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Persistence of the Thomas–Fermi approximation for ground states of the Gross–Pitaevskii equation supported by the nonlinear confinement

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ABSTRACT

We justify the Thomas–Fermi approximation for the stationary Gross–Pitaevskii equation with the repulsive nonlinear confinement, which was recently introduced in physics literature. The method is based on the resolvent estimates and the fixed-point iterations. The results cover the case of the algebraically growing nonlinear confinement.

Self-trapping of solitary waves in nonlinear physical media is a commonly known problem of profound significance [1,2]. An obvious condition is that attractive (alias self-focusing) nonlinearity is necessary for the creation of localized states. Recently, a radically different approach to this problem was proposed in Refs. [3–5]: *repulsive* (self-defocusing) nonlinearity that grows at infinity readily gives rise to the self-trapping of localized states, which are stable to weak and strong perturbations alike.

An advantage offered by models with the effective *nonlinear confinement* is a possibility to find particular solutions in an exact form [6–8], and to apply analytical methods for the qualitative approximation of various localized states [9]. The simplest method used for approximating the ground state of energy is the Thomas–Fermi (TF) approximation [10,11]. Comparison with numerical results has demonstrated that the TF approximation produces quite accurate results for the self-trapped modes with sufficiently large amplitudes [3,9]. The objective of the present work is to produce a rigorous estimate of the proximity of the TF approximation to true ground states in models with the spatially growing strength of the defocusing cubic nonlinearity.

In a similar context of the stationary Gross–Pitaevskii equation with the harmonic confinement and the defocusing cubic nonlinearity, the TF approximation was rigorously justified using calculus of variations [12] and reductions to the Painlevé-II equation [13,14] (see also earlier works on the Painlevé-II equation in physics literature [15,16]). The difficulty that arises in this context is that the TF approximation is compactly supported and the derivatives of the ground state diverge in a transitional layer near the boundary. Compared to this complication, we show that the justification of the TF approximation in the stationary Gross–Pitaevskii equation with the nonlinear confinement can be obtained from the standard resolvent estimates and fixed-point arguments.

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Following the main model used in Refs. [3–9], we consider the stationary Gross–Pitaevskii equation with the repulsive nonlinear confinement,

$$-\epsilon^2\Delta u + V(x)u^3 - u = 0, \quad x \in \mathbb{R}^d,\ d = 1, 2, 3, \tag{1}$$

where ϵ is a small parameter corresponding to the TF approximation, Δ is the d-dimensional Laplacian, u is a positive stationary state to be found, and, in accordance with what is said above, the strength of the nonlinear confinement V is supposed to satisfy the following properties: (i) $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^d$, and (ii) $V(x) \to \infty$ as $|x| \to \infty$. Further constraints on the smoothness of V and its growth at infinity will be needed for the main result. Note, however, that no symmetry assumptions on V are needed.

The formal TF solution of the elliptic problem is found for $\epsilon = 0$ and corresponds to the spatially decaying positive eigenfunction:

$$u_0(x) = \frac{1}{\sqrt{V(x)}}, \quad x \in \mathbb{R}^d. \tag{2}$$

If we require $u_0 \in L^2(\mathbb{R}^d)$, so that the stationary state can be normalized in the $L^2(\mathbb{R}^d)$ norm, then $1/V$ needs to be integrable. However, this requirement is not needed for the main persistence result formulated as follows.

Theorem 1. *Assume that $\nabla \log(V) \in H^2(\mathbb{R}^d)$ for $d = 1$ or $\nabla \log(V) \in H^3(\mathbb{R}^d)$ for $d = 2, 3$. There exist positive constants ϵ_0 and C_0 such that for every $\epsilon \in (0, \epsilon_0)$, there exists a unique solution $u = u_0 + U$ of the nonlinear elliptic problem (1) with $U \in H^1(\mathbb{R}^d)$ satisfying*

$$\|U\|_{H^1} \leq C_0\epsilon^2. \tag{3}$$

To study the persistence of the TF approximation, we set $u := w/\sqrt{V}$ and decompose $w := 1 + r$. A similar division representation is used in Refs. [12,14] and the main advantage of this trick is to transform the nonlinear terms of Eq. (1) to become independent of x. When the representation above is substituted in Eq. (1), the nonlinear elliptic problem can be rewritten for the perturbation function r:

$$L_\epsilon r = \epsilon^2 F + N(r), \tag{4}$$

where $N(r) = -3r^2 - r^3$ is the nonlinear term,

$$F = \sqrt{V}\Delta\frac{1}{\sqrt{V}} = -\frac{\Delta V}{2V} + \frac{3|\nabla V|^2}{4V^2} \tag{5}$$

is the source term, and

$$L_\epsilon = 2 - \epsilon^2\Delta + \epsilon^2\frac{1}{V}\nabla V \cdot \nabla - \epsilon^2 F \tag{6}$$

is the linearized operator at the TF approximation. Further, we write L_ϵ as a sum of two operators,

$$\tilde{L}_\epsilon := 2 - \epsilon^2\Delta - \frac{\epsilon^2|\nabla V|^2}{4V^2} \tag{7}$$

and

$$L_\epsilon - \tilde{L}_\epsilon := \epsilon^2\frac{\nabla V \cdot \nabla}{V} + \frac{\epsilon^2}{2}\nabla\left(\frac{\nabla V}{V}\right), \tag{8}$$

where the last term is a multiplicative potential. We establish invertibility of $\tilde{L}_\epsilon$ on any element of $L^2(\mathbb{R}^d)$ in the following lemma.

Lemma 1. *Assume that $\nabla \log(V) \in L^\infty(\mathbb{R}^d)$. There exists a positive constant ϵ_0 such that for every $\epsilon \in (0, \epsilon_0)$ and for every $f \in L^2(\mathbb{R}^d)$, the following is true:*

$$\|\tilde{L}_\epsilon^{-1} f\|_{L^2} \leq \|f\|_{L^2}, \qquad \epsilon\|\nabla\tilde{L}_\epsilon^{-1} f\|_{L^2} \leq \|f\|_{L^2}. \tag{9}$$

Additionally, if $\Delta \log(V) \in L^\infty(\mathbb{R}^d)$, then for every $f \in H^1(\mathbb{R}^d)$, the following is true as well:

$$\|\nabla\tilde{L}_\epsilon^{-1} f\|_{L^2} \leq \|f\|_{H^1}. \tag{10}$$

Proof. Under the condition of $\nabla \log(V) \in L^\infty(\mathbb{R}^d)$, the last term in $\tilde{L}_\epsilon$ is a small bounded negative perturbation added to the first positive term, whereas the second term, $-\epsilon^2\Delta$, is a nonnegative operator. The bilinear form

$$a(u, w) := \int_{\mathbb{R}^d}\left(2\bar{w}u + \epsilon^2\nabla\bar{w} \cdot \nabla u - \frac{\epsilon^2|\nabla V|^2}{4V^2}\bar{w}u\right)dx \tag{11}$$

satisfies the boundedness and coercivity assumptions in the $H^1(\mathbb{R}^d)$ space:

$$|a(u,w)| \leq (2+\epsilon^2 C(V))\|u\|_{H^1}\|w\|_{H^1}, \qquad a(u,u) \geq (2-\epsilon^2 C(V))\|u\|_{L^2}^2 + \epsilon^2\|\nabla u\|_{L^2}^2, \tag{12}$$

where the positive constant $C(V)$ depends on $\|\nabla \log(V)\|_{L^\infty}$. By the Lax–Milgram Theorem, for every $f \in L^2(\mathbb{R}^d)$, there is a unique $u \in H^1(\mathbb{R}^d)$ such that for every $w \in H^1(\mathbb{R}^d)$,

$$a(u,w) = \int_{\mathbb{R}^d} f\bar{w}dx.$$

Hence, u satisfies

$$\|u\|_{L^2}^2 + \epsilon^2\|\nabla u\|_{L^2}^2 \leq a(u,u) = \int_{\mathbb{R}^d} \bar{u} f dx. \tag{13}$$

By the Cauchy–Schwarz inequality, we obtain the bounds (9). Under the additional condition of $\Delta \log(V) \in L^\infty(\mathbb{R}^d)$, we apply operator ∇ to $\tilde{L}_\epsilon u = f$ and write the corresponding equation in the weak form for every $w \in H^1(\mathbb{R}^d)$,

$$a(\nabla u, \nabla w) = \int_{\mathbb{R}^d} \nabla\bar{w}\cdot\nabla f dx + \frac{\epsilon^2}{4}\int_{\mathbb{R}^d} u\nabla\bar{w}\cdot\nabla\left(\frac{|\nabla V|^2}{V^2}\right)dx, \tag{14}$$

where $a(\nabla u, \nabla w)$ is defined by a vector extension of the formula (11). From the Cauchy–Schwarz inequality, we obtain

$$(2-\epsilon^2 C_-(V))\|\nabla u\|_{L^2}^2 \leq a(\nabla u,\nabla u) \leq \|\nabla u\|_{L^2}\|\nabla f\|_{L^2} + \epsilon^2 C_+(V)\|u\|_{L^2}\|\nabla u\|_{L^2},$$

where the positive constants $C_\pm(V)$ depend on $\|\nabla \log(V)\|_{L^\infty}$ and $\|\Delta \log(V)\|_{L^\infty}$. Using the smallness of ϵ^2 and the bound (9), we obtain the bound (10). ∎

Using Lemma 1, the persistence problem (4) can be rewritten as the fixed-point equation:

$$r = \Phi_\epsilon(r) := \epsilon^2\tilde{L}_\epsilon^{-1}F + \tilde{L}_\epsilon^{-1}(\tilde{L}_\epsilon - L_\epsilon)r + \tilde{L}_\epsilon^{-1}N(r). \tag{15}$$

Using the contraction mapping principle, we prove the following result.

Lemma 2. *Assume that $\Delta\log(V) \in L^\infty(\mathbb{R}^d)$ and $\nabla\log(V) \in H^2(\mathbb{R}^d)$, with $d = 1, 2, 3$. There exist positive constants ϵ_0 and C_0 such that for every $\epsilon \in (0,\epsilon_0)$, there exists a unique solution $r \in H^1(\mathbb{R}^d)$ of Eq. (15) satisfying*

$$\|r\|_{H^1} \leq C_0\epsilon^2. \tag{16}$$

Proof. We will prove that under the assumptions of the lemma, operator Φ_ϵ is a contraction on the ball $B_\delta(H^1(\mathbb{R}^d))$ of radius δ if $\delta = C_0\epsilon^2$ for a positive ϵ-independent constant C_0.

From the assumption of $\nabla\log(V) \in H^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$, for $d = 1, 2, 3$, we realize that $F \in H^1(\mathbb{R}^d)$. Applying the bounds (9) and (10), we obtain that, for $\epsilon > 0$ sufficiently small, there is a positive constant $C_1(V)$ that depends on $\|F\|_{H^1}$ such that

$$\|\epsilon^2\tilde{L}_\epsilon^{-1}F\|_{H^1} \leq \epsilon^2 C_1(V). \tag{17}$$

By Sobolev's embedding of $H^1(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for any $p \geq 2$ if $d = 1$, $2 \leq p < \infty$ if $d = 2$, and $2 \leq p \leq 6$ if $d = 3$, and by the estimate (9), we obtain that, for $\epsilon > 0$ sufficiently small, there is a positive constant C_2 such that

$$\|\tilde{L}_\epsilon^{-1}N(r)\|_{H^1} \leq \sqrt{2}\epsilon^{-1}\|N(r)\|_{L^2} \leq \sqrt{2}\epsilon^{-1}(3\|r\|_{L^4}^2 + \|r\|_{L^6}^3) \leq C_2\epsilon^{-1}(\|r\|_{H^1}^2 + \|r\|_{H^1}^3). \tag{18}$$

Finally, under the conditions of $\nabla\log(V)$, $\Delta\log(V) \in L^\infty(\mathbb{R}^d)$, we have the bounds

$$\|(\tilde{L}_\epsilon - L_\epsilon)r\|_{L^2} \leq \epsilon^2\|\nabla\log(V)\|_{L^\infty}\|\nabla r\|_{L^2} + \frac{1}{2}\epsilon^2\|\Delta\log(V)\|_{L^\infty}\|r\|_{L^2}, \tag{19}$$

hence, using estimate (9), we obtain that, for $\epsilon > 0$ sufficiently small, there is a positive constant $C_3(V)$ that depends on $\|\nabla\log(V)\|_{L^\infty}$ and $\|\Delta\log(V)\|_{L^\infty}$ such that

$$\|\tilde{L}_\epsilon^{-1}(\tilde{L}_\epsilon - L_\epsilon)r\|_{H^1} \leq \sqrt{2}\epsilon^{-1}\|(\tilde{L}_\epsilon - L_\epsilon)r\|_{L^2} \leq \epsilon C_3(V)\|r\|_{H^1}. \tag{20}$$

From these three estimates, it is clear that Φ_ϵ maps a ball $B_\delta(H^1(\mathbb{R}^d))$ of radius $\delta = C_0\epsilon^2$ to itself, where $C_0 > C_1(V)$, independently of ϵ. Similar estimates on the Lipschitz continuous nonlinear term $N(r)$ and perturbation operator $\tilde{L}_\epsilon^{-1}(\tilde{L}_\epsilon - L_\epsilon)$ show that Φ_ϵ is a contraction on the ball $B_\delta(H^1(\mathbb{R}^d))$ of radius $\delta = C_0\epsilon^2$. Hence, the assertion of the theorem follows from the Banach fixed-point theorem. ∎

Remark 1. Sobolev's embedding of $H^s(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ for $s > \frac{d}{2}$ allows us to replace the three conditions of the lemma by only one condition: $\nabla\log(V) \in H^2(\mathbb{R})$ if $d = 1$ and $\nabla\log(V) \in H^3(\mathbb{R}^d)$ if $d = 2$ or $d = 3$. With this refinement, Theorem 1 follows from Lemma 2 after the decomposition $u = u_0(1 + r)$ is used.

Remark 2. Because $H^1(\mathbb{R}^d)$ is embedded into $L^\infty(\mathbb{R}^d)$ for $d = 1$, the solution $u = u_0(1 + r)$ in Theorem 1 with small $\|r\|_{H^1}$ is positive for $d = 1$. However, positivity of u is not proved for $d = 2, 3$, because the correction term r is not controlled in the supremum norm.

In the end of this article, we discuss several examples.

- If V grows algebraically at infinity with any rate $\alpha > 0$, that is, if

$$V(x) \sim |x|^\alpha \quad \text{as } |x| \to \infty$$

(the nonlinear confinement of this kind was adopted in Ref. [3]), then

$$|\nabla \log(V)| \sim |x|^{-1} \quad \text{and} \quad |\Delta \log(V)| \sim |x|^{-2}.$$

Assuming smoothness of V, these conditions provide $F \in H^1(\mathbb{R}^d)$ if $d = 1, 2, 3$, hence Theorem 1 holds for such potentials for any $\alpha > 0$. (Of course, $u \in L^2(\mathbb{R}^d)$ if and only if $\alpha > d$.) Some exact expressions are available for particular V and ϵ [8].

- If V grows like the exponential or Gaussian function (such as in the models introduced in Refs. [4,9]), then the assumption $F \in H^1(\mathbb{R}^d)$ fails for any $d = 1, 2, 3$. Nevertheless, if $V = (1 + \beta|x|^2)e^{\alpha|x|^2}$ with $\alpha, \beta > 0$, then the analytic expression is available [4] for a particular value of $\epsilon = \epsilon_0$:

$$u = \frac{\epsilon\alpha}{\sqrt{\beta}} e^{-\frac{\alpha}{2}|x|^2}, \qquad \epsilon_0 = \frac{\sqrt{\beta}}{\sqrt{\alpha^2 + d\alpha\beta}}. \tag{21}$$

However, because $F \notin H^1(\mathbb{R}^d)$ (F is not even bounded at infinity), it is not clear if there exists a family of stationary states for any $\epsilon \in (0, \epsilon_0)$ that connects the TF approximation (2) and the exact solution (21).

- If V is a symmetric double-well potential, then Theorem 1 justifies the construction of a symmetric stationary state u. Symmetry-breaking bifurcation may happen in double-well potentials, but it cannot happen to the symmetric state due to uniqueness arguments. Therefore, such a bifurcation may only happen to an anti-symmetric stationary state.

In conclusion, we have presented a rigorous proof of the proximity of the self-trapped states, produced by the TF (Thomas–Fermi) approximation in the recently developed models with the spatially growing local strength of the defocusing cubic nonlinearity, to the true ground state, in the space of any dimension, $d = 1, 2, 3$. As an extension of the analysis, it may be interesting to justify the empiric use of the TF approximation for the description of self-trapped modes with intrinsic vorticity (by themselves, they are not ground states, but may play such a role in the respective reduced radial models [7,9,12]). Another relevant extension can be developed for two-component Gross–Pitaevskii equations with the nonlinear confinement of the same type [5].

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