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Orbital stability in the cubic defocusing NLS equation: I. Cnoidal periodic waves

Thierry Gallay [a], Dmitry Pelinovsky [b,*]

[a] Institut Fourier, Université de Grenoble 1, 38402 Saint-Martin-d'Hères, France
[b] Department of Mathematics, McMaster University, Hamilton, Ontario, L8S 4K1, Canada

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Abstract

Periodic waves of the one-dimensional cubic defocusing NLS equation are considered. Using tools from integrability theory, these waves have been shown in [4] to be linearly stable and the Floquet–Bloch spectrum of the linearized operator has been explicitly computed. We combine here the first four conserved quantities of the NLS equation to give a direct proof that cnoidal periodic waves are orbitally stable with respect to subharmonic perturbations, with period equal to an integer multiple of the period of the wave. Our result is not restricted to the periodic waves of small amplitudes.

1. Introduction

We consider the cubic defocusing NLS (nonlinear Schrödinger) equation in one space dimension:

$$i\psi_t + \psi_{xx} - |\psi|^2\psi = 0, \tag{1.1}$$

where $\psi = \psi(x,t) \in \mathbb{C}$ and $(x,t) \in \mathbb{R} \times \mathbb{R}$. This equation arises in the study of modulational stability of small amplitude nearly harmonic waves in nonlinear dispersive systems [14]. In this context, monochromatic waves of the original system correspond to spatially homogeneous solutions of the cubic NLS equation (1.1) of the form $\psi(x,t) = ae^{-ia^2t}$, where the positive

* Corresponding author.
E-mail address: dmpeli@math.mcmaster.ca (D. Pelinovsky).

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parameter a can be taken equal to one without loss of generality, due to scaling invariance. According to the famous Lighthill criterion, these plane waves are spectrally stable with respect to sideband perturbations [16], because the nonlinearity in (1.1) is defocusing. Moreover, using energy methods, it can be shown that plane waves are also orbitally stable under perturbations in $H^1(\mathbb{R})$ [17, Section 3.3], where the orbit is defined with respect to arbitrary rotations of the complex phase of ψ.

More generally, it is important for the applications to consider spatially inhomogeneous waves of the form $\psi(x,t) = u_0(x)e^{-it}$, where the profile $u_0 : \mathbb{R} \to \mathbb{C}$ satisfies the second-order differential equation

$$\frac{d^2u_0}{dx^2} + \left(1 - |u_0|^2\right)u_0 = 0, \quad x \in \mathbb{R}. \tag{1.2}$$

Such solutions of the cubic NLS equation (1.1) correspond to slowly modulated wave trains of the original physical system. A complete list of all bounded solutions of the second-order equation (1.2) is known, see [4,7]. Most of them are quasi-periodic in the sense that $u_0(x) = r(x)e^{i\varphi(x)}$ for some real-valued functions r, φ such that r and φ' are periodic with the same period $T_0 > 0$. The corresponding solutions of the cubic NLS equation (1.1) are usually called "periodic waves", although strictly speaking they are not periodic functions of x in general. In addition, the second-order equation (1.2) has nonperiodic solutions such that r and φ' converge to a limit as $x \to \pm\infty$; these correspond to "dark solitons" of the cubic NLS equation. In the present paper, we focus on real-valued solutions of the second-order equation (1.2), which form a one-parameter family of periodic waves (often referred to as "cnoidal waves").

Several recent works addressed the stability of periodic waves for the cubic NLS equation (1.1). Using the energy method, it was shown in [6,7] that periodic waves are orbitally stable within a class of solutions which have the same periodicity properties as the wave itself. More precisely, if $u_0(x) = e^{ipx}q_0(x)$ where $p \in \mathbb{R}$ and q_0 is T_0-periodic, the wave $u_0(x)e^{-it}$ is orbitally stable among solutions of the form $\psi(x,t) = e^{i(px-t)}q(x,t)$, where $q(\cdot,t) \in H^1_{\text{per}}(0,T_0)$. Here the orbit is defined with respect to translations in space and rotations of the complex phase. The proof follows the general strategy proposed in [8] and relies on the fact that the periodic wave is a constrained minimizer of the energy

$$E(\psi) = \int_{\mathcal{I}} \left[|\psi_x|^2 + \frac{1}{2}\left(1 - |\psi|^2\right)^2 \right] \mathrm{d}x, \tag{1.3}$$

subject to fixed values of the charge Q and the momentum M given by

$$Q(\psi) = \int_{\mathcal{I}} |\psi|^2 \, \mathrm{d}x, \qquad M(\psi) = \frac{i}{2}\int_{\mathcal{I}} (\bar{\psi}\psi_x - \psi\bar{\psi}_x)\, \mathrm{d}x. \tag{1.4}$$

Here $\mathcal{I} = (0, T_0)$. On the other hand, if we consider the more general case of "subharmonic perturbations", which correspond to $q(\cdot,t) \in H^1_{\text{per}}(0, NT_0)$ for some integer $N \geq 2$, then the second variation of E at u_0 with $\mathcal{I} = (0, NT_0)$ contains additional negative eigenvalues, which cannot be eliminated by restricting the energy to the submanifold where Q and M are constant.

Generally speaking, in such an unfavorable energy configuration, there is no chance to establish orbital stability using the standard energy method [3]. However, the cubic defocusing NLS equation can (at least formally) be integrated using the inverse scattering transform method, and it admits therefore a countable sequence of independent conserved quantities. For instance, one can verify directly or with an algorithmic computation (see [15, Section 2.3] for a review of such techniques) that the higher-order functional

$$R(\psi) = \int_{\mathcal{I}} \left[|\psi_{xx}|^2 + 3|\psi|^2|\psi_x|^2 + \frac{1}{2}(\bar{\psi}\psi_x + \psi\bar{\psi}_x)^2 + \frac{1}{2}|\psi|^6 \right] \mathrm{d}x, \tag{1.5}$$

is also invariant under the time evolution defined by (1.1). These additional properties can be invoked to rescue the stability analysis of periodic waves. Indeed, using the eigenfunctions of Lax operators arising in the inverse scattering method, a complete set of Floquet–Bloch eigenfunctions satisfying the linearization of the cubic NLS equation (1.1) at the periodic wave with profile u_0 has been constructed in [4]. Moreover, it is shown in [4] that an appropriate linear combination of the energy E, the charge Q, the momentum M, and the higher order quantity R produces a functional for which the periodic wave with profile u_0 is a strict local minimizer, up to symmetries. This result holds for $q(\cdot, t) \in H^2_{\text{per}}(0, NT_0)$, for any $N \in \mathbb{N}$, where T_0 is the period of $|u_0|$. This easily implies that the periodic wave with profile u_0 is orbitally stable with respect to subharmonic perturbations.

The proof given in [4] that any periodic wave can be characterized as a local minimizer of a suitable higher-order conserved quantity is not direct. Indeed, the authors prove the positivity of the second variation at the periodic wave by evaluating the corresponding quadratic form on the basis of the Floquet–Bloch eigenfunctions associated with the linearized NLS flow. These, however, are *not* the eigenfunctions of the self-adjoint operator associated with the second variation itself, which would be more natural to use in the present context. In addition, many explicit computations are not transparent because they rely on nontrivial properties of the Jacobi elliptic functions and integrals that are used to represent the profile u_0 of the periodic wave. This is why we feel that it is worth revisiting the problem using more standard PDE techniques, which is the goal of the present work.

The idea of using higher-order conserved quantities to solve delicate analytical problems related to orbital stability of nonlinear waves in integrable evolution equations has become increasingly popular in recent years. Orbital stability of n-solitons in the Korteweg–de Vries (KdV) and the cubic focusing NLS equations was established in the space $H^n(\mathbb{R})$ by combining the first $(n+1)$ conserved quantities of these equations in [11] and [9], respectively. For the modified KdV equation, orbital stability of breathers in the space $H^2(\mathbb{R})$ was established in [2] by using two conserved quantities. For the massive Thirring model (a system of nonlinear Dirac equations), orbital stability of solitary waves was proved in the space $H^1(\mathbb{R})$ with the help of the first four conserved quantities [13].

As already mentioned, we consider in this paper periodic waves of the cubic defocusing NLS equation (1.1) which correspond to *real-valued* solutions of the second-order equation (1.2). In that case, the second-order equation (1.2) can be integrated once to obtain the first-order equation

$$\left(\frac{du_0}{dx}\right)^2 = \frac{1}{2}\left[\left(1 - u_0^2\right)^2 - \mathcal{E}^2\right], \quad x \in \mathbb{R}, \tag{1.6}$$

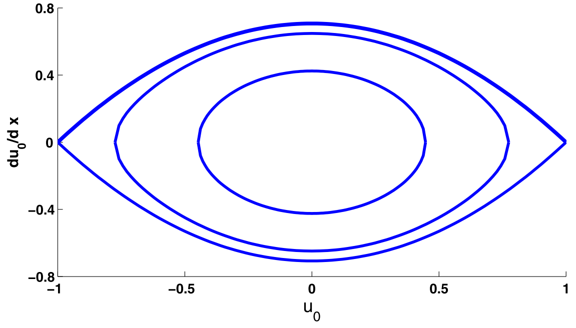

Fig. 1. The level set given by (1.6) on the phase plane (u_0, u_0') for $\mathcal{E} = 0, 0.4, 0.8$.

where the integration constant $\mathcal{E} \in [0, 1]$ can be used to parameterize all bounded solutions, up to translations. If $0 < \mathcal{E} < 1$, we obtain a periodic solution which has the explicit form

$$u_0(x) = \sqrt{1-\mathcal{E}}\,\mathrm{sn}\left(x\sqrt{\frac{1+\mathcal{E}}{2}}, \sqrt{\frac{1-\mathcal{E}}{1+\mathcal{E}}}\right), \tag{1.7}$$

where $\mathrm{sn}(\xi, k)$ denotes the Jacobi elliptic function with argument ξ and parameter k [10]. This solution corresponds to a closed orbit in the phase plane for (u_0, u_0'), which is represented in Fig. 1. When $\mathcal{E} \to 1$ the orbit shrinks to the center point $(0, 0)$, while in the limit $\mathcal{E} \to 0$ the solution u_0 approaches the black soliton

$$u_0(x) = \tanh\left(\frac{x}{\sqrt{2}}\right), \tag{1.8}$$

which corresponds to a heteroclinic orbit connecting the two saddle points $(-1, 0)$ and $(1, 0)$. If $\mathcal{E} \in (0, 1)$, the period of u_0 (which is exactly twice the period T_0 of the modulus $|u_0|$) is given by

$$2T_0 = 4\sqrt{\frac{2}{1+\mathcal{E}}}\,K\left(\sqrt{\frac{1-\mathcal{E}}{1+\mathcal{E}}}\right), \tag{1.9}$$

where $K(k)$ is the complete elliptic integral of the first kind. It can be verified that T_0 is a decreasing function of $\mathcal{E}$ which satisfies $T_0 \to +\infty$ as $\mathcal{E} \to 0$ and $T_0 \to \pi$ as $\mathcal{E} \to 1$ [7].

Now we study the stability of the periodic wave $\psi(x, t) = u_0(x)e^{-it}$, where u_0 is given by (1.7) for some $\mathcal{E} \in (0, 1)$. It is clear from (1.2) that the wave profile u_0 is a critical point of the energy functional E defined by (1.3). In addition, one can verify by explicit (but rather cumbersome) calculations that u_0 is also a critical point of the higher-order functional

$$S(\psi) = R(\psi) - \frac{1}{2}\left(3 - \mathcal{E}^2\right)Q(\psi), \tag{1.10}$$

where R is given by (1.5) and Q by (1.4). Using an idea borrowed from [4], we combine E and S by introducing the functional

$$\Lambda_c(\psi) = S(\psi) - cE(\psi), \tag{1.11}$$

where $c \in \mathbb{R}$ is a parameter that will be fixed below. Our first result is the following proposition, which establishes an *unconstrained* variational characterization for the periodic waves of the NLS equation (1.1), at least when their amplitude is small enough.

Proposition 1.1. *There exists $\mathcal{E}_0 \in (0, 1)$ such that, for all $\mathcal{E} \in (\mathcal{E}_0, 1)$, there exist values c_- and c_+ in the range $1 < c_- < 2 < c_+ < 3$ such that, for any $c \in (c_-, c_+)$, the second variation of the functional Λ_c at the periodic wave profile u_0 is nonnegative for perturbations in $H^2(\mathbb{R})$. Furthermore, we have*

$$c_\pm = 2 \pm \sqrt{2(1-\mathcal{E})} + \mathcal{O}(1-\mathcal{E}) \quad \text{as } \mathcal{E} \to 1. \tag{1.12}$$

Remark 1.2. The second variation of Λ_c at u_0 is the quadratic form associated with a fourth-order selfadjoint operator with T_0-periodic coefficients, which will be explicitly calculated in Section 2 below. Proposition 1.1 asserts that the Floquet–Bloch spectrum of that operator is nonnegative, if we consider it as acting on the whole space $H^4(\mathbb{R})$. In particular, the same operator has nonnegative spectrum when acting on $H^4_{\text{per}}(0, T)$, where T is any multiple of T_0. In fact, the proof of Proposition 1.1 shows that $\Lambda_c''(u_0)$ is positive except for two neutral directions corresponding to symmetries (translations in space and rotations of the complex phase). This key observation will allow us to prove orbital stability of the periodic wave with respect to subharmonic perturbations, see Theorem 1.8 below.

Our second result suggests a rather explicit formula for the limiting values $c_\pm$ that appear in Proposition 1.1.

Proposition 1.3. *For all $\mathcal{E} \in (0, 1)$ and all $c \geq 1$, the second variation of the functional Λ_c at the periodic wave profile u_0 is positive, except for two neutral directions due to symmetries, only if $c \in [c_-, c_+]$ with*

$$c_\pm = 2 \pm \frac{2k}{1+k^2}, \quad \text{where } k = \sqrt{\frac{1-\mathcal{E}}{1+\mathcal{E}}}. \tag{1.13}$$

Remark 1.4. Proposition 1.3 gives a necessary condition for the second variation $\Lambda_c''(u_0)$ to be positive except for two neutral directions due to translations and phase rotations. The condition is obtained by considering one particular band of the Floquet–Bloch spectrum of the fourth-order operator associated with $\Lambda_c''(u_0)$. That band touches the origin when the Floquet–Bloch wave number is equal to zero, is strictly convex near the origin if $c \in (c_-, c_+)$, and is strictly concave if $c \geq 1$ and $c \notin [c_-, c_+]$. In the latter case, the second variation $\Lambda_c''(u_0)$ has therefore negative directions. Interestingly enough, the alternative approach of Bottman et al. [4] suggests that, for any $\mathcal{E} \in (0, 1)$, the second variation $\Lambda_c''(u_0)$ is positive (except for neutral directions due to symmetries) whenever $c \in (c_-, c_+)$. Indeed, after adopting our definition of the functionals E and S, and performing explicit computations with Jacobi elliptic functions, one can show that the conditions implicitly defined in [4, Theorem 7] exactly correspond to choosing our parameter c in the interval (c_-, c_+) given by (1.13).

In Fig. 2, the values $c_\pm$ are represented as a function of the parameter $\mathcal{E}$ by a solid line. Note that the asymptotic expansion (1.12) is recovered from the analytical expressions (1.13) in the limit $k \to 0$, that is, $\mathcal{E} \to 1$. The asymptotic result (1.12) is shown by dashed lines.

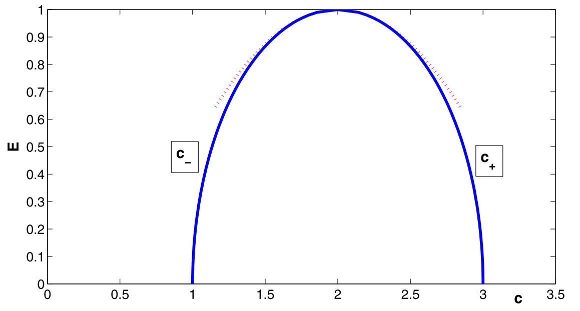

Fig. 2. The values $c_\pm$ given by the explicit expressions (1.13) are represented as a function of the parameter $\mathcal{E}$ (solid line). The asymptotic result (1.12) is shown by dashed lines.

The result of Proposition 1.1 relies on perturbation theory and is therefore restricted to periodic waves of small amplitude. Although the analytic formula (1.13) suggests that the conclusion of Proposition 1.1 should hold for all periodic waves, namely for all $\mathcal{E} \in (0, 1)$, the result of Proposition 1.3 is only a necessary condition for positivity of the functional Λ_c. In the next result, we fix $c = 2$ (the mean value in the interval $[c_-, c_+]$) and prove the positivity of the second variation of the functional $\Lambda_{c=2}$.

Proposition 1.5. *Fix $c = 2$. For every $\mathcal{E} \in (0, 1)$, the second variation of the functional $\Lambda_{c=2}$ at the periodic wave profile u_0 is positive, except for two neutral directions due to symmetries.*

Remark 1.6. In the proof of Proposition 1.5, we show that the quadratic form defined by the second variation $\Lambda''_{c=2}(u_0)$ restricted to purely imaginary perturbations of the periodic wave can be decomposed as a sum of squared quantities, hence is obviously nonnegative. In order to control the quadratic form for the real perturbations to the periodic wave, we use a continuation argument from the limit to the periodic waves of small amplitude, combined with the analysis of a pair of second-order Schrödinger operators with T_0-periodic coefficients.

Remark 1.7. Proposition 1.5 implies the spectral stability of the periodic wave profile u_0 for every $\mathcal{E} \in (0, 1)$, see the end of Section 5.

Our final result establishes orbital stability of the periodic wave (1.7) with respect to the subharmonic perturbations in $H^2_{\rm per}(0, T)$, where $T > 0$ is any integer multiple of the period of u_0. Therefore, we use $\mathcal{I} = (0, T)$ in the definition of all functionals (1.3)–(1.5). If we consider Λ_c as defined on $H^2_{\rm per}(0, T)$, we know from Proposition 1.5 that $\Lambda'_c(u_0) = 0$ and that the second variation $\Lambda''_{c=2}(u_0)$ is strictly positive, except for two neutral directions corresponding to symmetries. Since $\Lambda_{c=2}(\psi)$ is a conserved quantity under the evolution defined by the cubic NLS equation (1.1), we obtain the following orbital stability result.

Theorem 1.8. *Fix $\mathcal{E} \in (0, 1)$ and let T be an integer multiple of the period $2T_0$ of u_0. For any $\epsilon > 0$, there exists $\delta > 0$ such that, if $\psi_0 \in H^2_{\rm per}(0, T)$ satisfies*

$$\|\psi_0 - u_0\|_{H^2_{\rm per}(0,T)} \le \delta, \tag{1.14}$$

the unique global solution $\psi(\cdot, t)$ of the cubic NLS equation (1.1) with initial data ψ_0 has the following property. For any $t \in \mathbb{R}$, there exist $\xi(t) \in \mathbb{R}$ and $\theta(t) \in \mathbb{R}/(2\pi\mathbb{Z})$ such that

$$\left\| e^{i(t+\theta(t))} \psi\big(\cdot + \xi(t), t\big) - u_0 \right\|_{H^2_{\mathrm{per}}(0,T)} \le \epsilon. \tag{1.15}$$

Moreover ξ and θ are continuously differentiable functions of t which satisfy

$$\left|\dot{\xi}(t)\right| + \left|\dot{\theta}(t)\right| \le C\epsilon, \quad t \in \mathbb{R}, \tag{1.16}$$

for some positive constant C.

Remark 1.9. It is well known that the Cauchy problem for the cubic NLS equation (1.1) is globally well posed in the Sobolev space $H^s_{\mathrm{per}}(0, T)$ for any integer $s \ge 0$, see [5].

Remark 1.10. The proof of Theorem 1.8 shows that, when $\epsilon \le 1$, one can take $\delta = \epsilon/\mathcal{C}$ for some constant $\mathcal{C} \ge 1$ depending on $\mathcal{E}$ and on the ratio T/T_0. We emphasize, however, that $\mathcal{C} \to \infty$ as $T/T_0 \to \infty$. This indicates that, although a given periodic wave is orbitally stable with respect to perturbations with arbitrary large period T, the size of the stability basin becomes very small when the ratio T/T_0 is large.

Applying the same technique, we can also prove the orbital stability of the black soliton (1.8) with respect to perturbations in $H^2(\mathbb{R})$. The details of this analysis are given in Part II, which is a companion paper to this work.

The rest of this article is organized as follows. Section 2 contains the proof of Proposition 1.1. The sufficient condition of Proposition 1.3 is proved in Section 3. In Section 4, we provide a representation of the quadratic form associated with $\Lambda_c''(u_0)$ as a sum of squared quantities. Section 5 reports the continuation argument, which yields the proof of Proposition 1.5. Section 6 is devoted to the proof of Theorem 1.8. Appendix A summarizes some explicit computations with the use of Jacobi elliptic functions.

2. Positivity of $\Lambda_c''(u_0)$ for periodic waves of small amplitude

This section presents the proof of Proposition 1.1.

Let u_0 be the periodic wave profile defined by (1.7) for some $\mathcal{E} \in (0, 1)$. We consider perturbations of u_0 of the form $\psi = u_0 + u + iv$, where u, v are real-valued. Since u_0 is a critical point of both E and S defined by (1.3) and (1.10), the leading order contributions to the renormalized quantities $E(\psi) - E(u_0)$ and $S(\psi) - S(u_0)$ are given by the second variations

$$\frac{1}{2}\langle E''(u_0)[u, v], [u, v]\rangle = \int_{\mathcal{I}} \left[u_x^2 + \left(3u_0^2 - 1\right)u^2\right] \mathrm{d}x + \int_{\mathcal{I}} \left[v_x^2 + \left(u_0^2 - 1\right)v^2\right] \mathrm{d}x \tag{2.1}$$

and

$$\begin{aligned}\frac{1}{2}\langle S''(u_0)[u, v], [u, v]\rangle &= \int_{\mathcal{I}} \left[u_{xx}^2 + 5u_0^2 u_x^2 + \left(-5u_0^4 + 15u_0^2 - 4 + 3\mathcal{E}^2\right)u^2\right] \mathrm{d}x \\ &\quad + \int_{\mathcal{I}} \left[v_{xx}^2 + 3u_0^2 v_x^2 + \left(u_0^2 - 1\right)v^2\right] \mathrm{d}x. \end{aligned} \tag{2.2}$$

In the proof of the orbital stability theorem (Theorem 1.8) given in Section 6, we eventually take $\mathcal{I} = (0, T)$, where T is a multiple of the period $2T_0$ of the periodic wave profile u_0, and we assume that $u, v \in H^2_{\rm per}(0, T)$. In this case, the formulas (2.1) and (2.2) represent the second variations of the functionals E and S defined on the space $H^2_{\rm per}(0, T)$. However, here and in the following three sections, we only investigate the positivity properties of the second variations. For that purpose, it is more convenient to take $\mathcal{I} = \mathbb{R}$ and to assume that $u, v \in H^2(\mathbb{R})$.

As is clear from (2.1) and (2.2), the second variations $E''(u_0)$ and $S''(u_0)$ are block-diagonal in the sense that the contributions of u and v do not mix together (this is the main reason for which we restrict our analysis to *real-valued* wave profiles u_0). We can thus write

$$\frac{1}{2}\big\langle E''(u_0)[u, v], [u, v]\big\rangle = \langle L_+u, u\rangle_{L^2} + \langle L_-v, v\rangle_{L^2}$$

and

$$\frac{1}{2}\big\langle S''(u_0)[u, v], [u, v]\big\rangle = \langle M_+u, u\rangle_{L^2} + \langle M_-v, v\rangle_{L^2},$$

where $\langle \cdot, \cdot\rangle_{L^2}$ is the scalar product on $L^2(\mathbb{R})$ and the operators $L_\pm$ and $M_\pm$ are defined by

$$\begin{aligned} L_+ &= -\partial_x^2 + 3u_0^2 - 1, \qquad M_+ = \partial_x^4 - 5\partial_x u_0^2 \partial_x - 5u_0^4 + 15u_0^2 - 4 + 3\mathcal{E}^2, \\ L_- &= -\partial_x^2 + u_0^2 - 1, \qquad M_- = \partial_x^4 - 3\partial_x u_0^2 \partial_x + u_0^2 - 1. \end{aligned} \tag{2.3}$$

Note that $L_+u_0' = M_+u_0' = 0$, due to the translation invariance of the cubic NLS equation (1.1), and that $L_-u_0 = M_-u_0 = 0$, due to the gauge invariance $\psi \mapsto e^{i\theta}\psi$ with $\theta \in \mathbb{R}$.

We now fix $c \in \mathbb{R}$ and consider the functional $\Lambda_c(\psi) = S(\psi) - cE(\psi)$, as in (1.11). We have

$$\frac{1}{2}\big\langle \Lambda_c''(u_0)[u, v], [u, v]\big\rangle = \big\langle K_+(c)u, u\big\rangle_{L^2} + \big\langle K_-(c)v, v\big\rangle_{L^2}, \tag{2.4}$$

where $K_\pm(c) = M_\pm - cL_\pm$. By construction, $K_\pm(c)$ are selfadjoint, fourth-order differential operators on $\mathbb{R}$ with T_0-periodic coefficients, where T_0 is the period of $|u_0|$. Our goal is to show that these operators are nonnegative, at least if $\mathcal{E}$ is sufficiently close to 1 and if the parameter c is chosen appropriately. Equivalently, the quadratic forms in the right-hand side of (2.4) are nonnegative for all $u, v \in H^2(\mathbb{R})$ under the same assumptions on $\mathcal{E}$ and c.

Before going further, let us explain why a careful choice of the parameter c is necessary. Assume for simplicity that $\mathcal{E} = 1$, so that $u_0 = 0$. In that case, we have

$$\begin{aligned} \big\langle K_\pm(c)u, u\big\rangle_{L^2} &= \int_{\mathbb{R}} \big[u_{xx}^2 - cu_x^2 + (c-1)u^2\big]\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \Big(u_{xx} + \frac{c}{2}u\Big)^2 \mathrm{d}x - \Big(1 - \frac{c}{2}\Big)^2 \int_{\mathbb{R}} u^2\,\mathrm{d}x. \end{aligned} \tag{2.5}$$

This simple computation shows that the second variation $\Lambda_c''(0)$ is nonnegative if and only if $c = 2$. By a perturbation argument, we shall verify that $\Lambda_c''(u_0)$ remains nonnegative for $\mathcal{E}$ sufficiently close to 1, provided c is close enough to 2. More precisely, we shall prove that the operators $K_+(c)$ and $K_-(c)$ are nonnegative and have only the following zero modes

$$K_+(c)u_0' = 0, \quad \text{and} \quad K_-(c)u_0 = 0. \tag{2.6}$$

This means that the second variation $\Lambda_c''(u_0)$ is strictly positive, except along the subspace spanned by the eigenfunctions u_0' and iu_0, which correspond to symmetries of the NLS equation (1.1). Note that, when $\mathcal{E} = 1$, the second variation $\Lambda_c''(u_0)$ vanishes on a four-dimensional subspace, according to the representation (2.5), but the degeneracy disappears as soon as $\mathcal{E} < 1$.

The proof of Proposition 1.1 relies on perturbation theory for the Floquet–Bloch spectrum of the operators $K_\pm(c)$. First, we normalize the period of the profile u_0 to 2π by using the transformation $u_0(x) = U(\ell x)$, where $\ell = \pi/T_0$, so that $U(z + 2\pi) = U(z)$. The second-order differential equation satisfied by rescaled profile $U(z)$, as well as the associated first-order invariant, is given by

$$\ell^2 \frac{d^2U}{dz^2} + U - U^3 = 0 \quad \Rightarrow \quad \ell^2 \left(\frac{dU}{dz}\right)^2 = \frac{1}{2}\left[\left(1 - U^2\right)^2 - \mathcal{E}^2\right]. \tag{2.7}$$

In agreement with the exact solution (1.7) we assume that U is odd with $U'(0) > 0$, so that $U \in H^2_{\text{per}}(0, 2\pi)$ is entirely determined by the value of $\mathcal{E} \in (0, 1)$. As was already mentioned, it is known for the soft potential in (2.7) that the map $(0, 1) \ni \mathcal{E} \mapsto \ell \in (0, 1)$ is strictly increasing and onto [7]. The following proposition specifies the precise asymptotic behavior of the rescaled profile U as $\mathcal{E} \to 1$.

Proposition 2.1. *The map $(0, 1) \ni \mathcal{E} \mapsto (\ell, U) \in \mathbb{R} \times H^2_{\text{per}}(0, 2\pi)$ can be uniquely described, when $\mathcal{E} \to 1$, by a small parameter $a > 0$ in the following way:*

$$\mathcal{E} = 1 - a^2 + \mathcal{O}\left(a^4\right), \qquad \ell^2 = 1 - \frac{3}{4}a^2 + \mathcal{O}\left(a^4\right), \tag{2.8}$$

and

$$U(z) = aU_0(z) + \mathcal{O}_{H^2_{\text{per}}(0,2\pi)}\left(a^3\right),$$

where $U_0(z) = \sin(z)$.

Proof. The argument is rather standard, so we just mention here the main ideas. Since the wave profile $U(z)$ is an odd function of z, we work in the space

$$L^2_{\text{per,odd}}(0, 2\pi) = \left\{U \in L^2_{\text{loc}}(\mathbb{R}):\ U \text{ is odd and } 2\pi\text{-periodic}\right\}.$$

We use the Lyapunov–Schmidt decomposition $\ell^2 = 1 + \tilde{\ell}$, $U = aU_0 + \tilde{U}$, where the perturbation $\tilde{U} \in H^2_{\text{per,odd}}(0, 2\pi)$ is orthogonal to U_0 in $L^2_{\text{per}}(0, 2\pi)$, namely $\langle U_0, \tilde{U}\rangle_{L^2_{\text{per}}} = 0$. The quantities $\tilde{\ell}$ and $\tilde{U}$ can be determined by projecting Eq. (2.7) onto the one-dimensional subspace $\text{Span}\{U_0\} \subset L^2_{\text{per,odd}}(0, 2\pi)$ and its orthogonal complement. This gives the relations

$$a\tilde{\ell} = -\frac{\langle U_0, (aU_0 + \tilde{U})^3 \rangle_{L^2_{\text{per}}}}{\langle U_0, U_0 \rangle_{L^2_{\text{per}}}} \tag{2.9}$$

and

$$(1 + \tilde{\ell})\tilde{U}'' + \tilde{U} = (aU_0 + \tilde{U})^3 - \frac{\langle U_0, (aU_0 + \tilde{U})^3 \rangle_{L^2_{\text{per}}}}{\langle U_0, U_0 \rangle_{L^2_{\text{per}}}} U_0. \tag{2.10}$$

For any small $\tilde{\ell}$ and a, it is easy to verify (by inverting the linear operator in the left-hand side and using a fixed point argument) that Eq. (2.10) has a unique solution $\tilde{U} \in H^2_{\text{per,odd}}(0, 2\pi)$ such that $\langle U_0, \tilde{U} \rangle_{L^2_{\text{per}}} = 0$ and $\tilde{U} = \mathcal{O}_{H^2_{\text{per}}(0,2\pi)}(a^3)$ as $a \to 0$. This solution depends smoothly on $\tilde{\ell}$, so if we substitute it into the right-hand side of (2.9) we obtain an equation for $\tilde{\ell}$ only, which can in turn be solved uniquely for small $a > 0$. The result is

$$\tilde{\ell} = -a^2 \frac{\langle U_0, U_0^3 \rangle_{L^2_{\text{per}}}}{\langle U_0, U_0 \rangle_{L^2_{\text{per}}}} + \mathcal{O}\big(a^4\big) = -\frac{3}{4}a^2 + \mathcal{O}\big(a^4\big).$$

Finally the expression $\mathcal{E} = 1 - a^2 + \mathcal{O}(a^4)$ follows from the first-order invariant (2.7), if we use the above decompositions and the asymptotic formulas for $\tilde{\ell}$ and $\tilde{U}$. □

We next study the Floquet–Bloch spectrum of the operators $K_\pm(c) = M_\pm - cL_\pm$. Using the same rescaling $z = \ell x$ and the Floquet parameter κ, we write these operators in the following form

$$\begin{aligned}
P_-(c,\kappa) &= \ell^4(\partial_z + i\kappa)^4 - 3\ell^2(\partial_z + i\kappa)U^2(\partial_z + i\kappa) + c\ell^2(\partial_z + i\kappa)^2 + (c-1)\big(1 - U^2\big), \\
P_+(c,\kappa) &= \ell^4(\partial_z + i\kappa)^4 - 5\ell^2(\partial_z + i\kappa)U^2(\partial_z + i\kappa) + c\ell^2(\partial_z + i\kappa)^2 \\
&\quad - 5U^4 + (15 - 3c)U^2 - 4 + 3\mathcal{E}^2 + c.
\end{aligned}$$

Note that the operators $P_\pm(c,\kappa)$ have π-periodic coefficients, hence we can look for π-periodic Bloch wave functions so that κ can be defined in the Brillouin zone $[-1, 1]$. However, for computational simplicity of the perturbation expansions, it is more convenient to work with the 2π-periodic Bloch wave functions, in which case κ is defined in the Brillouin zone $\mathbb{T} = [-\frac{1}{2}, \frac{1}{2}]$. If $\kappa \in \mathbb{T}$ and if the function $w(\cdot,\kappa) \in H^4_{\text{per}}(0, 2\pi)$ satisfies

$$P_\pm(c,\kappa)w(z,\kappa) = \lambda(\kappa)w(z,\kappa), \quad z \in \mathbb{R}, \tag{2.11}$$

for some $\lambda(\kappa) \in \mathbb{R}$ and either sign, then defining $u(x,\kappa) = e^{i\kappa\ell x}w(\ell x,\kappa)$ we obtain a function $u(\cdot,\kappa) \in L^\infty(\mathbb{R}) \cap H^4_{\text{loc}}(\mathbb{R})$ such that

$$K_\pm(c)u(x,\kappa) = \lambda(\kappa)u(x,\kappa), \quad x \in \mathbb{R}.$$

This precisely means that $\lambda(\kappa)$ belongs to the Floquet–Bloch spectrum of $K_\pm(c)$.

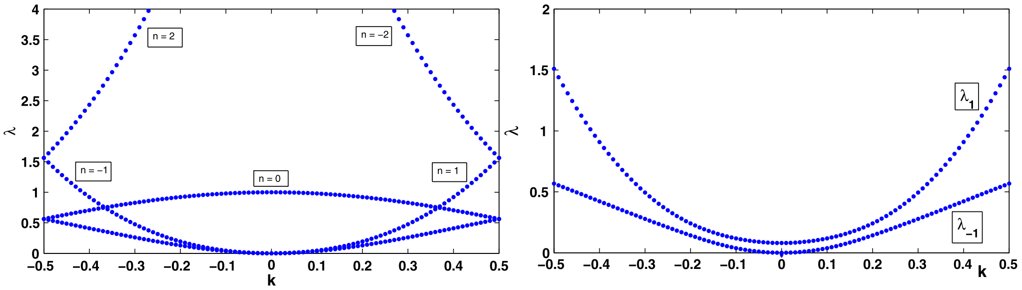

Fig. 3. Left: spectral bands given by (2.12) for $c = 2$ and $a = 0$. Right: spectral bands given by the matrix eigenvalue problem (2.16) for $c = 2$ and $a = 0.2$.

By Proposition 2.1, when $\mathcal{E}$ is close to 1, the operators $P_\pm(c,\kappa)$ can be expanded as

$$P_\pm(c,\kappa) = P^{(0)}(c,\kappa) + a^2 P_\pm^{(1)}(c,\kappa) + \mathcal{O}_{H^4_{\rm per}(0,2\pi)\to L^2_{\rm per}(0,2\pi)}\big(a^4\big),$$

where

$$P^{(0)}(c,\kappa) = (\partial_z + i\kappa)^4 + c(\partial_z + i\kappa)^2 + c - 1,$$
$$P_-^{(1)}(c,\kappa) = -\frac{3}{2}(\partial_z + i\kappa)^4 - 3(\partial_z + i\kappa)U_0^2(\partial_z + i\kappa) - \frac{3}{4}c(\partial_z + i\kappa)^2 + (1-c)U_0^2,$$
$$P_+^{(1)}(c,\kappa) = -\frac{3}{2}(\partial_z + i\kappa)^4 - 5(\partial_z + i\kappa)U_0^2(\partial_z + i\kappa) - \frac{3}{4}c(\partial_z + i\kappa)^2 + (15-3c)U_0^2 - 6.$$

The operator $P^{(0)}(c,\kappa)$ has constant coefficients, and its spectrum in the space $L^2_{\rm per}(0,2\pi)$ consists of a countable family of real eigenvalues $\{\lambda_n^{(0)}(\kappa)\}_{n\in\mathbb{Z}}$ given by

$$\lambda_n^{(0)}(\kappa) = (\kappa+n)^4 - c(\kappa+n)^2 + c - 1, \quad n \in \mathbb{Z}. \tag{2.12}$$

As was already observed, one has $\lambda_n^{(0)}(\kappa) \geq 0$ for all $n \in \mathbb{Z}$ and all $\kappa \in \mathbb{T}$ if and only if $c = 2$. This is the case represented in Fig. 3 (left), where it is clear that all spectral bands $\{\lambda_n^{(0)}(\kappa)\}_{\kappa\in\mathbb{T}}$ are strictly positive, except for two bands corresponding to $n = \pm 1$ which touch the origin at $\kappa = 0$.

For small $a > 0$, the eigenvalues of the perturbed operators $P_\pm(c,\kappa)$ are denoted by $\lambda_n^\pm(\kappa)$ with $n \in \mathbb{Z}$, and we number them in such a way that $\lambda_n^\pm(\kappa) \to \lambda_n^{(0)}(\kappa)$ as $a \to 0$ for $n \neq \pm 1$. By classical perturbation theory, we know that the eigenvalues $\lambda_n^\pm(\kappa)$ stay bounded away from zero for $n \neq \pm 1$, so it remains to study how the bands $\{\lambda_1^\pm(\kappa)\}_{\kappa\in\mathbb{T}}$ and $\{\lambda_{-1}^\pm(\kappa)\}_{\kappa\in\mathbb{T}}$ behave near $\kappa = 0$ as $a \to 0$. The following proposition indicates that these bands separate from each other when $a > 0$, so that one band still touches the origin at $\kappa = 0$ while the other one remains strictly positive for all $\kappa \in \mathbb{T}$. In other words, the degeneracy of the limiting case $c = 2$, $a = 0$ is unfold by the perturbation as soon as $a > 0$. This phenomenon is illustrated in Fig. 3 (right), which shows the solutions of the matrix eigenvalue problem (2.16) obtained below.

Proposition 2.2. *If $a > 0$ is sufficiently small and $c \in (c_-, c_+)$, where*

$$c_\pm = 2 \pm \sqrt{2}a + \mathcal{O}\big(a^2\big), \tag{2.13}$$

the operator $K_\pm(c)$ *has exactly one Floquet–Bloch band denoted by* $\{\lambda^\pm_{-1}(\kappa)\}_{\kappa\in\mathbb{T}}$ *that touches the origin at* $\kappa = 0$*, while all other bands are strictly positive. Moreover, for any* $\nu < \sqrt{2}$*, there exist positive constants* C_1*,* C_2*,* C_3 *(independent of* a*) such that, if* $|c-2| \le \nu a$*, one has*

$$\lambda^\pm_{-1}(\kappa) \ge C_1\kappa^2, \qquad \lambda^\pm_1(\kappa) \ge C_2\big(a^2+\kappa^2\big), \qquad \lambda^\pm_n(\kappa) \ge C_3, \quad n \in \mathbb{Z}\setminus\{+1,-1\}, \tag{2.14}$$

for all $\kappa \in \mathbb{T}$.

Proof. From (2.12) we know that, if $|c-2|$ is sufficiently small, there exists a constant $C > 0$ (independent of c) such that

$$0 < \lambda^{(0)}_n(\kappa)^{-1} \le C \quad \text{for all } n \in \mathbb{Z}\setminus\{+1,-1\} \text{ and all } \kappa \in \mathbb{T}. \tag{2.15}$$

By classical perturbation theory, this bound remains true (with possibly a larger constant C) for the perturbed eigenvalues $\lambda^\pm_n(\kappa)$ when $n \ne \pm 1$ and $a > 0$ is small enough. We thus obtain the third estimate in (2.14).

To control the critical bands corresponding to $n = \pm 1$, we concentrate on the operator $P_-(c,\kappa)$ (the argument for $P_+(c,\kappa)$ being similar, see below), and for simplicity we denote its eigenvalues by $\lambda_n(\kappa)$ instead of $\lambda^-_n(\kappa)$. The same perturbation argument as before shows that $\lambda_{\pm 1}(\kappa)$ is bounded away from zero if $|\kappa| \ge \kappa_0$ and a is sufficiently small, where $\kappa_0 > 0$ is an arbitrary positive number. On the other hand, for small values of a, $|c-2|$, and $|\kappa|$, solutions to the spectral problem (2.11) for $P_-(c,\kappa)$ are obtained by the Lyapunov–Schmidt decomposition

$$w(z,\kappa) = b_1(\kappa)e^{iz} + b_{-1}(\kappa)e^{-iz} + \tilde{w}(z,\kappa), \qquad \big\langle e^{\pm i\cdot}, \tilde{w}(\cdot,\kappa)\big\rangle_{L^2_{\text{per}}} = 0,$$

where all terms can be determined by projecting the spectral problem (2.11) onto the two-dimensional subspace $\text{Span}\{e^{i\cdot}, e^{-i\cdot}\} \subset L^2_{\text{per}}(0,2\pi)$ and its orthogonal complement in $L^2_{\text{per}}(0,2\pi)$. Using the bound (2.15), one can prove that $\tilde{w}(\cdot,\kappa) = \mathcal{O}_{H^4_{\text{per}}(0,2\pi)}(a^2)$, which allows us to find $\lambda(\kappa)$ near $\lambda^{(0)}_{\pm 1}(\kappa)$ as a solution of the matrix eigenvalue problem

$$\begin{bmatrix} \lambda^{(0)}_1(\kappa) + a^2 g_{1,1}(\kappa) + \mathcal{O}(a^4) & -a^2 g_{1,-1}(\kappa) + \mathcal{O}(a^4) \\ -a^2 g_{-1,1}(\kappa) + \mathcal{O}(a^4) & \lambda^{(0)}_{-1}(\kappa) + a^2 g_{-1,-1}(\kappa) + \mathcal{O}(a^4) \end{bmatrix} \begin{bmatrix} b_1 \\ b_{-1} \end{bmatrix} = \lambda(\kappa) \begin{bmatrix} b_1 \\ b_{-1} \end{bmatrix}, \tag{2.16}$$

where

$$g_{\pm 1,\pm 1}(\kappa) = -\frac{3}{2}(\kappa \pm 1)^4 + \frac{3}{4}c(\kappa \pm 1)^2 + \frac{1}{2}(1-c) + \frac{3}{2}(\kappa \pm 1)^2,$$
$$g_{\pm 1,\mp 1}(\kappa) = \frac{1}{4}(1-c) + \frac{3}{4}\big(\kappa^2 - 1\big).$$

Setting $c = 2 + \gamma$ with small $|\gamma|$, we have

$$\lambda_{\pm 1}^{(0)}(\kappa) = \mp 2\gamma\kappa + (4-\gamma)\kappa^2 \pm 4\kappa^3 + \kappa^4,$$
$$g_{\pm 1,\pm 1}(\kappa) = 1 + \frac{1}{4}\gamma \pm \frac{3}{2}\gamma\kappa - 6\kappa^2 + \frac{3}{4}\gamma\kappa^2 \mp 6\kappa^3 - \frac{3}{2}\kappa^4,$$
$$g_{\pm 1,\mp 1}(\kappa) = -1 - \frac{1}{4}\gamma + \frac{3}{4}\kappa^2.$$

If we denote by A the matrix in the left-hand side of (2.16), we thus obtain the expansions

$$\frac{1}{2}\mathrm{tr}(A) = a^2 + 4\kappa^2 + \mathcal{O}\big((a^2+\kappa^2)(|\gamma| + a^2 + \kappa^2)\big),$$
$$\det(A) = \big(a^2+4\kappa^2\big)^2 - a^4 - 4\gamma^2\kappa^2 + \mathcal{O}\big((a^2+\kappa^2)^2(|\gamma| + a^2 + \kappa^2)\big).$$

As a result, the eigenvalues $\lambda_{\pm 1}(\kappa)$ of A satisfy

$$\lambda_{\pm 1}(\kappa) = a^2 + 4\kappa^2 + \mathcal{O}\big((a^2+\kappa^2)(|\gamma| + a^2 + \kappa^2)\big) \\ \pm\sqrt{a^4 + 4\gamma^2\kappa^2 + \mathcal{O}\big((a^2+\kappa^2)^2(|\gamma| + a^2 + \kappa^2)\big)}. \tag{2.17}$$

It remains to analyze (2.17). If $a > 0$ is small, we obviously have

$$\lambda_1(\kappa) \geq a^2 + 4\kappa^2 + \mathcal{O}\big((a^2+\kappa^2)(|\gamma| + a^2 + \kappa^2)\big) > 0,$$

which implies the second bound in (2.14). To estimate $\lambda_{-1}(\kappa)$, we first consider the regime where $|\kappa| \leq a$. If $|\gamma| \leq \nu a$ for any $\nu > 0$ independently of a, further expansion of (2.17) yields

$$\lambda_{-1}(\kappa) = \mu + 4\kappa^2 - \frac{2\gamma^2\kappa^2}{a^2} + \mathcal{O}\big(\kappa^2(|\gamma| + a^2)\big), \tag{2.18}$$

where $\mu = \mathcal{O}(a^2(|\gamma| + a^2))$ does not depend on κ. But since $K_-(c)u_0 = 0$ for any c, we must have $\lambda_{-1}(0) = 0$ to all orders in a and γ, hence actually $\mu = 0$. Then (2.18) shows that $\lambda_{-1}(\kappa)$ has a nondegenerate minimum at $\kappa = 0$ if and only if

$$\gamma^2 < 2a^2 + \mathcal{O}\big(a^3\big). \tag{2.19}$$

Since $\gamma = c - 2$, this yields expansion (2.13) for $c_\pm$. From now on, we assume that $|\gamma| \leq \nu a$ for some $\nu \in (0, \sqrt{2})$, so that the inequality (2.19) certainly holds if a is sufficiently small. The expansion (2.18) shows that if $|\kappa| \leq a$, then

$$\lambda_{-1}(\kappa) = \big(4 - 2\nu^2\big)\kappa^2 + \mathcal{O}\big(\kappa^2(|\gamma| + a^2)\big).$$

On the other hand, if $|\kappa| \geq a$, we easily find from (2.17) that

$$\lambda_{-1}(\kappa) \geq 4\kappa^2 - 2|\gamma||\kappa| + \mathcal{O}\big(\kappa^2(|\gamma| + \kappa^2)^{1/2}\big) \geq \kappa^2 + \mathcal{O}\big(\kappa^2(|\gamma| + \kappa^2)^{1/2}\big),$$

because $2|\gamma||\kappa| \leq \kappa^2 + \gamma^2 \leq \kappa^2 + \nu^2 a^2 \leq 3\kappa^2$. Altogether, we obtain the first estimate in (2.14).

The spectral problem (2.11) for the operator $P_+(c,\kappa)$ can be studied in a similar way and results in the matrix eigenvalue problem (2.16) with

$$g_{\pm 1,\pm 1}(\kappa) = -\frac{3}{2}(\kappa \pm 1)^4 + \frac{3}{4}c(\kappa \pm 1)^2 + \frac{3}{2}(1-c) + \frac{5}{2}(\kappa \pm 1)^2,$$
$$g_{\pm 1,\mp 1}(\kappa) = \frac{3}{4}(5-c) + \frac{5}{4}\big(\kappa^2 - 1\big).$$

Although the matrix A has now different entries, the leading order terms for the quantities $\mathrm{tr}(A)$ and $\det(A)$ are unchanged, hence the eigenvalues $\lambda_{\pm 1}(\kappa)$ still satisfy (2.17). Consequently, the conclusion remains true for c in the same interval (2.13). □

Remark 2.3. In view of expansion (2.8), Proposition 1.1 is a direct consequence of Proposition 2.2.

3. Necessary condition for positivity of $\Lambda_c''(u_0)$

This section presents the proof of Proposition 1.3.

In Section 2, we only considered small amplitude periodic waves (1.7) with $\mathcal{E}$ close to 1. To get some information on the quadratic form $\Lambda_c''(u_0)$ for larger periodic waves, we recall that, for any $\mathcal{E} \in (0,1)$ and any $c \in \mathbb{R}$, the operators $P_\pm(c,\kappa)$ have at least one Floquet–Bloch spectral band that touches the origin at $\kappa = 0$, because we know from (2.6) that the kernel of $P_\pm(c,0)$ in $L^2_{\mathrm{per}}(0,2\pi)$ is nontrivial.

In what follows, we focus on the operator $P_-(c,\kappa)$. Assuming that $\ker(P_-(c,0))$ in $L^2_{\mathrm{per}}(0,2\pi)$ is one-dimensional, we compute an asymptotic expansion as $\kappa \to 0$ of the unique Floquet–Bloch band that touches the origin at $\kappa = 0$. By Proposition 2.2, the assumption on $\ker(P_-(c,0))$ is satisfied at least for the periodic waves of small amplitude, in which case the Floquet–Bloch band that touches the origin is actually the lowest band $\lambda_{-1}^-(\kappa)$.

Proposition 3.1. *Fix $\mathcal{E} \in (0,1)$ and assume that $U = u_0(\ell^{-1}\cdot)$ is the only 2π-periodic solution of the homogeneous equation $P_-(c,0)w = 0$ for some $c \in \mathbb{R}$. Denote by $\mu(\kappa)$ the Floquet–Bloch band of $P_-(c,\kappa)$ that touches zero at $\kappa = 0$. Then μ is C^2 near $\kappa = 0$, $\mu(0) = \mu'(0) = 0$, and*

$$\mu''(0) = \frac{2}{\|U\|^2_{L^2_{\mathrm{per}}}}\Big[-4\ell^4(c-2)^2\big\langle U', \big(P_-(c,0)\big)^{-1}U'\big\rangle_{L^2_{\mathrm{per}}} + 3\ell^4\big\|U'\big\|^2_{L^2_{\mathrm{per}}} + (3-c)\ell^2\|U\|^2_{L^2_{\mathrm{per}}}\Big], \tag{3.1}$$

where $W = (P_-(c,0))^{-1}U'$ is uniquely defined under the orthogonality condition $\langle U, W\rangle_{L^2_{\mathrm{per}}} = 0$.

Proof. We consider $P_-(c,\kappa)$ as a self-adjoint operator in $L^2_{\mathrm{per}}(0,2\pi)$ with domain $H^4_{\mathrm{per}}(0,2\pi)$. As $\kappa \to 0$, we have

$$P_-(c,\kappa) = P_0(c) + i\kappa P_1(c) - \kappa^2 P_2(c) + \mathcal{O}_{H^4_{\mathrm{per}}(0,2\pi)\to L^2_{\mathrm{per}}(0,2\pi)}\big(\kappa^3\big),$$

where

$$P_0(c) = \ell^4 \partial_z^4 - 3\ell^2 \partial_z U^2 \partial_z + c\ell^2 \partial_z^2 + (c-1)\big(1 - U^2\big),$$
$$P_1(c) = 4\ell^4 \partial_z^3 - 6\ell^2 U^2 \partial_z - 6\ell^2 U U' + 2c\ell^2 \partial_z,$$
$$P_2(c) = 6\ell^4 \partial_z^4 - 3\ell^2 U^2 + c\ell^2.$$

We note that $P_0(c)$ and $P_2(c)$ are self-adjoint, whereas $P_1(c)$ is skew-adjoint. Under the assumptions of the proposition, the Floquet–Bloch band $\mu(\kappa)$ that touches zero at $\kappa = 0$ is separated from all the other bands of $P_-(c, \kappa)$ locally near $\kappa = 0$. Thus, $\mu(\kappa)$ is smooth near $\kappa = 0$, and it is possible to choose a nontrivial solution $w(z, \kappa)$ of the eigenvalue equation $P_-(c, \kappa) w(z, \kappa) = \mu(\kappa) w(z, \kappa)$ which also depends smoothly on κ. We look for an expansion of the form

$$\mu(\kappa) = i\kappa \mu_1 - \kappa^2 \mu_2 + \mathcal{O}\big(\kappa^3\big)$$

and

$$w(z, \kappa) = U(z) + i\kappa w_1(z) - \kappa^2 w_2(z) + \mathcal{O}_{H^4_{\text{per}}(0,2\pi)}\big(\kappa^3\big),$$

where w_1, w_2, and the remainder term belong to the orthogonal complement of span$\{U\}$ in $L^2_{\text{per}}(0, 2\pi)$. This gives the following system for the correction terms

$$P_0(c) w_1 + P_1(c) U = \mu_1 U, \tag{3.2}$$
$$P_0(c) w_2 + P_1(c) w_1 + P_2(c) U = \mu_1 w_1 + \mu_2 U. \tag{3.3}$$

If we take the scalar product of (3.2) with U in $L^2_{\text{per}}(0, 2\pi)$ and use the fact that $P_0(c)$ is self-adjoint, $P_1(c)$ is skew-adjoint, and $P_0(c) U = 0$, we obtain $\mu_1 = 0$. Similarly, taking the scalar product of (3.3) with U gives a nontrivial equation for μ_2:

$$\mu_2 \|U\|^2_{L^2_{\text{per}}} = \big\langle U, P_1(c) w_1\big\rangle_{L^2_{\text{per}}} + \big\langle U, P_2(c) U\big\rangle_{L^2_{\text{per}}}.$$

We note that

$$P_1(c) U = 2\ell^2 \big(2\ell^2 U''' - 6U^2 U' + cU'\big) = 2\ell^2 (c-2) U',$$
$$P_2(c) U = \ell^2 \big(6\ell^2 U'' - 3U^3 + cU\big) = \ell^2 \big(3\ell^2 U'' + (c-3) U\big).$$

Setting $w_1 = -2\ell^2 (c-2) W$, where W is the unique solution of $P_0(c) W = U'$ subject to the orthogonality condition $\langle U, W\rangle_{L^2_{\text{per}}} = 0$, we obtain

$$\mu_2 \|U\|^2_{L^2_{\text{per}}} = 4\ell^4 (c-2)^2 \big\langle U', W\big\rangle_{L^2_{\text{per}}} - \big(3\ell^4 \big\|U'\big\|^2_{L^2_{\text{per}}} + (3-c)\ell^2 \|U\|^2_{L^2_{\text{per}}}\big),$$

which yields the result (3.1) since $\mu''(0) = -2\mu_2$. □

Note that the first term in the right-hand side of (3.1) is negative, whereas the other two are positive for $c \le 3$. In the particular case where $c = 2$, it follows from Lemma 4.1 below that $\ker(P_-(2,0)) = \mathrm{span}\{U\}$ for any value of the parameter $\mathcal{E} \in (0,1)$, so that the assumption of Proposition 3.1 is satisfied. In this case, the formula (3.1) shows that $\mu''(0) > 0$.

Next, we give an explicit expression for $\mu''(0)$ by evaluating the various terms in (3.1) using known properties of the Jacobi elliptic functions. These computations are performed in Appendix A, see Eqs. (A.8)–(A.12), and yield the explicit formula

$$\mu''(0) = \frac{2\ell^2 k^2 (4k^2 - (c-2)^2(1+k^2)^2)}{(1+k^2)(1 - \frac{E(k)}{K(k)})(2k^2 + (c-2)(1+k^2)(1 - \frac{E(k)}{K(k)}))}, \tag{3.4}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively, and the parameter $k \in (0,1)$ is given by (1.13). The denominator in (3.4) is strictly positive if $c \ge 1$. Indeed, since $K(k) > E(k)$ for all $k \in (0,1)$, thanks to Eq. (A.10) in Appendix A, the denominator in (3.4) is a strictly increasing function of c, and for $c = 1$ we have

$$2k^2 + (c-2)\big(1+k^2\big)\left(1 - \frac{E(k)}{K(k)}\right)\bigg|_{c=1} = k^2 - 1 + \big(k^2+1\big)\frac{E(k)}{K(k)} > 0.$$

The expression above is positive for all $k \in (0,1)$, thanks to Eq. (A.11) in Appendix A. Thus, for $c \ge 1$, the sign of $\mu''(0)$ is the sign of the numerator in (3.4). It follows that $\mu''(0) \ge 0$ if $c \in [c_-, c_+] \subset [1,3]$, where $c_\pm$ are given by (1.13). Similarly, we have $\mu''(0) < 0$ if $c \ge 1$ and $c \notin [c_-, c_+]$.

Remark 3.2. The computations above imply the conclusion of Proposition 1.3. Indeed, either the kernel of $P_-(c,0)$ in $L^2_{\mathrm{per}}(0,2\pi)$ is one-dimensional, in which case the perturbation argument of Proposition 3.1 applies and proves the existence of negative spectrum if $c \ge 1$ is outside $[c_-, c_+]$, or the kernel is higher-dimensional and the second variation $\Lambda_c''(u_0)$ has more neutral directions than the two directions due to the symmetries. Note that we do not claim that the second variation $\Lambda_c''(u_0)$ (or even the quadratic form associated with $K_-(c)$) is indeed positive if $c \in (c_-, c_+)$, although by Proposition 2.2 this is definitely the case for the periodic waves of small amplitudes.

Remark 3.3. If we compare the above results with the computations in [4], one advantage of our approach is that we clearly distinguish between the spectra of the two linear operators $K_+(c)$ and $K_-(c)$. In particular, the necessary condition in Proposition 1.3 is derived from the positivity of the Floquet–Bloch spectrum of $K_-(c)$. We expect that, for any $\mathcal{E} \in (0,1)$, the Floquet–Bloch spectrum of $K_+(c)$ is positive for c in a larger subset of $\mathbb{R}$ than (c_-, c_+). For instance, the operator $K_+(c)$ is positive in $L^2(\mathbb{R})$ for every $c \le 3$ in the case of the black soliton that corresponds to $\mathcal{E} = 0$, see Remark 4.6 below.

4. Positive representations of $\Lambda_c''(u_0)$

As a first step in the proof of Proposition 1.5, which claims that the quadratic forms associated with the linear operators $K_\pm(c)$ are nonnegative on $H^2(\mathbb{R})$ if $c = 2$, we look for representations of these quadratic forms as sums of squared quantities.

Our first result shows that, if $c = 2$, the quadratic form associated with $K_-(c)$ is always positive, for all $\mathcal{E} \in [0,1]$, including the black soliton for $\mathcal{E} = 0$ and the zero solution for $\mathcal{E} = 1$.

Lemma 4.1. *Fix $c = 2$. For any $\mathcal{E} \in [0, 1]$ and any $v \in H^2(\mathbb{R})$, we have*

$$\langle K_-(2)v, v\rangle_{L^2} = \|L_-v\|_{L^2}^2 + \|u_0 v_x - u_0' v\|_{L^2}^2. \tag{4.1}$$

Proof. Using the definition (2.3) of the operator L_- and integrating by parts, we obtain

$$\begin{aligned}\|L_-v\|_{L^2}^2 &= \int_{\mathbb{R}} \left[v_{xx}^2 + 2\left(1-u_0^2\right)v v_{xx} + \left(1-u_0^2\right)^2 v^2\right] \mathrm{d}x \\ &= \int_{\mathbb{R}} \left[v_{xx}^2 - 2\left(1-u_0^2\right)v_x^2 - 2\left(u_0 u_0'\right)' v^2 + \left(1-u_0^2\right)^2 v^2\right] \mathrm{d}x.\end{aligned}$$

Similarly, we obtain

$$\|u_0 v_x - u_0' v\|_{L^2}^2 = \int_{\mathbb{R}} \left[u_0^2 v_x^2 + \left(u_0 u_0'\right)' v^2 + \left(u_0'\right)^2 v^2\right] \mathrm{d}x.$$

As a consequence, we have

$$\|L_-v\|_{L^2}^2 + \|u_0 v_x - u_0' v\|_{L^2}^2 = \int_{\mathbb{R}} \left[v_{xx}^2 + \left(3u_0^2 - 2\right)v_x^2 + \left[\left(1-u_0^2\right)^2 - u_0 u_0''\right]v^2\right] \mathrm{d}x,$$

which yields the desired result since $(1-u_0^2)^2 - u_0 u_0'' = 1 - u_0^2$. □

Remark 4.2. It is easy to verify that the right-hand side of the representation (4.1) vanishes if and only if $v = Cu_0$ for some constant C. As $u_0 \notin H^2(\mathbb{R})$, this shows that $\langle K_-(2)v, v\rangle_{L^2} > 0$ for any nonzero $v \in H^2(\mathbb{R})$.

Unfortunately, we are not able to find a positive representation for the quadratic form associated with the operator $K_+(c)$. If we proceed as in the proof of Lemma 4.1, we obtain

$$\langle K_+(2)u, u\rangle_{L^2} = \|L_+u\|_{L^2}^2 - \int_{\mathbb{R}} \left[u_0^2 u_x^2 - 3u_0^2 u^2 + 5u_0^4 u^2\right] \mathrm{d}x. \tag{4.2}$$

Here the second term in the right-hand side has no definite sign, hence it is difficult to exploit the representation (4.2). In the following lemma, we give a partial result which shows that the quadratic form associated with $K_+(c)$ is positive for $c < 3$ at least on a subspace of $H^2(\mathbb{R})$.

Lemma 4.3. *For any $\mathcal{E} \in (0, 1)$, any $c \in \mathbb{R}$, and any $u \in H^2(\mathbb{R})$ such that $u(x) = 0$ whenever $u_0'(x) = 0$, we have*

$$\langle K_+(c)u, u\rangle_{L^2} = \|w_x\|_{L^2}^2 + (3-c)\|w\|_{L^2}^2 + 2\mathcal{E}^2 \left\|\frac{u_0 w}{u_0'}\right\|_{L^2}^2, \tag{4.3}$$

where $w = u_x - \frac{u_0''}{u_0'}u \in H^1(\mathbb{R})$ satisfies $\frac{w}{u_0'} \in L^2(\mathbb{R})$.

Proof. Since u_0' satisfies the second-order differential equation $L_+ u_0' = 0$, the zeros of u_0' are all simple, as can also be deduced from the explicit formula (1.7). Thus, if $u \in H^2(\mathbb{R})$ is such that $u(x) = 0$ whenever $u_0'(x) = 0$, we can write $u = u_0' \tilde{u}$ and it follows from Hardy's inequality that $\tilde{u} \in H^1(\mathbb{R})$. With this notation, we have

$$w := u_x - \frac{u_0''}{u_0'} u = u_x - u_0'' \tilde{u} = u_0' \tilde{u}_x,$$

so that $w \in H^1(\mathbb{R})$ and $\frac{w}{u_0'} \in L^2(\mathbb{R})$. As a consequence, all terms in right-hand side of (4.3) are well-defined, and the integrations by parts used in the computations below can easily be justified.

To prove the representation (4.3), we first note that

$$u_{xx} + \left(1 - 3u_0^2\right) u = u_{xx} - \frac{u_0'''}{u_0'} u = w_x + \frac{u_0''}{u_0'} w.$$

Integrating by parts, we thus obtain

$$\|L_+ u\|_{L^2}^2 = \left\| w_x + \frac{u_0''}{u_0'} w \right\|_{L^2}^2 = \|w_x\|_{L^2}^2 + \int_{\mathbb{R}} \left[\left(1 - 3u_0^2\right) w^2 + \frac{2(u_0'')^2}{(u_0')^2} w^2 \right] \mathrm{d}x.$$

On the other hand, we have

$$\|w\|_{L^2}^2 = \int_{\mathbb{R}} \left[u_x^2 + \left(3u_0^2 - 1\right) u^2 \right] \mathrm{d}x,$$

and

$$\|u_0 w\|_{L^2}^2 = \int_{\mathbb{R}} \left[u_0^2 u_x^2 + \left(5u_0^2 - 3\right) u_0^2 u^2 \right] \mathrm{d}x.$$

Thus, using the analogue of (4.2) for all $c \in \mathbb{R}$, we find

$$\begin{aligned} \left\langle K_+(c) u, u \right\rangle_{L^2} &= \|L_+ u\|_{L^2}^2 + (2 - c) \int_{\mathbb{R}} \left[u_x^2 - u^2 + 3u_0^2 u^2 \right] \mathrm{d}x - \int_{\mathbb{R}} \left[u_0^2 u_x^2 - 3u_0^2 u^2 + 5u_0^4 u^2 \right] \mathrm{d}x \\ &= \|w_x\|_{L^2}^2 + (3 - c) \|w\|_{L^2}^2 + 2 \int_{\mathbb{R}} \left[\frac{(u_0'')^2}{(u_0')^2} - 2u_0^2 \right] w^2 \, \mathrm{d}x, \end{aligned}$$

which yields the desired result since $\frac{(u_0'')^2}{(u_0')^2} - 2u_0^2 = \mathcal{E}^2 \frac{u_0^2}{(u_0')^2}$ holds by Eqs. (1.2) and (1.6). □

Remark 4.4. If $c \le 3$, the right-hand side of the representation (4.3) is nonnegative and vanishes if and only if $w = 0$, which is equivalent to $u = C u_0'$ for some constant C. However, this does not imply positivity of the quadratic form associated with $K_+(c)$, because the representation (4.3) only holds for u in a subspace of $H^2(\mathbb{R})$. As a matter of fact, the right-hand side of the representation (4.3) is positive for any $c \le 3$, whereas we know from the proof of Proposition 2.2 that, when $\mathcal{E}$ is close to 1, the operator $K_+(c)$ is positive if and only if $c \in (c_-, c_+)$ where $c_\pm \to 2$ as $\mathcal{E} \to 1$.

For the black soliton (1.8) corresponding to the case $\mathcal{E} = 0$, the proof of Lemma 4.3 yields a much stronger conclusion, because u_0' never vanishes so that we do not need to impose any restriction to $u \in H^2(\mathbb{R})$. Using the identity $u_0'' = -\sqrt{2}u_0 u_0'$ which holds for the black soliton (1.8) only, we obtain the following result.

Corollary 4.5. *Consider the black soliton (1.8), for which $\mathcal{E} = 0$. For any $c \in \mathbb{R}$ and any $u \in H^2(\mathbb{R})$, we have*

$$\langle K_+(c)u, u\rangle_{L^2} = \|w_x\|^2_{L^2} + (3-c)\|w\|^2_{L^2}, \tag{4.4}$$

where $w = u_x + \sqrt{2}u_0 u \in H^1(\mathbb{R})$.

Remark 4.6. If $c \le 3$, the right-hand side of the representation (4.4) is nonnegative and vanishes if and only if $w = 0$, which is equivalent to $u = Cu_0'$ for some constant C. Note that $u_0' \in H^2(\mathbb{R})$ in the present case. On the other hand, using definitions (2.3) and the fact that $u_0(x) \to \pm 1$ as $x \to \pm\infty$, it is easy to verify that $K_+(c)$ has some negative essential spectrum as soon as $c > 3$. Thus the representation (4.4) gives a sharp positivity criterion for the operator $K_+(c)$ in the case of the black soliton (1.8).

5. Positivity of $\Lambda''_{c=2}(u_0)$ for periodic waves of large amplitude

This section presents the proof of Proposition 1.5.

The energy functionals (1.3) and (1.10) generate two different flows in the hierarchy of integrable NLS equations, see [4]. If we consider E and S as functions of the complex variables ψ and $\bar\psi$, these flows are defined by the evolution equations

$$i\frac{\partial \psi}{\partial t} = \frac{\delta E}{\delta \bar\psi}, \qquad i\frac{\partial \psi}{\partial \tau} = \frac{\delta S}{\delta \bar\psi}, \tag{5.1}$$

where the symbol δ is used to denote the standard variational derivative. Here t is the time of the cubic defocusing NLS equation (1.1), whereas τ is the time of the higher-order NLS equation. Since the quantities E and S are in involution, the flows defined by both equations in (5.1) commute with each other.

In what follows, we fix some $\mathcal{E} \in (0, 1)$ and consider the periodic wave profile u_0 defined by (1.7). Using the real-valued variables u, v for the perturbations, as in the representations (2.1) and (2.2), we obtain the following evolution equations for the linearized flows of the cubic NLS equation and the higher-order NLS equation at the periodic wave profile u_0:

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}, \qquad \frac{\partial}{\partial \tau}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & M_- \\ -M_+ & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}, \tag{5.2}$$

where the operators $L_\pm$ and $M_\pm$ are given by (2.3). Because the linearized flows also commute with each other, the operators $L_\pm$ and $M_\pm$ satisfy the following intertwining relations

$$L_- M_+ = M_- L_+, \qquad L_+ M_- = M_+ L_-. \tag{5.3}$$

Of course, the relations (5.3) can also be verified by a direct calculation, using the differential equations (1.2) and (1.6) satisfied by the periodic wave profile u_0. It follows from the relation (5.3) that, for every $c \in \mathbb{R}$, we have

$$L_- K_+(c) = K_-(c) L_+, \qquad L_+ K_-(c) = K_+(c) L_-, \tag{5.4}$$

where $K_\pm(c) = M_\pm - cL_\pm$ as before.

Given the positivity of the operator $K_-(2)$ established in Lemma 4.1, we shall use the intertwining relations (5.4) to deduce the positivity of the operator $K_+(2)$. This is achieved by studying all bounded solutions of the homogeneous equations associated with operators $L_\pm$ and $K_\pm(2)$ and by applying a continuation argument from the limit $\mathcal{E} \to 1$, where positivity of the operator $K_+(2)$ is proved in Proposition 2.2.

Lemma 5.1. *If $u \in L^\infty(\mathbb{R}) \cap H^2_{\text{loc}}(\mathbb{R})$ satisfies $L_+ u = 0$, then $u = Cu_0'$ for some constant C. Moreover, there exists a unique odd, $2T_0$-periodic function $U \in H^2_{\text{per,odd}}(0, 2T_0)$ such that $L_+ U = u_0$, where $2T_0$ is the period of u_0.*

Proof. We know that $L_+ u_0' = 0$. Another linearly independent solution to the equation $L_+ v = 0$ can be obtained by differentiating the periodic wave profile u_0 with respect to the parameter $\mathcal{E} \in (0, 1)$, namely $v = \partial_{\mathcal{E}} u_0$. Indeed, if we differentiate the equation

$$u_0'' + u_0 - u_0^3 = 0$$

with respect to the parameter $\mathcal{E}$, we see that

$$L_+ v = -v'' + \big(3u_0^2 - 1\big) v = 0.$$

Moreover, $v(x)$ is an odd function of x that grows linearly as $|x| \to \infty$. The latter claim can be verified by differentiating the explicit formula (1.7) with respect to $\mathcal{E}$, but that calculation is not immediate because it involves the derivative of the Jacobi elliptic function $\text{sn}(\xi, k)$ with respect to the parameter k. Alternatively, we can use Floquet theory to deduce that v is either periodic of period $2T_0$, where $2T_0$ is the minimal period of u_0, or grows linearly at infinity. The first possibility is excluded by the following argument. If we denote $u_0(x) = u_0(x; \mathcal{E})$ and $T_0 = T_0(\mathcal{E})$ to emphasize the dependence upon the parameter $\mathcal{E}$, we have by construction

$$u_0(0; \mathcal{E}) = u_0\big(2T_0(\mathcal{E}); \mathcal{E}\big) = 0.$$

Differentiating that relation with respect to $\mathcal{E}$, we find

$$v(0) = 0 \quad \text{and} \quad v(2T_0) + 2u_0'(2T_0) T_0'(\mathcal{E}) = 0.$$

But we know that $u_0'(2T_0) = u_0'(0) > 0$ and that $T_0'(\mathcal{E}) < 0$, hence we deduce that $v(2T_0) > 0$, which implies that v is not periodic of period $2T_0$. This proves that the kernel of L_+ (in the space of bounded functions) is spanned by u_0', which is the first part of the statement.

For the second part of the statement, we look for solutions of the inhomogeneous equation $L_+U = u_0$ and note that the Fredholm solvability condition $\langle u_0', u_0\rangle_{L^2_{\rm per}} = 0$ is trivially satisfied in the space of $2T_0$-periodic functions. Hence, there exists a unique odd $2T_0$-periodic solution U of the inhomogeneous equation $L_+U = u_0$ in the domain of L_+, that is, $U \in H^2_{\rm per,odd}(0, 2T_0)$. □

Lemma 5.2. *If $v \in L^\infty(\mathbb{R}) \cap H^2_{\rm loc}(\mathbb{R})$ satisfies $L_-v = 0$, then $v = Cu_0$ for some constant C. Moreover, there exists a unique even, $2T_0$-periodic function $V \in H^2_{\rm per,even}(0, 2T_0)$ such that $L_-V = u_0'$, where $2T_0$ is the period of u_0.*

Proof. We know that $L_-u_0 = 0$. Another linearly independent solution to the equation $L_-v = 0$ is given by

$$v(x) = 2u_0'(x) - u_0(x)\int_0^x u_0(y)^2\,dy, \quad x \in \mathbb{R},$$

as is easily verified by a direct calculation. Clearly $v(x)$ is an even function of x that grows linearly as $|x| \to \infty$. This proves that the kernel of L_- (in the space of bounded functions) is spanned by u_0. The second part of the statement follows by the same argument as in the proof of Lemma 5.1. □

Remark 5.3. The solutions U and V of the inhomogeneous equations $L_+U = u_0$ and $L_-V = u_0'$ can be expressed explicitly in terms of the Jacobi elliptic functions, see Eqs. (A.5) and (A.17) in Appendix A.

Next, we establish analogues of Lemmas 5.1 and 5.2 for the operators $K_\pm(c)$ in the particular case $c = 2$.

Lemma 5.4. *If $v \in L^\infty(\mathbb{R}) \cap H^4_{\rm loc}(\mathbb{R})$ satisfies $K_-(2)v = 0$, then $v = Cu_0$ for some constant C.*

Proof. Using integration by parts as in the proof of Lemma 4.1, we obtain the following identity for any $v \in H^4(-NT_0, NT_0)$, where $N \in \mathbb{N}$ and $2T_0$ is the period of u_0:

$$\int_{-NT_0}^{NT_0} vK_-(2)v\,dx = \int_{-NT_0}^{NT_0} \left(|L_-v|^2 + |u_0v_x - u_0'v|^2\right)dx - \left[2\left(1-u_0^2\right)vv_x + u_0u_0'v^2\right]\Big|_{x=-NT_0}^{x=NT_0}.$$

Assume now that $v \in L^\infty(\mathbb{R}) \cap H^4_{\rm loc}(\mathbb{R})$ satisfies $K_-(2)v = 0$. By standard elliptic estimates, we know that v is smooth on $\mathbb{R}$ and that all derivatives of v are bounded. Moreover, since the operator $K_-(2)$ has T_0-periodic coefficients, it follows from Floquet theory that $v(x) = e^{i\gamma x}w(x)$, where $\gamma \in \mathbb{R}$ and w is smooth on $\mathbb{R}$ and T_0-periodic. Using the identity above, we thus obtain

$$0 = \frac{1}{N}\int_{-NT_0}^{NT_0} \left(|L_- v|^2 + \left|u_0 v_x - u_0' v\right|^2\right) \mathrm{d}x - \frac{1}{N}\left[2\left(1-u_0^2\right) v v_x + u_0 u_0' v^2\right]\Big|_{x=-NT_0}^{x=NT_0}$$

$$= \int_{-T_0}^{T_0} \left(|L_- v|^2 + \left|u_0 v_x - u_0' v\right|^2\right) \mathrm{d}x - \frac{1}{N}\left[2\left(1-u_0^2\right) v v_x + u_0 u_0' v^2\right]\Big|_{x=-NT_0}^{x=NT_0}.$$

Taking the limit $N \to \infty$ and using the boundedness of v and v_x, we obtain $L_- v = 0$ and $u_0 v_x - u_0' v = 0$ for all $x \in \mathbb{R}$. By Lemma 5.2, we conclude that $v = C u_0$ for some constant C. $\square$

Lemma 5.5. *If $u \in L^\infty(\mathbb{R}) \cap H^4_{\mathrm{loc}}(\mathbb{R})$ satisfies $K_+(2)u = 0$, then $u = C u_0'$ for some constant C.*

Proof. Assume that $u \in L^\infty(\mathbb{R}) \cap H^4_{\mathrm{loc}}(\mathbb{R})$ satisfies $K_+(2)u = 0$. By the intertwining relation (5.4), we have $K_-(2) L_+ u = L_- K_+(2) u = 0$. Using Lemma 5.4, we deduce that $L_+ u = B u_0$ for some constant B. Finally, Lemma 5.1 implies that $u = BU + C u_0'$ for some constant C. In particular, we have $0 = K_+(2)u = B K_+(2) U$, because $K_+(2) u_0' = 0$. Now an explicit computation that is carried out in Appendix A shows that $K_+(2) U = D u_0$ for some constant $D \neq 0$, see Eq. (A.19), so that $K_+(2)U$ is not identically zero. Thus $B = 0$, hence $u = C u_0'$. $\square$

Remark 5.6. The result of Lemma 5.5 yields the conclusion of Proposition 1.5. Indeed, in the limit $\mathcal{E} \to 1$, positivity of the operator $K_+(2)$ is proved in Proposition 2.2. All Floquet–Bloch bands are strictly positive, except for the lowest band that touches the origin because of the zero eigenvalue due to translational symmetry, see Fig. 3. When the parameter $\mathcal{E}$ is decreased from 1 to 0, the Floquet–Bloch spectrum of $K_+(2)$ evolves continuously, and positivity of the spectrum is therefore preserved as long as no other band touches the origin. Such an event would result in the appearance of another bounded solution to the homogeneous equation $K_+(2)u = 0$, besides the zero mode u_0' due to translation invariance. By Lemma 5.5, such a solution does not exist, hence $K_+(2)$ is a nonnegative operator for any $\mathcal{E} \in (0, 1)$.

To conclude this section, we note that the intertwining relations (5.4) and the positivity of the operators $K_\pm(2)$ established in Proposition 1.5 imply the spectral stability of the periodic wave. Consider the linearized operator with T_0-periodic coefficients given by

$$\mathcal{J}\mathcal{L} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}, \tag{5.5}$$

and acting on vectors in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. We say that the periodic wave is spectrally stable if the Floquet–Bloch spectrum of $\mathcal{J}\mathcal{L}$ is purely imaginary. Let $\lambda \in \mathbb{C}$ belong to the Floquet–Bloch spectrum, so that $\mathcal{J}\mathcal{L}\psi = \lambda\psi$ for some nonzero eigenfunction ψ. We know that $\psi(x) = e^{i\gamma x}\tilde{\psi}(x)$, where $\gamma \in \mathbb{R}$ and $\tilde{\psi}$ is T_0-periodic. We want to show that $\lambda \in i\mathbb{R}$.

Let $\mathcal{K} := \mathrm{diag}[K_+(2), K_-(2)]$. Then $\mathcal{J}\mathcal{L}\mathcal{J}\mathcal{K}\psi = \mathcal{J}\mathcal{K}\mathcal{J}\mathcal{L}\psi = \lambda\mathcal{J}\mathcal{K}\psi$, because the operators $\mathcal{J}\mathcal{L}$ and $\mathcal{J}\mathcal{K}$ commute due to the intertwining relations (5.4). As $\mathcal{J}$ is invertible, we thus have $\mathcal{L}\mathcal{J}\mathcal{K}\psi = \lambda\mathcal{K}\psi$. If we now take the scalar product of both sides with the eigenfunction ψ in the space $L^2(0, T_0) \times L^2(0, T_0)$, we obtain

$$\lambda\langle\psi, \mathcal{K}\psi\rangle_{L^2} = \langle\psi, \mathcal{L}\mathcal{J}\mathcal{K}\psi\rangle_{L^2} = -\langle\mathcal{J}\mathcal{L}\psi, \mathcal{K}\psi\rangle_{L^2} = -\bar{\lambda}\langle\psi, \mathcal{K}\psi\rangle_{L^2},$$

where we have used the fact that $\mathcal{L}$ is self-adjoint and $\mathcal{J}$ is skew-adjoint. If $\lambda \neq 0$, then ψ is not a linear combination of the two neutral eigenfunctions $(u_0', 0)$ and $(0, u_0)$. In that case, we have $\langle \psi, \mathcal{K}\psi \rangle_{L^2} > 0$ by Proposition 1.5, and the identity above shows that $\lambda = -\bar{\lambda}$, that is, $\lambda \in i\mathbb{R}$.

Remark 5.7. Spectral stability of the periodic wave is established in [4], where explicit expressions for the Floquet–Bloch spectrum of the operator $\mathcal{JL}$ and the associated eigenfunctions are obtained using Jacobi elliptic functions. In our approach, once positivity of the operator $\mathcal{K}$ is known, the spectral stability of the periodic wave follows from the commutativity of the operators $\mathcal{JL}$ and $\mathcal{JK}$ and is established by a general argument that does not use the specific form of the eigenfunctions.

6. Proof of orbital stability of a periodic wave

This section is devoted to the proof of Theorem 1.8.

We fix $\mathcal{E} \in (0, 1)$ and consider the periodic wave profile u_0 given by (1.7). Let T be a multiple of the period $2T_0$ of u_0, so that $T = 2NT_0$ for some integer $N \geq 1$. If $\psi_0 \in H^2_{\text{per}}(0, T)$ is close to u_0 in the sense of the initial bound (1.14), we claim that the solution $\psi \in C(\mathbb{R}, H^2_{\text{per}}(0, T))$ of the cubic NLS equation (1.1) with initial data ψ_0 can be characterized as follows.

For any $t \in \mathbb{R}$, there exist modulation parameters $\xi(t) \in \mathbb{R}$ and $\theta(t) \in \mathbb{R}/(2\pi\mathbb{Z})$ such that

$$e^{it+i\theta(t)}\psi\big(x + \xi(t), t\big) = u_0(x) + u(x, t) + iv(x, t), \quad x \in \mathbb{R}, \tag{6.1}$$

where $u(\cdot, t), v(\cdot, t) \in H^2_{\text{per}}(0, T)$ are real-valued functions satisfying the orthogonality conditions

$$\big\langle u_0', u(\cdot, t)\big\rangle_{L^2_{\text{per}}} = 0, \qquad \big\langle u_0, v(\cdot, t)\big\rangle_{L^2_{\text{per}}} = 0, \tag{6.2}$$

where $\langle \cdot, \cdot \rangle_{L^2_{\text{per}}}$ denotes the usual scalar product in $L^2_{\text{per}}(0, T)$. Note that the orthogonality conditions (6.2) are not symplectic orthogonality conditions for the NLS equation, in contrast with the conditions that are often used to study the asymptotic stability of nonlinear waves [12].

To prove the decomposition (6.1), we proceed in two steps. We first show that the representation (6.1) holds whenever $\psi(\cdot, t)$ is sufficiently close to the orbit of u_0 under translations and phase rotations.

Lemma 6.1. *There exists constants $\epsilon_0 \in (0, 1)$ and $C_0 \geq 1$ such that, for any $\psi \in H^2_{\text{per}}(0, T)$ satisfying*

$$d := \inf_{\xi, \theta \in \mathbb{R}} \big\| e^{i\theta}\psi(\cdot + \xi) - u_0 \big\|_{H^2_{\text{per}}} \leq \epsilon_0, \tag{6.3}$$

one can find modulation parameters $\xi \in \mathbb{R}$ and $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ such that

$$e^{i\theta}\psi(x + \xi) = u_0(x) + u(x) + iv(x), \quad x \in \mathbb{R}, \tag{6.4}$$

where $u, v \in H^2_{\text{per}}(0, T)$ satisfy the orthogonality conditions (6.2) and $d \leq \|u + iv\|_{H^2_{\text{per}}} \leq C_0 d$.

Proof. We consider the smooth function $\mathbf{f}: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{f}(\xi,\theta) = \begin{bmatrix} \langle u_0'(\cdot - \xi), \mathrm{Re}(e^{i\theta}\psi)\rangle_{L^2_{\mathrm{per}}} \\ \langle u_0(\cdot - \xi), \mathrm{Im}(e^{i\theta}\psi)\rangle_{L^2_{\mathrm{per}}} \end{bmatrix}.$$

We have $\mathbf{f}(\xi,\theta) = \mathbf{0}$ if and only if ψ can be represented as in the decomposition (6.4) with u, v satisfying the orthogonality conditions (6.2). Let $(\xi_0,\theta_0) \in \mathbb{R}^2$ denote the arguments of the infimum in (6.3) (note that one can restrict the values of (ξ,θ) to $[0,T] \times [0,2\pi]$, so that the minimum exists). Then assumption (6.3) implies that $\|\mathbf{f}(\xi_0,\theta_0)\| \le Cd$, for some constant C independent of ψ. On the other hand, the Jacobian matrix of the function $\mathbf{f}$ at the point (ξ_0,θ_0) is given by

$$D\mathbf{f}(\xi_0,\theta_0) = \begin{bmatrix} \|u_0'\|^2_{L^2_{\mathrm{per}}} & 0 \\ 0 & \|u_0\|^2_{L^2_{\mathrm{per}}} \end{bmatrix} + \begin{bmatrix} -\langle u_0'', \mathrm{Re}(e^{i\theta_0}\psi(\cdot+\xi_0) - u_0)\rangle_{L^2_{\mathrm{per}}} & -\langle u_0', \mathrm{Im}(e^{i\theta_0}\psi(\cdot+\xi_0) - u_0)\rangle_{L^2_{\mathrm{per}}} \\ -\langle u_0', \mathrm{Im}(e^{i\theta_0}\psi(\cdot+\xi_0) - u_0)\rangle_{L^2_{\mathrm{per}}} & \langle u_0, \mathrm{Re}(e^{i\theta_0}\psi(\cdot+\xi_0) - u_0)\rangle_{L^2_{\mathrm{per}}} \end{bmatrix}.$$

The first term in the right-hand side is a fixed invertible matrix and the second term is bounded in norm by Cd, hence $D\mathbf{f}(\xi_0,\theta_0)$ is invertible if ϵ_0 is small enough, with $\|(D\mathbf{f}(\xi_0,\theta_0))^{-1}\| \le C$ where C is a positive constant independent of ψ. Finally, it is straightforward to verify that the second order derivative of $\mathbf{f}$ is uniformly bounded if $\epsilon_0 < 1$. These observations together imply that there exists a unique pair (ξ,θ), in the $\mathcal{O}(d)$ neighborhood of the point (ξ_0,θ_0), such that $\mathbf{f}(\xi,\theta) = \mathbf{0}$. Thus, we have the decomposition (6.4) with these values of (ξ,θ), and

$$\begin{aligned} \|u+iv\|_{H^2_{\mathrm{per}}} &= \left\|e^{i\theta}\psi(\cdot+\xi) - u_0\right\|_{H^2_{\mathrm{per}}} = \left\|\psi - e^{-i\theta}u_0(\cdot-\xi)\right\|_{H^2_{\mathrm{per}}} \\ &\le \left\|\psi - e^{-i\theta_0}u_0(\cdot-\xi_0)\right\|_{H^2_{\mathrm{per}}} + \left\|e^{-i\theta_0}u_0(\cdot-\xi_0) - e^{-i\theta}u_0(\cdot-\xi)\right\|_{H^2_{\mathrm{per}}} \\ &\le C_0 d, \end{aligned}$$

where $C_0 \ge 1$ is independent of ψ. This concludes the proof. □

We next show that the solution $\psi(\cdot,t)$ of the cubic NLS equation (1.1) stays close to the orbit of u_0 for all times. To show this, we use the conserved quantity Λ_c given by (1.11), where it is understood that the integration domain $\mathcal{I} = (0,T)$ is used in the definitions of all functionals (1.3), (1.4), and (1.5). Because positivity of the second variation of Λ_c is only proved for $c = 2$ independently of the parameter $\mathcal{E}$, see Proposition 1.5, we assume henceforth that $c = 2$.

Lemma 6.2. *Assume that ψ is given by (6.4) for some $(\xi,\theta) \in \mathbb{R}^2$ and some real-valued functions $u, v \in H^2_{\mathrm{per}}(0,T)$ satisfying the orthogonality conditions (6.2). There exist positive constants C_1, C_2, and ϵ_1 such that, if $\|u+iv\|_{H^2_{\mathrm{per}}} \le \epsilon_1$, then*

$$C_1\|u+iv\|^2_{H^2_{\mathrm{per}}} \le \Lambda_{c=2}(\psi) - \Lambda_{c=2}(u_0) \le C_2\|u+iv\|^2_{H^2_{\mathrm{per}}}. \tag{6.5}$$

Proof. We first note that the functional Λ_c is invariant under translations and phase rotations in $H^2_{\rm per}(0,T)$, so that $\Lambda_c(\psi) = \Lambda_c(u_0 + u + iv)$ if ψ satisfies the representation (6.4). Therefore, recalling that u_0 is a critical point of Λ_c and using the same notations as in Section 2, we find

$$\Lambda_c(\psi) - \Lambda_c(u_0) = \langle K_+(c)u, u\rangle_{L^2_{\rm per}} + \langle K_-(c)v, v\rangle_{L^2_{\rm per}} + N_c(u,v), \tag{6.6}$$

where $N_c(u,v)$ collects all terms that are at least cubic in (u,v). In particular, there exists a constant $C > 0$ such that, if $\|u + iv\|_{H^2_{\rm per}} \le \epsilon_1$, we have the estimate

$$\big|N_c(u,v)\big| \le C\|u + iv\|^3_{H^2_{\rm per}}. \tag{6.7}$$

The upper bound in (6.5) holds from the expressions (2.1) and (2.2) for the quadratic part, the estimate (6.7) for the cubic and quartic parts, and the decomposition (6.6).

To bound the expression (6.6) from below, we use the spectral properties of the operators $K_\pm(c)$ established in Sections 2, 4, and 5.

For periodic waves of small amplitude and for c in the interval (c_-, c_+), we know from Propositions 1.1 and 2.2 that the spectrum of $K_\pm(c)$ in $L^2(\mathbb{R})$ is the union of the nonnegative Floquet–Bloch spectral bands. If $K_\pm(c)$ are considered as operators in $L^2_{\rm per}(0,T)$ with $T = 2NT_0$, the same result holds except that the Floquet parameter only takes discrete values. In view of the bounds (2.14), this discretization of the Floquet–Bloch spectral bands implies that both $K_+(c)$ and $K_-(c)$ have exactly one zero eigenvalue, and that the rest of the spectrum is positive and bounded away from zero. As was already observed, the kernels of $K_\pm(c)$ are due to the symmetries of the NLS equation, and we have the explicit formulas (2.6) for the eigenvectors. Thus, the orthogonality conditions (6.2) mean precisely that u is orthogonal in $L^2_{\rm per}(0,T)$ to the kernel of $K_+(c)$ and v to the kernel of $K_-(c)$.

Although the results of Propositions 1.1 and 2.2 hold for periodic waves of small amplitude where $\mathcal{E}$ is close to one, Proposition 1.5 implies that the same result holds for periodic waves of arbitrary amplitude independently of the parameter $\mathcal{E} \in (0,1)$ in the case $c = 2$. It then follows that there is a positive constant C such that

$$\langle K_+(2)u, u\rangle_{L^2_{\rm per}} \ge C\|u\|^2_{L^2_{\rm per}} \quad \text{and} \quad \langle K_-(2)v, v\rangle_{L^2_{\rm per}} \ge C\|v\|^2_{L^2_{\rm per}}.$$

Using in addition Gårding's inequality for the elliptic operators $K_\pm(c)$ we conclude that

$$\langle K_+(2)u, u\rangle_{L^2_{\rm per}} \ge C\|u\|^2_{H^2_{\rm per}}, \qquad \langle K_-(2)v, v\rangle_{L^2_{\rm per}} \ge C\|v\|^2_{H^2_{\rm per}}, \tag{6.8}$$

with a possibly smaller constant C. The lower bound in (6.5) is a direct consequence of (6.6), (6.7), and (6.8). □

Without loss of generality, we assume from now on that $C_0\epsilon_0 \le \epsilon_1$, where C_0, ϵ_0, and ϵ_1 are as in the previous lemmas. It then follows from Lemmas 6.1 and 6.2 that, if $\psi \in H^2_{\rm per}(0,T)$ is close to the orbit of u_0 in the sense of the bound (6.3), then

$$C_1 d^2 \le \Lambda_{c=2}(\psi) - \Lambda_{c=2}(u_0) \le C_2 C_0^2 d^2. \tag{6.9}$$

With this estimate at hand, it is now easy to prove that the decomposition (6.1) with the orthogonality conditions (6.2) holds for all $t \in \mathbb{R}$ if $\psi(\cdot, t)$ is the solution of the cubic NLS equation (1.1) with initial data $\psi_0 \in H^2_{\rm per}(0, T)$ satisfying the initial bound (1.14), where $\delta > 0$ is small enough so that

$$C_0 (C_2/C_1)^{1/2} \delta < \epsilon_0. \tag{6.10}$$

Indeed, let $d(t)$ be the distance in $H^2_{\rm per}(0, T)$ from $\psi(\cdot, t)$ to the orbit of u_0, in the sense of (6.3). Initially we have $d(0) \le \delta < \epsilon_0$ by (1.14) and (6.10). Let $\mathcal{J} \subset \mathbb{R}$ be the largest time interval containing the origin such that $d(t) \le \epsilon_0$ for all $t \in \mathcal{J}$. As $d(t)$ is a continuous function of time, it is clear that $\mathcal{J}$ is closed. On the other hand, for any $t \in \mathcal{J}$, we have by (6.9)

$$C_1 d(t)^2 \le \Lambda_{c=2}\big(\psi(\cdot, t)\big) - \Lambda_{c=2}(u_0) = \Lambda_{c=2}(\psi_0) - \Lambda_{c=2}(u_0) \le C_2 C_0^2 \delta^2,$$

where we have used the crucial fact that Λ_c is conserved under the evolution defined by the cubic NLS equation (1.1) in $H^2_{\rm per}(0, T)$. Thus $d(t) \le C_0 (C_2/C_1)^{1/2} \delta < \epsilon_0$, hence by continuity the interval $\mathcal{J}$ contains a neighborhood of t. So $\mathcal{J}$ is open, hence finally $\mathcal{J} = \mathbb{R}$. This shows that the decomposition (6.1) holds for all $t \in \mathbb{R}$ with real-valued functions $u(\cdot, t), v(\cdot, t) \in H^2_{\rm per}(0, T)$ satisfying the orthogonality conditions (6.2) as well as the uniform bound

$$\big\| u(\cdot, t) + i v(\cdot, t) \big\|_{H^2_{\rm per}} \le C_0 d(t) \le C_0^2 (C_2/C_1)^{1/2} \delta, \quad t \in \mathbb{R}.$$

This yields the bound (1.15) with $\epsilon = C_0^2 (C_2/C_1)^{1/2} \delta$. To conclude the proof of Theorem 1.8, it remains to show that the modulation parameters ξ and θ are continuously differentiable functions of time t and satisfy the bound (1.16).

Lemma 6.3. *Assume that the solution $\psi(\cdot, t)$ of the cubic NLS equation (1.1) satisfies $d(t) \le \epsilon \le \epsilon_1$ for all $t \in \mathbb{R}$, where $d(t)$ denotes as in (6.3) the distance to the orbit of u_0. Then the modulation parameters $\xi(t)$, $\theta(t)$ given by Lemma 6.1 are continuously differentiable functions of t satisfying (1.16).*

Proof. As $\psi \in C(\mathbb{R}, H^2_{\rm per}(0, T))$, the proof of Lemma 6.1 shows that $\xi(t)$ and $\theta(t)$ depend continuously on t. To prove differentiability, we first consider more regular solutions with initial data $\psi_0 \in H^4_{\rm per}(0, T)$, and then recover the general case by a density argument. For regular solutions, we can differentiate both sides of the decomposition (6.1) and use the cubic NLS equation (1.1) to obtain the evolution system

$$\begin{cases} u_t = L_- v + \dot{\xi}\big(u_0' + u_x\big) - \dot{\theta} v + \big(2u_0 u + u^2 + v^2\big) v, \\ -v_t = L_+ u - \dot{\xi} v_x - \dot{\theta}(u_0 + u) + \big(3u_0 u + u^2 + v^2\big) u + u_0 v^2, \end{cases}$$

where the operators $L_\pm$ are defined in (2.3). Using the orthogonality conditions (6.2), we eliminate the time derivatives u_t, v_t by taking the scalar product of the first line with u_0' and of the second line with u_0. This gives the following linear system for the derivatives $\dot{\xi}$ and $\dot{\theta}$:

$$B \begin{bmatrix} \dot{\xi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \langle u_0', L_- v \rangle_{L^2_{\rm per}} \\ \langle u_0, L_+ u \rangle_{L^2_{\rm per}} \end{bmatrix} + \begin{bmatrix} \langle u_0', (2u_0 u + u^2 + v^2) v \rangle_{L^2_{\rm per}} \\ \langle u_0, (3u_0 u + u^2 + v^2) u + u_0 v^2 \rangle_{L^2_{\rm per}} \end{bmatrix}, \tag{6.11}$$

where

$$B = \begin{bmatrix} -\|u_0'\|^2_{L^2_{\text{per}}} & 0 \\ 0 & \|u_0\|^2_{L^2_{\text{per}}} \end{bmatrix} + \begin{bmatrix} -\langle u_0', u_x\rangle_{L^2_{\text{per}}} & \langle u_0', v\rangle_{L^2_{\text{per}}} \\ \langle u_0, v_x\rangle_{L^2_{\text{per}}} & \langle u_0, u\rangle_{L^2_{\text{per}}} \end{bmatrix}. \tag{6.12}$$

Since $\|u(\cdot,t) + iv(\cdot,t)\|_{H^2_{\text{per}}} \le C_0 d(t) \le C_0\epsilon$ for all $t \in \mathbb{R}$, the second term in the right-hand side of (6.12) is of size $\mathcal{O}(\epsilon)$, hence the matrix B is invertible if ϵ is small enough. Inverting B in (6.11), we obtain a formula for the derivatives $\dot\xi, \dot\theta$ where the right-hand side is a continuous function of time under the mere assumption that $\psi \in C(\mathbb{R}, H^2_{\text{per}}(0,T))$. By a classical density argument, we conclude that ξ, θ are differentiable in the general case, and that their derivatives are given by (6.11). Finally, the first term in the right-hand side of (6.11) is of size $\mathcal{O}(\epsilon)$, whereas the second term is $\mathcal{O}(\epsilon^2)$, hence $|\dot\xi(t)| + |\dot\theta(t)| \le C\epsilon$ for all $t \in \mathbb{R}$, where the positive constant C is independent of t. □

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Appendix A. Explicit expressions involving Jacobi elliptic functions

In this appendix, we derive explicit formulas the generalized eigenvectors of the linearized operators in (5.2) by using Jacobi elliptic functions. In particular, we show how to compute the explicit expression (3.4).

Fix $\mathcal{E} \in (0,1)$ and let $k \in (0,1)$ be given by (1.13). The periodic wave profile u_0 defined in (1.7) can be rewritten in the explicit form

$$u_0(x) = \sqrt{\frac{2k^2}{1+k^2}}\,\text{sn}\left(\frac{x}{\sqrt{1+k^2}}, k\right) = \sqrt{\frac{2k^2}{1+k^2}}\,\jmath\left(\frac{x}{\sqrt{1+k^2}}\right), \quad x \in \mathbb{R},$$

where $\jmath(\xi) = \text{sn}(\xi, k)$ denotes the Jacobi elliptic function. To simplify the calculations below, it is convenient to use the space variable $\xi = x/\sqrt{1+k^2}$ instead of x.

Let us recall a few properties of the Jacobi elliptic functions $\text{sn}(\xi,k)$, $\text{cn}(\xi,k)$, and $\text{dn}(\xi,k)$ [10]. The functions $\text{sn}(\xi,k)$ and $\text{cn}(\xi,k)$ are periodic with period $T = 4K(k)$, where $K(k)$ denotes the complete elliptic integral of the first kind. On the other hand, the function $\text{dn}(\xi,k) = \sqrt{1 - k^2\,\text{sn}(\xi,k)^2}$ is periodic with period $2K(k)$.

We have the following expressions for the first-order derivatives of the Jacobi elliptic functions:

$$\frac{d}{d\xi}\begin{bmatrix} \text{sn}(\xi,k) \\ \text{cn}(\xi,k) \\ \text{dn}(\xi,k) \end{bmatrix} = \begin{bmatrix} \text{cn}(\xi,k)\,\text{dn}(\xi,k), \\ -\,\text{sn}(\xi,k)\,\text{dn}(\xi,k), \\ -k^2\,\text{sn}(\xi,k)\,\text{cn}(\xi,k). \end{bmatrix} \tag{A.1}$$

In particular, the function $\jmath(\xi) = \mathrm{sn}(\xi, k)$ satisfies the differential equation

$$\frac{d^2 \jmath}{d\xi^2} = -\left(1+k^2\right)\jmath + 2k^2 \jmath^3. \tag{A.2}$$

Let us also introduce the incomplete elliptic integral of the second kind

$$E(\xi, k) = \int_0^{\xi} \mathrm{dn}^2(y, k)\,\mathrm{d}y, \quad \xi \in \mathbb{R}. \tag{A.3}$$

This function is not periodic and we have the relation

$$E\big(\xi + 2K(k), k\big) = E(\xi, k) + 2E(k) \quad \text{for all } \xi \in \mathbb{R},$$

where $E(k) := \frac{1}{2}E(2K(k), k)$ is the complete elliptic integral of the second kind. This means that the function $\xi \mapsto E(\xi, k)$ is linearly growing at infinity with asymptotic rate $E(k)/K(k)$.

Using the chain rule for the operator $L_- = -\partial_x^2 + u_0^2(x) - 1$, we obtain $\mathcal{L}_- = (1+k^2)L_-$, where

$$\mathcal{L}_- = -\partial_\xi^2 - \left(1+k^2\right) + 2k^2 \jmath(\xi)^2.$$

Recall that $\mathcal{L}_- \jmath = 0$. Using the relations (A.1)–(A.3), it is easy to verify that

$$\mathcal{L}_-\big(\mathrm{cn}(\xi,k)\,\mathrm{dn}(\xi,k)\big) = -4k^2\,\mathrm{cn}(\xi,k)\,\mathrm{dn}(\xi,k)\,\mathrm{sn}^2(\xi,k),$$
$$\mathcal{L}_-\big(\mathrm{sn}(\xi,k)E(\xi,k)\big) = -2\,\mathrm{cn}(\xi,k)\,\mathrm{dn}(\xi,k)\big(1 - 2k^2\,\mathrm{sn}^2(\xi,k)\big),$$
$$\mathcal{L}_-\big(\xi\,\mathrm{sn}(\xi,k)\big) = -2\,\mathrm{cn}(\xi,k)\,\mathrm{dn}(\xi,k).$$

Therefore, the function

$$V(\xi) := \mathrm{cn}(\xi,k)\,\mathrm{dn}(\xi,k) + \mathrm{sn}(\xi,k)\left[E(\xi,k) - \frac{E(k)}{K(k)}\xi\right], \tag{A.4}$$

is periodic with period $T = 4K(k)$ and satisfies the inhomogeneous equation

$$\mathcal{L}_- V = -2\left(1 - \frac{E(k)}{K(k)}\right)\mathrm{cn}(\xi,k)\,\mathrm{dn}(\xi,k) = -2\left(1 - \frac{E(k)}{K(k)}\right)\jmath'. \tag{A.5}$$

Note that the numerical coefficient in (A.5) is nonzero because $K(k) > E(k)$ for all $k \in (0,1)$.

Using the chain rule for the operator $M_- = \partial_x^4 - 3\partial_x u_0^2 \partial_x + u_0^2 - 1$, we obtain $\mathcal{M}_- = (1+k^2)^2 M_-$, where

$$\mathcal{M}_- = \partial_\xi^4 - 6k^2 \partial_\xi \jmath(\xi)^2 \partial_\xi + 2k^2\left(1+k^2\right)\jmath(\xi)^2 - \left(1+k^2\right)^2.$$

A long but direct calculation using (A.1) shows that the same function V in (A.4) also satisfies

$$\mathcal{M}_- V = 4\left[k^2 - \left(1 - \frac{E(k)}{K(k)}\right)\left(1+k^2\right)\right]\jmath'. \tag{A.6}$$

Recall that $K_-(c) = M_- - cL_-$. Combining (A.5) and (A.6) and using the chain rule, we obtain

$$\big(\mathcal{M}_- - c(1+k^2)\mathcal{L}_-\big)V = \left[4k^2 + 2(c-2)(1+k^2)\left(1 - \frac{E(k)}{K(k)}\right)\right]\jmath'. \tag{A.7}$$

Note that $\jmath$ and V are orthogonal with respect to the scalar product $\langle\cdot,\cdot\rangle_{L^2_{\rm per}}$ in $L^2_{\rm per}(-2K(k), 2K(k))$ because V is even and $\jmath$ is odd.

Remark A.1. The fact that both quantities $\mathcal{L}_-V$ and $\mathcal{M}_-V$ are proportional to the same function $\jmath'$ is not an accident. Associated with the neutral mode $(u_0', 0)$, we have $L_-v = u_0'$ arising in the solutions of the linearized evolution operator at u_0:

$$\begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}\begin{bmatrix} u_0' \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}\begin{bmatrix} 0 \\ v \end{bmatrix} = \begin{bmatrix} u_0' \\ 0 \end{bmatrix},$$

hence $(0, v)$ is the generalized neutral mode. Now the higher-order operators $M_\pm$ are associated with the linearization of another flow in the hierarchy of the integrable NLS equation, which commutes with the original flow of (1.1), see Section 5. As is easily verified, this implies that the same function v satisfies $M_-v = Au_0'$ for some constant $A \in \mathbb{R}$, in agreement with (A.5) and (A.6) after the scaling transformation from x to ξ.

We can now obtain the explicit expression (3.4) from the formula (3.1). Recall that $z = \ell x$ and $U = u_0(\ell^{-1}\cdot)$. If $W = w(\ell^{-1}\cdot)$ satisfies $P_-(c, 0)W = U'$, then

$$\big(P_-(c,0)W\big)(z) = \big(1+k^2\big)^{-2}\big(\mathcal{M}_- - c\big(1+k^2\big)\mathcal{L}_-\big)w(\xi), \quad \xi = \frac{z}{\ell\sqrt{1+k^2}}.$$

Using the chain rule, we rewrite the formula (3.1) in the equivalent form

$$\begin{aligned}\mu''(0) = \frac{2\ell^2}{\|\jmath\|^2_{L^2_{\rm per}}}\Big[&-4(c-2)^2\big(1+k^2\big)\big\langle \jmath', \big(\mathcal{M}_- - c\big(1+k^2\big)\mathcal{L}_-\big)^{-1}\jmath'\big\rangle_{L^2_{\rm per}} \\ &+ 3\big(1+k^2\big)^{-1}\|\jmath'\|^2_{L^2_{\rm per}} + (3-c)\|\jmath\|^2_{L^2_{\rm per}}\Big].\end{aligned} \tag{A.8}$$

It follows from (A.7) that

$$\big(\mathcal{M}_- - c\big(1+k^2\big)\mathcal{L}_-\big)^{-1}\jmath' = \frac{V}{4k^2 + 2(c-2)(1+k^2)(1 - \frac{E(k)}{K(k)})}. \tag{A.9}$$

It remains to compute the norms and the scalar products in the right-hand side of Eq. (A.8). Using the notations above, we find for all $k \in (0, 1)$,

$$\|\jmath\|^2_{L^2_{\rm per}} = \frac{4K(k)}{k^2}\left[1 - \frac{E(k)}{K(k)}\right] > 0, \tag{A.10}$$

$$\|\jmath'\|^2_{L^2_{\rm per}} = \frac{4K(k)}{3k^2}\left[k^2 - 1 + \big(k^2+1\big)\frac{E(k)}{K(k)}\right] > 0, \tag{A.11}$$

and

$$\langle \jmath', V\rangle_{L^2_{\mathrm{per}}} = \frac{2K(k)}{k^2}\left[k^2 - 1 + 2\frac{E(k)}{K(k)} - \frac{E(k)^2}{K(k)^2}\right]. \tag{A.12}$$

Substituting these expressions into (A.8) and finding a common denominator for all terms, we obtain the expression (3.4).

Next, using the chain rule for the operator $L_+ = -\partial_x^2 + 3u_0^2(x) - 1$, we obtain $\mathcal{L}_+ = (1+k^2)L_+$, where

$$\mathcal{L}_+ = -\partial_\xi^2 - \left(1+k^2\right) + 6k^2 \jmath(\xi)^2.$$

Recall that $\mathcal{L}_+\jmath' = 0$. Using the relations (A.1)–(A.3), it is easy to verify that

$$\mathcal{L}_+\big(\mathrm{sn}(\xi,k)\big) = 4k^2\,\mathrm{sn}^3(\xi,k),$$

$$\mathcal{L}_+\left(\mathrm{cn}(\xi,k)\,\mathrm{dn}(\xi,k)\int_0^\xi \frac{\mathrm{sn}^2(y,k)}{\mathrm{dn}^2(y,k)}\,\mathrm{d}y\right) = -2\,\mathrm{sn}(\xi,k)\big(1 - 2\,\mathrm{sn}^2(\xi,k)\big),$$

$$\mathcal{L}_+\big(\xi\,\mathrm{cn}(\xi,k)\,\mathrm{dn}(\xi,k)\big) = 2\,\mathrm{sn}(\xi,k)\big(1 + k^2 - 2k^2\,\mathrm{sn}^2(\xi,k)\big).$$

Therefore, the function

$$\begin{aligned} U(\xi) := {} & \left(1-k^2\right)\left(1+bk^2\right)\mathrm{sn}(\xi,k) \\ & - k^2\left(1-k^2\right)\mathrm{cn}(\xi,k)\,\mathrm{dn}(\xi,k)\left[\int_0^\xi \frac{\mathrm{sn}^2(y,k)}{\mathrm{dn}^2(y,k)}\,\mathrm{d}y - b\xi\right], \end{aligned} \tag{A.13}$$

satisfies the inhomogeneous equation

$$\mathcal{L}_+U = 2k^2\left(1-k^2\right)\left(1+b\left(1+k^2\right)\right)\mathrm{sn}(\xi,k) = 2k^2\left(1-k^2\right)\left(1+b\left(1+k^2\right)\right)\jmath, \tag{A.14}$$

for an arbitrary coefficient $b \in \mathbb{R}$.

We shall find the value of b from the condition that U is periodic with period $T = 4K(k)$. To do so, we recall the identity (see 16.26.6 in [1]):

$$\left(1-k^2\right)\int_0^\xi \frac{\mathrm{d}y}{\mathrm{dn}^2(y,k)} = E(\xi,k) - k^2\frac{\mathrm{sn}(\xi,k)\,\mathrm{cn}(\xi,k)}{\mathrm{dn}(\xi,k)}.$$

Using this identity, we rewrite the function U given by (A.13) in the equivalent form

$$\begin{aligned} U(\xi) = {} & \left(1-k^2\right)\left(1+bk^2\right)\mathrm{sn}(\xi,k) + k^2\,\mathrm{sn}(\xi,k)\,\mathrm{cn}^2(\xi,k) \\ & - \mathrm{cn}(\xi,k)\,\mathrm{dn}(\xi,k)\big[E(\xi,k) - \left(1-k^2\right)\left(1+bk^2\right)\xi\big], \end{aligned} \tag{A.15}$$

which is periodic if and only if $(1-k^2)(1+bk^2) = \frac{E(k)}{K(k)}$. Substituting this expression into (A.15), we finally obtain the $4K(k)$-periodic solution

$$U(\xi) = \frac{E(k)}{K(k)} \operatorname{sn}(\xi,k) + k^2 \operatorname{sn}(\xi,k) \operatorname{cn}^2(\xi,k) - \operatorname{cn}(\xi,k) \operatorname{dn}(\xi,k) \left[E(\xi,k) - \frac{E(k)}{K(k)} \xi \right] \tag{A.16}$$

of the inhomogeneous equation

$$\mathcal{L}_+ U = 2\left(k^2 - 1 + (1+k^2) \frac{E(k)}{K(k)} \right) \jmath. \tag{A.17}$$

Note that the numerical coefficient in (A.17) is nonzero for every $k \in (0,1)$, thanks to (A.11).

Using the chain rule for the operator $M_+ = \partial_x^4 - 5\partial_x u_0^2 \partial_x - 5u_0^4 + 15u_0^2 - 4 + 3\mathcal{E}^2$, we obtain $\mathcal{M}_+ = (1+k^2)^2 M_+$, where

$$\mathcal{M}_+ = \partial_\xi^4 - 10k^2 \partial_\xi \jmath(\xi)^2 \partial_\xi - 20k^4 \jmath(\xi)^4 + 30k^2(1+k^2)\jmath(\xi)^2 - (1 + 14k^2 + k^4).$$

After a long but direct calculation, we obtain that the same function U in (A.16) also satisfies

$$\mathcal{M}_+ U = 4\left[2k^4 - k^2 - 1 + (1 + 4k^2 + k^4) \frac{E(k)}{K(k)} \right] \jmath. \tag{A.18}$$

Combining (A.17) and (A.18) into $K_+(c) = M_+ - cL_+$ for $c = 2$ and using the chain rule, we obtain

$$\left(\mathcal{M}_+ - 2(1+k^2)\mathcal{L}_+ \right) U = 4k^2 \left[k^2 - 1 + \frac{2E(k)}{K(k)} \right] \jmath. \tag{A.19}$$

Since $2 > 1 + k^2$, the numerical coefficient in front of $\jmath$ is positive for all $k \in (0,1)$, thanks to (A.11).

Remark A.2. Again, we observe that both quantities $\mathcal{L}_+ U$ and $\mathcal{M}_+ U$ are proportional to the same function $\jmath$. This is due to the generalized neutral mode $(u, 0)$ associated with the neutral mode $(0, u_0)$, which arise in the solution of $L_+ u = u_0$. See also Remark A.1.

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