Distribution of eigenfrequencies for oscillations of the ground state in the Thomas-Fermi limit

P. G. Kevrekidis[1] and D. E. Pelinovsky[2]

[1]*Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts 01003, USA*
[2]*Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1*

In this work, we present a systematic derivation of the distribution of eigenfrequencies for oscillations of the ground state of a repulsive Bose-Einstein condensate in the semi-classical (Thomas-Fermi) limit. Our calculations are performed in one, two, and three-dimensional settings. Connections with the earlier work of Stringari, with numerical computations, and with theoretical expectations for invariant frequencies based on symmetry principles are also given.

I. INTRODUCTION

The Bose-Einstein condensation is one of the most exciting achievements within the physics community in the last two decades. Its experimental realization in 1995 by two experimental groups using vapors of Rb [1] and Na [2] marked the formation of a new state of matter consisting of a cloud of atoms within the same quantum state, creating a giant matter wave. However, in addition to its impact on the physical side, this development had a significant influence on mathematical studies of such Bose-Einstein condensates (BEC's) [3–7]. In considering typical BEC experiments and in exploring the unprecedented control of the condensates through magnetic and optical "knobs," a mean-field theory is applied to reduce the quantum many-atom description to a scalar nonlinear Gross-Pitaevskii equation (GPE). This is a variant of the famous nonlinear Schrödinger (NLS) equation [8,9] of the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi + g|\psi|^2\psi + V_{\text{ext}}(\mathbf{r})\psi, \qquad (1)$$

where $\psi = \psi(\mathbf{r}, t)$ is the BEC wave function (the atomic density is proportional to $|\psi(\mathbf{r}, t)|^2$), ∇^2 is the Laplacian in $\mathbf{r} = (x, y, z)$, m is the atomic mass, the prefactor g is proportional to the atomic scattering length (e.g., $g > 0$ for Rb and Na, while $g < 0$ for Li atoms), and $V_{\text{ext}}(\mathbf{r})$ is the external potential for magnetic or optical traps.

The nonlinear Schrödinger (NLS) equation is a well established model in optical and plasma physics as well as in fluid mechanics, where it emerges out of entirely different physical considerations [8,9]. In particular, for instance in optics, it emerges due to the so-called Kerr effect, where the material refractive index depends linearly on the intensity of incident light. The widespread use of the NLS equation stems from the fact that it describes, to the lowest order, the nonlinear dynamics of envelope waves.

One of the particularly desirable features of GPE is that the external potential $V_{\text{ext}}(\mathbf{r})$ can assume a multiplicity of forms, based on the type of trapping used to confine the atoms. Arguably, however, the most typical magnetic trapping imposes a parabolic potential [5,10]

$$V_{\text{ext}} = \frac{m}{2}\left(\Omega_x^2 x^2 + \Omega_y^2 y^2 + \Omega_z^2 z^2\right), \qquad (2)$$

where, in general, the trap frequencies $\Omega_{x,y,z}$ are different. In what follows, however, for simplicity, we will restrict our

consideration to the isotropic case of equal frequencies $\Omega_x = \Omega_y = \Omega_z \equiv \Omega$ along the different directions.

If the wave function is decomposed according to

$$\psi(\mathbf{r}, t) = e^{-i\mu t/\hbar}u(\mathbf{r}, t), \qquad (3)$$

then $u(\mathbf{r}, t)$ solves the GPE with the chemical potential μ in the form

$$i\hbar \frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 u + g|u|^2 u + V_{\text{ext}}(\mathbf{r})u - \mu u. \qquad (4)$$

One of the most extensively discussed limits in the case of self-repulsive nonlinearity $g > 0$ and in the presence of the parabolic potential (2) is the limit $\mu \to \infty$. This limit is referred to as the Thomas-Fermi limit.

If the kinetic (Laplacian) term is neglected, the approximate ground-state solution $u(\mathbf{r}, t) = U(\mathbf{r})$ is obtained in the form

$$U(\mathbf{r}) = (\max\{g^{-1}[\mu - V_{\text{ext}}(\mathbf{r})], 0\})^{1/2}. \qquad (5)$$

For this limit, the seminal work of Stringari [11] suggested a computation of the corresponding eigenfrequencies of oscillations of perturbations around the ground state using a hydrodynamic approach. This approach has become popular in the physics literature for more complicated problems involving anisotropic traps [12], dipole-dipole interactions [13], and rotating dipolar condensates [14].

The aim of the present work is to derive these eigenfrequencies systematically not only in the three-dimensional context, but also in the two-dimensional and one-dimensional cases. We will relate these eigenfrequencies to the eigenvalues discussed in the recent work [15] in the one-dimensional setting. The relevant eigenfrequencies of the perturbations around the ground state are then directly compared with numerical computations in the two-dimensional case.

In the numerical computations, the eigenfrequencies are obtained systematically as a function of the chemical potential μ starting from the low-amplitude limit (when the ground state is approximately that of the parabolic potential) all the way to the large values of the chemical potential. Earlier, these eigenfrequencies were approximated numerically near the low-amplitude limit by Zezyulin *et al.* [16] in the one-dimensional case and by Zezyulin [17] in the two-dimensional case. Our numerical computations also allow us to identify eigenfrequencies that remain invariant under changes in μ and to connect them to underlying symmetries of the GPE.

Our presentation will be structured as follows. In Sec. II, we present the mathematical setup of the problem. In Sec. III, we compute corresponding eigenfrequencies (around the Thomas-Fermi ground state) in the one, two, and three-dimensional settings. In Sec. IV, we compare these results to direct numerical computations in the two-dimensional case. Note that the one-dimensional case was considered in some detail in our earlier work [18]. Lastly, a brief summary of our findings and some interesting directions for future study are offered in Sec. V.

II. MATHEMATICAL SETUP

Using rescaling of variables, one can normalize the GPE (4) into two equivalent forms. One form corresponds to the semiclassical limit and it arises if $\hbar = \varepsilon, m = \frac{1}{2}, g = 1, \Omega = 2$, and $\mu = 1$, or equivalently, in the form

$$i\varepsilon u_t + \varepsilon^2 \nabla^2 u + (1 - |x|^2 - |u|^2)u = 0, \quad (6)$$

where $u(x,t) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ is a wave function, $\nabla^2 = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2$ is the Laplacian operator in d-spatial dimensions, and ε is a small parameter. On the other hand, if

$$v = \mu^{1/2} u, \quad \xi = (2\mu)^{1/2} x, \quad \tau = 2t, \quad \varepsilon = \frac{1}{2\mu}, \quad (7)$$

then Eq. (6) can be translated to the form

$$i v_\tau = -\tfrac{1}{2} \nabla_\xi^2 v + \tfrac{1}{2} |\xi|^2 v + |v|^2 v - \mu v, \quad (8)$$

which corresponds to the GPE (4) with $\hbar = 1$, $m = 1$, $g = 1$, and $\Omega = 1$. The semiclassical limit $\varepsilon \to 0$ of the GPE (6) corresponds to the Thomas-Fermi limit $\mu \to \infty$ of the GPE (8).

Let η_ε be a real positive solution of the stationary problem

$$\varepsilon^2 \nabla^2 \eta_\varepsilon + \left(1 - |x|^2 - \eta_\varepsilon^2\right)\eta_\varepsilon = 0, \quad x \in \mathbb{R}^d. \quad (9)$$

According to Gallo and Pelinovsky [19], for any sufficiently small $\varepsilon > 0$ there exists a smooth radially symmetric solution $\eta_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$ that decays to zero as $|x| \to \infty$ faster than any exponential function. This solution converges pointwise as $\varepsilon \to 0$ to the compact Thomas-Fermi cloud

$$\eta_0 := \lim_{\varepsilon \to 0} \eta_\varepsilon = \begin{cases} (1 - |x|^2)^{1/2}, & \text{for} \quad |x| < 1, \\ 0, & \text{for} \quad |x| > 1. \end{cases} \quad (10)$$

The solution η_ε with the properties above is generally referred to as the *ground state* of the GPE (6).

The spectral stability problem (often referred to as the Bogolyubov-de Gennes problem in the context of BEC's) for the ground state η_ε is written as the eigenvalue problem

$$L_+ u = -\lambda \varepsilon w, \quad L_- w = \lambda \varepsilon u, \quad (11)$$

associated with the two Schrödinger operators

$$\begin{cases} L_+ = -\varepsilon^2 \nabla^2 + |x|^2 + 3\eta_\varepsilon^2 - 1, \\ L_- = -\varepsilon^2 \nabla^2 + |x|^2 + \eta_\varepsilon^2 - 1. \end{cases}$$

A naive approximation of the eigenvalues λ of the spectral stability problem (11) arises if we replace η_ε^2 by η_0^2. Because L_+ is invertible, the eigenvalue problem can then be written in

the form

$$\left(-\varepsilon^2 \nabla^2 + |x|^2 + \eta_0^2 - 1\right)w$$
$$= \gamma \varepsilon^2 \left(-\varepsilon^2 \nabla^2 + |x|^2 + 3\eta_0^2 - 1\right)^{-1} w, \quad x \in \mathbb{R}^d, \quad (12)$$

where $\gamma = -\lambda^2$. The formal limit $\varepsilon \to 0$ gives a restricted problem in the unit ball

$$\text{LI:} \quad -2(1 - |x|^2)\nabla^2 w = \gamma w, \quad (13)$$
$$x \in B_0 = \{x \in \mathbb{R}^d : |x| < 1\},$$

subject to the Dirichlet boundary condition on the sphere $|x| = 1$. Convergence of eigenvalues of Eq. (12) to eigenvalues of the limiting problem (13) was rigorously justified by Gallo and Pelinovsky [15] in one spatial dimension $d = 1$.

Because L_+ is invertible for any small $\varepsilon > 0$, the original eigenvalue problem (11) can also be written in the form

$$\left(-\varepsilon^2 \nabla^2 + |x|^2 + \eta_\varepsilon^2 - 1\right)w$$
$$= \gamma \varepsilon^2 \left(-\varepsilon^2 \nabla^2 + |x|^2 + 3\eta_\varepsilon^2 - 1\right)^{-1} w, \quad x \in \mathbb{R}^d. \quad (14)$$

Because

$$|x|^2 + \eta_\varepsilon^2 - 1 = \frac{\varepsilon^2 \nabla^2 \eta_\varepsilon}{\eta_\varepsilon},$$

the formal limit $\varepsilon \to 0$ now gives a different problem in the unit ball

$$\text{LII:} \quad -2(1 - |x|^2)\left(\nabla^2 w - \frac{\nabla^2 \eta_0}{\eta_0} w\right) = \gamma w, \quad x \in B_0, \quad (15)$$

subject to Dirichlet boundary conditions on the sphere $|x| = 1$. The justification of the convergence of eigenvalues of (14) to eigenvalues of the limiting problem (15) is still an open problem in analysis.

The limiting eigenvalue problem (13) can be written in the vector form

$$\text{LI:} \quad -2\nabla(1 - |x|^2)\nabla \mathbf{v} = \gamma \mathbf{v}, \quad x \in B_0, \quad (16)$$

where $\mathbf{v} = \nabla w \in \mathbb{R}^d$. On the other hand, the limiting eigenvalue problem (15) can be rewritten in the equivalent scalar form

$$\text{LII:} \quad -2\nabla(1 - |x|^2)\nabla v = \gamma v, \quad x \in B_0, \quad (17)$$

where $v = \frac{w}{\eta_0}$ and η_0 is given by Eq. (10). It was exactly the representation (17) of the limiting eigenvalue problem LII, which was derived by Stringari [11] from the hydrodynamical formulation of the Gross-Pitaevskii equation (6) in three dimensions $d = 3$.

Comparison of the two representations (16) and (17) implies that the two limiting eigenvalue problems have *identical* nonzero eigenvalues in the space of one dimension $d = 1$ but may have *different* nonzero eigenvalues for $d \geqslant 2$. We will show in the next section that it is exactly the case. We will illustrate numerically for $d = 2$ that the eigenvalues of the second limiting problem (17) are detected in the limit $\varepsilon \to 0$ from the eigenvalues of the original problem (11).

We note that the anisotropic traps (2) with $\omega_x \neq \omega_y \neq \omega_z$ can be included in the consideration with the new definition of

$$|x|^2 := \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2, \quad (18)$$

where $\alpha_{1,2,3} > 0$. In particular, the limiting eigenvalue problems (13) and (15) or (16) and (17) are derived in the same way with the new definition of $|x|^2$.

III. EIGENVALUES OF THE LIMITING PROBLEMS

Case $d = 1$: Both representations (16) and (17) of the limiting eigenvalue problems LI and LII reduce to the Legendre equation

$$(x^2 - 1)v''(x) + 2xv'(x) = \tfrac{1}{2}\gamma v(x), \quad x \in (-1, 1). \quad (19)$$

For LI given by Eq. (13), the correspondence of eigenfunctions is $v(x) = w'(x)$. The only nonsingular solutions of this equation at the regular singular points $x = \pm 1$ are Legendre polynomials $v(x) \in \{P_n(x)\}_{n \geqslant 0}$, which correspond to eigenvalues $\gamma \in \{2n(n + 1)\}_{n \geqslant 0}$. The zero eigenvalue must be excluded from the set since it corresponds to $w(x) = x$, which violates Dirichlet boundary conditions at $x = \pm 1$ for $w(x)$. On the other hand, all nonzero eigenvalues are present because the corresponding eigenfunction $w(x) = C_{n+1}^{-1/2}(x)$ constructed from $v(x) = P_n(x)$ thanks to identities 8.936, 8.938, and 8.939 in [20]

$$\frac{d}{dx} C_{n+1}^{-1/2}(x) = C_n^{1/2}(x) = P_n(x),$$

also satisfies the Dirichlet boundary conditions thanks to the identity

$$C_{n+1}^{-1/2}(x) = \frac{x^2 - 1}{n(n + 1)} \frac{d^2}{dx^2} C_{n+1}^{-1/2}(x).$$

For LII given by Eq. (15), the correspondence of eigenfunctions is $v(x) = \frac{w(x)}{\sqrt{1 - x^2}}$. The zero eigenvalue should now be included for $n = 0$, since the eigenfunction $w(x) = \sqrt{1 - x^2}$ corresponds to the ground state η_ε in the limit $\varepsilon \to 0$, which is known to be the eigenfunction of operator L_-. All nonzero eigenvalues are the same as for the limiting problem (13), but the eigenfunctions are now different. For the eigenvalue $\gamma = 2n(n + 1)$, the eigenfunction is $w(x) = \sqrt{1 - x^2} P_n(x)$.

We note that the obtained eigenvalue distribution $\{2n(n + 1)\}_{n \geqslant 0}$ was numerically examined in Ref. [18] and was found to be in very good agreement with the true eigenvalues of system (11).

Case $d = 2$: We consider the limiting eigenvalue problem LII in the form (17) and use the polar coordinates

$$\begin{cases} x = r \cos(\theta), \\ y = r \sin(\theta), \end{cases} \quad r \geqslant 0, \quad \theta \in [0, 2\pi].$$

After the separation of variables $v(r, \theta) = V(r)e^{im\theta}$ for $m \in \mathbb{Z}$, we obtain an infinite set of eigenvalue problems for amplitudes of cylindrical harmonics

$$-(1 - r^2)\left[V''(r) + \frac{1}{r}V'(r) - \frac{m^2}{r^2}V(r) \right] + 2rV'(r)$$
$$= \frac{1}{2}\gamma V(r), \quad r \in (0, 1). \quad (20)$$

Let $m \geqslant 0$. We are looking for solutions of Eq. (20), which behave like $V(r) \sim r^m$ as $r \to 0$. Let us transform Eq. (20) to a hypergeometric equation with the substitution $V(r) = r^m F(z)$,

$z = r^2$. Direct computations show that $F(z)$ solves

$$z(1 - z)F''(z) + [1 + m - (2 + m)z]F'(z) + \left(\tfrac{1}{8}\gamma - \tfrac{1}{2}m\right)F(z)$$
$$= 0, \quad z \in (0, 1).$$

A nonsingular solution at $z = 0$ is the hypergeometric function $F(z) = \mathcal{F}(a, b, c; z)$ where

$$c = 1 + m, \quad a + b = 1 + m, \quad ab = \frac{m}{2} - \frac{\gamma}{8}.$$

Because $a + b - c = 0$, the hypergeometric function is singular at $z = 1$ unless it becomes a polynomial for $a = -k$ with an integer $k \geqslant 0$. The eigenvalues of the limiting problem LII are then given by

$$\gamma \in \left\{\gamma_{m,k}^{(2)}\right\}_{m \geqslant 0, k \geqslant 0}, \quad \gamma_{m,k}^{(2)} = 4[m + 2k(1 + m) + 2k^2]. \quad (21)$$

Let us now consider the limiting eigenvalue problem LI in the form (13) and use the same polar coordinates. The corresponding eigenvalue problem is

$$-2(1 - r^2)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = \gamma w,$$
$$r \in (0, 1), \quad \theta \in [0, 2\pi]. \quad (22)$$

Let $w(r, \theta) = W(r)e^{im\theta}$ for $m \in \mathbb{Z}$ and obtain an infinite set of eigenvalue problems for cylindrical harmonics

$$-(1 - r^2)\left[W''(r) + \frac{1}{r}W'(r) - \frac{m^2}{r^2}W(r) \right] = \frac{1}{2}\gamma W(r),$$
$$r \in (0, 1). \quad (23)$$

Let $m \geqslant 0$. Equation (23) can also be transformed to a hypergeometric equation after the substitution $W(r) = r^m F(z)$, $z = r^2$. Direct computations show that $F(z)$ solves

$$z(1 - z)F''(z) + (1 + m)(1 - z)F'(z) + \tfrac{1}{8}\gamma F(z) = 0,$$
$$z \in (0, 1).$$

A nonsingular solution at $z = 0$ is the hypergeometric function $F(z) = \mathcal{F}(a, b, c; z)$ where

$$c = 1 + m, \quad a + b = m, \quad ab = -\frac{\gamma}{8}.$$

Because $a + b - c = -1$, the hypergeometric function is bounded at $z = 1$. However, we need $F(1) = 0$ to satisfy the Dirichlet boundary conditions for $w(r, \theta)$ at $r = 1$. From this condition, $F(z)$ has to be a polynomial [or $F'(z)$ and $F''(z)$ are singular at $z = 1$ with $F(1) \neq 0$]. The polynomial arises for $a = -k$ with an integer $k \geqslant 0$. The eigenvalues of the limiting problem LI are then given by

$$\gamma \in \left\{\gamma_{m,k}^{(1)}\right\}_{m \geqslant 0, k \geqslant 0}, \quad \gamma_{m,k}^{(1)} = 8k(m + k). \quad (24)$$

A comparison of Eqs. (21) and (24) shows that $\gamma_{0,k}^{(2)} = \gamma_{1,k}^{(1)}$ for all $k \geqslant 0$, but the sets $\{\gamma_{m,k}^{(2)}\}_{m \geqslant 0, k \geqslant 0}$ and $\{\gamma_{m,k}^{(1)}\}_{m \geqslant 0, k \geqslant 0}$ are different. For instance, $\gamma_{1,0}^{(2)} = 4$ is not present in the set $\{\gamma_{m,k}^{(1)}\}_{m \geqslant 0, k \geqslant 0}$.

We will show in the next section that the limiting problem LI with eigenvalues $\{\gamma_{m,k}^{(1)}\}_{m \geqslant 0, k \geqslant 0}$ is not relevant for the approximations of eigenvalues of the original problem

(11), whereas the limiting problem LII with eigenvalues $\{\gamma_{m,k}^{(2)}\}_{m\geqslant 0,k\geqslant 0}$ gives correct asymptotic limits of the corresponding eigenvalues.

Case $d = 3$: We consider the limiting eigenvalue problem LII in the form (17) and use the spherical coordinates

$$\begin{cases} x = r\cos(\theta)\cos(\varphi), \\ y = r\sin(\theta)\cos(\varphi), \quad r\geqslant 0, \quad \theta\in[0,2\pi], \quad \varphi\in[0,\pi]. \\ z = r\sin(\varphi), \end{cases}$$

After the separation of variables $v(r,\theta,\varphi) = V(r)Y_{l,m}(\theta,\varphi)$ for $m\in\mathbb{Z}$ and $l\in\mathbb{N}$, where $Y_{l,m}(\theta,\varphi)$ are spherical harmonics, we obtain an infinite set of eigenvalue problems for amplitudes of the spherical harmonics

$$-(1-r^2)\left(V''(r) + \frac{2}{r}V'(r) - \frac{l(l+1)}{r^2}V(r)\right) + 2rV'(r)$$
$$= \frac{1}{2}\gamma V(r), \quad r\in(0,1). \tag{25}$$

Using a similar reduction $V(r) = r^l F(z)$, $z = r^2$ to the hypergeometric equation, we obtain the eigenvalues of the limiting problem LII in the form

$$\gamma\in\{\gamma_{l,k}^{(2)}\}_{l\geqslant 0,k\geqslant 0}, \quad \gamma_{l,k}^{(2)} = 4(l + 3k + 2kl + 2k^2). \tag{26}$$

This distribution was obtained by Stringari [11] from the balance of the leading powers in polynomial solutions of Eq. (25).

Using the same algorithm, the eigenvalues of the limiting problem LI are found in the form

$$\gamma\in\{\gamma_{l,k}^{(1)}\}_{l\geqslant 0,k\geqslant 0}, \quad \gamma_{l,k}^{(1)} = 4k(1 + 2l + 2k). \tag{27}$$

This distribution is different from Eq. (26). In particular, it does not include eigenvalue $\gamma_{1,0}^{(2)} = 4$.

Our preliminary numerical results for $d = 3$ (not included in this article) show that the set of eigenvalues $\{\gamma_{l,k}^{(2)}\}_{l\geqslant 0,k\geqslant 0}$ of the limiting problem LII gives a correct limit of eigenvalues of the original problem (11), whereas the set of eigenvalues $\{\gamma_{l,k}^{(1)}\}_{l\geqslant 0,k\geqslant 0}$ of the limiting problem LI is not accurate as an asymptotic approximation of the corresponding eigenvalues.

IV. NUMERICAL RESULTS

As indicated previously in the one-dimensional case, good agreement was observed between the predicted spectrum and the numerical computations of Ref. [18]; for this reason, we now turn our attention to the two-dimensional case. Eigenvalues of the original spectral problem (11) for $d = 2$ are computed numerically and shown on Fig. 1 (solid lines) together with the limiting eigenvalues (21) of the reduced spectral problem LII (dash-dotted lines). Notice that the results are presented in the context of the rescaled variant of the GPE (8) commonly used in the physical literature, illustrating the relevant eigenvalues as a function of the chemical potential $\mu = 1/(2\varepsilon)$.

The ground state η_ε exists for any $\varepsilon < \frac{1}{2}$ (i.e., $\mu > 1$) and the limit $\varepsilon\to\frac{1}{2}$ can be obtained via small-amplitude bifurcation theory [17]. All eigenvalues $\gamma = -\lambda^2$ in the spectral problem (11) in this limit occur at the integers $4(n + m)^2$ with $n, m\geqslant 0$
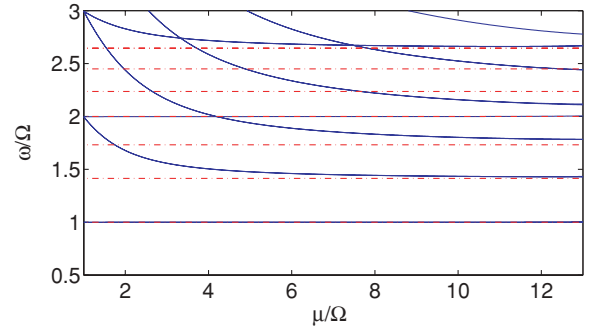

FIG. 1. (Color online) Eigenfrequencies of the two-dimensional spectral problem (11) (solid blue lines) and the limiting eigenvalues of the reduced problem (15) (dash-dotted red lines). Notice that for practical numerical reasons the computation was performed for a parabolic trap of frequency $\Omega = 0.3$ and the corresponding eigenfrequencies and chemical potentials were appropriately rescaled by Ω.

occur at the integers $4(n + m)^2$ with $n, m\geqslant 0$. When $\varepsilon < \frac{1}{2}$, all eigenvalues become smaller as ε gets smaller (or μ increases) besides the double eigenvalue $\gamma = 4$ and the simple eigenvalue $\gamma = 16$. Notice that the eigenvalues γ are related to the eigenfrequencies ω on Fig. 1 by $\omega = \sqrt{\gamma}/2$ (accounting for the time rescaling $\tau = 2t$).

Persistence of ε-independent eigenvalues $\gamma = 4$ and $\gamma = 16$ of the spectral problem (11) is explained by the symmetries of the GPE (6). One symmetry is given by the explicit transformation of solutions

$$u(x,y,t) = e^{ip(t)x + is(t)y + i\omega(t)}\tilde{u}(\tilde{x},\tilde{y},t),$$
$$\tilde{x} = x - q(t), \quad \tilde{y} = y - k(t). \tag{28}$$

If $\tilde{u}(\tilde{x},\tilde{y},t)$ is a solution of Eq. (6) rewritten in tilded variables and (p,s,q,k,ω) satisfy

$$\begin{cases} \dot{q} = 2\varepsilon p, \quad \varepsilon\dot{p} + 2q = 0, \\ \dot{k} = 2\varepsilon s, \quad \varepsilon\dot{s} + 2k = 0, \end{cases} \quad \omega = -\frac{1}{2}(qp + ks),$$

then $u(x,y,t)$ is also a solution of Eq. (6). Therefore, both q and k satisfy the linear oscillator equations with eigenvalue $\gamma = 4$, which gives the double degeneracy of eigenfrequency $\omega = 1$ in Fig. 1. This eigenmode corresponding to the eigenvalue $\omega = 1$ is generally referred to as the Kohn mode [21]. The corresponding symmetry is present in the space of one and three dimensions $d = 1$ and $d = 3$. The order of the mode's degeneracy is equal to the system's dimensionality.

The other symmetry of the GPE (6) with $d = 2$ is given by the conformal transformation

$$u(x,y,t) = a(t)\tilde{u}(\tilde{x},\tilde{y},\tilde{t})e^{ic(t)x^2 + ic(t)y^2 + i\omega(t)},$$
$$\tilde{x} = a(t)x, \quad \tilde{y} = a(t)y, \quad \tilde{t} = b(t), \tag{29}$$

where (a,b,c,ω) satisfy the first-order differential equations

$$\dot{b} = a^2, \quad \varepsilon\dot{\omega} = 1 - a^2, \quad \dot{a} + 4\varepsilon ac = 0,$$
$$\varepsilon\dot{c} + 4c^2\varepsilon^2 + 1 = a^4.$$

Excluding c and denoting $a(t) = z^{-1}(t)$, we obtained the nonlinear oscillator equation for $z(t)$

$$\ddot{z} + 4z(1 - z^{-4}) = 0.$$

There is a unique critical point $z = 1$ and it is a center with eigenvalue $\gamma = 16$, corresponding to the eigenfrequency $\omega = 2$ in Fig. 1. The eigenmode corresponding to the eigenvalue $\omega = 2$ is generally referred to as the breathing mode [22]. The corresponding symmetry is *not* present in the space of one and three dimensions $d = 1$ and $d = 3$. It is related to the fact that the cubic NLS equation in two dimensions $d = 2$ has an additional conformal symmetry.

The two symmetries (28) and (29) explain the μ independent eigenfrequencies ω on Fig. 1. On the other hand, the figure shows that all eigenvalues γ approach to the limiting eigenvalues (21) as $\varepsilon \to 0$ (i.e., as $\mu \to \infty$). This output confirms the robustness of the asymptotic distributions presented herein.

It is worth noting that in all the cases shown on Fig. 1 the eigenvalues were confirmed including also their multiplicities. For instance the eigenfrequency associated with $\omega = 2$ ($\gamma = 16$) is associated with two eigenvalues in the set (21), namely with $k = 1$ and $m = 0$, as well as with $k = 0$ and $m = 4$. One of these corresponds to the conformal symmetry (29), while the other one can be observed on Fig. 1 to asymptote to $\omega = 2$ in the large μ limit, as expected.

V. CONCLUSION AND FUTURE DIRECTIONS

In the present work, we offered a systematic approach toward identifying the eigenfrequencies of oscillations of the perturbations around the ground state of a BEC in an arbitrary number of dimensions (our calculations were given in one, two, and three-dimensions). This spectrum is important because it corresponds to the excitations that can be (and have been) experimentally observed once the condensate is perturbed appropriately.

Part of the rationale for attempting to understand the details of this spectrum is that when fundamental nonlinear excitations are additionally considered on top of the ground state, then the spectrum contains both a "ghost" of the spectrum of the ground state and the so-called negative energy modes that pertain to the nonlinear excitation itself. Relevant examples of this sort can be found both for the case of one-dimensional dark solitons (and multisolitons) as analyzed in Ref. [23] and in the case of two-dimensional vortices, as examined in Ref. [24]. It is then of particular interest to try to understand eigenfrequencies of excitations of these structures in the Thomas-Fermi limit, as well as those of their three-dimensional generalizations bearing line or ring vortices. Such studies will be especially interesting for future work.

[1] M. H. J. Anderson, J. R. Ensher, M. R. Matthews, and C. E. Wieman, Science **269**, 198 (1995).

[2] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).

[3] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge, England, 2002).

[4] L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation*, (Oxford University Press, Oxford, 2003).

[5] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).

[6] *Emergent Nonlinear Phenomena in Bose-Einstein Condensates: Theory and Experiment* edited by P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-González, Springer Series on Atomic, Optical, and Plasma Physics, Vol. 45 (Springer, Heidelberg, 2008).

[7] R. Carretero-González, D. J. Frantzeskakis, and P. G. Kevrekidis, Nonlinearity **21**, R139 (2008).

[8] C. Sulem and P. L. Sulem, *The Nonlinear Schrödinger Equation*, (Springer-Verlag, New York, 1999).

[9] M. J. Ablowitz, B. Prinari, and A. D. Trubatch, *Discrete and Continuous Nonlinear Schrödinger Systems* (Cambridge University Press, Cambridge, England, 2004).

[10] P. G. Kevrekidis and D. J. Frantzeskakis, Mod. Phys. Lett. B **18**, 173 (2004).

[11] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).

[12] M. Fliesser, A. Csordas, P. Szepfalusy, and R. Graham, Phys. Rev. A **56**, R2533 (1997).

[13] C. Eberlein, S. Giovanazzi, and D. H. J. O'Dell, Phys. Rev. A **71**, 033618 (2005).

[14] R. M. W. van Bijnen, A. J. Dow, D. H. J. O'Dell, N. G. Parker, and A. M. Martin, Phys. Rev. A **80**, 033617 (2009).

[15] C. Gallo and D. Pelinovsky, J. Math. Anal. Appl. **355**, 495 (2009).

[16] D. A. Zezyulin, G. L. Alfimov, V. V. Konotop, and V. M. Pérez–García, Phys. Rev. A **78**, 013606 (2008).

[17] D. A. Zezyulin, Phys. Rev. A **79**, 033622 (2009).

[18] D. E. Pelinovsky and P. G. Kevrekidis, Contemp Math. **473**, 159 (2008).

[19] C. Gallo and D. Pelinovsky, e-print arXiv:0911.3913.

[20] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 6th ed. (Academic Press, New York, 2005).

[21] W. Kohn, Phys. Rev. **123**, 1242 (1961).

[22] L. P. Pitaevskii and A. Rosch, Phys. Rev. A **55**, R853 (1997).

[23] G. Theocharis, A. Weller, J. P. Ronzheimer, C. Gross, M. K. Oberthaler, P. G. Kevrekidis, and D. J. Frantzeskakis, e-print arXiv:0909.2122.

[24] S. Middelkamp, P. G. Kevrekidis, D. J. Frantzeskakis, R. Carretero-González, and P. Schmelcher, e-print arXiv:0911.3308.