Nonlocal Models for Envelope Waves in a Stratified Fluid

By Dmitry E. Pelinovsky and Roger H. J. Grimshaw

A new, nonlocal evolution equation similar to the nonlinear Schrödinger equation is derived for envelope waves in a continuously stratified fluid by means of a multiscale perturbation technique. This new equation governs propagation of quasi-harmonic wave packets having length scales much longer than the depth of the density variations and much shorter than the total depth of fluid. Generalizations of the nonlocal evolution equation for a description of two-dimensional wave modulations are also presented. The modulational stability of small-amplitude waves is then investigated in the framework of the derived equations. It is shown that quasi-harmonic waves with the scales under consideration are unstable with respect to oblique perturbations at certain angles.

1. Introduction

For the last 30 years the problem of self-modulation of small-amplitude nonlinear waves has been investigated intensively both for surface water waves [1–4] and for internal waves in a density-stratified fluid [5–9]. It was shown that the cubic nonlinear Schrödinger (NLS) equation is a universal

Address for correspondence: Dmitry E. Pelinovsky, Department of Mathematics, Monash University, Clayton, Victoria 3168, Australia.

STUDIES IN APPLIED MATHEMATICS 97:369–391 369

Published by Blackwell Publishers, 238 Main Street, Cambridge, MA 02142, USA and 108 Cowley Road, Oxford, OX4 1JF, UK.

model for a description of wave propagation when there is no resonance between the main quasi-harmonic wave and the second or zero (mean flow) harmonics induced by nonlinear effects. If the coefficients of the linear dispersive term and the cubic nonlinear term have the same sign, then small-amplitude waves are unstable with respect to subharmonic generation and this effect results in the formation of envelope solitary waves from a localized initial perturbation. Otherwise, small-amplitude waves are stable and the localized wave packets disperse. A critical situation can arise when either the coefficient of the linear dispersive term or that of the cubic nonlinear term become zero. Such situations were discovered for both surface and internal waves along special curves in parameter space [4, 5].

A correct description of wave propagation and of the instability criterion in these critical cases can be found if one takes into account higher-order dispersive and/or nonlinear terms in the governing evolution equation. For instance, the evolution of surface gravity waves in a vicinity of the critical depth $kh \approx 1.363$ is described by a modified NLS equation with fifth-order nonlinearity and also derivative third-order nonlinear terms [10]. For this case, it was shown by Kakutani and Michihiro [11] that quasi-harmonic waves may become unstable even in the subcritical region if their amplitudes are large enough. This result coincides with a full analysis of periodic wave stability of surface waves carried out by McLean [12]. A similar situation appears for interfacial waves between two layers of different densities but the marginal stability curve depends now on three parameters, the depths of the upper and lower layers and the ratio of their densities [5].

Besides the aforementioned situation, interfacial waves cannot also be described by the cubic NLS equation for a certain case, the so-called *shallow–deep* limit of a stratified fluid, when the upper layer is shallow and the lower one is deep compared to the scales of quasi-harmonic wave packets. For this case, it was shown by Tanaka [5] and Grimshaw [13] that the lowest-order part of the cubic nonlinear coefficient becomes zero because the nonlinear effects induced by the second harmonics exactly compensate those induced by the mean flow. Since the shallow–deep limit is very important for internal waves on a real oceanic pycnocline, the present problem is to derive a new type of NLS equation describing quasi-harmonic wave packets for this situation.

A first step to the solution of this problem has been recently achieved by Pelinovsky [14] where a new evolution equation was derived by means of asymptotic multiscale reduction of the nonlocal intermediate long-wave equation. The latter equation governs evolution of long internal waves to the first order in the shallow–deep limit. The new equation is also nonlocal and it was referred to as the intermediate NLS equation. In addition, it was shown that this new equation is integrable and supports N-soliton solutions on a modulationally stable wave background. The inverse scattering transform for the intermediate NLS equation was constructed in our previous paper [15].

However, the applicability region of this intermediate NLS equation is limited by that of the nonlocal long-wave equation. Second-order effects in the long-wave expansions are not described by this latter equation while they might influence the behavior of quasi-harmonic wave packets. The aim of the present article is to take into account these second-order effects and to derive a universal evolution equation of the NLS type in the shallow–deep limit of a continuously stratified fluid. This equation is finally found in the form

$$i\Psi_t = \alpha\Psi_{xx} + \beta\Psi(i+\mathbf{T}_h)(|\Psi|^2)_x - \gamma|\Psi|^2\Psi, \tag{1.1}$$

where Ψ is the slowly varying amplitude of quasi-harmonic waves, $\mathbf{T}_h$ is nonlocal operator

$$\mathbf{T}_h(u) = \frac{1}{2h}\text{p.v.}\int_{-\infty}^{\infty}\coth\left[\frac{\pi(z-x)}{2h}\right]u(z)\,dz, \tag{1.2}$$

p.v. stands for principal value of the integral, and the positive coefficients α, β, and γ are expressed by the parameters of the fluid stratification. For $\gamma \to 0$ (1.1) transforms to the intermediate NLS equation, while for $\beta \to 0$ it transforms to the cubic NLS equation. These special limits can arise for special relationships between the wavelength of quasi-harmonic wave packets and the two characteristic scales of density stratification.

Our strategy is as follows. In Section 2 we obtain a higher-order nonlocal long-wave equation from the original Euler equations when there are rapid variations of the density stratification compared to the wavelength. In addition to the well-known analysis of Benjamin [16] and Davis and Acrivos [17], we keep not only the lowest-order terms of the long-wave expansions but also the second-order terms. A similar second-order expansion was recently analyzed for surface and interfacial waves in a general form by Kraenkel et al. [18] and Matsuno [19] as well as for internal steady-state waves by Grimshaw [20]. Then, in Section 3 we consider the propagation of modulated quasi-harmonic long waves whose modulation scale is much shorter than the total depth of the stratified fluid. As a result, we find the desired evolution equation (1.1). The generalizations of our approach for two-dimensional modulations of quasi-harmonic waves are discussed in Section 4. Our approach is based on a two-dimensional nonlocal long-wave equation derived for internal waves by Ablowitz and Segur [21]. We find two types of two-dimensional evolution equations of the NLS type. The first describes perturbations with equal scales in the longitudinal and transverse directions and was first derived by Grimshaw and Pullin [6]. The other, essentially nonlocal two-dimensional evolution equation is found for smooth transverse modulations. Using both these models we then investigate the problem of small-amplitude wave stability in Section 5 and show that a quasi-harmonic wave is unstable with respect to oblique perturbations at certain angles. These results are in agreement with recent results of Spector and Miloh [22] and clarify the physical mechanism of this instability. The final Section 6 is devoted to discussion. Appendices A and B present the explicit form of coefficients of Eq. (1.1) calculated for two conventional representations of the density stratification.

2. A higher-order nonlocal long-wave equation

We consider an incompressible and inviscid fluid that is density stratified along the vertical coordinate z. In Sections 2 and 3 we restrict ourselves only to two-dimensional motion of the stratified fluid, which is described by the Euler equations,

$$\rho(u_t + uu_x + wu_z) + p_x = 0, \tag{2.1a}$$

$$\rho(w_t + uw_x + ww_z) + p_z + \rho g = 0, \tag{2.1b}$$

$$\rho_t + u\rho_x + w\rho_z = 0, \tag{2.1c}$$

as well as by the incompressibility equation

$$u_x + w_z = 0. \tag{2.1d}$$

Here u, w are horizontal and vertical components of fluid velocity; ρ, p are its density and pressure; and g is the gravity constant. The fluid is bounded by upper and lower surfaces, which are supposed to be plane and rigid. Under these conditions we impose the boundary conditions to (2.1a)–(2.1d) in the form

$$w\,|_{z=0} = 0, \tag{2.2a}$$

$$w\,|_{z=-h} = 0, \tag{2.2b}$$

Let us consider a basic continuous stratification of the fluid, which is presented by a density profile $\rho = R(z)$. We suppose that the stratification has two characteristic scales. The first scale describes a rapid density variation in the near-surface shallow layer (for $-d < z < 0$, where $d \ll h$). The second one corresponds to a nearly nonstratified deep layer (for

$-h < z < -d$), where $R \approx \rho_\infty$. Long internal waves are localized in the upper layer and, therefore, the shallow scale is responsible for the nonlinear effects of wave propagation. On the other hand, the deep layer is responsible for the dispersive effects essential for the problem.

According to the profile of density stratification assumed above we introduce two vertical scales, an inner scale described by variable z and an outer scale described by a slow variables $\zeta = \epsilon z$ so that $h = H/\epsilon$. Here ϵ is a small parameter that specifies both the small amplitude and the smooth modulation of long internal waves according to the following asymptotic expansions,

$$u = \epsilon\left(u^{(0)}(\xi,\tau;z,\zeta) + \sum_{n=1}^{\infty} \epsilon^n u^{(n)}(\xi,\tau;z,\zeta)\right), \quad (2.3a)$$

$$w = \epsilon^2\left(w^{(0)}(\xi,\tau;z,\zeta) + \sum_{n=1}^{\infty} \epsilon^n w^{(n)}(\xi,\tau;z,\zeta)\right), \quad (2.3b)$$

$$\rho = R(z) + \epsilon\left(\rho^{(0)}(\xi,\tau;z,\zeta) + \sum_{n=1}^{\infty} \epsilon^n \rho^{(n)}(\xi,\tau;z,\zeta)\right), \quad (2.3c)$$

$$p = P(z) + \epsilon\left(p^{(0)}(\xi,\tau;z,\zeta) + \sum_{n=1}^{\infty} \epsilon^n p^{(n)}(\xi,\tau;z,\zeta)\right), \quad (2.3d)$$

where $\xi = \epsilon(x - ct)$, $\tau = (\tau_2, \tau_3, \ldots)$, $\tau_n = \epsilon^n t$, c is the linear long-wave phase speed and the hydrostatic pressure $P(z)$ is given by $P_z = -gR$.

Substitution of (2.3a)–(2.3d) into (2.1a)–(2.1d) reduces the original Euler equations into a set of equations for each order in ϵ. It is convenient to express the leading-order terms in the form

$$u^{(0)} = f(\xi,\tau;\zeta)W_z(z), \quad (2.4a)$$

$$w^{(0)} = -f_\xi(\xi,\tau;\zeta)W(z), \quad (2.4b)$$

$$\rho^{(0)} = -\frac{1}{c}f(\xi,\tau;\zeta)R_z(z)W(z), \quad (2.4c)$$

$$p^{(0)} = cf(\xi,\tau;\zeta)R(z)W_z(z). \quad (2.4d)$$

In this representation the inner and outer variables z and ζ are separated from each other. Note that such a representation is different from the conventional one [20] where asymptotic expansions similar to (2.3a)–(2.3d) are constructed independently for both regions and then are matched for

$z \to -\infty$, $\zeta \to 0$. However, this modification of an asymptotic multiscale technique gives the same results as the standard scheme.

The function $f(\xi, \tau; \zeta)$ in (2.4a)–(2.4d) represents the varying amplitude of long internal waves, while $W(z)$ is found from the inner eigenvalue problem

$$\mathbf{L}W \equiv (R(z)W_z(z))_z - \frac{g}{c^2}R_z(z)W(z) = 0 \tag{2.5}$$

with boundary conditions

$$W(0) = 0, \tag{2.6a}$$

$$W_z(-\infty) = 0. \tag{2.6b}$$

Moreover, we specify the normalization of the modal function $W(z)$ as $W(-\infty) = 1$. This can always be done because of the linear properties of (2.5). Note that the boundary condition (2.2b) is not fulfilled for the fast variable z for the modal function $W(z)$. However, if the function $f(\xi, \tau; \zeta)$ vanishes at $\zeta = -H$, the function $w^{(0)}$ satisfies (2.2b) according to (2.4b).

Next, we consider the restrictions to be imposed on the function $f(\xi, \tau; \zeta)$ to obtain a valid asymptotic series (2.3a)–(2.3d). As is well known, such restrictions reduce in the leading order of ϵ to the intermediate long-wave (ILW) equation for the amplitude of internal waves localized in a stratified layer, $A(\xi, \tau) = f(\xi, \tau; \zeta = 0)$. However, here we do not confine ourselves to the leading order of asymptotic expansions but present a general scheme for the calculation of the second-order corrections to the ILW equation.

First, we deal with the first-order approximation and find the corrections $u^{(1)}$, $\rho^{(1)}$, $p^{(1)}$ in terms of $w^{(1)}$. As a result, Eqs. (2.1a)–(2.1d) reduce at the first order to the linear equation

$$\begin{aligned} \mathbf{L}w^{(1)} = & -\frac{2}{c}f_{\tau_2}(RW_z)_z + f_{\zeta\xi}[(RW)_z + RW_z] \\ & + \frac{1}{c}ff_\xi\left[(RWW_{zz} - RW_z^2 - R_zWW_z)_z + W(RW_z)_{zz} - W_z(RW_z)_z\right]. \end{aligned} \tag{2.7}$$

The solution of (2.7) with boundary conditions $w^{(1)}(z = 0)$ and $w_z^{(1)}(z \to -\infty) = 0$ can be found only if a certain, *compatibility condition* is met. To obtain this condition we multiply (2.7) by $W(z)$ and then perform an integration with respect to z. As a result, we get an evolution equation for the function f,

$$\sigma f_{\tau_2} - a_1 f_{\zeta\xi} + b_1 f f_\xi = 0, \tag{2.8}$$

where

$$\sigma = 2\int_{-\infty}^{0} RW_z^2\, dz, \qquad a_1 = c\rho_\infty, \qquad b_1 = 3\int_{-\infty}^{0} RW_z^3\, dz.$$

Next, using this restriction on f_{τ_2} we can solve the linear inhomogeneous equation (2.7) in the form

$$w^{(1)} = -\tilde{f}_\xi W(z) + f_{\zeta\xi} W_{11}(z) + \frac{1}{c} f f_\xi W_{12}(z). \tag{2.9}$$

Here the function $\tilde{f}(\xi, \tau; \zeta)$ describes a correction to the amplitude of the long waves and the functions $W_{11}(z)$, $W_{12}(z)$ satisfy equations

$$\mathbf{L} W_{1n} = H_{1n}, \qquad \text{for } n = 1,2,$$

where

$$H_{11} = (RW)_z + RW_z - \frac{2\rho_\infty}{\sigma}(RW_z)_z,$$

$$H_{12} = (RWW_{zz} - RW_z^2 - R_z WW_z)_z + W(RW_z)_{zz} - W_z(RW_z)_z + \frac{2b_1}{\sigma}(RW_z)_z.$$

Furthermore, the functions $W_{1n}(z)$ can be explicitly expressed in terms of the modal function $W(z)$ according to the formula

$$W_{1n} = W(z)\int_0^z \frac{dz'}{R(z')W^2(z')}\int_0^{z'} dz''\, W(z'') H_{1n}(z''). \tag{2.10}$$

Now we turn to the second-order approximation and derive a correction to the evolution equation (2.8). To do this, we obtain the inhomogeneous linear equation for the correction term $w^{(2)}$ and again apply a compatability condition. As a result of direct but laborious calculations, we find finally the following equation for the function $\tilde{f}$,

$$\sigma\tilde{f}_{\tau_2} - a_1\tilde{f}_{\zeta\xi} + b_1(f\tilde{f})_\xi + \sigma f_{\tau_3} + a_{21}f_{\zeta\tau_2} + a_{22}f_{\zeta\zeta\xi} + + a_{23}\int f_{\tau_2\tau_2}\,d\xi$$

$$+ b_{21}(ff_\zeta)_\xi + b_{22}ff_{\zeta\xi} + b_{23}ff_{\tau_2} + b_{24}f_\xi\int f_{\tau_2}\,d\xi + c_2f^2f_\xi = 0. \quad (2.11)$$

The coefficients of this equation are presented by the following expressions

$$a_{21} = \rho_\infty - 2\int_{-\infty}^0 RW_zW_{11z}\,dz,$$

$$a_{22} = -c\int_{-\infty}^0 (R_zWW_{11} + 2RWW_{11z})\,dz,$$

$$a_{23} = \frac{\sigma}{2c},$$

$$b_{21} = \int_{-\infty}^0 (2RW^2W_{zz} - 3RW_z^2W_{11z} - R_zWW_{12} - 2RWW_{12z})\,dz,$$

$$b_{22} = \int_{-\infty}^0 (RWW_{zz}W_{11z} - RW_zW_{zz}W_{11} - RW^2W_{zz})\,dz,$$

$$b_{23} = \frac{1}{c}\int_{-\infty}^0 (-2RW_zW_{12z} + 5RW_z^3 + 6RWW_zW_{zz})\,dz,$$

$$b_{24} = \frac{b_1}{3c},$$

$$c_2 = \frac{1}{c}\int_{-\infty}^0 \Big(RWW_{zz}W_{12z} - RW_zW_{zz}W_{12} - \frac{9}{2}RW_z^2W_{12z}$$

$$-2RW^2W_{zz}^2 + 6RW_z^4 + 12RWW_z^2W_{zz}\Big)\,dz.$$

Thus, for the function $\tilde{f}$ we obtain an evolution equation whose linear part is the linearization of (2.8) and whose inhomogeneous part contains the function f and its higher-order time derivative f_{τ_3} [18, 20].

Let us next consider the functions $f(\xi,\tau;\zeta)$, $\tilde{f}(\xi,\tau;\zeta)$ with respect to the vertical coordinate ζ. It is readily checked that the condition (2.11) removes only a linear secular divergence for $w^{(2)}$ in z for $z \to -\infty$. However, the inhomogeneous equation for the correction term $w^{(2)}$ might also generate a quadratic secular growth ($w^{(2)} \sim z^2$ as $z \to -\infty$) unless the function $f(\xi, \tau; \zeta)$ satisfies the outer problem

$$f_{\zeta\zeta} + f_{\xi\xi} = 0, \tag{2.12}$$

with boundary conditions

$$f_{|\zeta=0} = A(\xi, \tau), \tag{2.13a}$$

$$f_{|\zeta=-H} = 0. \tag{2.13b}$$

It is convenient for our analysis to introduce the function $g(\xi, \tau; \zeta)$, which is expressed through the first-order correction term $w^{(1)}$ continued to the nonstratified layer as $z \to -\infty$,

$$w^{(1)} \to -\tilde{f}_\xi + \overline{W}_{11} f_{\zeta\xi} + \frac{1}{c}\overline{W}_{12} ff_\xi \equiv -g_\xi.$$

Here we have denoted $\overline{W}_{1n} = \lim_{z \to -\infty} W_{1n}(z)$. Then, at the order of $O(\epsilon^3)$ we find that the function $g(\xi, \tau; \zeta)$ satisfies the same problem (2.12) but with boundary conditions

$$g_{|\zeta=0} = \tilde{A}(\xi, \tau) - \overline{W}_{11} f_\zeta \,|_{\zeta=0} - \frac{\overline{W}_{12}}{2c} A^2, \tag{2.14a}$$

$$g_{|\zeta=-H} = 0, \tag{2.14b}$$

where $\tilde{A}(\xi, \tau) = \tilde{f}(\xi, \tau; \zeta = 0)$. Note that the second boundary condition [see (2.13b) and (2.14b)] provides vanishing of the vertical velocity at $z = -h$.

The solutions to the outer problem (2.12) with (2.13a)–(2.13b) and (2.14a)–(2.14b) can be found by Fourier transforms [20]. Using this analysis, we can relate the functions f_ζ, $\tilde{f}_\zeta$ for $\zeta = 0$ to the amplitudes A, $\tilde{A}$ as follows,

$$f_\zeta \,|_{\zeta=0} = -\mathbf{T}_H(A_\xi), \tag{2.15a}$$

$$\tilde{f}_\zeta \,|_{\zeta=0} = -\mathbf{T}_H\big(\tilde{A}_\xi\big) - \overline{W}_{11}\mathbf{T}_H^2(A_{\xi\xi}) - \overline{W}_{11} A_{\xi\xi}$$
$$+ \frac{\overline{W}_{12}}{c}\mathbf{T}_H(AA_\xi) - \frac{\overline{W}_{12}}{c} A\mathbf{T}_H(A_\xi), \tag{2.15b}$$

where the integral operator $\mathbf{T}_H$ is given by formula (1.2).

Finally, using relations (2.15a)–(2.15b) we rewrite Eqs. (2.8) and (2.11) at $\zeta = 0$ in the form of a higher-order long-wave equation for the sum amplitude $A'(\xi, \tau) \equiv A + \epsilon\tilde{A} + O(\epsilon^2)$,

$$A_\tau + \alpha_1 \mathbf{T}_H(A_{\xi\xi}) + \beta_1 A A_\xi$$
$$= \epsilon\Big[\alpha_{21} A_{\xi\xi\xi} - \alpha_{22}\mathbf{T}_H^2(A_{\xi\xi\xi}) + \beta_{21}(A\mathbf{T}_H(A_\xi))_\xi + \beta_{22} A\mathbf{T}_H(A_{\xi\xi})$$
$$+ \beta_{23}\mathbf{T}_H(AA_\xi)_\xi + \gamma_2 A^2 A_\xi\Big] + O(\epsilon^2). \quad (2.16)$$

Here we have omitted the superscript prime for the variable A and introduced the following coefficients

$$\alpha_1 = \frac{a_1}{\sigma},$$

$$\alpha_{21} = \frac{a_{22} - a_1\overline{W}_{11}}{\sigma},$$

$$\alpha_{22} = \frac{a_1(a_{21} + \sigma\overline{W}_{11} + \alpha_1 a_{23})}{\sigma^2},$$

$$\beta_1 = \frac{b_1}{\sigma},$$

$$\beta_{21} = \frac{b_{21} - \rho_\infty\overline{W}_{12} + \alpha_1 b_{24}}{\sigma},$$

$$\beta_{22} = \frac{b_{22} + \alpha_1(b_{23} - b_{24} - \beta_1 a_{23})}{\sigma},$$

$$\beta_{23} = \frac{\rho_\infty\overline{W}_{12} - \beta_1(a_{21} + \alpha_1 a_{23})}{\sigma},$$

$$\gamma_2 = \frac{-c_2 + \beta_1(b_{23} + b_{24}/2 - \beta_1 a_{23})}{\sigma}.$$

This equation governs the evolution of long internal waves in a deep fluid with a thin stratification layer. Note that for the infinitely deep fluid ($H = \infty$) and for stationary solitary wave solutions the higher-order long-wave equation (2.16) can be simplified and reduced to the equations derived by Grimshaw [20]. In this case, solutions to (2.16) can be constructed by means of perturbation theory and the higher-order derivative term A_{τ_3} can be found from the dispersion relation for small-amplitude waves [18, 20].

However, for our problem it is more convenient to keep the higher-order long-wave equations in the form of a unique equation (2.16) and then expand its solution in a series of the wave amplitude.

3. Nonlocal NLS equation for modulated waves

3.1. Derivation

Now we turn to the problem of wave packet self-modulation for the shallow–deep limit of a stratified fluid, which is described by our higher-order long-wave equation (2.16). We suppose that the length of the wave packets is much longer than the stratification scale but much shorter than the total depth of fluid. Hence we introduce a new small parameter μ so that $H = \delta / \mu$. This parameter also determines the small amplitude of the wave packets as $O(\sqrt{\mu})$ and their slow modulation being $O(\mu)$ according to the following asymptotic expansion,

$$A = \sum_{n=1}^{\infty} \mu^{n/2} A_n(X,T;\Theta), \tag{3.1}$$

where the leading term is presented in the form of a slowly modulated quasi-harmonic wave

$$A_1 = \Psi(X,T)\exp(i\Theta) + \Psi^*(X,T)\exp(-i\Theta), \tag{3.2}$$

depending on a fast phase $\Theta = k(\xi - V_p \tau)$ $(k > 0)$ and on slow variables $X = \mu(\xi - V_g \tau)$ and $T = \mu^2 \tau$. Here V_p determines the higher-order corrections of the phase velocity of a quasi-harmonic wave with respect to the limiting long-wave phase speed c, while V_g corresponds to those of the group velocity of the wave packet. The linear part of the higher-order long-wave equation (2.16) allows us to find the first terms of the phase and group velocities from the parameters of the fluid stratification, $V_p = -\alpha_1 k + \epsilon(\alpha_{21} + \alpha_{22})k^2 + O(\epsilon^2)$ and $V_g = -2\alpha_1 k + 3\epsilon(\alpha_{21} + \alpha_{22})k^2 + O(\epsilon^2)$.

The second-order term A_2 is generated by nonlinear effects and includes second harmonics and mean flow terms as

$$A_2 = \frac{\beta_1}{2\alpha_1 k}\left[n + \Psi_2 \exp(2i\Theta) + \Psi_2^* \exp(-2i\Theta)\right], \tag{3.3}$$

where $n(X,T)$, $\Psi_2(X,T)$ designate amplitudes of the mean flow and the second harmonics respectively. Substitution of (3.1)–(3.3) into the long-wave equation (2.16) allows us to relate n, Ψ_2 to the amplitude of the carrier wave Ψ,

$$n = -|\Psi|^2\left[1+\epsilon k\left(\frac{2(\beta_{21}+\beta_{22})}{\beta_1}+\frac{3(\alpha_{21}+\alpha_{22})}{2\alpha_1}\right)+O(\epsilon^2)\right] \quad (3.4a)$$

$$\Psi_2 = \Psi^2\left[1+\epsilon k\left(\frac{2\beta_{21}+\beta_{22}+2\beta_{23}}{\beta_1}+\frac{3(\alpha_{21}+\alpha_{22})}{\alpha_1}\right)+O(\epsilon^2)\right]. \quad (3.4b)$$

It is remarkable that the second and zero harmonics completely compensate each other in the leading order of our asymptotic expansions and do not influence the nonlinear self-modulation of quasi-harmonic wave packets. As a result, in the third-order approximation in μ where the cubic NLS equation usually appears, we have no restrictions imposed on the amplitude Ψ. Therefore, we have to extend the asymptotic expansion (3.1) to higher-order approximations and find the higher-order corrections,

$$A_3 = \left(\frac{\beta_1}{2\alpha_1 k}\right)^2[\Psi_3\exp(3i\Theta)+\Psi_3^*\exp(-3i\Theta)] \quad (3.5)$$

$$A_4 = \frac{\beta_1}{4\alpha_1 k^2}\left[\tilde{n}+\tilde{\Psi}_2\exp(2i\Theta)+\tilde{\Psi}_2^*\exp(-2i\Theta)\right]$$

$$+\left(\frac{\beta_1}{2\alpha_1 k}\right)^3[\Psi_4\exp(4i\Theta)+\Psi_4^*\exp(-4i\Theta)]. \quad (3.6)$$

Substituting these expressions into the long-wave equation (2.16) we consecutively find the coefficients $\tilde{n}, \tilde{\Psi}_2, \Psi_3$, and Ψ_4,

$$\tilde{n} = i(\Psi\Psi_X^* - \Psi_X\Psi^*) + \mathbf{T}_\delta(|\Psi|^2)_X - \frac{\beta_1^2}{2\alpha_1^2 k}|\Psi|^4 + O(\epsilon)$$

$$\tilde{\Psi}_2 = 2i\Psi\Psi_X + O(\epsilon), \qquad \Psi_3 = \Psi^3 + O(\epsilon), \qquad \Psi_4 = \Psi^4 + O(\epsilon).$$

Finally, at the order of $\mu^{5/2}$ we get the desired equation of NLS type, which governs the evolution of the amplitude Ψ,

$$i\Psi_T = \alpha\Psi_{XX} + \beta\Psi(i+\mathbf{T}_\delta)(|\Psi|^2)_X - \frac{\epsilon}{\mu}\gamma|\Psi|^2\Psi, \quad (3.7)$$

where

$$\alpha = \alpha_1,$$

$$\beta = \frac{\beta_1^2}{4\alpha_1 k},$$

$$\gamma = k\left(\gamma_2 - \frac{3\beta_1^2(\alpha_{21}+\alpha_{22})}{4\alpha_1^2} - \frac{\beta_1(2\beta_{21}+\beta_{22}+2\beta_{23})}{2\alpha_1}\right).$$

3.2. Discussion

Equation (3.7) contains a balance of local and nonlocal cubic nonlinear terms and we call it a mixed NLS equation. It is difficult to evaluate the coefficients α, β, γ in the general case. Therefore, we consider two conventional representations of the density stratification (two-layer fluid and constant stratification layer) to find the coefficients of the higher-order long-wave equation (2.16) and those of the mixed NLS equation (3.7) in explicit form. The corresponding results are presented in Appendices A and B. For both cases we find that all coefficients of the mixed NLS equation are positive. The consequences of this fact for the problem of small-amplitude wave stability are discussed in Section 5.

It is obvious that Eq. (3.7) reduces in physical variables to the form (1.1), where the formal small parameters ϵ, μ are absent. However, the form (3.7) makes clear the range of applicability of the mixed NLS equation for the description of internal quasi-harmonic waves. In physical variables, the factor ϵ/μ is proportional to the quantity $S = k^2 dh$ and, therefore, the mixed NLS equation is valid when this quantity is $O(1)$.

Now we discuss solutions to the mixed NLS equation. First, let us consider the limiting cases when the parameter S is small or large. For these limiting cases, the mixed NLS equation can be reduced to the integrable intermediate or cubic NLS equations respectively.

Indeed, if $S \ll 1$ the local nonlinear term in (3.7) is negligible and we get the intermediate NLS equation [14, 15]. Solitary envelope waves of this equation are described by a family of so-called *dark solitons*, which are the amplitude dips propagating along the quasi-harmonic wave,

$$|\Psi|^2 = a^2 - \frac{\alpha K \sin[K\delta]}{\beta(\cosh[K(X-VT)] + \cos[K\delta])}, \tag{3.8a}$$

where a is the amplitude of the carrier quasi-harmonic wave, and K, V are parameters of solitons, which are related by the equation

$$V(V+2\beta a^2) + \alpha^2 K^2 = 2\alpha\beta a^2 K \cot[K\delta].$$

Note that the dark solitons (3.8a) are anisotropic; i.e., solitons with equal amplitudes propagate with different velocities to the left and to the right in the reference frame moving with the group velocity V_g. In a general case, the range of soliton velocity is limited by the interval

$$-\sqrt{\beta^2 a^4 + 2\alpha\beta/\delta} - \beta a^2 \le V \le \sqrt{\beta^2 a^4 + 2\alpha\beta/\delta} - \beta a^2.$$

In the opposite limit ($S \gg 1$), we can neglect the nonlocal nonlinear term in (3.7) and, therefore, get the cubic NLS equation with dark solitons in the form

$$|\Psi|^2 = a^2 - \frac{2\alpha K^2}{\gamma \cosh^2[K(X - VT)]}, \tag{3.8b}$$

where the soliton dispersion relation takes the form

$$V^2 + 4\alpha^2 K^2 = 2\alpha\gamma a^2.$$

For this case, solitary waves are isotropic and their velocities belong to the interval $-\sqrt{2\alpha\gamma}\,a \le V \le \sqrt{2\alpha\gamma}\,a$.

In the analysis presented above we considered the case of finite δ when both solitary waves (3.8a)–(3.8b) have exponentially decaying tails. However, for an infinitely deep fluid ($\delta \to \infty$) the difference between the solitary waves (3.8a)–(3.8b) becomes greater because the solitary waves (3.8a) transform to algebraically decaying solitons described by the rational function,

$$|\Psi|^2 = a^2 - \frac{2\alpha\hat{K}}{\beta\left[1 + \hat{K}^2(X - VT)^2\right]}. \tag{3.8c}$$

Here $\hat{K}$ appears in the limiting transition $\delta \to \infty$, so that $K\delta = \pi[1 - 1/(\hat{K}\delta)]$. The velocity of the algebraic solitons V can be found from the equation

$$V(V + 2\beta a^2) + 2\alpha\beta a^2 \hat{K} = 0.$$

It is obvious that the algebraic solitons always have the negative velocity, $-2\beta a^2 \le V \le 0$.

In the general case, when S is $O(1)$, we may conjecture that the mixed NLS equation (3.7) possesses dark-soliton solutions that have features of both limiting solutions (3.8a)–(3.8b). Because for $X \to \infty$ the nonlocal nonlinear term dominates over the local term, such soliton solutions are expected to resemble the asymptotic solutions (3.8a) or (3.8c) in the far-field region, $KX \gg 1$ or $\hat{K}X \gg 1$. On the other hand, in the near-field region ($KX \ll 1$) the local nonlinear term is more essential and the solitons of the mixed NLS equation (3.7) are locally close to the dark solitons (3.8b). Thus, the balance of the local and nonlocal nonlinearities is crucially important for a correct description of quasi-harmonic wave packets in the shallow–deep limit of a stratified fluid.

4. Two-dimensional nonlocal evolution equations

In this section we generalize our analysis to three-dimensional motion of a stratified shallow–deep fluid and present local and nonlocal models describing quasi-harmonic internal waves modulated in both the transverse and longitudinal directions with respect to the wave propagation direction. It is important to emphasize that the form of the evolution model essentially depends on a ratio of the perturbation scales in both directions.

It is sufficient for our purposes to find only the first correction to the higher-order long-wave equation (2.16) induced by fluid motion in the transverse direction, whose coordinate is y. This correction was derived by Ablowitz and Segur [21] under the conditions of a balance between nonlinear, dispersive, and diffractive effects. Therefore, we refer to their article for details and obtain a two-dimensional (2D) generalization of the long-wave equation in the form

$$\left(A_\tau + \alpha_1 \mathbf{T}_H(A_{\xi\xi}) + \beta_1 A A_\xi\right)_\xi + \frac{c}{2} A_{\eta\eta} = O(\epsilon), \tag{4.1}$$

where $\eta = \epsilon^{3/2} y$. It is remarkable that after substitution of (4.1) into a modified equation (2.7) for the first-order correction term $w^{(1)}$, which contains now an additional term responsible for transverse effects, we get the same form of $w^{(1)}$ as before [see (2.9)]. Therefore, 2D modulations of long internal waves do not lead to a change of the coefficients α_{21}, α_{22}, β_{21}, β_{22}, β_{23}, and γ_2 of the second-order terms of Eq. (2.16).

First, let us consider evolution of quasi-harmonic internal waves under the action of perturbations with comparable scales in the transverse and longitudinal directions. Therefore, we introduce the asymptotic expansion,

$$A = \sum_{n=1}^{\infty} \mu^n A_n(X, Y, T; \Theta), \tag{4.2}$$

with functions A_1 and A_2 given by (3.2) and (3.3) but now depending also on a slow transverse variable $Y = \mu\eta$. Then, in the orders of $O(\mu^3)$ and

$O(\mu^4)$ we get a standard set of evolution equations for amplitude of quasi-harmonic waves $\Psi(X,Y,T)$ and self-consistent mean flow $n(X,Y,T)$,

$$i\Psi_T + \frac{c}{2k}\Psi_{YY} = \alpha_1\Psi_{XX} + \frac{\beta_1^2}{2\alpha_1}(n+|\Psi|^2)\Psi, \quad (4.3a)$$

$$n_{XX} + \frac{c}{4\alpha_1 k}n_{YY} + (|\Psi|^2)_{XX} = 0. \quad (4.3b)$$

This system has the form of the Davey–Stewartson equations [3] describing gravity-capillary wave packets on the surface of homogeneous fluid. On the other hand, the system (4.3a)–(4.3b) with parameters presented in Appendix A follows also from the general equations for quasi-harmonic interfacial waves [6–9] in the shallow–deep limit of the fluid depths. Note that for arbitrary fluid depths the coupling of mean flow n and amplitude Ψ of the interfacial waves is given by a fourth-order differential equation that is different from the Davey–Stewartson system.

It is easy to see that Eqs. (4.3a)–(4.3b) transform to a cubic NLS equation for any plane-wave solutions that depend on a combination $k_x x + k_y y$ excepting the case $k_y \to 0$. For the latter case, the leading-order nonlinear effects completely disappear according to the analysis presented above. Therefore, for wave packets with $k_y \ll k_x$ the model (4.3a)–(4.3b) becomes incorrect and we should modify the asymptotic expansions (4.2). In this case we introduce a modified expansion (3.1) where all terms A_n depend also on a transverse coordinate, which is now given by the variable $Y = \mu^{3/2}\eta$. Using the same asymptotic technique as that in Section 3, we finally get a set of nonlocal equations

$$i\Psi_T = \alpha\Psi_{XX} + \beta\Psi(i+\mathrm{T}_\delta)(|\Psi|^2)_X - \gamma|\Psi|^2\Psi + \beta\tilde{n}\Psi, \quad (4.4a)$$

$$\tilde{n}_{XX} = \frac{c}{2\alpha}(|\Psi|^2)_{YY}. \quad (4.4b)$$

Here $\tilde{n}$ is the Y-induced part of the higher-order mean flow [see (3.6)], which is consistent with amplitude of a carrier quasi-harmonic wave, while the parameters α, β, and γ are the same as those in (3.7). It is clear that for Y-independent perturbations the system (4.4a)–(4.4b) transforms to the mixed NLS equation (3.7). However, for X-independent perturbations the system (4.4a)–(4.4b) fails. Therefore, this new nonlocal model can be applied to describe only almost longitudinal perturbations of quasi-harmonic internal waves.

5. Modulational stability of small-amplitude waves

Here we use the evolution equations derived in the previous section to discuss the problem of modulational stability of small-amplitude internal waves in the shallow–deep stratified fluid. This problem has been partially considered earlier. Tanaka [5] found that the shallow–deep limit of a two-layer fluid yields a stable region with respect to longitudinal modulations of interfacial waves. However, this case is very special and a correct description can be carried out only in the framework of the mixed NLS equation (3.7). Later, Grimshaw and Pullin [6] considered the stability of small-amplitude interfacial waves with respect to oblique perturbations and showed that the small-amplitude waves in the shallow–deep limit become unstable due to a quartet resonance with oblique wave satellites. However, their analysis was restricted by a set of amplitude–mean flow equations, which reduce to the cubic NLS equation for plane-wave perturbations. Recently, similar results were rediscovered by Spector and Miloh [22] who considered a more general problem of nonlinear periodic wave stability in the framework of the 2D long-wave model (4.1) for $H=\infty$. However, terms of the order of $O(\epsilon)$ were not analyzed in this article, although they might be important for this problem. Now we present a general solution of the stability problem in the shallow–deep limit of a stratified fluid.

We start our analysis from the system (4.3a)–(4.3b), which is correct for comparable scales of 2D perturbations of small-amplitude waves. We linearize these equations about a background carrier wave with amplitude a, as

$$\begin{aligned} \Psi &= \left[a+(u_r+iu_j)\exp(\lambda T+iK_X X+iK_Y Y)\right]\exp\left(-i\frac{\beta_1^2 a^2}{2\alpha_1}T\right), \\ n &= u_n\exp(\lambda T+iK_X X+iK_Y Y), \end{aligned} \tag{5.1}$$

where $u_r, u_j, u_n \ll a$. From this linear analysis, we obtain the growth rate λ as a function of the modulation wavenumbers K_X, K_Y,

$$\lambda^2 = \frac{a^2\beta_1^2}{2\alpha_1 k}\left(\frac{2\alpha_1 kK_X^2 - cK_Y^2}{4\alpha_1 kK_X^2 + cK_Y^2}\right)K_Y^2 - \left(\alpha_1 K_X^2 - \frac{cK_Y^2}{2k}\right)^2. \tag{5.2}$$

The first term of (5.2) coincides with the result of Spector and Miloh (see their formulas (4.9), (4.18) in [22]), while the full expression (5.2) can also be deduced from the analysis of Grimshaw and Pullin [6]. It follows from this expression that an instability occurs for wavenumbers K_X, K_Y limited from above by a critical value,

$$\frac{K_Y}{K_X} \leq \left(\frac{K_Y}{K_X}\right)_u = \sqrt{\frac{2\alpha_1 k}{c}}, \tag{5.3a}$$

where we have supposed that $\alpha_1 > 0$ according to the calculations presented in Appendices A and B. The critical ratio $(K_Y/K_X)_u$ determines a slope of the resonance curve in (K_X, K_Y)-space for a quartet–wave resonant interaction (see [12]),

$$2\omega(k,0) = \omega(k+K_X, K_Y) + \omega(k-K_X, -K_Y),$$

where $\omega(k_x, k_y)$ is the frequency of small-amplitude waves that follows from (4.1) in the limit $H \to \infty$,

$$\omega = -\alpha_1 k_x^2 + \frac{ck_y^2}{2k_x}.$$

It is important to note that the instability region of quartet resonance does not disappear in the shallow–deep limit of a stratified fluid, which is different from the shallow–shallow limit [6].

The lower boundary of the instability region can be found from (5.2) in the limit $K_Y \ll K_X$. It is expressed by the quadratic curve

$$\frac{K_Y}{K_X^2} \geq \left(\frac{K_Y}{K_X^2}\right)_l = \frac{2\alpha_1\sqrt{\alpha_1 k}}{a\beta_1}. \tag{5.3b}$$

However, it was emphasized in Section 4 that the model (4.3a)–(4.3b) loses validity for such small scales of transverse wavenumbers. Therefore, a correct description of the lower boundary of the instability region can be realized only in the framework of model (4.4a)–(4.4b).

Next, we turn to the nonlocal model (4.4a)–(4.4b) and use a linearization technique similar to that described for (5.1). The result is presented by the following dispersion relation for the growth rate λ:

$$(\lambda - i\beta a^2 K_X)^2 = c\beta a^2 K_Y^2 - (\beta^2 a^2 + 2\gamma\alpha) a^2 K_X^2 - 2\alpha\beta a^2 K_X^3 \coth[K_X\delta] - \alpha^2 K_X^4. \tag{5.4}$$

It is readily found in the limit $K_X, K_Y \to 0$ that the lower boundary of the instability region is a linear rather than quadratic curve. For longitudinal perturbations ($K_Y = 0$) we see that small-amplitude waves are stable because $\alpha, \beta, \gamma > 0$. This result is in agreement with the analysis of Tanaka [5]. Thus, the instability region is presented by the angular region,

$$\frac{(\beta^2 a^2 + 2\gamma\alpha + 2\alpha\beta/\delta)}{c\beta} \leq \left(\frac{K_Y}{K_X}\right)^2 \leq \frac{2\alpha k}{c}. \tag{5.5}$$

If the cubic nonlinear term can be neglected ($\gamma \to 0$) and the fluid is infinitely deep ($\delta \to \infty$), formula (5.5) transforms to that obtained by Spector and Miloh [22] by direct analysis of periodic wave stability in the framework of the long-wave model (4.1) at $H \to \infty$. Here we have obtained a more general formula for the instability region using another approach, which is based on the short-wave nonlocal models for small-amplitude waves. It is important to mention that for different ratios of perturbation wavenumbers we should use different short-wave models, namely Eqs. (4.3a)–(4.3b) or (4.4a)–(4.4b).

6. Conclusion

In this article we have shown that a correct description of short internal (interfacial) waves in a thin-layer stratified fluid can be realized within the framework of higher-order nonlocal evolution equations. We have derived a new evolution equation that we call the mixed NLS equation. We believe that this equation is universal for the theory of shallow–deep stratified fluid and it governs the evolution not only of internal and interfacial waves but also for wave perturbations in shear and shear-stratified flows where the corresponding long-wave equation takes the form of the ILW equation [23]. Note that usually a cubic NLS equation is considered as an adequate model for envelope waves in stratified shear flows [24, 25].

However, it is important to note that two-dimensional generalizations of this mixed NLS equation will be different for stratified and shear-stratified fluids. This difference is a consequence of the anisotropy induced by shear flows, which modifies the dispersion relation for 2D wave perturbations. Therefore, the problem of finding new 2D nonlocal models for description of envelope waves in stratified shear flows remains open for further studies.

Appendix A: Two-layer fluid

Here we suppose that the fluid is represented by two layers of constant but different densities, ρ_0 for $-d < z \leq 0$ and ρ_∞ for $-h \leq z < -d$. Therefore, the density profile can be expressed by means of the generalized functions,

$$R = \rho_0 \Theta(z+d) + \rho_\infty[1-\Theta(z+d)],$$

where $\Theta(z)$ is the unit Heaviside function, $\Theta(z)=0$ for $z<0$ and $\Theta(z)=1$ for $z>0$. For this profile we can find the modal function $W(z)$ and eigenvalue c of the problem (2.5), (2.6a)–(2.6b) in explicit form,

$$W = \begin{cases} -z/d, & \text{for } -d<z<0 \\ 1, & \text{for } -\infty<z<-d \end{cases}, \qquad c^2 = \frac{g(\rho_\infty - \rho_0)d}{\rho_0}.$$

Next, we find the coefficients $\sigma = 2\rho_0/d$ and $b_1 = -3\rho_0/d$ and the functions $W_{11}(z)$ and $W_{12}(z)$,

$$W_{11} = \begin{cases} -z^2/d, & \text{for } -d<z<0 \\ -d, & \text{for } -\infty<z<-d \end{cases}, \qquad W_{12} = \frac{2}{d}(1-\Theta(z+d)),$$

so that $\overline{W}_{11} = -d$, $\overline{W}_{12} = 2/d$. Using the properties of the generalized functions, we evaluate the integrals entering into the coefficients of Eqs. (2.8) and (2.11). As a result, we obtain the following coefficients of the higher-order long-wave equation (2.16),

$$\alpha_1 = \frac{c\rho_\infty d}{2\rho_0}, \qquad \alpha_{21} = -\frac{cd^2}{6}, \qquad \alpha_{22} = \frac{3cd^2\rho_\infty^2}{8\rho_0^2},$$

$$\beta_1 = -\frac{3}{2d}, \beta_{21} = -\frac{\rho_\infty + 6\rho_0}{4\rho_0}, \beta_{22} = -\frac{\rho_\infty}{8\rho_0}, \beta_{23} = \frac{17\rho_\infty + 12\rho_0}{8\rho_0}, \gamma_2 = 0.$$

Finally, we find the coefficients of the mixed NLS equation (3.7),

$$\alpha = \frac{c\rho_\infty d}{2\rho_0} > 0,$$

$$\beta = \frac{9\rho_0}{8ckd^3\rho_\infty} > 0,$$

$$\gamma = \frac{9k\rho_0^2}{8cd^2\rho_\infty^2}\left(1 + \frac{31\rho_\infty^2}{12\rho_0^2}\right) > 0.$$

Note that coefficient γ coincides with the results of Tanaka [5] and Grimshaw and Pullin [6].

Appendix B: Constant stratification layer

For this case we suppose that $R(z)=\rho_\infty \exp[-N^2(z+d)/g]$ for $-d \le z \le 0$ and $R(z)=\rho_\infty$ for $-h \le z \le -d$, where the constant quantity N^2 is referred to as the Brunt–Vaisala frequency. In addition, we use the Boussinesq approximation so that the function $R(z)$ in the problem (2.5) and all integral coefficients is considered to be constant. Then, we find a simple solution to the problem (2.5),

$$W = \begin{cases} (-1)^{n+1}\sin\left[\dfrac{\pi(1+2n)z}{2d}\right], & \text{for } -d<z<0 \\ 1, & \text{for } -\infty<z<-d \end{cases},$$

$$c \equiv \pm c_n = \pm\frac{2Nd}{\pi(1+2n)},$$

where $n=0,1,2,\ldots$. Thus, for this case there is an infinite countable set of modes of long internal waves. We can derive the higher-order long-wave equation for each internal mode. First, we evaluate the parameters $\sigma = \pi^2(1+2n)^2/(4d)$ and $b_1 = -\pi^2(1+2n)^2/(2d^2)$ and solve the linear inhomogeneous equations for the functions $W_{11}(z)$ and $W_{12}(z)$,

$$W_{11} = \begin{cases} (-1)^{n+1}z\left(\sin\left[\dfrac{\pi(1+2n)z}{2d}\right]-\dfrac{2}{\pi(1+2n)}\cos\left[\dfrac{\pi(1+2n)z}{2d}\right]\right), \\ \qquad \text{for } -d<z<0 \\ -d, \qquad \text{for } -\infty<z<-d \end{cases}$$

$$W_{12} = \begin{cases} \dfrac{(-1)^n\pi(1+2n)}{d^2}z\cos\left[\dfrac{\pi(1+2n)z}{2d}\right], & \text{for } -d<z<0 \\ 0, & \text{for } -\infty<z<-d. \end{cases}$$

Using these expressions we can evaluate coefficients of Eq. (2.16),

$$\alpha_1 = \frac{4cd}{\pi^2(1+2n)^2}, \qquad \alpha_{21} = -\frac{8cd^2}{\pi^4(1+2n)^4},$$

$$\alpha_{22} = \frac{40cd^2}{\pi^4(1+2n)^4}\left(1-\frac{\pi^2(1+2n)^2}{20}\right),$$

$$\beta_1 = -\frac{2}{d}, \qquad \beta_{21} = -\left(1+\frac{40}{3\pi^2(1+2n)^2}\right), \qquad \beta_{22} = \frac{8}{\pi^2(1+2n)^2},$$

$$\beta_{23} = 1+\frac{20}{\pi^2(1+2n)^2}, \qquad \gamma_2 = \frac{2}{cd^2},$$

and of the mixed NLS equation (3.7),

$$\alpha = \frac{4cd}{\pi^2(1+2n)^2} > 0,$$

$$\beta = \frac{\pi^2(1+2n)^2}{4ckd^3} > 0,$$

$$\gamma = \frac{4k}{3cd^2}\left(1+\frac{9\pi^2(1+2n)^2}{32}\right) > 0.$$

Acknowledgments

The authors are indepted to P. Christodoulides, Yu. A. Stepanyants, and J.-M. Vanden-Broeck for useful discussions. The paper was supported in part by Grant No. INTAS-93-1373 and by ARC, Grant No. A-892306.

References

1. D. J. Benney and A. C. Newell, The propagation of nonlinear wave envelopes, *J. Math. Phys.* 46:133–139 (1967).
2. H. Hasimoto and H. Ono, Nonlinear modulation of gravity waves, *J. Phys. Soc. Jpn.* 33:805–811 (1972).
3. A. Davey and K. Stewartson, On three-dimensional packets of surface waves, *Proc. Roy. Soc. London Ser. A* 338:101–110 (1974).
4. V. D. Djordjevic and L. G. Redekopp, On two-dimensional packets of capillary gravity waves, *J. Fluid Mech.* 79:703–714 (1977).
5. M. Tanaka, Nonlinear self-modulation of interfacial waves, *J. Phys. Soc. Jpn.* 51:2016–2023 (1982).
6. R. H. J. Grimshaw and D. I. Pullin, Stability of finite-amplitude interfacial waves. Part 1. Modulational instability for small-amplitude waves, *J. Fluid Mech.* 160:297–315 (1985).
7. A. Dixon, A stability analysis for interfacial waves using a Zakharov equation, *J. Fluid Mech.* 214:185–210 (1990).
8. A. K. Dhar and K. P. Das, Stability analysis from fourth order evolution equation for small but finite amplitude interfacial waves in the presence of a basic current shear, *J. Austral. Math. Soc. B* 35:348–365 (1994).
9. P. Christodoulides and F. Dias, Stability of capillary–gravity interfacial waves between two bounded fluids, *Phys. Fluids* 7:3013–3027 (1995).
10. R. S. Johnson, On the modulation of water waves in the neighbourhood of $kh \approx 1.363$, *Proc. Roy. Soc. London Ser. A* 357:131–141 (1977).
11. T. Kakutani and K. Michihiro, Marginal state of modulational instability. Note on Benjamin–Feir instability, *J. Phys. Soc. Jpn.* 52:4129–4137 (1983).

12. J. M. McLean, Instability of finite-amplitude gravity waves on water of finite depth, *J. Fluid Mech.* 114:331-341 (1982).

13. R. H. J. Grimshaw, Solitary waves in slowly varying environments: wave packets, in *Advances in Nonlinear Waves* (L. Debnath, Ed.), Pitman Research Notes in Mathematics, Vol. 95, 1984, pp. 38–58.

14. D. Pelinovsky, Intermediate nonlinear Schrödinger equation for internal waves in a fluid of finite depth, *Phys. Lett. A* 197:401–406 (1995).

15. D. E. Pelinovsky and R. H. J. Grimshaw, A spectral transform for the intermediate nonlinear Schrödinger equation, *J. Math. Phys.* 36(8):4203–4219 (1995).

16. T. B. Benjamin, Internal waves of permanent form in fluids of great depth, *J. Fluid Mech.* 29:559–592 (1967).

17. R. E. Davis and A. Acrivos, Solitary internal waves, *J. Fluid Mech.* 29:593–608 (1967).

18. R. A. Kraenkel, M. A. Manna, and J. G. Pereira, The Korteweg–de Vries hierarchy and long water-waves, *J. Math. Phys.* 36:307–320 (1995).

19. Y. Matsuno, Higher-order nonlinear evolution equation for interfacial waves in a two-layer fluid system, *Phys. Rev. E* 49:2091–2095 (1994).

20. R. Grimshaw, A second-order theory for solitary waves in deep fluids, *Phys. Fluids* 24:1611–1618 (1981).

21. M. J. Ablowitz and H. Segur, Long internal waves in fluids of great depth, *Stud. Appl. Math.* 62:249–262 (1980).

22. M. D. Spector and T. Miloh, Stability of nonlinear periodic internal waves in a deep stratified fluid, *SIAM J. Appl. Math.* 54:688–707 (1994).

23. S. A. Maslowe and L. G. Redekopp, Long nonlinear waves in stratified shear flows, *J. Fluid Mech.* 101:321–348 (1980).

24. R. H. J. Grimshaw, Modulation of an internal gravity wave packet in a stratified shear flow, *Wave Motion* 3:81–103 (1981).

25. A. K. Liu and D. J. Benney, The evolution of nonlinear wave trains in stratified shear flows, *Stud. Appl. Math.* 64:247–269 (1981).

Monash University

(Received July 21, 1995)