function as $|x| \to \infty$ cannot be simply represented as series of eigenfunctions of the linearized operator $\partial_x L$. To explain the importance of such representations, we set $v(x, t) := v_G(x) + w(x, t)$ and obtain the equivalent log-KdV equation

$$w_t = \partial_x L w - \partial_x N(w), \tag{1.14}$$

where the nonlinear term $N(w)$ is given by

$$N(w) := w \log\left(1 + \frac{w}{v_G}\right) + v_G\left[\log\left(1 + \frac{w}{v_G}\right) - \frac{w}{v_G}\right].$$

It is clear that the nonlinear term $N(w)$ does not behave uniformly in x unless w decays at least as fast as v_G in (1.3). On the other hand, if $w(x, t) = v_G(x)h(x, t)$, where h is a bounded function in its variables, then $N(w) = v_G n(h)$, where $n(h) := h \log(1 + h) + \log(1 + h) - h$ is analytic in h for any $h \in (-1, 1)$. This observation on the nonlinear term $N(w)$ inspires us to consider solutions of the linearized log-KdV equation (1.7) starting with the initial data $u_0 \in X_c$ such that $u_0 = v_G h_0$ for $h_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$.

We have undertaken numerical simulations of this linear Cauchy problem to illustrate that solutions $u(x, t)$ of the linearized log-KdV equation (1.7) with Gaussian initial data do not spread out as the time variable evolves. Nevertheless, they produce visible radiation at the left slope of the Gaussian solutions. Further studies are needed to figure out if the nonlinear orbital stability of the Gaussian solitary wave v_G can be proved in the framework of the nonlinear evolution problem (1.14) with initial data $w_0 = v_G h_0$ for small $h_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$.

The rest of the paper consists of the following. Theorem 1.2 is proved in section 2. Theorem 1.4 is proved in section 3. Section 4 contains the proof of theorem 1.1 (which is alternative to the one given in [10]) and the numerical simulations of the time evolution of the linearized log-KdV equation (1.7) starting with Gaussian initial data.

2. Global solutions of the log-KdV equation

To prove theorem 1.2, we follow the algorithm developed in [4,6] for the log-NLS equation (see also [5, section 9.3]), which shares common features with the approach of Ginibre and Velo [9]. We slightly simplify the functional framework from [4,6], by using Strichartz estimates instead of arguments coming from convex analysis. In the context of KdV equations, these tools were developed by Kenig, Ponce, and Vega [13].

The strategy of the proof is the following. We first regularize the logarithmic nonlinearity near the origin and construct a sequence of approximating solutions via a contraction principle as in [13]. Then, we derive uniform estimates on the $H^1(\mathbb{R})$ norm of the approximating solutions. This allows us to pass to the limit and obtain a global solution v in the energy space X with non-increasing $\|v\|_{L^2}$ and energy $E(v)$. In the last subsection, we establish uniqueness of the global solutions v of the log-KdV equation (1.1) with bounded $\partial_x \log |v|$ by using special properties of the logarithmic nonlinearity.

2.1. Approximating solutions

We shall regularize the behaviour of the logarithmic nonlinearity $f(v) = v \log |v|$ near $v = 0$. Compared to the approximation considered in [4, 6], we will work with a different (much simpler) approximation of the logarithmic nonlinearity.

For any fixed $\varepsilon > 0$, let us define the family of regularized nonlinearities in the form

$$f_\varepsilon(v) = \begin{cases} f(v), & |v| \geqslant \varepsilon, \\ p_\varepsilon(v), & |v| \leqslant \varepsilon, \end{cases} \tag{2.1}$$

where p_ε is an odd polynomial of degree $2m+1$ such that $\partial_v^k p_\varepsilon(\varepsilon) = \partial_v^k f(\varepsilon)$ for all $0 \leqslant k \leqslant m$. In this way, we construct $f_\varepsilon \in C^m(\mathbb{R})$ for any $m \in \mathbb{N}$. For instance, for $m = 1$, we can compute

$$p_\varepsilon(v) = \left(\log(\varepsilon) - \frac{1}{2} \right) v + \frac{1}{2\varepsilon^2} v^3, \tag{2.2}$$

which yields $f_\varepsilon(v) \in C^1(\mathbb{R})$. For $m = 2$, we compute

$$p_\varepsilon(v) = \left(\log(\varepsilon) - \frac{3}{4} \right) v + \frac{1}{\varepsilon^2} v^3 - \frac{1}{4\varepsilon^4} v^5, \tag{2.3}$$

which yields $f_\varepsilon(v) \in C^2(\mathbb{R})$. As pointed out in remark 2.3 below, for the proof of theorem 1.2, it is actually sufficient to consider the approximation $f_\varepsilon(v)$ with $m = 2$.

Global behaviour of the function $f_\varepsilon(v)$ is still determined by the logarithmic nonlinearity $f(v)$. If $m \geqslant 1$, the function $f_\varepsilon(v)$ is globally Lipschitz and for any fixed $\varepsilon > 0$ there exists a positive constant C_ε such that

$$|f_\varepsilon(v) - f_\varepsilon(u)| \leqslant C_\varepsilon(|v| + |u|)|v - u|, \quad \text{for every } v, u \in \mathbb{R}. \tag{2.4}$$

If $m \geqslant 2$, we also have

$$|f'_\varepsilon(v)v_x - f'_\varepsilon(u)u_x| \leqslant C_\varepsilon \left((|v| + |u|)|v_x - u_x| + (|v_x| + |u_x|)|v - u| \right), \tag{2.5}$$

for every $v, u, v_x, u_x \in \mathbb{R}$, where C_ε is another constant, which may change from one line to another line. Of course, we realize from examples (2.2) and (2.3) that $C_\varepsilon \to \infty$ as $\varepsilon \to 0$.

For a given initial data v_0, we shall now consider a sequence of the approximating Cauchy problems associated with the generalized KdV equations

$$\begin{cases} v^\varepsilon_t + v^\varepsilon_{xxx} + f'_\varepsilon(v^\varepsilon)v^\varepsilon_x = 0, & t > 0, \\ v^\varepsilon|_{t=0} = v_0, \end{cases} \tag{2.6}$$

Using the linear estimates and the contraction principle from the work of Kenig, Ponce and Vega [13], we have the following local well-posedness result.

Theorem 2.1. *Fix $s > \frac{3}{4}$ and assume that $f_\varepsilon \in C^2(\mathbb{R})$ satisfy the global Lipschitz estimates (2.4) and (2.5). For any $v_0 \in H^s(\mathbb{R})$, there exists a time $T(\|v_0\|_{H^s}) > 0$ and a unique solution of the generalized KdV equation (2.6) satisfying*

$$\begin{aligned} &(1) \quad v^\varepsilon \in C([-T, T], H^s(\mathbb{R})), \\ &(2) \quad v^\varepsilon_x \in L^4([-T, T], L^\infty(\mathbb{R})), \\ &(3) \quad \|D^s_x v^\varepsilon_x\|_{L^\infty_x L^2_T} < \infty, \\ &(4) \quad \|v^\varepsilon\|_{L^2_x L^\infty_T} < \infty, \end{aligned}$$

where

$$\|f\|_{L^p_x L^q_T} := \left(\int_\mathbb{R} \left(\int_{-T}^T |f(t, x)|^q \, \mathrm{d}t \right)^{p/q} \mathrm{d}x \right)^{1/p}.$$

Moreover, the solution v^ε depends continuously on the initial data v_0 in $H^s(\mathbb{R})$ and

$$\|v(t)\|_{L^2} = \|v_0\|_{L^2}, \quad \text{for every } t \in [-T, T]. \tag{2.7}$$

If in addition $s \geqslant 1$, then the energy is conserved:

$$E_\varepsilon(v^\varepsilon(t)) = E_\varepsilon(v_0), \quad \text{for every } t \in [-T, T], \tag{2.8}$$

where

$$E_\varepsilon(v) := \frac{1}{2} \int_\mathbb{R} (v_x)^2 \, \mathrm{d}x - \int_\mathbb{R} W_\varepsilon(v) \, \mathrm{d}x, \quad W_\varepsilon(v) := \int_0^v f_\varepsilon(v) \, \mathrm{d}v. \tag{2.9}$$

Remark 2.2. Although theorem 2.1 was proved for the KdV equation with $f(v) = v^2$ in [13], Duhamel's principle was used to write the Cauchy problem (2.6) in the integral form

$$v^\varepsilon(0) = S(t)v_0 - \int_0^t S(t - t') \left(f'_\varepsilon(v^\varepsilon(t'))v^\varepsilon_x(t') \right) \mathrm{d}t', \tag{2.10}$$

where $S(t)$ is the solution operator associated with the group $\mathrm{e}^{-t\partial^3_x}$. Contraction principle is now applied in the same spirit as in [13] provided the regularized nonlinearity f_ε satisfies the global Lipschitz estimates (2.4) and (2.5).

Remark 2.3. The solution is extended globally for any integer $s \in \mathbb{N}$ thanks to the a priori energy estimates provided that the nonlinearity f_ε is sufficiently smooth. We only use this construction for $s = 1$, when $f_\varepsilon \in C^2(\mathbb{R})$ is sufficient. In this context, the approximation (2.1) with (2.3) can be used for the rest of this work.

2.2. Uniform energy estimates

We work with local solutions of theorem 2.1 for $s = 1$. Besides conservation of the L^2 norm in (2.7), the generalized KdV equation (2.6) admits conservation of the energy in (2.8), as it is stated in theorem 2.1.

For the purpose of construction of global solutions in $H^1(\mathbb{R})$, we need to bound $E_\varepsilon(v)$ from below by the squared $H^1(\mathbb{R})$ norm. To do so, we only need to bound positive values of $W_\varepsilon(v)$ from above.

Since $f'(v) = \log|v| + 1 \to -\infty$ as $v \to 0$, we realize that $f'_\varepsilon(v) = p'_\varepsilon(v) \ll -1$ for $|v| \leqslant \varepsilon$ if $\varepsilon > 0$ is sufficiently small. Indeed, it follows from the explicit example (2.3) that

$$f'_\varepsilon(v) = p'_\varepsilon(v) = \log(\varepsilon) + \mathcal{O}(1) \quad \text{for all } |v| \leqslant \varepsilon \quad \text{as } \varepsilon \to 0.$$

Therefore, if $\varepsilon > 0$ is sufficiently small, we have

$$W_\varepsilon(v) = \tfrac{1}{2}\left[\log(\varepsilon) + \mathcal{O}(1)\right] v^2 \leqslant 0, \quad |v| \leqslant \varepsilon. \tag{2.11}$$

Let us define

$$W(v) := \tfrac{1}{2} v^2 \log|v| - \tfrac{1}{4} v^2, \tag{2.12}$$

and denote the positive part of W by $[W]_+$. Note that $W(v) \geqslant 0$ for $|v| \geqslant \sqrt{e}$. Because $f_\varepsilon(v) = f(v)$ for $|v| \geqslant \varepsilon$, there exists a positive constant C_m, which only depends on m such that

$$W_\varepsilon(v) = W(v) + C_m \varepsilon^2, \quad |v| \geqslant \varepsilon. \tag{2.13}$$

For instance, integration of the explicit example (2.3) yields $C_2 = \tfrac{1}{12}$. It follows from (2.11), (2.12), and (2.13) that there exists a positive ε-independent constant C such that

$$[W_\varepsilon(v)]_+ \leqslant [W(v)]_+ + C_m v^2 \leqslant C|v|^3, \quad \text{for every } v \in \mathbb{R}. \tag{2.14}$$

By Sobolev's embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$, we obtain from (2.9) and (2.14)

$$E_\varepsilon(v) \geqslant \tfrac{1}{2}\|v_x\|^2_{L^2} - C\|v\|_{L^\infty}\|v\|^2_{L^2}$$
$$\geqslant \tfrac{1}{2}\|v_x\|^2_{L^2} - C\|v\|_{H^1} P(v),$$

where $P(v) := \tfrac{1}{2}\|v\|^2_{L^2}$. Because $E_\varepsilon(v^\varepsilon)$ and $P(v^\epsilon)$ are constants of motion for the solution v^ϵ of the generalized KdV equation (2.6) in theorem 2.1 with $s = 1$, we obtain the time-uniform bound on the H^1 norm of v^ε:

$$\|v^\varepsilon(t)\|_{H^1} \leqslant C P(v_0) + \sqrt{2E_\varepsilon(v_0) + 2P(v_0) + C^2 P^2(v_0)} < \infty, \quad t \in [-T, T]. \tag{2.15}$$

By using bound (2.15), we use a standard continuation argument for solutions of the integral equation (2.10) obtained via a contraction principle and continue the local solution $v^\varepsilon \in C([-T, T], H^1(\mathbb{R}))$ to a global solution $v^\varepsilon \in C(\mathbb{R}, H^1(\mathbb{R}))$. The global solution satisfies the time-uniform bound (2.15) extended for every $t \in \mathbb{R}$.

Additionally, because of (2.14), there exists an ε-independent constant C such that

$$\|[W_\varepsilon(v^\varepsilon(t))]_+\|_{L^1} \leqslant C\|v^\varepsilon(t)\|_{L^\infty}\|v^\varepsilon(t)\|^2_{L^2}$$
$$\leqslant C P(v_0)\left[C P(v_0) + \sqrt{2E_\varepsilon(v_0) + 2P(v_0) + C^2 P^2(v_0)}\right]. \tag{2.16}$$

For the negative part of W_ε, we obtain

$$\|[W_\varepsilon(v^\varepsilon(t))]_-\|_{L^1} = \|[W_\varepsilon(v^\varepsilon(t))]_+\|_{L^1} - \int_\mathbb{R} W_\varepsilon(v^\varepsilon(t))\,\mathrm{d}x$$

$$= \|[W_\varepsilon(v^\varepsilon(t))]_+\|_{L^1} + E_\varepsilon(v^\varepsilon(t)) - \frac{1}{2}\|v^\varepsilon_x\|^2_{L^2}$$

$$\leqslant C P(v_0)\left[C P(v_0) + \sqrt{2E_\varepsilon(v_0) + 2P(v_0) + C^2 P^2(v_0)}\right] + E_\varepsilon(v_0). \tag{2.17}$$

Bounds (2.16) and (2.17) allow us to control $\|W_\varepsilon(v^\varepsilon(t))\|_{L^1}$ for every $t \in \mathbb{R}$.

2.3. Passage to the limit

We shall now consider the limit $\varepsilon \to 0$ for the sequence of global approximating solutions $v^\varepsilon \in C(\mathbb{R}, H^1(\mathbb{R}))$ satisfying the generalized KdV equations (2.6). We recall definition (1.9) of the energy space X for the log-KdV equation (1.1). Assume that $v_0 \in X$ so that $E(v_0) < \infty$. Clearly $E_\varepsilon(v_0) < \infty$ for any $v_0 \in H^1(\mathbb{R})$.

Since $f_\varepsilon(0) = f(0) = 0$, we have the pointwise limit $f_\varepsilon(v) \to f(v)$ as $\varepsilon \to 0$ for every $v \in \mathbb{R}$ for the sequence of regularized nonlinearities in (2.1). Consequently, we have the pointwise limit of $W_\varepsilon(v)$ to the potential $W(v)$ given by (2.12):

$$W_\varepsilon(v) \to W(v) \quad \text{as} \quad \varepsilon \to 0 \quad \text{for every } v \in \mathbb{R}.$$

In view of (2.3), there exists a positive ε-independent constant C such that

$$\left| W_\varepsilon(v) - \tfrac{1}{2}(\log \varepsilon) v^2 \right| \leqslant C v^2, \quad |v| \leqslant \varepsilon.$$

We infer from the explicit expression of W given by (2.12) that

$$|W_\varepsilon(v)| \leqslant |W(v)| + C v^2, \quad |v| \leqslant \varepsilon.$$

On the other hand, a similar estimate holds in the region $|v| \geqslant \varepsilon$, thanks to the relation (2.13), so there exists a positive ε-independent constant C such that

$$|W_\varepsilon(v)| \leqslant |W(v)| + C v^2, \quad \text{for every } v \in \mathbb{R}.$$

By Lebesgue's dominated convergence theorem, we have

$$\int_{\mathbb{R}} W_\varepsilon(v) \, \mathrm{d}x \to \int_{\mathbb{R}} W(v) \, \mathrm{d}x \quad \text{as } \varepsilon \to 0, \quad \text{for every } v \in X.$$

Therefore, $E_\varepsilon(v_0) \to E(v_0)$ as $\varepsilon \to 0$.

By using the above estimates and the bounds (2.15), (2.16), and (2.17), we obtain the following ε- and t-uniform bound for the sequence of approximating solutions $v^\varepsilon \in C(\mathbb{R}, H^1(\mathbb{R}))$ starting with the initial data $v_0 \in X$:

$$\|v^\varepsilon(t)\|_{H^1} + \|W_\varepsilon(v^\varepsilon(t))\|_{L^1} \leqslant C(P(v_0), E(v_0)) < \infty, \quad \forall t \in \mathbb{R}, \quad \forall \varepsilon \in (0, 1], \tag{2.18}$$

where the positive constant C depends on the initial values $P(v_0)$ and $E(v_0)$ only. Therefore, the sequence of approximating solutions v^ε is bounded in space $L^\infty(\mathbb{R}, X)$. It also follows from the generalized KdV equation (2.6) that the sequence v_t^ε is bounded in space $L^\infty(\mathbb{R}, H^{-2}(\mathbb{R}))$. From Arzela–Ascoli Theorem, there exist $v \in L^\infty(\mathbb{R}, H^1(\mathbb{R}))$ and a subsequence of $v^\varepsilon \in L^\infty(\mathbb{R}, H^1(\mathbb{R}))$, still denoted by v^ε, such that

$$v^\varepsilon \to v \quad \text{strongly in } L^\infty_{\text{loc}}(\mathbb{R}, H^s_{\text{loc}}(\mathbb{R})) \quad \text{as } \varepsilon \to 0, \quad \text{for all } s < 1. \tag{2.19}$$

Because the limit (2.19) includes the range $(\tfrac{1}{2}, 1)$ for s, up to subtracting another subsequence, we may assume that

$$v^\varepsilon(x, t) \to v(x, t) \quad \text{as } \varepsilon \to 0, \quad \text{for almost every } x \in \mathbb{R} \text{ and for every } t \in \mathbb{R}. \tag{2.20}$$

In addition, because the upper bound (2.18) also controls $\|W_\varepsilon(v^\varepsilon(t))\|_{L^1}$, Fatou's lemma shows that the limiting function v belongs to $L^\infty(\mathbb{R}, X)$.

To show that $P(v(t)) \leqslant P(v_0)$ and $E(v(t)) \leqslant E(v_0)$ for every $t \in \mathbb{R}$, we use the weak lower semicontinuity of the H^1 norm and Fatou's lemma for the potential term $[W_\varepsilon]_-$ to obtain

$$\|v(t)\|_{H^1}^2 \leqslant \lim_{\varepsilon \to 0} \|v^\varepsilon(t)\|_{H^1}^2, \quad \|[W_\varepsilon(v(t))]_-\|_{L^1} \leqslant \lim_{\varepsilon \to 0} \|[W_\varepsilon(v^\varepsilon(t))]_-\|_{L^1}, \quad \text{for every } t \in \mathbb{R}.$$

On the other hand, using the relation (2.13), the limit (2.20), and the fact that $W(v) \geqslant 0$ for $|v| \geqslant \sqrt{e}$, we obtain

$$\|[W(v(t))]_+\|_{L^1} = \lim_{\varepsilon \to 0} \|[W_\varepsilon(v^\varepsilon(t))]_+\|_{L^1}, \quad \text{for every } t \in \mathbb{R}.$$

Then, it follows from the conservation

$$P(v^\varepsilon(t)) = P(v_0) \quad \text{and} \quad E_\varepsilon(v^\varepsilon(t)) = E_\varepsilon(v_0) \quad \text{forevery } t \in \mathbb{R}$$

and the convergence $E_\varepsilon(v_0) \to E(v_0)$ as $\varepsilon \to 0$ that

$$P(v(t)) \leqslant P(v_0) \quad \text{and} \quad E(v(t)) \leqslant E(v_0).$$

It remains to show that the limiting function $v \in L^\infty(\mathbb{R}, X)$ is a weak solution of the log-KdV equation (1.1). To do so, we first write a weak formulation of the generalized KdV equation (2.6). If $v^\varepsilon \in C(\mathbb{R}, H^1(\mathbb{R}))$ is a solution in theorem 2.1 for $T = \infty$, then for any test functions $\psi \in C_0^\infty(\mathbb{R}_x)$ and $\varphi \in C_0^\infty(\mathbb{R}_t)$, we have

$$\int_\mathbb{R} \left[\langle v^\varepsilon, \psi \rangle_{L^2} \varphi'(t) + \langle v^\varepsilon, \psi''' \rangle_{L^2} \varphi(t) \right] \, \mathrm{d}t + \int_\mathbb{R} \int_\mathbb{R} f_\varepsilon(v^\varepsilon) \psi'(x) \varphi(t) \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{2.21}$$

Since $f_\varepsilon(v) \to f(v)$ pointwise in v as $\varepsilon \to 0$, we apply the limit (2.19) to the integral formulation (2.21) and obtain the following integral equation for the limiting function v,

$$\int_\mathbb{R} \left[\langle v, \psi \rangle_{L^2} \varphi'(t) + \langle v, \psi''' \rangle_{L^2} \varphi(t) \right] \, \mathrm{d}t + \int_\mathbb{R} \int_\mathbb{R} f(v) \psi'(x) \varphi(t) \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{2.22}$$

where ψ and φ are any test functions. Therefore, $v \in L^\infty(\mathbb{R}, X)$ is a weak solution of the log-KdV equation (1.1), in particular, $v_t \in L^\infty(\mathbb{R}, H^{-2}(\mathbb{R}))$. The existence part of theorem 1.2 is now proven.

Remark 2.4. Since v_t is in $H^{-2}(\mathbb{R})$ (as opposed to the NLS case, where it is defined in $H^{-1}(\mathbb{R})$, see [6, 9]), we are not able to establish even conservation of the L^2 norm for the weak solutions of the log-KdV equation (1.1), since we cannot prove rigorously that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|v(t)\|_{L^2}^2 = 2\langle v_t, v \rangle_{L^2} = 0.$$

Nevertheless, we have proved that $\|v(t)\|_{L^2}^2$ is a non-increasing function of time t.

2.4. Uniqueness of solutions

We will show uniqueness of the solution $v \in L^\infty(\mathbb{R}, X)$ under the additional condition

$$(\log |v|)_x \in L^\infty((-t_0, t_0) \times \mathbb{R}).$$

Provided that uniqueness is proven, continuous dependence on the initial data $v_0 \in X$, the L^2 norm and energy conservation, and continuity of the solution $v \in C((-t_0, t_0), X)$ in time t follow from the arguments identical to the case of the log-NLS equation [5, section 9.3].

Assume existence of two local solutions v and u of the log-KdV equation (1.1) starting with the same initial data v_0. Set $w := v - u$ such that $w|_{t=0} = 0$. Then w satisfies a weak formulation similar to the integral equation (2.22) for the partial differential equation

$$w_t + w_{xxx} + (v \log |v| - u \log |u|)_x = 0.$$

Multiplying this equation by w, integrating over x, and formally neglecting the values of w and its derivatives as $|x| \to \infty$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} P(w) = -\int_\mathbb{R} (v_x \log |v| - u_x \log |u|) w \, \mathrm{d}x,$$

where $P(w) = \frac{1}{2} \|w\|_{L^2}^2$. We use the following bound for the log-nonlinearity [5]:

$$|\log |v| - \log |u|| \leqslant \frac{|v - u|}{\min(|v|, |u|)}. \tag{2.23}$$

We then write
$$\frac{\mathrm{d}}{\mathrm{d}t}P(w) = -\int_{|v|<|u|}(v_x\log|v| - u_x\log|u|)w\,\mathrm{d}x - \int_{|v|\geqslant|u|}(v_x\log|v| - u_x\log|u|)w\,\mathrm{d}x$$
$$= -\int_{|v|<|u|}v_x(\log|v| - \log|u|)w\,\mathrm{d}x - \int_{|v|<|u|}\log|u|ww_x\,\mathrm{d}x$$
$$- \int_{|v|\geqslant|u|}u_x(\log|v| - \log|u|)w\,\mathrm{d}x - \int_{|v|\geqslant|u|}\log|v|ww_x\,\mathrm{d}x.$$

Applying (2.23), we obtain
$$\left|\int_{|v|<|u|}v_x(\log|v| - \log|u|)w\,\mathrm{d}x + \int_{|v|\geqslant|u|}u_x(\log|v| - \log|u|)w\,\mathrm{d}x\right|$$
$$\leqslant \left(\left\|\frac{v_x}{v}\right\|_{L^\infty} + \left\|\frac{u_x}{u}\right\|_{L^\infty}\right)2P(w).$$

Integrating by parts, we also have
$$-\int_{|v|<|u|}\log|u|ww_x\,\mathrm{d}x - \int_{|v|\geqslant|u|}\log|v|ww_x\,\mathrm{d}x$$
$$= \frac{1}{2}\int_{|v|<|u|}(\log|u|)_x w^2\,\mathrm{d}x + \frac{1}{2}\int_{|v|\geqslant|u|}(\log|v|)_x w^2\,\mathrm{d}x,$$

hence
$$\left|\frac{\mathrm{d}}{\mathrm{d}t}P(w)\right| \leqslant 3\left(\left\|\frac{v_x}{v}\right\|_{L^\infty} + \left\|\frac{u_x}{u}\right\|_{L^\infty}\right)P(w).$$

Gronwall's inequality implies that if $w|_{t=0} = 0$, then $P(w) = 0$ for all $t \in (-t_0, t_0)$, for which the solutions v, u of the log-KdV equation (1.1) satisfy $(\log|v|)_x, (\log|u|)_x \in L^\infty((-t_0, t_0)\times\mathbb{R})$. The uniqueness part of theorem 1.2 is now proven.

3. Spectral stability of Gaussian solitary waves

To prove theorem 1.4, we study the spectrum of the linearized operator $A := \partial_x L$ given by
$$A = -\partial_x^3 + \frac{1}{4}(x^2 - 6)\partial_x + \frac{1}{2}x. \tag{3.1}$$
The domain of this operator with a range in $L^2(\mathbb{R})$ is given by
$$D(A) = \left\{u \in H^3(\mathbb{R}) : \quad x^2\partial_x u \in L^2(\mathbb{R}), \quad xu \in L^2(\mathbb{R})\right\}. \tag{3.2}$$
We note that the eigenfunctions of the linearized operator $\partial_x L$ associated with the KdV-type spectral problem $Af = \lambda f$ is defined in the function space $X_A := D(A) \cap \dot{H}^{-1}(\mathbb{R})$, hence the anti-derivative of the eigenfunction f is required to be squared integrable [11, 16].

We shall employ the Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by
$$\hat{u}(k) := \mathcal{F}(u)(k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}u(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, \quad k \in \mathbb{R}. \tag{3.3}$$
The Fourier transform $\mathcal{F}$ is helpful in the study of the spectrum of A because of dualism between derivatives and multiplications. After an application of the Fourier transform $\mathcal{F}$, the third-order differential operator A in x-space is mapped to the second-order differential operator $\hat{A}$ in k-space, where
$$\hat{A} = \frac{\mathrm{i}}{4}k\left(-\partial_k^2 + 4k^2 - 6\right). \tag{3.4}$$
The function space $X_A = D(A) \cap \dot{H}^{-1}(\mathbb{R})$ is now mapped to the function space $\hat{X}_A$ given by
$$\hat{X}_A = \left\{\hat{u} \in H^1(\mathbb{R}) : \quad k\partial_k^2\hat{u} \in L^2(\mathbb{R}), \quad k^3\hat{u} \in L^2(\mathbb{R}), \quad k^{-1}\hat{u} \in L^2(\mathbb{R})\right\}. \tag{3.5}$$
Denoting $\hat{B} := k(-\partial_k^2 + 4k^2 - 6)$, we have $\sigma(\hat{A}) = \frac{1}{4}\sigma(\hat{B})$.

3.1. Double zero eigenvalue of $\hat{A}$

Let $\hat{v}_G(k) := e^{-k^2}$ be the Fourier transform of the Gaussian solitary wave v_G (up to the constant multiplicative factor). We check by direct computation that

$$\hat{B}\partial_k \hat{v}_G = 0 \quad \text{and} \quad \hat{B}\hat{v}_G = -4k\hat{v}_G = 2\partial_k \hat{v}_G. \tag{3.6}$$

Since $\hat{B} = 4k\hat{L}$, where $\hat{L}$ is the Fourier image of the Schrödinger operator L with a harmonic potential (1.5), we conclude that $\text{null}(\hat{B}) = \text{span}\{\partial_k \hat{v}_G\}$, whereas the generalized null space of $\hat{B}$ includes the two-dimensional subspace $\hat{X}_0 = \text{span}\{\partial_k \hat{v}_G, \hat{v}_G\}$.

Moreover, since $\hat{B}^*\hat{v}_G = 0$ and $\|\hat{v}_G\|_{L^2} \neq 0$, no solution $\hat{u} \in \hat{X}_A$ of the inhomogeneous equation $\hat{B}\hat{u} = \hat{v}_G$ exists. Therefore, the Jordan chain for the zero eigenvalue of $\hat{B}$ is two-dimensional and the generalized null space of $\hat{B}$ is exactly $\hat{X}_0$. With the inverse Fourier transform, these arguments conclude consideration of the zero eigenvalue in theorem 1.4.

3.2. Nonzero eigenvalues of $\hat{A}$

Let us now consider nonzero values for the spectral parameter λ in the spectral problem $\hat{A}\hat{f} = \lambda\hat{f}$. Recall that $\sigma(\hat{A}) = \frac{i}{4}\sigma(\hat{B})$. Therefore, let $\lambda = \frac{i}{4}E$ and consider the spectral problem $\hat{B}\hat{u} = E\hat{u}$ rewritten as the following differential equation

$$\frac{\mathrm{d}^2\hat{u}}{\mathrm{d}k^2} + \left(\frac{E}{k} + 6 - 4k^2\right)\hat{u}(k) = 0, \quad k \in \mathbb{R}. \tag{3.7}$$

We employ the theory of differential equations to study solutions of the spectral problem (3.7).

If $E \neq 0$, the point $k = 0$ is a regular singular point of the differential equation (3.7) with indices 0 and 1. By Frobenius' method [17, chapter 4], there exist two linearly independent solutions of this differential equation. The first solution is given by the power series expansion

$$\hat{u}_1(k) = k - \frac{1}{2}Ek^2 + \left(\frac{1}{12}E^2 - 1\right)k^3 + \mathcal{O}(k^4) \quad \text{as} \quad k \to 0 \tag{3.8}$$

and the other solution is given by the logarithm-modified power series expansion

$$\hat{u}_2(k) = 1 - 3\left(1 + \frac{1}{4}E^2\right)k^2 + \mathcal{O}(k^3) - Ek\log(|k|)\left(1 - \frac{1}{2}Ek + \mathcal{O}(k^2)\right) \quad \text{as} \quad k \to 0. \tag{3.9}$$

Because the other singular point of the differential equation (3.7) is infinity, the expansions (3.8) and (3.9) converge for every $k \in \mathbb{R}$.

Recall again the function space $\hat{X}_A$ defined by (3.5). Because $k^{-1}\hat{u} \in L^2(\mathbb{R})$ is required in $\hat{X}_k$, we have $\hat{u} \in \hat{X}_A$ if and only if $\hat{u}$ is constant proportional to the solution $\hat{u}_1$. Hence, both for $k > 0$ and $k < 0$, the solution of the differential equation (3.7) is uniquely defined by the solution $\hat{u}_1$ (up to the constant multiplicative factor) for every $E \in \mathbb{C}\backslash\{0\}$. The admissible values of E are determined from the behaviour of the solution $\hat{u}(k)$ as $|k| \to \infty$.

By the WKB method without turning points [14, chapter 7.2], there exist two linearly independent solutions of the differential equation (3.7): one solution diverges and the other one decays to zero as $|k| \to \infty$. The decaying solution is defined by the asymptotic behaviour

$$\hat{u}(k) \sim ke^{-k^2}\left(1 - \frac{E}{4k} + \mathcal{O}\left(\frac{1}{k^2}\right)\right) \quad \text{as} \quad |k| \to \infty. \tag{3.10}$$

It becomes now clear that two decay conditions at $k \to +\infty$ and $k \to -\infty$ over-determine the spectral problem (3.7) because the behaviour of $\hat{u}$ both for $k > 0$ and $k < 0$ is uniquely determined by only one spectral parameter E (up to the constant multiplicative factors) and no parity symmetry exists for $k > 0$ and $k < 0$.

The way around this obstacle is to consider a weak formulation for the solutions of the differential equation (3.7) in the function space $\hat{X}_A$. Since the differential equation with $E \neq 0$ has singularities at $\{-\infty, 0, \infty\}$, we split $\mathbb{R}$ into two sets $I_- := (-\infty, 0)$ and $I_+ := (0, \infty)$ and look for piecewisely defined eigenfunctions $\hat{u}_\pm$ supported on $I_\pm$ only.

Let $\hat{B}_\pm$ denote the operator $\hat{B}$ restricted on $I_\pm$ subject to the Dirichlet boundary condition at $k = 0$. By theorem 4 in [8, p 1438], the continuous spectrum of B is the union of the continuous spectra of operators B_+ and B_-.

We shall first characterize the spectrum of the operator $\hat{B}_+$ on I_+ and prove that this spectrum is purely discrete. By using a substitution $\hat{u}_+(k) = k^{1/2}\hat{v}_+(k)$ for $k > 0$, we reduce the spectral problem $\hat{B}_+\hat{u}_+ = E\hat{u}_+$ to the symmetric form

$$k^{1/2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}k^2} + 4k^2 - 6\right)k^{1/2}\hat{v}_+(k) = E\hat{v}_+(k), \quad k \in I_+. \tag{3.11}$$

Since $k = 0$ is singular, we require $|\hat{v}_+(0)| < \infty$, although the solution $\hat{u}_1$ in (3.8) implies more precisely that $\hat{v}_+(k) = \mathcal{O}(k^{1/2})$ as $k \to 0$. The spectral problem (3.11) in the function space with $|\hat{v}_+(0)| < \infty$ is self-adjoint, hence the admissible values of E are real.

We shall now claim that $\sigma(\hat{B}_+) = \{E_n\}_{n \in \mathbb{N}_+}$, where $0 = E_0 < E_1 < E_2 < \ldots$ and $E_n \to +\infty$ as $n \to \infty$. First, because the potential $4k^2 - 6$ is confining, the resolvent operator is compact, and the spectrum of $\hat{B}_+$ is purely discrete. Next, $\hat{v}_+ = k^{1/2}\mathrm{e}^{-k^2}$ is the exact solution of the spectral problem (3.11) for $E = 0$ and this eigenfunction is strictly positive for all $k \in I_+$. By Sturm's Theorem, $E_0 = 0$ is at the bottom of the spectrum of $\hat{B}_+$, which is bounded from below, and hence $E_n > 0$ for every $n \in \mathbb{N}$. Finally, the eigenfunctions are uniquely determined by the expansion (3.8) as $k \to 0$, hence each eigenvalue is simple.

As a consequence of the reduction to the self-adjoint problem (3.11), we have orthogonality and normalization of the real-valued eigenfunctions $\hat{u}_{+n}(k) = k^{1/2}\hat{v}_{+n}(k)$ corresponding to the real eigenvalues E_n for all $n \in \mathbb{N}_+$:

$$\delta_{n,m} = \int_0^\infty \hat{v}_{+n}(k)\hat{v}_{+m}(k)\,\mathrm{d}k = \int_0^\infty \frac{\hat{u}_{+n}(k)\hat{u}_{+m}(k)}{k}\,\mathrm{d}k, \tag{3.12}$$

where $\delta_{n,m}$ is the Kronecker's symbol.

The same analysis applies to the spectrum of the operator $\hat{B}_-$ on I_- with the only difference that the substitution $\hat{u}_-(k) = (-k)^{1/2}\hat{v}_-(k)$ for $k < 0$ reduces the spectral problem $\hat{B}_-\hat{u}_- = E\hat{u}_-$ to the symmetric form

$$(-k)^{1/2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}k^2} + 4k^2 - 6\right)(-k)^{1/2}\hat{v}_-(k) = -E\hat{v}_-(k), \quad k \in I_-, \tag{3.13}$$

subject to the boundary behaviour $|\hat{v}_-(0)| < \infty$. Therefore, $\sigma(\hat{B}_-) = \{-E_n\}_{n \in \mathbb{N}_+}$, where E_n are the same eigenvalues as in the spectral problem (3.11). Similarly to (3.12), it follows from the self-adjoint problem (3.13) that the real-valued eigenfunctions $\hat{u}_{-n}(k) = (-k)^{1/2}\hat{v}_{-n}(k)$ corresponding to the real eigenvalues $-E_n$ for all $n \in \mathbb{N}_+$ are orthogonal and can be normalized by

$$\delta_{n,m} = \int_{-\infty}^0 \hat{v}_{-n}(k)\hat{v}_{-m}(k)\,\mathrm{d}k = -\int_{-\infty}^0 \frac{\hat{u}_{-n}(k)\hat{u}_{-m}(k)}{k}\,\mathrm{d}k. \tag{3.14}$$

The statement about nonzero eigenvalues of the operator A in theorem 1.4 is now proven with the correspondence $\omega_n = \frac{1}{4}E_n$ for all $n \in \mathbb{N}$.

Remark 3.1. The double zero eigenvalue associated with the two-dimensional subspace $\hat{X}_0 = \mathrm{span}\{\partial_k\hat{v}_G, \hat{v}_G\}$ for the operator $\hat{A}$ is mapped into a semi-simple zero eigenvalue associated with the simple zero eigenvalues of the operators $\hat{B}_+$ and $\hat{B}_-$. There is no ambiguity here because if $E = 0$, then $k = 0$ is an ordinary point of the differential equation (3.7) and the splitting of $\mathbb{R}$ into I_- and I_+ does not make sense.

3.3. Eigenfunctions of the spectral problem

We now confirm that the eigenfunction $\hat{u}$ defined piecewise as

$$\text{either} \quad \hat{u}(k) = \begin{cases} \hat{u}_+(k), & k > 0, \\ 0, & k < 0, \end{cases} \quad \text{or} \quad \hat{u}(k) = \begin{cases} 0, & k > 0, \\ \hat{u}_-(k), & k < 0, \end{cases}$$

belongs to the space $\hat{X}_A$. We note that $\hat{u}$ is continuous on $\mathbb{R}$ and piecewise C^1 with the jump discontinuity of the first derivative at $k = 0$. Moreover, $\hat{u}(k) = \mathcal{O}(k)$ as $k \to 0$. Therefore, $\hat{u} \in H^1(\mathbb{R})$ and $k^{-1}\hat{u} \in L^2(\mathbb{R})$. Furthermore, $\hat{u}(k)$ decays to zero as $|k| \to \infty$ according to the asymptotic behaviour (3.10). Therefore, $k^3\hat{u} \in L^2(\mathbb{R})$. Finally, $\partial_k^2 \hat{u}$ is proportional to the Dirac delta distribution δ at $k = 0$ and is smooth for $k \neq 0$, hence $\partial_k^2 \hat{u} \notin L^2(\mathbb{R})$ but $k\partial_k^2 \hat{u} \in L^2(\mathbb{R})$. Thus, $\hat{u} \in \hat{X}_A$.

The Fourier transform $\mathcal{F}$ is dual between the smoothness and decay of the eigenfunctions. Since the eigenfunctions $\hat{u}(k)$ decays fast as $|k| \to \infty$, according to the Gaussian decay (3.10), their inverse Fourier transform $u(x)$ is smooth in x. Moreover, $u_\pm$ defined by

$$u_+(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{u}_+(k)e^{ikx}\, dk, \quad u_-(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \hat{u}_-(k)e^{ikx}\, dk \quad (3.15)$$

are analytically extended off the real line. On the other hand, $\hat{u}(k)$ have jump discontinuity in the first derivative across $k = 0$, therefore, $u(x)$ decay only algebraically as $|x| \to \infty$. The only exception is the double zero eigenvalue, since one can glue $\hat{u}_+(k)$ and $\hat{u}_-(k)$ for the eigenvalue $E_0 = 0$ to a smooth eigenfunction $\hat{u}(k) = ke^{-k^2}$, which corresponds to the eigenfunction $u(x) = xe^{-x^2/4}$ with the fast decay as $|x| \to \infty$.

The statement about eigenfunctions of the operator A in theorem 1.4 is now proven.

3.4. Numerical illustrations

Figure 1 shows the first three eigenfunctions $\hat{u}_+(k) = k^{1/2}\hat{v}_+(k)$ of the spectral problem (3.11) for the first three eigenvalues $E_0 = 0$, $E_1 \approx 5.4109$, and $E_2 \approx 12.3080$. These eigenfunctions and eigenvalues were computed numerically by means of the central difference approximation of the derivatives in the spectral problem (3.11) and the MATLAB eigenvalue solver. The fast decay of the eigenfunctions $\hat{u}_+(k)$ as $k \to \infty$ and the Sturm's nodal properties of the eigenfunctions for $k > 0$ are obvious from the figure.

Figure 2 shows the real (left) and imaginary (right) parts of the eigenfunctions u_+ versus x after the inverse Fourier transform (3.15). The imaginary part of the first eigenfunction decays fast as $|x| \to \infty$, according to the exact eigenfunction $v_G'(x) \sim xe^{-x^2/4}$, which is also shown by a dotted line (invisible from the numerical dots).

To inspect the slow (algebraic) decay of the eigenfunctions, we multiply the eigenfunctions by the factor $(1+x^2)^{p/2}$ with $p = 2$ and $p = 3$. It follows from the numerical data that the real parts of the eigenfunctions decay like $\mathcal{O}(|x|^{-2})$ as $|x| \to \infty$, whereas the imaginary parts of the eigenfunctions (except for the first eigenfunction) decay like $\mathcal{O}(|x|^{-3})$ as $|x| \to \infty$. This corresponds to the finite jump discontinuity of the first derivative of $\hat{u}(k)$ at $k = 0$.

4. Cauchy problem for the linearized log-KdV equation

We consider the Cauchy problem for the linearized log-KdV equation (1.7) at the Gaussian solitary wave v_G. For convenience, we rewrite the Cauchy problem again:

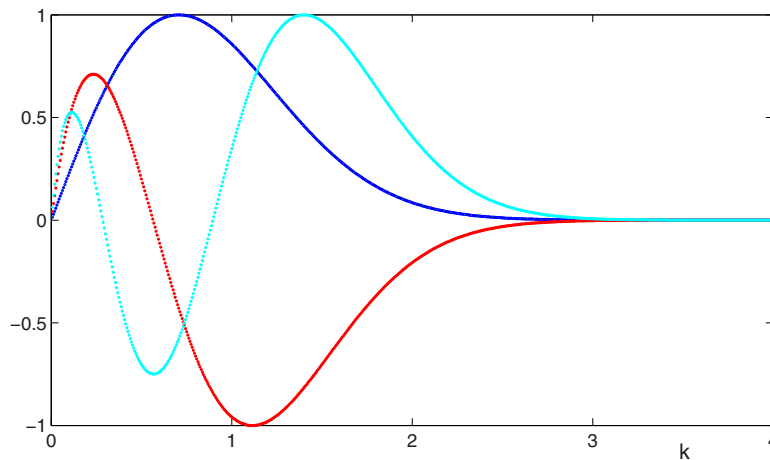$$\begin{cases} u_t = \partial_x L u, & t > 0, \\ u|_{t=0} = u_0, \end{cases} \quad (4.1)$$

Figure 1. Eigenfunctions $\hat{u}$ of the spectral problem (3.7) versus k for the first three eigenvalues $0 = E_0 < E_1 < E_2$. Eigenfunctions for E_1 and E_2 have one and two zeros for $k > 0$, respectively.
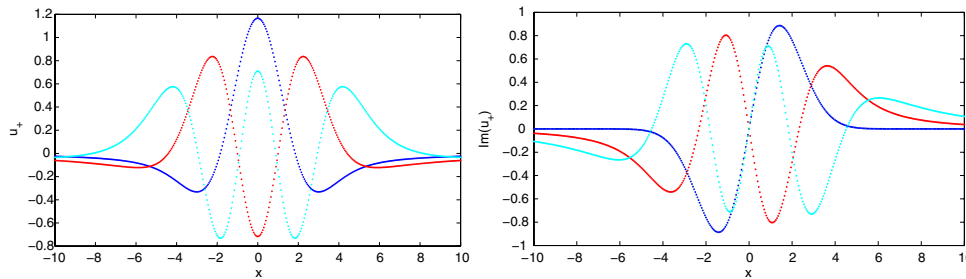

Figure 2. Real (left) and imaginary (right) parts of the three eigenfunctions u versus x for the first three eigenvalues E.

We shall first make use of theorem 1.4 to give a quick proof of theorem 1.1. This proof is alternative to the arguments given in [10], which rely on the symplectic projections and the energy method. Next, we shall approximate the Cauchy problem numerically, starting with the Gaussian initial data. We conclude the paper with a short discussion.

4.1. Proof of theorem 1.1

Let $\{u_{\pm n}(x)\}_{n\in\mathbb{N}}$ be a sequence of eigenfunctions of $\partial_x L$ for the sequence of nonzero imaginary eigenvalues $\{\pm i\omega_n\}_{n\in\mathbb{N}}$ constructed in theorem 1.4. It follows from the orthogonality and normalization conditions (3.12) and (3.14) that the eigenfunctions are orthogonal to each other and normalized with respect to the symplectic inner product:

$$\langle \partial_x^{-1} u_{\pm n}, u_{\pm m} \rangle_{L^2} := \int_{-\infty}^{\infty} \frac{\bar{\hat{u}}_{\pm n}(k)\hat{u}_{\pm m}(k)}{ik} \, dk = \mp i\delta_{n,m}, \tag{4.2}$$

where $\partial_x^{-1} u(x) := \int_{-\infty}^{x} u(x')\,dx'$ and the Fourier transform (3.3) is used. Note that $u_{\pm n} \in L^1(\mathbb{R})$ if $u_{\pm n} \in L_1^2(\mathbb{R}) \subset D(A)$. Also note that $\hat{u}_{\pm n}(0) = 0$ if $u_{\pm n} \in \dot{H}^{-1}(\mathbb{R}) \subset X_A$.

Let us consider the decomposition of the solution u of the linearized log-KdV equation (4.1) as the series of eigenfunctions of the operator A:

$$u(x,t) = b\left[v_G(x) - tv'_G(x)\right] + a_0 v'_G(x) + \sum_{n\in\mathbb{N}} a_{+n} u_{+n}(x) e^{i\omega_n t} + \sum_{n\in\mathbb{N}} a_{-n} u_{-n}(x) e^{-i\omega_n t}, \quad (4.3)$$

where the coefficients b and $\{a_n\}_{n\in\mathbb{Z}}$ are found uniquely from the conditions of symplectic orthogonality and normalization (4.2) applied to the initial data $u|_{t=0} = u_0$:

$$b = \frac{\langle v_G, u_0\rangle_{L^2}}{\langle v_G, v_G\rangle_{L^2}}, \quad a_0 = \frac{1}{\langle v_G, v_G\rangle_{L^2}}\left[\frac{b}{2}\left(\int_{-\infty}^{\infty} v_G(x)dx\right)^2 - \langle\partial_x^{-1} v_G, u_0\rangle_{L^2}\right], \quad (4.4)$$

and

$$a_{\pm n} = \pm i\langle\partial_x^{-1} u_{\pm n}, u_0\rangle_{L^2}, \quad n\in\mathbb{N}. \quad (4.5)$$

If $u_0 \in X_c$ defined in (1.6), then $b = 0$. The coefficients $\{a_n\}_{n\in\mathbb{Z}}$ are well defined for $u_0 \in X_c$. Moreover, by the spectral theorem and the reduction of the non-self-adjoint operator A to the two self-adjoint operators in section 3.2, the series (4.3) converges in the L^2 sense for every $t \in \mathbb{R}$.

Define now the conserved energy of the linearized log-KdV equation,

$$E_c(u) := \tfrac{1}{2}\langle Lu, u\rangle_{L^2} = \text{const}, \quad t \in \mathbb{R}. \quad (4.6)$$

Recall that $E_c(u_0) \geqslant 0$ if $u_0 \in X_c$. It follows from (4.3) and (4.6) that if $u_0 \in X_c$, then $u(\cdot, t) \in X_c$ and $E_c(u(\cdot, t)) \geqslant 0$ for every $t \in \mathbb{R}$. Let $y(x, t) := u(x, t) - a_0 v'_G(x)$ and recall that $b = 0$ for $u_0 \in X_c$. Under these conditions, we have

$$E_c(y(\cdot, t)) = E_c(u(\cdot, t)) = E_c(u_0), \quad t \in \mathbb{R}$$

and

$$E_c(y(\cdot, t)) = \tfrac{1}{2}\sum_{n\in\mathbb{N}} \omega_n\left(|a_{+,n}|^2 + |a_{-n}|^2\right),$$

where the orthogonality and normalization conditions (4.2) have been used.

Because y belongs to a subspace of X_c spanned by $\{u_{\pm n}\}_{n\in\mathbb{N}}$ associated with the nonzero eigenvalues $\{\pm i\omega_n\}_{n\in\mathbb{N}}$, where each $\omega_n > 0$, the energy functional $E_c(y)$ is equivalent to the squared $H^1(\mathbb{R})\cap L_1^2(\mathbb{R})$ norm. Therefore, there is a t-independent positive constant C such that

$$\|y(\cdot, t)\|_{H^1}^2 + \|y(\cdot, t)\|_{L_1^2}^2 \leqslant C E_c(y(\cdot, t)) = C E_c(u_0) \leqslant C\left(\frac{1}{2}\|u_0\|_{H^1}^2 + \frac{1}{8}\|u_0\|_{L_1^2}^2\right). \quad (4.7)$$

By the decomposition (4.3), the bound (4.7) and the triangle inequality, we obtain the bound (1.8) for every $t \in \mathbb{R}$. Theorem 1.1 is now proven.

4.2. Numerical illustrations

We truncate the spatial domain on $[-L, L]$ for sufficiently large L ($L = 40$ was used in our numerical results) and discretize $u(x, t)$ at equally spaced grid points $\{x_k\}_{k=0}^{N+1}$ subject to the periodic boundary conditions $u(x_0, t) = u(x_{N+1}, t)$. Using the central difference approximation and the Heun's method, we rewrite the linearized log-KdV equation (4.1) in the iterative form

$$\left(I - \frac{\Delta t}{2} D L_D\right) u_{m+1} = \left(I + \frac{\Delta t}{2} D L_D\right) u_m \quad (4.8)$$

where Δt is the time step, u_m is the vector of discretized solution at the time t_m, I is an identity matrix, D is the matrix for the first derivative, and L_D is the matrix for the second-order differential operator L.
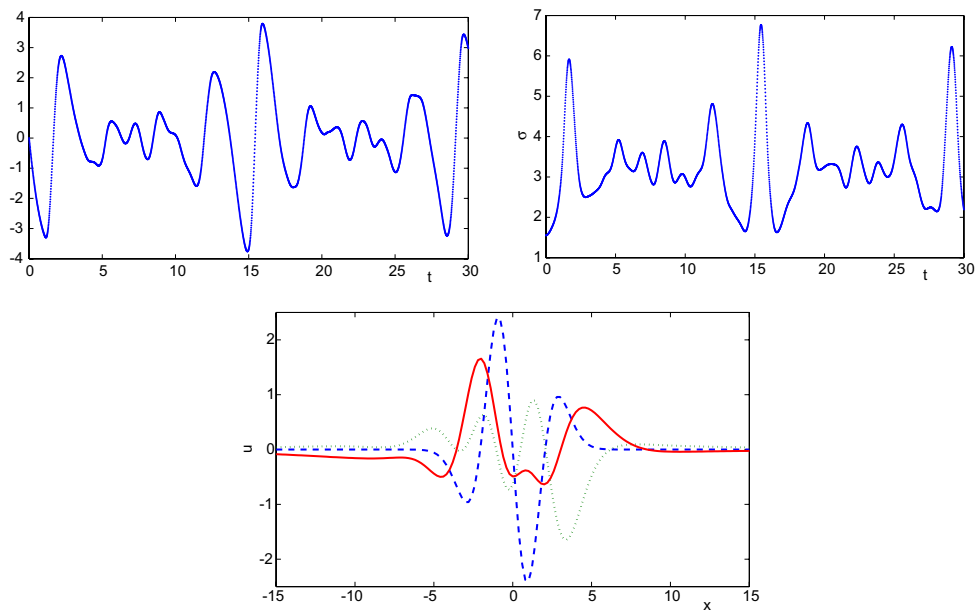
Figure 3. Numerical solution of the linearized log-KdV equation (4.1) with an odd initial data (4.9): centre of mass $\bar{x}$ (top left) and standard deviation σ (top right) versus time; the profile $u(x, t)$ versus x (bottom) for $t = 0$ (blue dashed), $t = 2.5$ (green dotted), and $t = 5$ (red solid).

For a positive parameter α, we consider the odd initial data

$$u_0(x) = x \left(x^2 - \frac{5 + 6\alpha}{(1 + \alpha)(1 + 2\alpha)} \right) e^{-\frac{(1+2\alpha)x^2}{4}}, \qquad (4.9)$$

which satisfies the symplectic orthogonality conditions $\langle v_G, u_0 \rangle_{L^2} = \langle \partial_x^{-1} v_G, u_0 \rangle_{L^2} = 0$. These constraints ensure that $b = 0$ and $a_0 = 0$ in the decomposition (4.3) and the solution u of the linearized log-KdV equation (4.1) is spanned by eigenfunctions of the linearized operator $A = \partial_x L$ for nonzero eigenvalues.

Figure 3 reports numerical computations of the iterative scheme (4.8) for the Gaussian initial data (4.9) with $\alpha = 0.1$. Besides the profiles of the solution shown on the spatial interval $[-15, 15]$ (bottom panel), we also show the centre of mass (top left panel)

$$\bar{x}(t) = \frac{\int_{\mathbb{R}} x u^2(x, t) \mathrm{d}x}{\int_{\mathbb{R}} u^2(x, t) \, \mathrm{d}x},$$

and the standard deviation (top right panel)

$$\sigma(t) = \left(\frac{\int_{\mathbb{R}} (x - \bar{x}(t))^2 u^2(x, t) \, \mathrm{d}x}{\int_{\mathbb{R}} u^2(x, t) \, \mathrm{d}x} \right)^{1/2}.$$

We can see from the top right panel of figure 3 that the standard deviation $\sigma(t)$ oscillates periodically with a large amplitude, which is still much smaller than the half-size of the computational domain $L = 40$. Therefore, the odd Gaussian data (4.9) evolves into a solution, which does not spread out in the time evolution of the linearized log-KdV equation (4.1). Nevertheless, a visible radiation appears on the left slope of the Gaussian pulse.
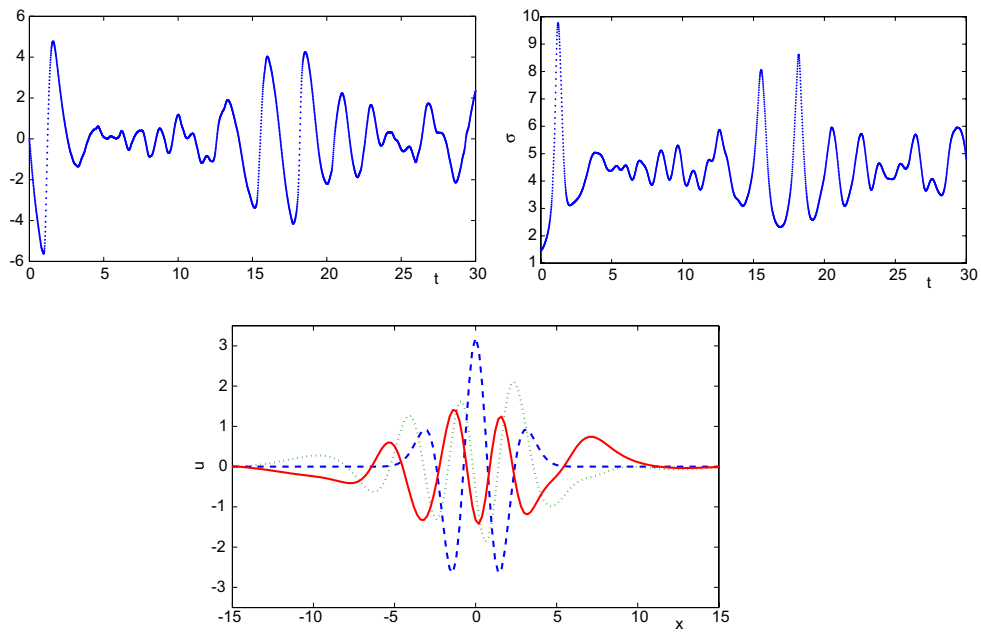
Figure 4. Numerical solution of the linearized log-KdV equation (4.1) with the even initial data (4.10): centre of mass $\bar{x}$ (top left) and standard deviation σ (top right) versus time; the profile $u(x, t)$ versus x (bottom) for $t = 0$ (blue dashed), $t = 2.5$ (green dotted), and $t = 5$ (red solid).

Figure 4 reports numerical computations for the even initial data

$$u_0(x) = \left(x^4 - \frac{3(3 + 4\alpha)}{(1 + \alpha)(1 + 2\alpha)}x^2 + \frac{6}{(1 + \alpha)(1 + 2\alpha)}\right) e^{-\frac{(1+2\alpha)x^2}{4}}, \qquad (4.10)$$

with $\alpha = 0.25$. The even function (4.10) also satisfies the symplectic orthogonality conditions $\langle v_G, u_0 \rangle_{L^2} = \langle \partial_x^{-1} v_G, u_0 \rangle_{L^2} = 0$ for any $\alpha \geqslant 0$. The results are qualitatively similar to the case of odd initial data. The preserved localization of the solution coexists with the radiation at the left slope of the Gaussian pulse.

4.3. Discussion

The numerical results of Section 4.2 support the conjecture that if the initial data $u_0 \in X_c$ of the linearized log-KdV equation (1.7) satisfies the constraint

$$u_0(x) = v_G(x)h_0(x) \quad \text{with} \quad h_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}),$$

then there exists a unique solution of the linearized log-KdV equation (1.7) in the form

$$u(x, t) = v_G(x)h(x, t),$$

where $h(\cdot, t) \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ for every $t \in \mathbb{R}$ and $h(\cdot, 0) = h_0$.

If this result can be established rigorously, then one can analyse the linearized log-KdV equation with a source term to complete analysis of the equivalent log-KdV equation (1.14) with the initial data $w_0 = v_G h_0$, where $h_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\|h_0\|_{L^\infty \cap L^2}$ is sufficiently small. This route may lead to the proof of orbital stability of the Gaussian solitary wave v_G for the log-KdV equation (1.1) in the class of functions with initial data $v_0 = (1 + h_0)v_G$,

where $h_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\|h_0\|_{L^\infty \cap L^2}$ is sufficiently small. However, this work is still to be done, hence the problem of nonlinear orbital stability of the Gaussian solitary wave v_G remains opened for further studies.

Acknowledgments

References

[1] Angulo Pava J 2009 *Nonlinear Dispersive Equations: Existence and Stability of Solitary and Periodic Travelling Wave Solutions Mathematical Surveys and Monographs* vol 156 (Providence, RI: American Mathematical Society)

[2] Carles R and Gallo C 2011 Finite time extinction by nonlinear damping for the Schrödinger equation *Commun. Part. Diff. Eqns* **36** 961–75

[3] Carles R and Ozawa T 2014 Finite time extinction for nonlinear Schrödinger equation in 1D and 2D *Commun. Part. Diff. Eqns* submitted (arXiv:1405.0995)

[4] Cazenave T 1983 Stable solutions of the logarithmic Schrödinger equation *Nonlinear Anal.* **7** 1127–40

[5] Cazenave T 2003 Semilinear Schrödinger equations *Courant Lecture Notes in Mathematics* vol 10 (New York: New York University Courant Institute of Mathematical Sciences)

[6] Cazenave T and Haraux A 1980 Équations d'évolution avec non linéarité logarithmique *Ann. Fac. Sci. Toulouse Math. (5)* **2** 21–51

[7] Chatterjee A 1999 Asymptotic solution for solitary waves in a chain of elastic spheres *Phys. Rev.* E **59** 5912–9

[8] Dunford N and Schwartz J T 1963 *Linear operators. Part 2: Spectral theory. Self adjoint operators in Hilbert space* ed W G Bade, R G Bartle (New York: Wiley)

[9] Ginibre J and Velo G 1985 The global Cauchy problem for the nonlinear Schrödinger equation revisited *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **2** 309–27

[10] James G and Pelinovsky D 2014 Gaussian solitary waves and compactons in Fermi–Pasta–Ulam lattices with Hertzian potentials *Proc. Roy. Soc.* A **470** 20130465

[11] Kapitula T and Stefanov A 2014 A Hamiltonian–Krein (instability) index theory for KdV-like eigenvalue problems *Stud. Appl. Math.* **132** 183–211

[12] Kato T 1979 On the Korteweg–de Vries equation *Manuscr. Math.* **28** 89–99

[13] Kenig C E, Ponce G and Vega L 1993 Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle *Commun. Pure Appl. Math.* **46** 527–620

[14] Miller P D 2006 Applied asymptotic analysis *Graduate Studies in Mathematics* vol 75 (Providence, RI: American Mathematical Society)

[15] Nesterenko V F 2001 *Dynamics of Heterogeneous Materials* (New York: Springer)

[16] Pelinovsky D E 2014 Spectral stability of nonlinear waves in KdV-type evolution equations *Spectral Analysis, Stability, and Bifurcation in Modern Nonlinear Physical Systems* ed O Kirillov and D E Pelinovsky (London: Wiley–ISTE) pp 377–400

[17] Teschl G 2012 Ordinary differential equations and dynamical systems *Graduate Studies in Mathematics* vol 140 (Providence, RI: American Mathematical Society)

[18] Zhidkov P E 2001 Korteweg–de Vries, nonlinear Schrödinger equations: qualitative theory *Lecture Notes in Mathematics* vol 1756 (Berlin: Springer)