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Survey on global existence in the nonlinear Dirac equations in one spatial dimension

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Abstract

We consider the nonlinear Dirac equations in one spatial dimension and review various results on global existence of solutions in H^1. Depending on the character of the nonlinear terms, existence of the large-norm solutions can be extended for all times. Global existence of the small-norm solutions is proved for the most general nonlinear Dirac equations with cubic and higher-order nonlinear terms. Integrability is used to find conditions that no solitons occur in the Cauchy problem for the massive Thirring model with small initial data in L^2.

1 Introduction

The goal of this article is to survey a number of recent results on global well-posedness of the nonlinear Dirac equations in the space of one dimension. The nonlinear Dirac equations are known for long time in quantum mechanics and relativity theory [7, 22]. Recently these equations were used to model other physical systems such as photonic crystals and Bose–Einstein condensates in optical lattices [1, 17, 21].

The nonlinear Dirac equations are similar to the nonlinear Klein–Gordon equation on one hand and to the nonlinear Schrödinger equation on the other hand. In the former case, the reduction to the nonlinear Klein–Gordon equation is possible for a special form of nonlinear terms in the nonlinear Dirac equations as the linear dispersion relation between the two models are identical. In the latter case, the reduction to the nonlinear Schrödinger equation holds in the asymptotic limit using an envelope wave approximation near one branch of the wave spectrum in the nonlinear Dirac equations. Since analysis of global well-posedness in the nonlinear Klein-Gordon and nonlinear Schrödinger equations has been booming in the last ten years, it is not surprising that the interest of harmonic analysts turns recently to the nonlinear Dirac equations.

The organization of this article is as follows. Section 2 sets up the nonlinear Dirac equations and reviews a number of physically relevant models. Global well-posedness in H^1 is studied in Section 3 using apriori estimates as in Goodman *et al.* [6]. We show that the nonlinear Dirac equations with a special structure of nonlinear terms are globally well-posed even for large powers of the nonlinear terms. This result is different from the behavior of the nonlinear Schrödinger equation.

Section 4 deals with global solutions for small initial data in H^1 using analysis in Strichartz spaces as in Pelinovsky & Stefanov [18]. We prove the global existence of small-norm solutions for the nonlinear Dirac equations with quintic and higher-order nonlinear terms.

The nonlinear Dirac equations with cubic terms are considered in Section 5. The decay of small-norm solutions to zero is proved using the recent results of Hayashi & Naumkin [8, 9]. In Section 6, we discuss the special role of the integrable version of the nonlinear Dirac equations known as the massive Thirring model [22], for which global well-posedness in L^2 was proved recently by Candy [2]. Using the formalism of the inverse scattering transform, we show that small initial data in L^2 are associated to purely continuous spectrum of the Lax operator and admits no solitons in the long-time asymptotics.

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2 Model

Let us consider the nonlinear Dirac equations

$$\left\{ \begin{array}{l} i(u_t + u_x) + v = \partial_{\bar{u}} W(u,v), \\ i(v_t - v_x) + u = \partial_{\bar{v}} W(u,v), \end{array} \right. \tag{2.1}$$

where $(x,t) \in \mathbb{R}^2$, $(u,v) \in \mathbb{C}^2$, and $W(u,v) : \mathbb{C}^2 \to \mathbb{R}$ is a nonlinear function which satisfies the following three conditions:

- symmetry $W(u,v) = W(v,u)$;
- gauge invariance $W(e^{i\theta}u, e^{i\theta}v) = W(u,v)$ for any $\theta \in \mathbb{R}$;
- polynomial in (u,v) and $(\bar{u},\bar{v})$.

The nonlinear Dirac equations can be rewritten in the abstract evolutionary form,

$$i\partial_t \mathbf{u} = \mathcal{H}\mathbf{u} + \mathbf{f}(\mathbf{u}), \quad \mathcal{H} = \begin{bmatrix} -i\partial_x & -1 \\ -1 & i\partial_x \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{bmatrix} \partial_{\bar{u}} W(u,v) \\ \partial_{\bar{v}} W(u,v) \end{bmatrix}. \tag{2.2}$$

A homogeneous quartic polynomial $W(u,v)$ satisfying the three properties above is characterized by Chugunova & Pelinovsky [3],

$$W = \alpha_1(|u|^4 + |v|^4) + \alpha_2|u|^2|v|^2 + \alpha_3(\bar{u}v + u\bar{v})^2 + \alpha_4(|u|^2 + |v|^2)(\bar{u}v + u\bar{v}), \tag{2.3}$$

where $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^4$ are numerical coefficients.

The standard example of the nonlinear Dirac equations with

$$W = \alpha(|u|^2 + |v|^2)^2 + 2\alpha|u|^2|v|^2, \quad \alpha \in \mathbb{R}, \tag{2.4}$$

occurs in the context of periodic dielectric materials under the Bragg resonance [21]. The account of a zero-mean periodic modulation of the nonlinear refractive index gives the nonlinear Dirac equations with

$$W = \alpha(\bar{u}v + u\bar{v})(|u|^2 + |v|^2) + \beta((\bar{u}v + u\bar{v})^2 - 2|u|^2|v|^2), \tag{2.5}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ are proportional to Fourier coefficients of the nonlinear refractive index [1].

Two models are relevant for general relativity: the massive Thirring model with $W = 4|u|^2|v|^2$ [22] and the massive Gross–Neveu model with $W = 2(\bar{u}v + u\bar{v})^2$ [7]. Under the following change of variables,

$$\boldsymbol{\psi} = T\mathbf{u}, \quad \mathbf{g} = T\mathbf{f}, \quad T = \begin{bmatrix} 1 & -1 \\ -i & -i \end{bmatrix}, \tag{2.6}$$

the nonlinear Dirac equation (2.2) can be written in the equivalent form

$$i\partial_t \boldsymbol{\psi} = \mathcal{M}\boldsymbol{\psi} + \mathbf{g}, \quad \mathcal{M} = \begin{bmatrix} 1 & \partial_x \\ -\partial_x & -1 \end{bmatrix}. \tag{2.7}$$

If $\boldsymbol{\psi} = (\psi, \phi)$, system (2.7) can be written as follows: the massive Thirring model is

$$\begin{cases} i\psi_t - \psi - \phi_x = (\psi^2 + \phi^2)\bar{\psi}, \\ i\phi_t + \phi + \psi_x = (\psi^2 + \phi^2)\bar{\phi}, \end{cases} \tag{2.8}$$

and the massive Gross–Neveu model is

$$\begin{cases} i\psi_t - \psi - \phi_x = (\psi^2 - \phi^2)\psi, \\ i\phi_t + \phi + \psi_x = (\phi^2 - \psi^2)\phi, \end{cases} \tag{2.9}$$

Thus, we see that all coefficients of the general quartic potential (2.3) have physically relevant applications. Note also that W may have sixth-order and higher-order terms, as in the context of the Feshbach resonance for Bose–Einstein condensates [19], where

$$W = \alpha(|u|^2 + |v|^2)|u|^2|v|^2, \quad \alpha \in \mathbb{R}.$$

Local existence of solutions of the nonlinear Dirac equations (2.2) in Sobolev space $H^s(\mathbb{R})$ can be proved with standard methods using the Duhamel formulation and the fixed-point arguments [4, 6]. If $\mathbf{u}_0 \in H^s(\mathbb{R})$ for a fixed $s > \frac{1}{2}$, then there exists a $T > 0$ such that the nonlinear Dirac equations (2.2) admits a unique solution

$$\mathbf{u}(t) \in C([0,T], H^s(\mathbb{R})) \cap C^1([0,T], H^{s-1}(\mathbb{R})),$$

where $\mathbf{u}(t)$ depends continuously on the initial data $\mathbf{u}(0) = \mathbf{u}_0$.

In what follows, we review results on global well-posedness of general nonlinear Dirac equations (2.1) in some subspaces of $H^1(\mathbb{R})$ or $L^2(\mathbb{R})$.

3 Global well-posedness in H^1

There exist three conserved quantities of the nonlinear Dirac equations (2.1) corresponding to *Hamiltonian* H, *momentum* P, and *charge* Q,

$$H = \frac{i}{2}\int_{\mathbb{R}} (u_x\bar{u} - u\bar{u}_x - v_x\bar{v} + v\bar{v}_x)\, dx + \int_{\mathbb{R}} (v\bar{u} + u\bar{v} - W(u,v))\, dx, \tag{3.1}$$

$$P = \frac{i}{2}\int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} + v\bar{v}_x - v_x\bar{v})\, dx, \tag{3.2}$$

and

$$Q = \int_{\mathbb{R}} \left(|u|^2 + |v|^2\right) dx. \tag{3.3}$$

These conserved quantities are well defined for a local solution in $H^1(\mathbb{R})$ thanks to the Banach algebra of $H^1(\mathbb{R})$ with respect to multiplication.

Unlike the nonlinear Schrödinger equation, the Hamiltonian H is not useful for analysis of global well-posedness because the quadratic part of H is sign-indefinite. Nevertheless, for a special nonlinear function $W(u,v)$, local solutions can be extended to global solutions for all $t \in \mathbb{R}$. The following theorem generalizes the result of Delgado [4] for the massive Thirring model (2.8) and the result of Goodman *et al.* [6] for the nonlinear Dirac equations with W in (2.4).

Theorem 3.1 *Assume that W is a polynomial in variables $|u|^2$ and $|v|^2$. Let $\mathbf{u}(t) \in C([0,T], H^1(\mathbb{R}))$ be a local solution of the nonlinear Dirac equations (2.2) for some $T > 0$. Then, the solution is extended globally as $\mathbf{u}(t) \in C(\mathbb{R}_+, H^1(\mathbb{R}))$.*

Proof. To extend the local solution $\mathbf{u}(t) \in C([0,T], H^1(\mathbb{R}))$ to all $t \in \mathbb{R}_+$, it is sufficient to prove that the H^1-norm of the solution $\mathbf{u}(t)$ satisfies the estimate

$$\sup_{t\in[0,T]} \|\mathbf{u}(t)\|_{H^1} \leq C(T), \tag{3.4}$$

where the constant $C(T)$ is finite for $T < \infty$ but may grow as $T \to \infty$. By the conservation of Q in (3.3), we have

$$\|\mathbf{u}(t)\|_{L^2} = \|\mathbf{u}(0)\|_{L^2}, \quad t \in \mathbb{R}. \tag{3.5}$$

To consider $\|\mathbf{u}(t)\|_{L^{2p+2}}$ for a fixed $p > 0$, we multiply the first equation of system (2.1) by $|u|^{2p}\bar{u}$ and the second equation by $|v|^{2p}\bar{v}$, add the two equations, and take the imaginary part. If W depends only on $|u|^2$ and $|v|^2$, the nonlinear function is cancelled out and we obtain

$$\frac{1}{p+1}\partial_t \left(|u|^{2p+2} + |v|^{2p+2}\right) + \frac{1}{p+1}\partial_x \left(|u|^{2p+2} - |v|^{2p+2}\right) = i(v\bar{u} - \bar{v}u)(|u|^{2p} - |v|^{2p}).$$

Integrating this balance equation on $x \in \mathbb{R}$ for a local solution in $C([0,T], H^1(\mathbb{R}))$ and using inequality $|u||v| \leq \frac{1}{2}(|u|^2 + |v|^2)$, we obtain apriori estimate

$$\frac{d}{dt}\|\mathbf{u}(t)\|^{2p+2}_{L^{2p+2}} \leq 4(p+1)\|\mathbf{u}(t)\|^{2p+2}_{L^{2p+2}}. \tag{3.6}$$

By Gronwall's inequality, we have

$$\|\mathbf{u}(t)\|_{L^{2p+2}} \leq e^{2|t|}\|\mathbf{u}(0)\|_{L^{2p+2}}, \quad t \in [0,T]. \tag{3.7}$$

Since the estimate holds for any $p > 0$, it holds for $p \to \infty$ and gives apriori estimate on the L^∞-norm of the local solution $\mathbf{u}(t)$. The bound on the L^∞-norm is needed to control the growth rate of the L^2-norm of $\mathbf{u}_x(t)$ as $t \to \infty$.

Taking x-derivatives and performing a similar computation, we obtain the balance equation

$$\partial_t \left(|u_x|^2 + |v_x|^2\right) + \partial_x \left(|u_x|^2 - |v_x|^2\right) = i \left(u_x \partial_x \partial_{\bar{u}} + v_x \partial_x \partial_{\bar{v}} - \bar{u}_x \partial_x \partial_u - \bar{v}_x \partial_x \partial_v\right) W(u,v).$$

Let $N \geq 1$ be the degree of the polynomial W in variables $|u|^2$ and $|v|^2$. Integrating the previous equation over $x \in \mathbb{R}$ and using bound (3.7), we obtain the estimate

$$\frac{d}{dt}\|\mathbf{u}_x(t)\|_{L^2}^2 \leq C_W e^{4(N-1)|t|}\|\mathbf{u}_x(t)\|_{L^2}^2,$$

where the constant $C_W > 0$ depends on the coefficients of the polynomial W. By Gronwall's inequality again, we obtain

$$\|\mathbf{u}_x(t)\|_{L^2}^2 \leq e^{\frac{C_W}{4(N-1)}\left(e^{4(N-1)|t|}-1\right)}\|\mathbf{u}_x(0)\|_{L^2}^2, \quad t \in [0,T]. \tag{3.8}$$

The exponential factor remains bounded for any finite time $T > 0$. Therefore, $C(T) < \infty$ if $T < \infty$ and bound (3.4) gives global well-posedness of the nonlinear Dirac equations (2.2) in the H^1-norm. $\square$

Remark 3.2 *The result of Theorem 3.1 is very different from the behavior of the nonlinear Schrödinger equations, where global solutions may not exist for nonlinear terms generated by a polynomial W in variables $|u|^2$ and $|v|^2$ of degree $N \geq 3$.*

If W also depends on $(\bar{u}v + u\bar{v})$, apriori estimates of the L^p-norm include nonlinear terms which may lead to the finite-time blow-up of solutions in L^∞ and H^1 norms. That is, there may exist $T_{\max} < \infty$ such that

$$\lim_{t \uparrow T_{\max}} \|\mathbf{u}\|_{H^1} = \infty. \tag{3.9}$$

The next sections address the question if the finite-time blow-up (3.9) can be excluded at least for small-norm solutions in the general nonlinear Dirac equations (2.1).

4 Global well-posedness in Strichartz spaces

Strichartz spaces $L_t^p L_x^q$ and $L_x^q L_t^p$ are defined for $1 \leq p, q \leq \infty$ by the norms

$$\|f\|_{L_t^p L_x^q} := \left(\int_0^T \|f(\cdot,t)\|_{L_x^q}^p dt\right)^{1/p}, \quad \|f\|_{L_x^q L_t^p} := \left(\int_{\mathbb{R}} \|f(x,\cdot)\|_{L_t^p}^q dx\right)^{1/q}, \tag{4.1}$$

where $T > 0$ is an arbitrary time including $T = \infty$. Strichartz spaces have become popular in the context of global scattering and asymptotic stability of solitary waves. Nakanishi [16] applied these spaces to the proof of global well-posedness for small initial data in the nonlinear Klein–Gordon and Schrödinger equations. Here we consider the nonlinear Dirac equations and modify arguments from a more general work of Pelinovsky & Stefanov [18] on the asymptotic stability of small solitary waves.

Let $\mathcal{H}$ be the one-dimensional Dirac operator in (2.2) and $R_{\mathcal{H}}(\lambda) = (\mathcal{H} - \lambda I)^{-1}$ be the resolvent operator, defined as a bounded operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ for any $\lambda \notin \sigma(\mathcal{H})$, where

$$\sigma(\mathcal{H}) \equiv (-\infty, -1] \cup [1, \infty).$$

Using Fourier transform for 2-vectors,

$$\mathrm{f}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\mathrm{f}}(k) e^{ikx} dk, \quad \hat{\mathrm{f}}(k) = \int_{\mathbb{R}} \mathrm{f}(x) e^{-ikx} dx,$$

the resolvent operator $R_{\mathcal{H}}(\lambda)$ can be expressed in the Green's function form,

$$(R_{\mathcal{H}}(\lambda)\mathrm{f})(x) = \int_{\mathbb{R}} G_\lambda(x-y)\mathrm{f}(y)dy, \quad \lambda \notin \sigma(\mathcal{H}), \tag{4.2}$$

where

$$G_\lambda(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{ikx}}{\lambda^2 - 1 - k^2} \begin{bmatrix} -(k+\lambda) & 1 \\ 1 & (k-\lambda) \end{bmatrix} dx. \tag{4.3}$$

Let $\kappa \in \mathbb{C}$ be a solution of the algebraic equation $\kappa^2 + \lambda^2 = 1$ for $\lambda \notin \sigma(\mathcal{H})$ such that $\mathrm{Re}(\kappa) > 0$. After computations of the Fourier integral (4.3), we obtain

$$G_\lambda(x) = \frac{1}{2\kappa} \begin{bmatrix} \lambda + i\kappa \mathrm{sign}(x-y) & -1 \\ -1 & -\lambda - i\kappa \mathrm{sign}(x-y) \end{bmatrix} e^{-\kappa|x-y|}. \tag{4.4}$$

The representation (4.2) with Green's function (4.4) shows that the resolvent operator $R_{\mathcal{H}}(\lambda)$ can be extended to the continuous spectrum $\sigma(\mathcal{H})$ as a bounded operator from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$ for any $\lambda \in \sigma(\mathcal{H}) \backslash \{1, -1\}$ excluding the end points ± 1 of the continuous spectrum,

$$R^{\pm}_{\mathcal{H}}(\lambda) := \lim_{\epsilon \downarrow 0} R_{\mathcal{H}}(\lambda \pm i\epsilon), \quad \lambda \in (-\infty, -1) \cup (1, \infty).$$

It follows from (4.2) and (4.4) that for any $\lambda_0 > 1$, there is $C(\lambda_0) > 0$ such that

$$\sup_{|\lambda| \geq \lambda_0} \|R^{\pm}_{\mathcal{H}}(\lambda)\|_{L^1 \to L^\infty} \leq C(\lambda_0). \tag{4.5}$$

Strichartz estimates for the nonlinear Dirac equations were derived in [18] using analysis of Green's function (4.4). The behavior of the semi-group $e^{-it\mathcal{H}}$ for large Fourier wave numbers was controlled by bound (4.5).

Definition 4.1 *We say that a pair (q, r) is Strichartz admissible for the nonlinear Dirac equations in Strichartz space $L_t^q L_x^r$ if*

$$q \geq 2, \quad r \geq 2 \quad \text{and} \quad \frac{2}{q} + \frac{1}{r} \leq \frac{1}{2}.$$

In particular, $(q, r) = (4, \infty)$ and $(q, r) = (\infty, 2)$ are end-point Strichartz pairs.

Lemma 4.2 *Let (q, r) be a Strichartz admissible pair. There are constants $C > 0$ such that*

$$\|e^{-it\mathcal{H}}\mathbf{f}\|_{L_t^4 L_x^\infty} \leq C\|\mathbf{f}\|_{H_x^1}, \tag{4.6}$$

$$\|e^{-it\mathcal{H}}\mathbf{f}\|_{L_t^\infty H_x^1} \leq C\|\mathbf{f}\|_{H_x^1}, \tag{4.7}$$

$$\left\| \int_0^t e^{-i(t-\tau)\mathcal{H}}\mathbf{F}(\tau, \cdot)d\tau \right\|_{L_t^4 L_x^\infty \cap L_t^\infty H_x^1} \leq C\|\mathbf{F}\|_{L_t^1 H_x^1}. \tag{4.8}$$

Proof. See Lemma 4 in [18]. □

The following theorem simplifies the main result of Pelinovsky & Stefanov [18] to the global small-norm solutions of the nonlinear Dirac equations with quintic and higher-order nonlinear terms.

Theorem 4.3 *Consider the nonlinear Dirac equations (2.2) with homogeneous* $\mathbf{f}$ *such that there is an integer $n \geq 2$ such that*

$$\mathbf{f}(a\mathbf{u}) = a^{2n+1}\mathbf{f}(\mathbf{u}), \quad a \in \mathbb{R}.$$

Assume that $\mathbf{u}(0) \in H^1(\mathbb{R})$ and $\|\mathbf{u}(0)\|_{H^1}$ is sufficiently small. The nonlinear Dirac equations (2.2) admit a global solution

$$\mathbf{u}(t) \in C(\mathbb{R}_+, H^1(\mathbb{R})) \cap L^4(\mathbb{R}_+, L^\infty(\mathbb{R})).$$

Proof. By Duhamel's principle, we can rewrite the Cauchy problem for the nonlinear Dirac equations (2.2) in the integral form,

$$\mathbf{u}(t) = e^{-it\mathcal{H}}\mathbf{u}(0) + \int_0^t e^{-i(t-s)\mathcal{H}}\mathbf{f}(\mathbf{u}(s))ds. \tag{4.9}$$

By Lemma 4.2, solutions of the integral equation (4.9) satisfy the bound,

$$\|\mathbf{u}\|_{L_t^4 L_x^\infty \cap L_t^\infty H_x^1} \leq C\|\mathbf{u}_0\|_{H^1} + C\|\mathbf{f}(\mathbf{u})\|_{L_t^1 H_x^1}, \tag{4.10}$$

for some $C > 0$. We set up the problem of solving the integral equation (4.9) as an iteration scheme, where we look for a fixed point in a small ball in normed space $L_t^4 L_x^\infty \cap L_t^\infty H_x^1$. Because $\mathbf{f}(\mathbf{u})$ is a homogeneous polynomial of degree $2n+1$, we obtain

$$\|\mathbf{f}(\mathbf{u})\|_{L_t^1 H_x^1} \leq C\|(|\mathbf{u}| + |\mathbf{u}_x|)|\mathbf{u}|^{2n}\|_{L_t^1 L_x^2} \leq C\|\mathbf{u}\|_{L_t^\infty H_x^1}\|\mathbf{u}\|_{L_t^{2n} L_x^\infty}^{2n}.$$

By Sobolev embedding and the log convexity of the $L^{2n}(\mathbb{R})$ norms for any $n \geq 2$, we have

$$\|\mathbf{u}\|_{L_t^{2n} L_x^\infty} \leq \|\mathbf{u}\|_{L_t^4 L_x^\infty}^{2/n} \|\mathbf{u}\|_{L_t^\infty L_x^\infty}^{1-2/n} \leq C\|\mathbf{u}\|_{L_t^4 L_x^\infty \cap L_t^\infty H_x^1}.$$

As a result, we obtain

$$\|\mathbf{f}(\mathbf{u})\|_{L_t^1 H_x^1} \leq C\|\mathbf{u}\|_{L_t^4 L_x^\infty \cap L_t^\infty H_x^1}^{2n+1},$$

and the fixed point argument is closed for small $\mathbf{u}(0) \in H^1(\mathbb{R})$. □

Remark 4.4 *Because $\|\mathbf{u}(t)\|_{L^\infty}$ is a continuous function of $t \in \mathbb{R}_+$ and $\|\mathbf{u}(t)\|_{L^\infty} \in L^4(\mathbb{R}_+)$, we have*

$$\lim_{t\to\infty} \|\mathbf{u}(t)\|_{L^\infty} = 0.$$

Moreover, $\|\mathbf{u}(t)\|_{L^\infty} = \mathcal{O}(t^{-1/4-\nu})$ as $t \to \infty$ for some $\nu > 0$.

5 Decay of small solutions for cubic Dirac equations

Here we consider the nonlinear Dirac equations (2.1) with the cubic nonlinear terms, which are generated by the quartic function W in (2.3). If W is a function of $|u|^2$ and $|v|^2$, results of Theorem 3.1 show that the local solutions in H^1 are globally well-posed. On the other hand, decay of small solutions to zero is not covered by the results of Theorem 4.3 because the cubic nonlinear terms with $n = 1$ can not be treated by the nonlinear analysis in Strichartz spaces. Additional constraints must be imposed to ensure that small solutions in H^1 decay to zero.

A similar question has been addressed in the context of the nonlinear Klein–Gordon equation,

$$u_{tt} - u_{xx} + u + |u|^{p-1}u = 0, \tag{5.1}$$

Let us consider the semi-group of the linear Klein–Gordon equation,

$$S(t) := e^{-it\langle\partial_x\rangle}, \quad t > 0,$$

where $\langle x\rangle = \sqrt{1+x^2}$. The $L^1 \to L^\infty$ norm of the semi-group $S(t)$ decays like $\mathcal{O}(t^{-1/2})$ as $t \to \infty$. As a result, the term $\|u\|_{L^\infty}^{p-1} = \mathcal{O}(t^{-(p-1)/2})$ is absolutely integrable in t for $p > 3$. The dispersive decay of small initial data for $p > 3$ was proven by Georgiev & Lecente [5]. In the critical case $p = 3$ of the cubic nonlinear terms, the decay of small solutions to zero was recently obtained by Hayashi & Naumkin [8, 9].

We will show that the results of Hayashi & Naumkin [8, 9] can be equally applied to the nonlinear Dirac equations (2.1). Using the Fourier transform, we rewrite the system as

$$\left\{ \begin{array}{l} i\hat{u}_t - k\hat{u} + \hat{v} = \hat{f}, \\ i\hat{v}_t + k\hat{v} + \hat{u} = \hat{g}, \end{array} \right. \tag{5.2}$$

Using the projection matrix

$$\hat{P} = \left[\begin{array}{cc} 1 & -\sqrt{1+k^2} - k \\ \sqrt{1+k^2} + k & 1 \end{array} \right], \quad \hat{P}^{-1} = \frac{1}{2\sqrt{1+k^2}} \left[\begin{array}{cc} \sqrt{1+k^2} - k & 1 \\ -1 & \sqrt{1+k^2} - k \end{array} \right],$$

we can write (5.2) in the equivalent form

$$\left\{ \begin{array}{l} i\hat{a}_t - \langle k \rangle \hat{a} = \frac{1}{2\langle k \rangle}(-\hat{f} + (\langle k \rangle - k)\hat{g}), \\ i\hat{b}_t + \langle k \rangle \hat{b} = \frac{1}{2\langle k \rangle}(\hat{g} + (\langle k \rangle - k)\hat{f}), \end{array} \right. \tag{5.3}$$

In the physical space, this system takes the form

$$\left\{ \begin{array}{l} a_t + i\langle i\partial_x \rangle \hat{a} = \frac{1}{2i}\langle i\partial_x \rangle^{-1}(-\tilde{f}(a,b) + (\langle i\partial_x \rangle + i\partial_x)\tilde{g}(a,b)), \\ b_t - i\langle i\partial_x \rangle b = \frac{1}{2i}\partial_x \rangle^{-1}(\tilde{g}(a,b) + (\langle i\partial_x \rangle + i\partial_x)\tilde{f}(a,b)), \end{array} \right. \tag{5.4}$$

where

$$\begin{aligned} \tilde{f}(a,b) &= f(b + (i\partial_x - \langle i\partial_x \rangle)a, a + (\langle i\partial_x \rangle - i\partial_x)b), \\ \tilde{g}(a,b) &= g(b + (i\partial_x - \langle i\partial_x \rangle)a, a + (\langle i\partial_x \rangle - i\partial_x)b). \end{aligned}$$

If f and g are homogeneous cubic polynomials in variables u and v, then $\tilde{f}$ and $\tilde{g}$ are cubic polynomials in variables a, b, $\partial_x a$, $\partial_x b$, $\langle i\partial_x \rangle a$, and $\langle i\partial_x \rangle b$. By Remark 1.1 in [9], all these cubic coefficients can be treated in spaces $H_s^m(\mathbb{R})$ equipped with the norm,

$$\|u\|_{H_s^m} := \|\langle x \rangle^s \langle i\partial_x \rangle^m u\|_{L^2}.$$

The method of Hayashi & Naumkin [8, 9] yields the following theorem.

Theorem 5.1 *Fix small $\epsilon > 0$ and assume that $\mathbf{u}(0) \in H_1^4(\mathbb{R})$ with $\|\mathbf{u}(0)\|_{H_1^4} \leq \epsilon$. There exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, the Cauchy problem for the nonlinear Dirac equations (2.2) with the quartic function W in (2.3) admit a unique global solution $\mathbf{u}(t) \in C(\mathbb{R}_+, H_1^4(\mathbb{R}))$ satisfying the time decay estimate*

$$\|\langle i\partial_x \rangle \mathbf{u}(t)\|_{L^\infty} \leq C\epsilon(1+t)^{-1/2}, \quad t \in \mathbb{R}_+.$$

Remark 5.2 *Small norm $\|\mathbf{u}(0)\|_{H_1^4}$ implies small H^1 and L^1 norms of the initial data $\mathbf{u}(0)$.*

6 Massive Thirring model in L^2

We consider the integrable case of the nonlinear Dirac equations (2.1) with $W = 2|u|^2|v|^2$, which is referred to as the massive Thirring model (MTM). Theorem 3.1 implies global existence of solutions of (MTM) in H^1. Theorem 5.1 implies that these solutions decay to zero in the L^∞ norm. More results were obtained for the massive Thirring model recently.

Selberg and Tesfahun [20] proved local well-posedness of (MTM) in $H^s(\mathbb{R})$ for $s > 0$ and global well-posedness in $H^s(\mathbb{R})$ for $s > \frac{1}{2}$. Machihara *et al.* [14] proved for similar nonlinear Dirac equations with quadratic nonlinear terms that local well-posedness holds in $H^s(\mathbb{R})$ for $s > -\frac{1}{2}$ and that the Cauchy problem is ill-posed in $H^{-1/2}(\mathbb{R})$. Using ideas from [20] and [14], Candy [2] proved local and global well-posedness of (MTM) in $L^2(\mathbb{R})$.

In characteristic coordinates,

$$\xi = \frac{x-t}{2}, \quad \tau = \frac{x+t}{2},$$

the nonlinear Dirac equations (2.1) with $W = 2|u|^2|v|^2$ are written explicitly by

$$\begin{cases} iu_\tau + v = 2|v|^2 u, \\ -iv_\xi + u = 2|u|^2 v. \end{cases} \tag{6.1}$$

Let us introduce the change of variables,

$$u(\xi,\tau) = \frac{1}{2}\, w(\xi,\tau)\, \exp\left(-\frac{i}{2}\int_\xi^\infty |w|^2(\xi',\tau)d\xi'\right). \tag{6.2}$$

The second equation of system (6.1) can be solved with

$$v(\xi,\tau) \;=\; -\frac{i}{2}\partial_\xi^{-1} w(\xi,\tau)\exp\left(-\frac{i}{2}\int_\xi^\infty |w|^2(\xi',\tau)d\xi'\right), \tag{6.3}$$

where

$$\partial_\xi^{-1} w(\xi,\tau) := -\int_\xi^\infty w(\xi',\tau)d\xi'.$$

If $v(\cdot,\tau) \in H^1(\mathbb{R})$, then $w(\cdot,\tau) \in L^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$ satisfies the zero-mass constraint $\int_{\mathbb{R}} w(\xi,\tau)d\xi = 0$.

With the substitutions (6.2)–(6.3) to (6.1), the massive Thirring model becomes the scalar evolution equation,

$$w_\tau - \partial_\xi^{-1} w + i|\partial_\xi^{-1} w|^2 w = 0. \tag{6.4}$$

The scalar equation (6.4) is invariant under the following change of variables,

$$w = \delta W(X,T), \quad X = \delta^2\xi, \quad T = \delta^{-2}\tau, \quad \delta > 0, \tag{6.5}$$

which implies that the massive Thirring model is the L^2-critical model with $\|w\|_{L^2} = \|W\|_{L^2}$. Therefore, it is natural to expect that the dispersive decay to zero can occur already for a smooth initial data with a small L^2-norm. To deal with this question, we shall review the inverse scattering transform method for the massive Thirring model (6.1).

The scalar equation in characteristic coordinates (6.4) appears as a solvability condition [11, 12] of the spectral problem,

$$\partial_\xi \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} -i\lambda^2 & \lambda w \\ -\lambda \bar{w} & i\lambda^2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \tag{6.6}$$

and the linear time-evolution problem,

$$\partial_\tau \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = i \begin{bmatrix} \eta^2 - \frac{1}{2}|\partial_\xi^{-1} w|^2 & -\eta \partial_\xi^{-1} w \\ -\eta \partial_\xi^{-1} w & -\eta^2 + \frac{1}{2}|\partial_\xi^{-1} w|^2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \tag{6.7}$$

where $\lambda \in \mathbb{C}$ is a (ξ, τ)-independent spectral parameter and $\eta = \frac{1}{2\lambda}$. Note that the massive Thirring model in the laboratory coordinates can also be represented as a solvability condition of the Lax system [10, 13].

The spectral problem (6.6) has some symmetries. If $(\psi_1(x;\lambda), \psi_2(x;\lambda))$ is a solution of (6.6), then

$$(\psi_1(x;-\lambda), -\psi_2(x;-\lambda)), \quad (\bar{\psi}_2(x;-\bar{\lambda}), \bar{\psi}_1(x;-\bar{\lambda})), \quad (\bar{\psi}_2(x;\bar{\lambda}), -\bar{\psi}_1(x;\bar{\lambda})) \tag{6.8}$$

are also solutions of (6.6).

The continuous spectrum of the spectral problem (6.6) is located for $\lambda \in \mathbb{R} \cup \{i\mathbb{R}\}$, whereas isolated eigenvalues are located symmetrically in quartets $(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda})$ in all quadrants of the complex plane for λ [12]. Isolated eigenvalues are associated with solitons that occur in the long-time dynamics of the solution $w(\xi, \tau)$ thanks to the independence of λ from τ and the inverse scattering transform technique. We shall prove that solitons are absent if $w(\cdot, \tau)$ for a frozen τ has a small norm in $L^2(\mathbb{R})$. For clarity of presentation, we do not write τ in the arguments of w and $\psi_{1,2}$.

Theorem 6.1 *Fix small $\epsilon > 0$ and assume that $w \in L^2(\mathbb{R})$ with $\|w\|_{L^2} \leq \epsilon$. There is $C > 0$ such that the spectral problem (6.6) admits no solutions in $L^2(\mathbb{R})$ for any $\lambda \in \mathbb{C}$ with $\arg(\lambda) \in \left(C\epsilon^2, \frac{\pi}{2} - C\epsilon^2\right)$. If in addition, $w \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\partial_\xi w \in L^1(\mathbb{R})$ with*

$$\|w\|_{L^1}(\|w\|_{L^\infty} + \|\partial_\xi w\|_{L^1}) \leq \epsilon, \tag{6.9}$$

then the spectral problem (6.6) admits no solutions in $L^2(\mathbb{R})$ for any $\lambda \in \mathbb{C}$.

Proof. Let us choose $\lambda \in \mathbb{C}$ in the first quadrant of the complex plane. Because of the symmetry (6.8) of eigenvectors, the results are valid in all four quadrants of the complex plane. If λ is in the first quadrant of $\mathbb{C}$, then $\text{Im}(\lambda^2) > 0$ and we can parameterize λ by $\lambda = |\lambda| e^{i\theta}$ with $\theta \in \left(0, \frac{\pi}{2}\right)$.

If $\text{Im}(\lambda^2) > 0$, then $e^{-i\lambda^2\xi}$ decays to zero as $\xi \to -\infty$, and we can introduce $\psi(\xi) = e^{-i\lambda^2\xi}\varphi(\xi)$ with boundary conditions $\lim_{\xi\to-\infty}\varphi(\xi) = (1,0)^T$ for eigenvectors of the spectral problem (6.6).

Integrating system (6.6) from $-\infty$ to ξ under the boundary conditions for $\varphi(\xi)$, we obtain the integral equations,

$$\varphi_1(\xi) = 1 + \lambda \int_{-\infty}^{\xi} w(\xi')\varphi_2(\xi')d\xi', \quad \varphi_2(\xi) = -\lambda \int_{-\infty}^{\xi} e^{2i\lambda^2(\xi-\xi')}\bar{w}(\xi')\varphi_1(\xi')d\xi'. \quad (6.10)$$

Using the exact integral for $\lambda = |\lambda|e^{i\theta}$ with $\theta \in \left(0, \frac{\pi}{2}\right)$,

$$I(\lambda) := |\lambda^2| \int_0^\infty e^{-2\mathrm{Im}(\lambda^2)x}dx = \frac{|\lambda|^2}{2\mathrm{Im}(\lambda^2)} = \frac{1}{2\sin(2\theta)},$$

the Schwarz inequality, and Young's inequality for convolution integrals, we obtain

$$\|\varphi_1 - 1\|_{L^\infty} \leq \|w\|_{L^2}\|\lambda\varphi_2\|_{L^2}, \quad \|\lambda\varphi_2\|_{L^2} \leq I(\lambda)\|w\|_{L^2}\|\varphi_1\|_{L^\infty}.$$

Closing the inequalities and using fixed-point arguments, we can see that if

$$I(\lambda)\|w\|_{L^2}^2 < 1,$$

then there is a unique solution of system (6.10) for $\varphi_1 \in L^\infty(\mathbb{R})$ and $\lambda\varphi_2 \in L^2(\mathbb{R})$ such that $\|\varphi_1 - 1\|_{L^\infty} < 1$. Therefore, $\varphi_1(\xi) \not\to 0$ as $\xi \to +\infty$, and so $\psi_1(\xi)$ grows exponentially as $\xi \to +\infty$. Eigenvectors in L^2 may only exist if $I(\lambda)\|w\|_{L^2}^2 \geq 1$, that is, if either $\theta \in (0, C\epsilon^2)$ or $\theta \in \left(\frac{\pi}{2} - C\epsilon^2, \frac{\pi}{2}\right)$ for some $C > 0$.

To eliminate eigenvectors everywhere in the first quadrant of $\mathbb{C}$, we add now the condition (6.9). Integrating the second equation of system (6.10) by parts and using the first equation, we obtain

$$\begin{aligned}
\lambda\varphi_2(\xi) &= -\lambda^2 \int_{-\infty}^{\xi} e^{2i\lambda^2(\xi-\xi')}\bar{w}(\xi')\varphi_1(\xi')d\xi' \\
&= \frac{1}{2i}\bar{w}(\xi)\varphi_1(\xi) - \frac{1}{2i}\int_{-\infty}^{\xi} e^{2i\lambda^2(\xi-\xi')}\left(\partial_{\xi'}\bar{w}(\xi')\varphi_1(\xi') + \bar{w}(\xi')\partial_{\xi'}\varphi_1(\xi')\right)d\xi' \\
&= \frac{1}{2i}\bar{w}(\xi)\varphi_1(\xi) - \frac{1}{2i}\int_{-\infty}^{\xi} e^{2i\lambda^2(\xi-\xi')}\partial_{\xi'}\bar{w}(\xi')\varphi_1(\xi')d\xi' \\
&\qquad - \frac{1}{2i}\int_{-\infty}^{\xi} e^{2i\lambda^2(\xi-\xi')}|w(\xi')|^2\lambda\varphi_2(\xi')d\xi'.
\end{aligned}$$

Using Hölder's inequality and Young's inequality for convolution integrals, we obtain

$$\begin{aligned}
\|\varphi_1 - 1\|_{L^\infty} &\leq \|w\|_{L^1}\|\lambda\varphi_2\|_{L^\infty}, \\
\|\lambda\varphi_2\|_{L^\infty} &\leq \frac{1}{2}\left(\|w\|_{L^\infty} + \|\partial_\xi w\|_{L^1}\right)\|\varphi_1\|_{L^\infty} + \frac{1}{2}\|w\|_{L^2}^2\|\lambda\varphi_2\|_{L^\infty}.
\end{aligned}$$

If $\|w\|_{L^2}^2 < 2$, then there is $C > 0$ such that

$$\begin{aligned}
\|\varphi_1 - 1\|_{L^\infty} &\leq C\|w\|_{L^1}(\|w\|_{L^\infty} + \|\partial_\xi w\|_{L^1})\|\varphi_1\|_{L^\infty}, \\
\|\lambda\varphi_2\|_{L^\infty} &\leq C(\|w\|_{L^\infty} + \|\partial_\xi w\|_{L^1})\|\varphi_1\|_{L^\infty}.
\end{aligned}$$

Under the condition (6.9), there is a unique solution of system (6.10) for $\varphi_1 \in L^\infty(\mathbb{R})$ and $\lambda\varphi_2 \in L^\infty(\mathbb{R})$ such that $\|\varphi_1 - 1\|_{L^\infty} < 1$. Repeating the arguments above, we conclude the proof that no eigenvector in L^2 exists for any $\lambda \in \mathbb{C}$ under the condition (6.9). $\square$

Remark 6.2 *Using the scaling transformation (6.5), we can see that both $\|w\|_{L^2}$ and $\|w\|_{L^1}(\|w\|_{L^\infty} + \|\partial_\xi w\|_{L^1})$ are invariant with respect to parameter δ. Soliton solutions of the massive Thirring model (6.1) correspond to particular values for these quantities. If $\|w\|_{L^2}$ and $\|w\|_{L^1}(\|w\|_{L^\infty} + \|\partial_\xi w\|_{L^1})$ are below these particular values, no solitons can occur in the long-time evolution of the massive Thirring model (6.1).*

Further analysis of the inverse scattering transform using the time-evolution problem (6.7) may give an analogue of Theorem 5.1 for the massive Thirring model, perhaps, with relaxed assumptions on the initial data $\mathbf{u}_0$. Another interesting open problem is to explore global existence of the massive Thirring model in L^2 [2] and obtain L^2-orbital stability of MTM solitons. A similar task was recently achieved by Mizumachi & Pelinovsky [15] in the context of the nonlinear Schrödinger equation. The massive Thirring model is more interesting for orbital stability analysis of solitary waves. Because it is associated with the sign-indefinite Hamiltonian function (3.1), no orbital stability in H^1 can be extracted from the standard energy analysis. These open problems will likely to attract interests of researchers in near future.

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