NONLINEAR WAVES ON METRIC GRAPHS

Nonlinear waves on metric graphs

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Abstract

We study the nonlinear Schrödinger (NLS) equation on star graphs with the Neumann-Kirchhoff (NK) boundary conditions at the vertex. We analyze the stability of standing wave solutions of the NLS equation by using different techniques.

We consider a half-soliton state of the NLS equation, and by using normal forms, we prove it is nonlinearly unstable due to small perturbations that grow slowly in time. Moreover, under certain constraints on parameters of the generalized NK conditions, we show the existence of a family of shifted states, which are parametrized by a translational parameter. We obtain the spectral stability/instability result for shifted states by using the Sturm theory for counting the Morse indices of the shifted states. For the spectrally stable shifted states, we show that the momentum of the NLS equation is not conserved which results in the irreversible drift of the family of shifted states towards the vertex of the star graph. As a result, the spectrally stable shifted states are nonlinearly unstable.

We also study the NLS equation on star graphs with a δ interaction at the vertex. The presence of the interaction modifies the NK boundary conditions by adding an extra parameter. Depending on the value of the parameter, the NLS equation admits symmetric and asymmetric standing waves with either monotonic or non-monotonic structure on each edge. By using the Sturm theory approach, we prove the orbital instability of the standing waves.

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To my wife, Zhanara, and my daughter, Alana

Contents

Chapter 1

Introduction

1.1 Differential equations on star graphs

One of the first justifications of differential equations on graph models was published in 1953 in the Journal of Chemical Physics [65]. The study was based on the analysis of the free-electron model for the conjugated system of naphthalene molecule, which has alternating single and double bonds. Each atom in the conjugated system is associated with two "fixed" σ electrons and one "free" π electron. Under the effect of the charge potential, "free" π electrons move close to the network $\mathcal{N}$ constructed by the "fixed" σ electrons connection. Since the electrons could possibly transport along the entire network, the natural interest was to descibe the electronic motion around the vertices where three edges meet, see Figure 1.1. Hence, in [65] authors considered an ϵ-thin three-dimensional neighborhood (tube) around the vertex, and approximated the electronic motion in the neighborhood by a molecular orbital function $\Phi(x_j, y_j, z_j)$ satisfying the three-dimensional Schrödinger equation inside the tube as

$$\Phi(x_j, y_j, z_j) \approx \phi(x_j)\sin(\pi y_j/\epsilon)\sin(2\pi z_j/\epsilon), \tag{1.1.1}$$

where $0 \le y_j \le \epsilon$, $-\frac{\epsilon}{2} \le z_j \le -\frac{\epsilon}{2}$, and $\phi(x_j)$ is the scalar molecular orbital with x_j being the space coordinate of the branch j emerging from the vertex. According to the representation (1.1.1), the electronic motion of the π electrons can be described by the scalar functions $\phi(x_j)$, which turns out to be the limiting case of the three-dimensional model. As the ϵ-tube squeezes, the domain near the vertex p approaches a graph with three edges (Figure 1.1), and $\phi(x_j)$ satisfies the stationary Schrödinger equation on the branch j. The boundary conditions for $\Phi(x_j, y_j, z_j)$ imply connection formulas for ϕ_1, ϕ_2 and ϕ_3 at the vertex p given as

$$\begin{cases} \phi_1(p) = \phi_2(p) = \phi_3(p), & \text{the continuity condition,} \\ \phi_1'(p) + \phi_2'(p) + \phi_3'(p) = 0, & \text{the flux condition.} \end{cases} \tag{1.1.2}$$

The graph with three edges in Figure 1.1 is the example of a *metric graph* which can be analyzed in the Hilbert and Sobolev spaces. The well-posedness of the Cauchy

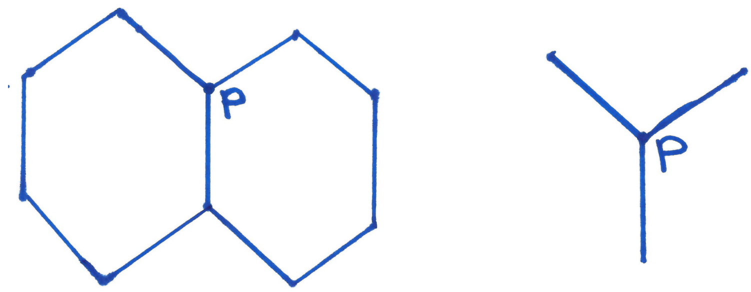

FIGURE 1.1: Left: the network $\mathcal{N}$ constructed by the "fixed" σ electrons. The "free" π electron moves along $\mathcal{N}$. Right: the local region in $\mathcal{N}$ around the vertex p. This is an example of the graph with three branches.

problem associated with the differential equations and the existence of particular solutions heavily depend on the boundary conditions at the vertex (1.1.2). The connection formula (1.1.2) is the simpliest example of so-called *classical Kirchhoff* conditions. Justifications of Kirchhoff conditions on other types of metric graphs has been obtained in many realistic physical experiments involving wave propagation in thin waveguides and quantum nanowires, where multi-dimensional models were approximated by scalar partial differential equations (PDEs) on graphs, see [11, 16, 34, 38, 39, 49] and references therein.

It is relatively less known that the classical Kirchhoff conditions similar to (1.1.2) are not the only possible boundary conditions arising when the narrow waveguides shrinks to a metric graph. By working with different values of the thickness parameters vanishing at the same rate, it was shown in [50, 64] (see also [27, 31, 32, 48, 53]) that *generalized Kirchhoff* boundary conditions can also arise in the asymptotic limit. In the generalized Kirchhoff boundary conditions, the wave functions have finite jumps across the vertex points and these jumps are compensated reciprocally in the sum of the first derivatives of the wave function. The nature of the jumps at the vertex points is related with the coefficients which appear when the thickness parameter converges to zero. As an example, we refer to [50] and consider a graph Γ with three edges and its neighborhood M^ϵ as in Figure 1.2.

The quadratic form for the Laplace operator Δ_ϵ in $L^2(\Gamma^\epsilon)$ is given by

$$Q^\epsilon(u,u) = -\int_{\Gamma^\epsilon} |\nabla u|^2 dA,$$

where u is in the appropriate H^1 Sobolev space, and dA stands for the integration over the area. In such configuration, it has been proven in [50] that, as $\epsilon \to 0$, the discrete eigenvalues $\lambda_n(-\Delta_\epsilon)$ converge towards the discrete eigenvalues $\lambda_n(-\Delta_0)$ of the self-adjoint Laplace operator Δ_0 defined on the graph Γ by

$$\Delta_0 \Psi = (\psi_1'', \psi_2'', \psi_3''),$$

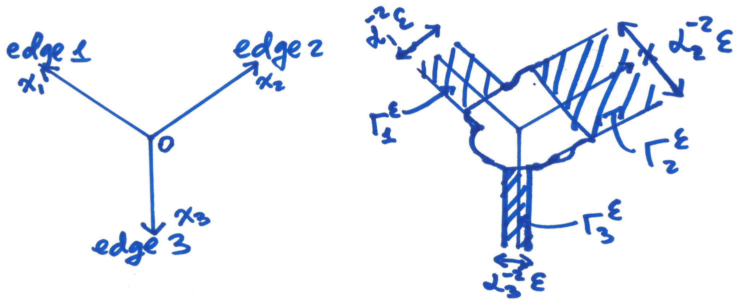

FIGURE 1.2: Left: the graph Γ with three edges. The vertex is assumed to be the origin, and the space coordinate takes positive values. Right: the ϵ-thin neighborhood Γ^ϵ of Γ. The bold region around the edge j is called to be the tube Γ_j^ϵ, and the width of the tube Γ_j^ϵ is equal to $\alpha_j^{-2}\epsilon$ with $\alpha_j > 0$. The neighborhood of the vertex partially bounded by the tubes Γ_j^ϵ is allowed to have almost arbitrary boundaries in between Γ_j^ϵ's.

where $\Psi = (\psi_1, \psi_2, \psi_3)$ with ψ_j defined on the edge j only, and the derivatives in ψ_j'' are computed with respect to the space parameter x_j, see Figure 1.2. The domain of the operator Δ_0, $\mathcal{D}(\Delta_0)$ is given by H^2 functions ψ_j satisfying

$$\begin{cases} \psi_1(0) = \psi_2(0) = \psi_3(0), \quad \text{the continuity condition,} \\ \alpha_1^{-2}\psi_1'(0) + \alpha^{-2}\psi_2'(0) + \alpha^{-2}\psi_3'(0) = 0, \quad \text{the generalized flux condition,} \end{cases} \tag{1.1.3}$$

where the coefficients $(\alpha_1, \alpha_2, \alpha_3)$ are defined in Figure 1.2. The domain $\mathcal{D}(\Delta_0)$ is the subspace of the *weighted* L^2 space $L^2(\Gamma, \alpha_j^{-2}dx)$, where the quadratic form of Δ_0 is

$$Q(u,u) = -\sum_{j=1}^{3} \int_{\Gamma_j} \alpha_j^{-2} |u_j'(x)|^2 dx \tag{1.1.4}$$

with Γ_j denoting the edge j of the graph Γ in Figure 1.2.

The natural simplification of the above structure on the graph Γ is to normalize the weighted L^2 space by mapping the element $\Psi = (\psi_1, \psi_2, \psi_3) \in L^2(\Gamma, \alpha_j^{-2}dx)$ into the element $\Phi = (\phi_1, \phi_2, \phi_3) \in L^2(\Gamma, dx)$ via the transformation

$$\phi_j = \alpha_j^{-1}\psi_j. \tag{1.1.5}$$

Then, under the transformation (1.1.5), the quadratic form (1.1.4) of the Laplacian Δ_0 becomes

$$Q(u,u) = -\sum_{j=1}^{3} \int_{\Gamma_j} |u_j'(x)|^2 dx,$$

and the boundary conditions in the domain of Δ_0 become

$$\begin{cases} \alpha_1\phi_1(0) = \alpha_2\phi_2(0) = \alpha_3\phi_3(0), & \text{the generalized continuity condition,} \\ \alpha_1^{-1}\phi_1'(0) + \alpha^{-1}\phi_2'(0) + \alpha^{-1}\phi_3'(0) = 0, & \text{the generalized flux condition.} \end{cases} \tag{1.1.6}$$

We refer to the boundary conditions (1.1.6) as to *generalized Kirchhoff* conditions.

Throught the limiting process above, when $\Gamma^\epsilon \to \Gamma$ and $\lambda_n(-\Delta_\epsilon) \to \lambda_n(-\Delta_0)$ as $\epsilon \to 0$, we assumed that the vertex neighborhood (Figure 1.2) decays at the same rate as the ϵ-thin tubes Γ_j^ϵ. In general, one can remove such assumption [49], and obtain the boundary conditions on the graph Γ with δ interaction at the vertex given as

$$\begin{cases} \alpha_1\phi_1(0) = \alpha_2\phi_2(0) = \alpha_3\phi_3(0), \\ \alpha_1^{-1}\phi_1'(0) + \alpha^{-1}\phi_2'(0) + \alpha^{-1}\phi_3'(0) = \gamma\phi_1(0). \end{cases} \tag{1.1.7}$$

The parameter γ is real, and defines the strength of the δ interaction.

Repeating the process described above for graphs with higher number of edges justifies the choice of the boundary conditions used in the thesis. Numerical confirmations of validity of the classical and generalized Kirchhoff boundary conditions are reported in a number of recent publications in physics literature [19, 71, 77].

1.2 Background literature

Spectral properties of Laplacian and other linear operators on graphs have been intensively studied in the past twenty year [15, 30]. The time evolution of linear PDEs on graphs is well defined by the standard semi-group theory, once a self-adjoint extension of the graph Laplacian is constructed. On the other hand, the time evolution of nonlinear PDEs on graphs is a more challenging problem involving interplay between nonlinear analysis, geometry, and the spectral theory of non-self-adjoint operators. The nonlinear PDEs on graphs, mostly the nonlinear Schrödinger equation (NLS), has been studied in the past decade in the context of existence, stability, and propagation of solitary waves [54].

Among the limitless amount of possible graph models, we are particularly interested in the class of star graphs which we define as follows:

Definition 1.1. *We call a graph Γ to be a* **star graph** *if it is constructed by attaching N edges of finite or infinite length at a common vertex.*

The example of the star graph Γ with $N = 3$ edges is given in Figure 1.1.

Below, we overview the recent results related to existence and stability of stationary states for the nonlinear Schrödinger (NLS) equation on star graphs with boundary conditions of type (1.1.2), (1.1.6) and (1.1.7). Further works on stationary states on unbounded star graphs have been developed in the context of the logarithmic NLS equation

[9, 61], the power NLS equation with δ' interactions [60], the power NLS equation on the tadpole [55, 18] and dumbbell [35, 51], double-bridge [56] and periodic ring graphs [29, 33, 58, 63]. A variational characterization of standing waves was developed for general metric graphs [8, 17, 28] and graphs with compact nonlinear core in [67, 68, 75].

1.2.1 Classical Kirchhoff conditions

The recent works of Adami, Serra, and Tilli [6, 7] were devoted to the existence of ground states on the unbounded graphs that are connected to infinity after removal of any edge. It was proven that if the infimum of the constrained NLS energy on the unbounded graph coincides with the infimum of the constrained NLS energy on the infinite line, then it is not achieved (that is, no ground state exists) for every such a graph with the exception of graphs isometric to the real line [6]. The reason why the infimum is not achieved is a possibility to minimize the constrained NLS energy by a family of NLS solitary waves escaping to infinity along one edge of the graph.

The star graph Γ with classical Kirchhoff conditions (1.1.2) is an example of the unbounded graphs with no ground states. When the number of edges in Γ is odd, there is only one stationary state of the NLS equation on the star graph [1]. This state is represented by the *half-solitons* along each edge glued by their unique maxima at the vertex. By using a one-parameter deformation of the NLS energy constrained by the fixed mass, it was shown that the half-soliton state is a saddle point of the constrained NLS energy [2]. The study in [6] provides a general argument of the computations in [2], where it is shown that the one-parameter deformation of the half-soliton state with the fixed mass reduces the NLS energy and connects the half-soliton state with the solitary wave escaping along one edge of the star graph. The saddle point geometry of energy at the half-soliton state was not related in [2] to the instability of the half-soliton state in the time evolution of the NLS equation.

It is known that the saddle point geometry does not necessarily imply instability of stationary states in Hamiltonian systems. In the linearized Hamiltonian systems, eigenvalues of the negative energy may be accounted in the neutrally stable modes that are bounded for all times [46]. Nonlinear instability of such states may still appear in the nonlinear Hamiltonian systems due to resonant coupling between neurally stable modes of negative energy and the continuous spectrum [47], however, this coupling can be avoided in some Hamiltonian systems [26].

1.2.2 Generalized Kirchhoff conditions

In a series of papers [66, 72, 73], it was shown that if the parameters $(\alpha_1, \alpha_2, \dots, \alpha_N)$ in the generalized Kirchhoff conditions (1.1.6) on a star graph are related to the parameters of the nonlinear evolution equations and satisfy a constraint

$$\sum_{j=1}^{K} \frac{1}{\alpha_j^2} = \sum_{j=K+1}^{N} \frac{1}{\alpha_j^2} \tag{1.2.1}$$

for some K, then the nonlinear evolution equation on the star graph can be reduced to the homogeneous equation on the infinite line. In other words, singularities of the star graph are unfolded in the transformation and the vertex points become regular points on the line. In this case, we partition the graph Γ into two sets of $\{K, N-K\}$ edges with coefficients $(\alpha_1, \alpha_2, \dots, \alpha_N)$ satisfying the constraint (1.2.1). In the transmission problems, it is natural to think that K edges represent **incoming** bonds for the solitary wave propagation whereas the remaining $N-K$ edges represent **outgoing** bonds for the solitary wave propagation. Under the constraint (1.2.1), a transmission of a solitary wave through the vertex point is reflectionless [73].

In such configuration of a star graph, there exists a family of *shifted states* parametrized by a translational parameter. The shifted states appear naturally in the case of classical Kirchhoff boundary conditions when the number of edges is even [4]. These states can be considered to be translations of the half-soliton states, which exist for any number of edges. In the variational characterization of the NLS stationary states on a star graph, such shifted states were mentioned in Remarks 5.3 and 5.4 in [4], where it was conjectured that all shifted states are saddle points of the action functional and are thus unstable for all star graphs with even number of edges exceeding two.

1.2.3 Kirchhoff conditions in the presence of δ interaction

The existence, stability and variational properties of stationary states for the NLS on the star graph with a δ interaction at the vertex were analyzed in [1, 3, 4, 5]. In particular, in case of focusing delta interaction in the Kirchhoff conditions (1.1.7) with $\gamma < 0$, it was proven in [3] that there exists a global minimizer of the constrained NLS energy for a sufficiently small mass below the critical mass. This minimizer coincides with the N-tail state symmetric under exchange of edges, which has monotonically decaying tails and which becomes the half-soliton state if the delta interaction vanishes. In [5], it was proven that although the constrained minimization problem admits no global minimizers for a sufficiently large mass above the critical mass, the N-tail state symmetric under exchange of edges is still a local minimizer of the constrained NLS energy on the star graph, when a delta interaction is added on the vertex. Due to local minimization property, the N-tail state symmetric under exchange of edges is orbitally stable in the time evolution of the NLS in the presence of the focusing delta impurity. Although the second variation of the constrained energy was mentioned in the first work [1], the authors obtained all the variational results in [3, 4, 5] from the energy formulation avoiding the linearization procedure.

Besides the N-tail symmetric state, the NLS equation on the star graph with focusing delta interaction admits the family of asymmetric states which is the combination of monotonic and non-monotonic components on the edges. The asymmetric states are not the constrained energy minimizers, and their instability has been conjectured in [54] and studied in [59].

1.3 The outline of the thesis

Here, we give the brief description of the main results in the next chapters.

- In **Chapter 2**, we define the NLS equation on the star graph Γ, and set the Hilbert and Sobolev spaces on Γ. We also review the well-posedness of the Cauchy problem for the NLS, the existence of standing wave solutions, and the tools required to study their spectral and orbital stability.

- In **Chapter 3**, we consider a half-soliton state of the stationary NLS equation on a star graph Γ with N edges. For the subcritical power nonlinearity, the half-soliton state is a degenerate critical point of the action functional under the mass constraint such that the second variation is nonnegative. By using normal forms, we prove that the degenerate critical point is a nonlinear saddle point, for which the small perturbations to the half-soliton state grow slowly in time resulting in the nonlinear instability of the half-soliton state. The result holds for any $N \geq 3$ and arbitrary subcritical power nonlinearity. It gives a precise dynamical characterization of the previous result in [2], where the half-soliton state was shown to be a saddle point of the action functional under the mass constraint for $N = 3$ and for cubic nonlinearity.

  The content of Chapter 3 is based on [41]:

  A. Kairzhan and D. Pelinovsky, "Nonlinear instability of half-solitons on star graphs", J. Diff. Eqs. **264** *(2018) 7357–7383.*

- In **Chapter 4**, we consider the NLS equation with the subcritical power nonlinearity on a star graph consisting of N edges under generalized Kirchhoff conditions (1.1.6). Under the constraint (1.2.1), the stationary NLS equation admits a family of solitary waves parameterized by a translational parameter, which we call the shifted states. We obtain the general counting results on the Morse index of the shifted states, from which we prove that the shifted states with $1 < K < N$ in (1.2.1) are saddle points of the action functional which are spectrally unstable under the NLS flow. Since the NLS equation on a star graph with shifted states can be reduced to the homogeneous NLS equation on an infinite line, the spectral instability of shifted states is due to the perturbations breaking this reduction.

  We also prove that the shifted states with the monotone profiles in the $N-1$ edges (for $K = 1$ case) are spectrally stable. We give a simple argument suggesting that the spectrally stable shifted states are nonlinearly unstable under the NLS flow due to the perturbations breaking the reduction to the homogeneous NLS equation.

  The content of Chapter 4 is based on [42]:

  A. Kairzhan and D. Pelinovsky, "Spectral stability of shifted states on star graphs", J. Phys. A: Math. Theor. **51** *(2018) 095203*

- In **Chapter 5**, we prove that the spectrally stable states obtained in Chapter 4 with $N-1$ monotonic tails are nonlinearly unstable because of the irreversible drift along the incoming edge towards the vertex of the star graphs. When the shifted states reach the vertex as a result of the drift, they become saddle points of the action functional, in which case the nonlinear instability leads to their destruction.

The content of Chapter 5 is based on [43]:

A. Kairzhan, D. Pelinovsky and R. Goodman, "Drift of Spectrally Stable Shifted States on Star Graphs", SIAM J. Appl. Dyn. Syst. **18** *(2019), 1723–1755.*

- In **Chapter 6**, we consider the NLS equation on a star graph Γ with a δ interaction at the vertex. The strength of the interaction is defined by a fixed value $\gamma \in \mathbb{R}$. In [1, 5], it was shown that for $\gamma \neq 0$ the NLS equation on Γ admits the unique symmetric standing wave and all other standing waves are asymmetric. Also, it was proven that for $\gamma < 0$, the unique symmetric standing wave is orbitally stable.

  We analyze stability of asymmetric standing waves for an arbitrary $\gamma \neq 0$. By extending the Sturm theory to Schrödinger operators on the star graph, we give the explicit count of the Morse index for each standing wave, from which the orbital instability result follows for every $\gamma \neq 0$.

  The content of Chapter 6 is based on [44]:

  A. Kairzhan, "Orbital instability of standing waves for NLS equation on star graphs", Proc. Amer. Math. Soc. **147** *(2019), 2911–2924.*

- In **Chapter 7**, we describe the set of possible research questions which are related to the results obtained in the thesis.

Chapter 2

The NLS equation on star graphs

2.1 The domain of the graph Laplacian

We denote a star graph consisting of N half-lines to be Γ, see Figure 2.1. All N half-lines are connected at a common vertex, which we chose to be the origin, and each edge of the star graph is parameterized by $\mathbb{R}^+$.

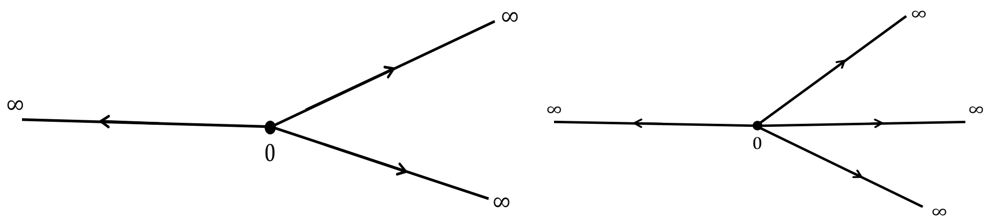

FIGURE 2.1: Left: The star graph with $N = 3$ edges. Right: The star graph with $N = 4$ edges.

The Hilbert space of squared integrable functions on the graph Γ is given by

$$L^2(\Gamma) = \oplus_{j=1}^N L^2(\mathbb{R}^+).$$

Elements in $L^2(\Gamma)$ are represented in the componentwise sense as vectors

$$\Psi = (\psi_1, \psi_2, \dots, \psi_N)^T$$

with each component $\psi_j \in L^2(\mathbb{R}^+)$ defined on the j-th edge. The inner product and the squared norm of such $L^2(\Gamma)$-functions are given by

$$\langle \Psi, \Phi \rangle_{L^2(\Gamma)} := \sum_{j=1}^N \int_{\mathbb{R}^+} \psi_j(x) \overline{\phi_j(x)} dx, \quad \|\Psi\|^2_{L^2(\Gamma)} := \sum_{j=1}^N \|\psi_j\|^2_{L^2(\mathbb{R}^+)},$$

for every $\Psi = (\psi_1, \psi_2, \dots, \psi_N)^T$ and $\Phi = (\phi_1, \phi_2, \dots, \phi_N)^T$ in $L^2(\Gamma)$.

Similarly, we define the L^2-based Sobolev spaces on the graph Γ to be

$$H^k(\Gamma) = \oplus_{j=1}^N H^k(\mathbb{R}^+), \quad k \in \mathbb{N}$$

and equip them with suitable boundary conditions at the vertex. We also define the squared $H^k(\Gamma)$-norm as

$$\|\Psi\|^2_{H^k(\Gamma)} := \sum_{j=1}^{N} \|\psi_j\|^2_{H^k(\mathbb{R}^+)}.$$

Throughout the thesis we are mainly interested in the Sobolev spaces with $k = 1$ and $k = 2$. In what follows, for $k = 1$, we set the generalized continuity boundary condition at the vertex as in

$$H^1_\Gamma := \{\Psi \in H^1(\Gamma) : \quad \alpha_1\psi_1(0) = \alpha_2\psi_2(0) = \cdots = \alpha_N\psi_N(0)\}, \tag{2.1.1}$$

where α_1, α_2, ..., α_N are positive coefficients. These coefficients arise naturally, according to Section 1.1, when the one-dimensional star graph is obtained as a limit of a narrow two-dimensional waveguide with different values of the thickness parameters that go to zero at the same rate, [31, 32, 64].

For $k = 2$, we set an additional generalized Kirchhoff boundary condition as follows:

$$H^2_\Gamma := \left\{\Psi \in H^2(\Gamma) \cap H^1_\Gamma : \quad \sum_{j=1}^{N} \frac{1}{\alpha_j}\psi_j'(0) = 0\right\}, \tag{2.1.2}$$

where derivatives are defined $\lim_{x\to 0^+}$.

One advantage of the generalized boundary conditions (2.1.1)–(2.1.2) is related to self-adjointness of **the graph Laplacian** operator Δ defined as

$$\Delta\Psi = (\psi_1'', \psi_2'', \ldots, \psi_N'')^T$$

for every $\Psi \in H^2_\Gamma \subset L^2(\Gamma)$. Indeed, the following result is a consequence of Theorem 1.4.4 in [15].

Proposition 2.1. *The Laplacian operator*

$$\Delta : H^2_\Gamma \subset L^2(\Gamma) \to L^2(\Gamma)$$

is self-adjoint.

Proof. We only include integral computations below to show the necessity of both conditions in (2.1.2). The full proof is given in the original theorem in [15].

If $\Psi \in H^2(\Gamma)$, then each $\psi_j \in H^2(\mathbb{R}^+)$. By Sobolev embedding theorem, this requires $\psi_j(x), \psi_j'(x) \to 0$ as $x \to \infty$ for every j. Therefore, for every $\Psi, \Phi \in H^2(\Gamma)$, integration by parts and the generalized boundary conditions (2.1.1)–(2.1.2) yield

$$\begin{aligned} \langle \Delta\Psi, \Phi\rangle_{L^2(\Gamma)} &= \langle \Psi, \Delta\Phi\rangle_{L^2(\Gamma)} + \sum_{j=1}^N \psi_j(0)\overline{\phi_j'(0)} - \sum_{j=1}^N \psi_j'(0)\overline{\phi_j(0)} \\ &= \langle \Psi, \Delta\Phi\rangle_{L^2(\Gamma)} + \alpha_1\psi_1(0)\sum_{j=1}^N \alpha_j^{-1}\overline{\phi_j'(0)} - \alpha_1\overline{\phi_1(0)}\sum_{j=1}^N \alpha_j^{-1}\psi_j'(0) \\ &= \langle \Psi, \Delta\Phi\rangle_{L^2(\Gamma)}. \end{aligned}$$

□

2.2 Well-posedness of the Cauchy problem

Throughout the thesis we consider the nonlinear Schrödinger (NLS) equation on the star graph Γ with the power-type nonlinearity given as:

$$i\frac{\partial\Psi}{\partial t} = -\Delta\Psi - (p+1)\alpha^{2p}|\Psi|^{2p}\Psi, \quad x\in\Gamma, \quad t\in\mathbb{R}, \tag{2.2.1}$$

where $\Psi = \Psi(t,x) = (\psi_1, \psi_2, \dots, \psi_N)^T \in \mathbb{C}^N$, $\Delta : H^2_\Gamma \subset L^2(\Gamma) \to L^2(\Gamma)$ is the Laplacian operator in Proposition 2.1, $\alpha \in L^\infty(\Gamma)$ is a piecewise constant function with components $(\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{R}^N_+$ defined on the edges of Γ, and the nonlinear term $\alpha^{2p}|\Psi|^{2p}\Psi$ is interpreted as a symbol for

$$(\alpha_1^{2p}|\psi_1|^{2p}\psi_1, \alpha_2^{2p}|\psi_2|^{2p}\psi_2, \dots, \alpha_N^{2p}|\psi_N|^{2p}\psi_N)^T.$$

That is, the NLS equation (2.2.1) on each edge j can be written as

$$i\frac{\partial\psi_j}{\partial t} = -\psi_j'' - (p+1)\alpha_j^{2p}|\psi_j|^{2p}\psi_j, \quad x\in\mathbb{R}^+, \quad t\in\mathbb{R}. \tag{2.2.2}$$

Remark 2.2. *The constant coefficients $(\alpha_1, \alpha_2, \dots, \alpha_N)$ in (2.2.2) coincide with the coefficients in the generalized boundary conditions (2.1.1) and (2.1.2).*

The local well-posedness for the Cauchy problem associated with (2.2.1) with $\alpha = 1$ was initially given in Proposition 2.2 in [4]. In fact, the result and its proof in [4] hold for every α.

Proposition 2.3. *For every $p > 0$ and every $\Psi(0) \in H^1_\Gamma$, there exists $t_0 \in (0, \infty]$ and a local solution*

$$\Psi(t) \in C((-t_0, t_0), H^1_\Gamma) \cap C^1((-t_0, t_0), H^{-1}(\Gamma)) \tag{2.2.3}$$

to the Cauchy problem associated with the NLS equation (2.2.1).

Proof. Local well-posedness of the NLS equation (2.2.1) in H^1_Γ is proved by using a standard contraction method thanks to the isometry of the semi-group $e^{it\Delta}$ in H^1_Γ and the Sobolev embedding of H^1_Γ into $L^\infty(\Gamma)$. □

We note that the NLS equation (2.2.1) is invariant under the phase rotation $\Psi \mapsto e^{i\theta}\Psi$ and under the time translation $\Psi(t,x) \mapsto \Psi(t+t_0,x)$ with $\theta \in \mathbb{R}$ and $t_0 \in \mathbb{R}$. This motivates to consider the energy and mass functionals, which are defined as

$$E(\Psi) := \|\Psi'\|^2_{L^2(\Gamma)} - \|\alpha^{\frac{p}{p+1}}\Psi\|^{2p+2}_{L^{2p+2}(\Gamma)}, \quad Q(\Psi) := \|\Psi\|^2_{L^2(\Gamma)}, \tag{2.2.4}$$

respectively. In the case of $\alpha = 1$, Proposition 2.3 in [4] shows that these functionals are constant under the time flow of the NLS equation (2.2.1). The result and its proof in [4] hold for every α. Below we state the energy and mass conservation, and provide the alternative proof of this result.

Proposition 2.4. *Let $p > 0$. For every solution Ψ in Proposition 2.3 the mass and energy functionals in (2.2.4) are constant under the time flow of the NLS equation (2.2.1).*

Proof. Here, we give an alternative proof of the mass and energy conservation under simplifying assumptions $p > 1/2$ and $p \geq 1$ respectively.

If $p > 1/2$ and $\Psi(0) \in H^2_\Gamma$, it follows from the contraction method that there exists $t_0 > 0$ and a local strong solution

$$\Psi(t) \in C((-t_0,t_0), H^2_\Gamma) \cap C^1((-t_0,t_0), L^2(\Gamma)) \tag{2.2.5}$$

to the NLS equation (2.2.1). Applying time derivative to $Q(\Psi)$ and using the NLS equation (2.2.1) yield the mass balance equation:

$$\begin{aligned} \frac{d}{dt}Q(\Psi) &= -i\langle -\Delta\Psi - (p+1)\alpha^{2p}|\Psi|^{2p}\Psi, \Psi\rangle_{L^2(\Gamma)} + i\langle \Psi, -\Delta\Psi - (p+1)\alpha^{2p}|\Psi|^{2p}\Psi\rangle_{L^2(\Gamma)} \\ &= i\langle \Delta\Psi, \Psi\rangle_{L^2(\Gamma)} - i\langle \Psi, \Delta\Psi\rangle_{L^2(\Gamma)} = 0, \end{aligned}$$

where the last equality is obtained by Proposition 2.1. Thus, the mass conservation is proven for $\Psi(0) \in H^2_\Gamma$.

If $p > 1/2$ and $\Psi(0) \in H^1_\Gamma$ but $\Psi(0) \notin H^2_\Gamma$, then in order to prove the mass conservation, we define an approximating sequence $\{\Psi^{(n)}(0)\}_{n\in\mathbb{N}}$ in H^2_Γ such that $\Psi^{(n)}(0) \to \Psi(0)$ in H^1_Γ as $n \to \infty$. For each $\Psi^{(n)}(0) \in H^2_\Gamma$, there exists a local strong solution $\Psi^{(n)}(t)$ given by (2.2.5) for $t \in (-t_0^{(n)}, t_0^{(n)})$. By Gronwall's inequality, there exists a positive constant K which only depends on the $H^1(\Gamma)$ norm of the $\Psi(0)$ such that

$$\|\Psi^{(n)\prime\prime}(t)\|_{L^2(\Gamma)} \leq K\|\Psi^{(n)\prime\prime}(0)\|_{L^2(\Gamma)}, \quad t \in (-t_0^{(n)}, t_0^{(n)}),$$

hence, the local existence time $t_0^{(n)}$ is determined by the $H^1(\Gamma)$ norm of the initial data $\Psi^{(n)}(0)$. Due to the convergence $\Psi^{(n)}(0) \to \Psi(0)$ in H^1_Γ, this implies that there is $t_0 > 0$

that depends on the $H^1(\Gamma)$ norm of $\Psi(0)$ such that $t_0^{(n)} \geq t_0$ for every $n \in \mathbb{N}$. Moreover, $\Psi^{(n)}(t) \to \Psi(t)$ in H^1_Γ as $n \to \infty$ for every $t \in (-t_0, t_0)$. Since $Q(\Psi^{(n)}(t)) = Q(\Psi^{(n)}(0))$ for every $t \in (-t_0, t_0)$, the limit $n \to \infty$ and the strong convergence in H^1_Γ implies that $Q(\Psi(t)) = Q(\Psi(0))$ for every $t \in (-t_0, t_0)$.

In order to prove the energy conservation, let us define the space H^3_Γ compatible with the NLS flow:

$$H^3_\Gamma := \left\{ \Psi \in H^3(\Gamma) \cap H^2_\Gamma : \quad \alpha_1 \psi_1''(0) = \alpha_2 \psi_2''(0) = \dots = \alpha_N \psi_N''(0) \right\}. \tag{2.2.6}$$

If $p \geq 1$ and $\Psi(0) \in H^3_\Gamma$, it follows from the contraction method that there exists $t_0 > 0$ and a local strong solution if $\Psi(0) \in H^3_\Gamma$

$$\Psi(t) \in C((-t_0, t_0), H^3_\Gamma) \cap C^1((-t_0, t_0), H^1_\Gamma) \tag{2.2.7}$$

to the NLS equation (2.2.1). Applying time derivative to $E(\Psi)$ and using the NLS equation (5.1.1) yield the energy balance equation:

$$\begin{aligned}
\frac{d}{dt} E(\Psi) &= i\langle \Psi''', \Psi' \rangle_{L^2(\Gamma)} - i\langle \Psi', \Psi''' \rangle_{L^2(\Gamma)} \\
&\quad + i(p+1)\langle \alpha^{2p}(|\Psi|^{2p})'\Psi, \Psi' \rangle_{L^2(\Gamma)} - i(p+1)\langle \Psi', \alpha^{2p}(|\Psi|^{2p})'\Psi \rangle_{L^2(\Gamma)} \\
&\quad + i(p+1)\langle \Psi^{p+1}, \alpha^{2p}\Psi^p \Delta\Psi \rangle_{L^2(\Gamma)} - i(p+1)\langle \alpha^{2p}\Psi^p \Delta\Psi, \Psi^{p+1} \rangle_{L^2(\Gamma)} \\
&= i \sum_{j=1}^N \psi_j'(0) \left[\overline{\psi}_j''(0) + (p+1)\alpha_j^{2p} |\psi_j(0)|^{2p} \overline{\psi}_j(0) \right] \\
&\quad - i \sum_{j=1}^N \overline{\psi}_j'(0) \left[\psi_j''(0) + (p+1)\alpha_j^{2p} |\psi_j(0)|^{2p} \psi_j(0) \right],
\end{aligned}$$

where the decay of $\Psi(x)$, $\Psi'(x)$, and $\Psi''(x)$ to zero at infinity has been used for the solution in H^3_Γ. Due to the boundary conditions in (2.1.1), (2.1.2), and (2.2.6), we obtain $\frac{d}{dt}E(\Psi) = 0$, that is, the energy conservation of (2.2.4) is proven for $\Psi(0) \in H^3_\Gamma$. The proof for $p \geq 1$ and $\Psi(0) \in H^1_\Gamma$ but $\Psi(0) \notin H^3_\Gamma$ is achieved by using an approximating sequence similarly to the argument above.

Finally, the proof can be extended for all values of $p > 0$ by using other approximation techniques, see, e.g., Theorems 3.3.1, 3.3.5, and 3.39 in [20]. $\square$

Remark 2.5. *Due to the validity of the energy and mass conservation laws in Proposition 2.4, it is natural to ask for existence of other conserved quantities. However, the translational symmetry of the infinite line $\mathbb{R}$ is broken in the star graph Γ due to the vertex at $x = 0$. As a result, a momentum functional is not generally conserved under the NLS flow, see also Section 4.4 below.*

Global existence of solutions under the NLS flow only holds in the subcritical case $p \in (0, 2)$, see Corollary 2.1 in [4].

Proposition 2.6. *For every $p \in (0,2)$, the local solution (2.2.3) in Proposition 2.3 is extended globally with $t_0 = \infty$.*

Proof. This follows by the energy conservation and the Gagliardo-Nirenberg inequality

$$\|\alpha^{\frac{p}{p+1}}\Psi\|_{L^{2p+2}(\Gamma)}^{2p+2} \leq C_{p,\alpha}\|\Psi'\|_{L^2(\Gamma)}^{p}\|\Psi\|_{L^2(\Gamma)}^{p+2},$$

for every $\alpha \in L^\infty(\Gamma)$, $\Psi \in H^1_\Gamma$, $p > 0$, where the constant $C_{p,\alpha} > 0$ depends on p and α but does not depend on Ψ. □

Remark 2.7. *For $p = 2$, the H^1-norm of the solution (2.2.3) is bounded uniformly by the energy and mass functionals only if the initial datum $\Psi(0) \in H^1_\Gamma$ in Proposition 2.3 has sufficiently small L^2-norm. In this case, since the energy and mass are conserved under the NLS flow, the solution (2.2.3) can be extended globally with $t_0 = \infty$.*

2.3 Stationary states

Stationary states of the NLS are given by the solutions of the form

$$\Psi(t,x) = e^{i\omega t}\Phi_\omega(x),$$

where $(\omega, \Phi_\omega) \in \mathbb{R} \times H^2_\Gamma$ is a real-valued solution of the stationary NLS equation,

$$-\Delta\Phi_\omega - (p+1)\alpha^{2p}|\Phi_\omega|^{2p}\Phi_\omega = -\omega\Phi_\omega. \tag{2.3.1}$$

No solution $\Phi_\omega \in H^2_\Gamma$ to the stationary NLS equation (2.3.1) exist for $\omega \leq 0$ because $\sigma(-\Delta) \geq 0$ in $L^2(\Gamma)$ and $\Phi_\omega(x), \Phi'_\omega(x) \to 0$ as $x \to \infty$ if $\Phi_\omega \in H^2_\Gamma$ by Sobolev's embedding theorem. Therefore, we only consider $\omega > 0$ in the stationary NLS equation (2.3.1). Since Γ consists of edges with the parametrization on $\mathbb{R}^+$, the scaling transformation

$$\Phi_\omega(x) = \omega^{\frac{1}{2p}}\Phi(z), \quad z = \omega^{\frac{1}{2}}x \tag{2.3.2}$$

can be used to scale the positive parameter ω to unity. The normalized profile Φ is now a solution of the stationary NLS equation

$$-\Delta\Phi + \Phi - (p+1)\alpha^{2p}|\Phi|^{2p}\Phi = 0, \quad \Phi \in H^2_\Gamma. \tag{2.3.3}$$

For every N and $\alpha \in \mathbb{R}^N_+$, the stationary NLS equation (2.3.3) has a particular solution

$$\Phi(x) = \begin{bmatrix} \alpha_1^{-1} \\ \alpha_2^{-1} \\ \vdots \\ \alpha_N^{-1} \end{bmatrix} \phi(x), \quad \text{with } \phi(x) = \operatorname{sech}^{\frac{1}{p}}(px). \tag{2.3.4}$$

Throughtout the thesis, this solution is labeled as the **half-soliton state**. We are also interested in the families of solitary waves parameterized by a translational parameter, which are labeled as the **shifted states**. Such families exist if $(\alpha_1, \alpha_2, \dots, \alpha_N)$ satisfy the constraint

$$\sum_{j=1}^{K} \frac{1}{\alpha_j^2} = \sum_{j=K+1}^{N} \frac{1}{\alpha_j^2} \tag{2.3.5}$$

with integer K satisfying $0 < K < N$. The origin of the constraint (2.3.5) was discussed in Chapter 1, and we refer to the K edges corresponding to coefficients $\alpha_1, \alpha_2, \dots, \alpha_K$ as incoming, whereas the remaining edges are thought to be outgoing.

Remark 2.8. *If $N = 2$ and $K = 1$, then the constraint (2.3.5) is only satisfied if $\alpha_1 = \alpha_2$. In this case, the NLS equation (2.2.1) on the graph Γ is equivalent to the homogeneous NLS equation on the infinite line $\mathbb{R}$.*

The following lemma gives the existence of a family of shifted states under the constraint (2.3.5)

Lemma 2.9. *For every $p > 0$ and every $(\alpha_1, \alpha_2, \dots, \alpha_N)$ satisfying the constraint (2.3.5), there exists a one-parameter family of solutions to the stationary NLS equation (2.3.3) with any $p > 0$ given by $\Phi(x; a) = (\phi_1, \dots, \phi_N)^T$ with components*

$$\phi_j(x; a) = \begin{cases} \alpha_j^{-1}\phi(x + a), & j = 1, \dots, K \\ \alpha_j^{-1}\phi(x - a), & j = K+1, \dots, N, \end{cases} \tag{2.3.6}$$

where $\phi(x) = \operatorname{sech}^{\frac{1}{p}}(px)$ and $a \in \mathbb{R}$ is arbitrary.

Proof. A general solution to the stationary NLS equation (2.3.3) decaying to zero at infinity is given by $\Phi = (\phi_1, \dots, \phi_N)^T$ with components

$$\phi_j(x; a) = \alpha_j^{-1}\phi(x + a_j), \quad 1 \le j \le N,$$

where $(a_1, \dots, a_N) \in \mathbb{R}^N$ are arbitrary parameters. The generalized continuity boundary condition in H^2_Γ imply that $|a_1| = \dots = |a_N|$. Hence for every $j = 1, \dots, N$, there exists $m_j \in \{0, 1\}$, such that $a_j = (-1)^{m_j}|a|$ for some $a \in \mathbb{R}$. The generalized Kirchhoff boundary condition in H^2_Γ is equivalent to

$$\phi'(|a|) \sum_{j=1}^{N} \frac{(-1)^{m_j}}{\alpha_j^2} = 0. \tag{2.3.7}$$

If $a = 0$, the equation (2.3.7) holds since $\phi'(0) = 0$ and this yields the half-soliton state in the form (2.3.4). If $a \neq 0$, then the equation (2.3.7) holds due to the constraint (2.3.5) if either

$$m_j = \begin{cases} 1 & \text{for} \quad 1 \le j \le K \\ 0 & \text{for} \quad K+1 \le j \le N \end{cases} \quad \text{or} \quad m_j = \begin{cases} 0 & \text{for} \quad 1 \le j \le K \\ 1 & \text{for} \quad K+1 \le j \le N \end{cases} \tag{2.3.8}$$

In both cases, the shifted state appears in the form (2.3.6) with either $a < 0$ or $a > 0$. $\square$

Remark 2.10. *The half-soliton state $\Phi(x)$ in (2.3.4) corresponds to the shifted state $\Phi(x; a)$ of Lemma 2.9 with $a = 0$, that is $\Phi(x) \equiv \Phi(x; 0)$.*

Remark 2.11. *By using the scaling transformation (2.3.2), we can convert the shifted states $\Phi(x; a)$ in Lemma 2.9 into the ω-dependent shifted states $\Phi_\omega(x; a)$ which solve the stationary NLS equation (2.3.1).*

Remark 2.12. *Besides the two choices specified in the proof of Lemma 2.9, there might be other N-tuples $(m_1, m_2, \dots, m_N) \in \{0, 1\}^N$ such that the bracket in (2.3.7) becomes zero. Such N-tuples generate new one-parameter families different from the one given by Lemma 2.9 under the same constraint (2.3.5). For instance, if $\alpha_j = 1$ for all j and $K = N/2$, there exist C_N different shifted states given by Lemma 2.13 below with C_N computed in (2.3.9).*

The following lemma gives a full classification of families of shifted states in case $\alpha = 1$, see also Theorem 5 in [4].

Lemma 2.13. *For $\alpha = 1$ and for even N, there exists C_N one-parameter families of solutions to the stationary NLS equation (2.3.3) with any $p > 0$, where*

$$C_N = \frac{N!}{2[(N/2)!]^2}. \tag{2.3.9}$$

Each family is generated from the simplest state $\Phi(x; a) = (\phi_1, \dots, \phi_N)^T$ with components

$$\phi_j(x; a) = \begin{cases} \phi(x + a), & j = 1, \dots, \frac{N}{2} \\ \phi(x - a), & j = \frac{N}{2} + 1, \dots, N \end{cases} \tag{2.3.10}$$

where $\phi(x) = \operatorname{sech}^{\frac{1}{p}}(px)$ and $a \in \mathbb{R}$ is arbitrary, after rearrangements between $N/2$ edges with $+a$ shifts and $N/2$ edges with $-a$ shifts.

Proof. A general solution to the stationary NLS equation (2.3.3) decaying to zero at infinity is given by

$$(\phi(x + a_1), \dots, \phi(x + a_N))^T,$$

where $\phi(x) = \operatorname{sech}^{\frac{1}{p}}(px)$, and $(a_1, \dots, a_N) \in \mathbb{R}^N$ are arbitrary parameters. The continuity condition in H^2_Γ imply that $|a_1| = \dots = |a_N|$. The Kirchhoff condition in H^2_Γ is equivalent to

$$\phi(a_1) \sum_{j=1}^{N} \tanh(a_j) = 0, \tag{2.3.11}$$

which together with the continuity condition implies that the set $(a_1, \dots, a_N)$ has exactly $\frac{N}{2}$ positive elements and exactly $\frac{N}{2}$ negative elements.

The number of all possible solutions is given by rearrangements of $N/2$ edges from N edges,

$$C_N = \frac{1}{2}\binom{N}{N/2},$$

where the factor $\frac{1}{2}$ is due to the double count of rearrangements with $a > 0$ and $a < 0$. $\square$

Remark 2.14. *For $\alpha = 1$ and odd N, the half-soliton state (2.3.4) is the unique stationary solution to (2.3.3). Indeed, in this case the equation (2.3.11) holds if and only if $a_1 = a_2 = \cdots = a_N = 0$.*

Remark 2.15. *If $N = 2$, then $C_2 = 1$. The only branch of shifted states in Lemma 2.13 corresponds to the NLS solitary wave translated along an infinite line $\mathbb{R}$, see Remark 2.8.*

Remark 2.16. *If $N = 4$, then $C_4 = 3$. The three branches of shifted states in Lemma 2.13 correspond to the three NLS solitary waves translated along an infinite line $\mathbb{R}$ defined by the union of either $(1,2)$ or $(1,3)$, or $(1,4)$ edges of the star graph Γ, with mirror-symmetric NLS solitary waves translated along another line $\mathbb{R}$ defined by the two complementary edges of the star graph Γ.*

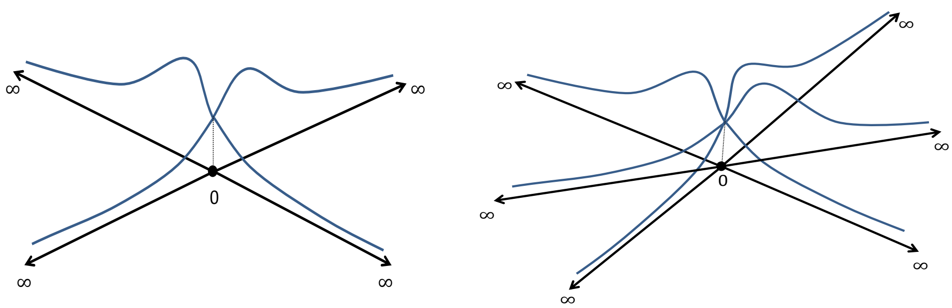

FIGURE 2.2: Schematic representation of the shifted states (2.3.10) with $a \neq 0$ for $N = 4$ (left) and $N = 6$ (right).

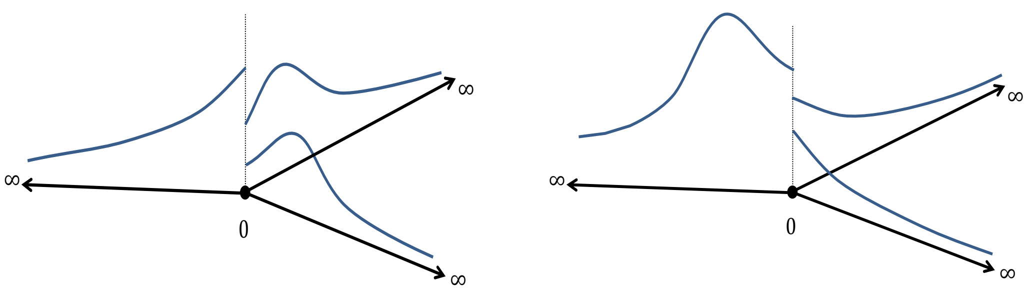

FIGURE 2.3: Schematic representation of the shifted states (2.3.6) with $K = 1$, $N = 3$, and either $a > 0$ (left) or $a < 0$ (right).

For graphical illustrations, we present some of the shifted states on Figures 2.2 and 2.3. Figure 2.2 shows the shifted states corresponding to Lemma 2.13 with $N = 4$ (left) and $N = 6$ (right). If $a \neq 0$, the profile of Φ contains $N/2$ monotonic and $N/2$ non-monotonic tails in different edges of the star graph Γ. Figures 2.3 shows the shifted states corresponding to Lemma 2.9 with $N = 3$ and $K = 1$. If $a > 0$ (left), the profile of Φ contains 1 monotonic and 2 non-monotonic tails whereas if $a < 0$ (right), the profile of Φ contains 2 monotonic and 1 non-monotonic tails.

2.4 The action functional $\Lambda(\Psi)$ and its Hessian

Every stationary state $\Phi_\omega(x; a)$ satisfying the stationary NLS equation (2.3.1) is a critical point of the action functional

$$\Lambda_\omega(\Psi) := E(\Psi) + \omega Q(\Psi), \quad \Psi \in H^1_\Gamma, \tag{2.4.1}$$

where Q and E are conserved mass and energy in (2.2.4) under the NLS flow, see Proposition 2.3.

Substituting $\Psi = \Phi_\omega(\cdot; a) + U + iW$ with real-valued $U, W \in H^1_\Gamma$ into $\Lambda_\omega(\Psi)$ and expanding in U, W yield

$$\Lambda_\omega(\Psi) = \Lambda_\omega(\Phi_\omega(\cdot; a)) + \langle L_+(\omega)U, U\rangle_{L^2(\Gamma)} + \langle L_-(\omega)W, W\rangle_{L^2(\Gamma)} + N(U, W), \tag{2.4.2}$$

where

$$\begin{aligned}
\langle L_+(\omega)U, U\rangle_{L^2(\Gamma)} &:= \int_\Gamma \left[(\nabla U)^2 + \omega U^2 - (2p+1)(p+1)\alpha^{2p}\Phi_\omega(\cdot; a)^{2p}U^2\right] dx, \\
\langle L_-(\omega)W, W\rangle_{L^2(\Gamma)} &:= \int_\Gamma \left[(\nabla W)^2 + \omega W^2 - (p+1)\alpha^{2p}\Phi_\omega(\cdot; a)^{2p}W^2\right] dx,
\end{aligned}$$

and $N(U, W) = \mathrm{o}(\|U + iW\|^2_{H^1(\Gamma)})$ for every $p > 0$. The quadratic forms are defined by the two Hessian operators

$$L_+(\omega) = -\Delta + \omega - (2p+1)(p+1)\alpha^{2p}\Phi_\omega(\cdot; a)^{2p} : H^2_\Gamma \subset L^2(\Gamma) \to L^2(\Gamma), \tag{2.4.3}$$

$$L_-(\omega) = -\Delta + \omega - (p+1)\alpha^{2p}\Phi_\omega(\cdot; a)^{2p} : \quad H^2_\Gamma \subset L^2(\Gamma) \to L^2(\Gamma), \tag{2.4.4}$$

Using the scaling transformation similar to (2.3.2), we simplify the consideration to $\omega = 1$. We also assume that $\Phi \equiv \Phi_{\omega=1}(\cdot; a)$ is the shifted state defined in Lemma 2.9 for an arbitrary $a \in \mathbb{R}$.

We denote $L_+ := L_+(\omega = 1)$ and $L_- := L_-(\omega = 1)$ as in (2.4.3) and (2.4.4), respectively. With the use of Proposition 2.1, we observe that the operators L_+ and L_- are self-adjoint in $L^2(\Gamma)$. The spectrum $\sigma(L_\pm) \subset \mathbb{R}$ consists of the continuous and discrete parts denoted by $\sigma_c(L_\pm)$ and $\sigma_p(L_\pm)$, respectively. Since the bounded and exponentially decaying potential $\alpha^{2p}\Phi^{2p}$ is a relatively compact perturbation to the unbounded operator $L_0 := -\Delta + 1$, the absolutely continuous spectra of $L_\pm$, by Weyl's

Theorem, is

$$\sigma_c(L_\pm) = \sigma(L_0) = [1, \infty). \tag{2.4.5}$$

Therefore, we are only interested in the eigenvalues of $\sigma_p(L_\pm)$ in $(-\infty, 1)$.

Definition 2.17. *The number of negative eigenvalues of $L_\pm$ is called the* ***Morse index*** *of $L_\pm$. Also, the multiplicity of the zero eigenvalue is called the* **degeneracy index**.

2.4.1 The operator L_-

The following result shows that $\sigma_p(L_-)$ is nonnegative, $0 \in \sigma_p(L_-)$ is a simple eigenvalue with the eigenvector Φ, and all other eigenvalues in $\sigma_p(L_-)$ are bounded away from zero. In other words, the Morse index of L_- is zero, see also Theorem 3.12 in [60].

Lemma 2.18. *For any $W \in H^1_\Gamma$,*

$$\langle L_- W, W\rangle_{L^2(\Gamma)} = 0 \quad \text{if and only if} \quad W \in \text{span}\{\Phi\}.$$

Moreover, there exists $C > 0$ such that

$$\langle L_- W, W\rangle_{L^2(\Gamma)} \geq C\|W\|^2_{H^1(\Gamma)}, \tag{2.4.6}$$

for every $W \in H^1_\Gamma \cap L^2_c$, where L^2_c is defined by

$$L^2_c := \left\{ V \in L^2(\Gamma) : \quad \langle V, \Phi\rangle_{L^2(\Gamma)} = 0 \right\}, \tag{2.4.7}$$

Proof. By using (2.3.6), we write for every $W = (w_1, w_2, \dots, w_N)^T \in H^1_\Gamma$,

$$\langle L_- W, W\rangle_{L^2(\Gamma)} = \sum_{j=1}^N \int_0^{+\infty} \left[\left(\frac{dw_j}{dx}\right)^2 + w_j^2 - (p+1)\varphi_j^{2p} w_j^2 \right] dx, \tag{2.4.8}$$

where

$$\varphi_j(x) = \begin{cases} \phi(x+a), & j = 1, \dots, K \\ \phi(x-a), & j = K+1, \dots, N, \end{cases} \tag{2.4.9}$$

with $\phi(x) = \text{sech}^{\frac{1}{p}}(px)$. By using $\varphi_j'' = \varphi_j - (p+1)\varphi_j^{2p+1}$, $(\varphi_j')^2 = \varphi_j^2 - \varphi_j^{2p+2}$, integration by parts, the boundary conditions in (2.1.1) and the constraint (2.3.5), we obtain

$$\int_0^{+\infty} p w_j^2 \varphi_j^{2p} dx = \int_0^{+\infty} 2w_j \frac{dw_j}{dx} \frac{\varphi_j'}{\varphi_j} dx$$

and

$$\int_0^{+\infty} \left(w_j^2 - \varphi_j^{2p} w_j^2 \right) dx = \int_0^{+\infty} w_j^2 \left(\frac{\varphi_j'}{\varphi_j} \right)^2 dx,$$

so that the representation (2.4.8) is formally equivalent to

$$\langle L_- W, W\rangle_{L^2(\Gamma)} = \sum_{j=1}^{N} \int_0^{+\infty} \varphi_j^2 \left| \frac{d}{dx}\left(\frac{w_j}{\varphi_j}\right)\right|^2 dx \geq 0. \tag{2.4.10}$$

Since $\varphi_j(x) > 0$ for every $x \in \mathbb{R}^+$ and $\partial_x \log \varphi_j \in L^\infty(\mathbb{R})$, the representation (2.4.10) is justified for every $W \in H^1_\Gamma$. It follows from (2.4.10) that $\langle L_- W, W\rangle_{L^2(\Gamma)} = 0$ if and only if $W \in H^1_\Gamma$ satisfies

$$\frac{d}{dx}\left(\frac{w_j}{\varphi_j}\right) = 0 \quad \text{almost everywhere and for every } \; j. \tag{2.4.11}$$

Sobolev's embedding of $H^1(\mathbb{R}^+)$ into $C(\mathbb{R}^+)$ and equation (2.4.11) imply that $w_j = c_j \varphi_j$ for some constant c_j. The generalized continuity boundary conditions in (2.1.1) and the equality $\phi(a) = \phi(-a)$ for (2.4.9) then yield

$$c_1\alpha_1 = c_2\alpha_2 = \cdots = c_N\alpha_N$$

which means that 0 is a simple eigenvalue of the operator L_- in (2.4.4) with the eigenvector $W \in \operatorname{span}\{\Phi\}$. Since eigenvalues of $\sigma_p(L_-) \in (-\infty, 1)$ are isolated, the variational characterization of eigenvalues implies the $L^2(\Gamma)$-coercivity

$$\langle L_- W, W\rangle_{L^2(\Gamma)} \geq C_0 \|W\|^2_{L^2(\Gamma)}$$

for every $W \in H^1_\Gamma \cap L^2_c$ and some $C_0 > 0$. The $H^1(\Gamma)$-coercivity comes by contradiction. There exists no sequence $\{W_n\}_{n\in\mathbb{N}} \subset H^1_\Gamma \cap L^2_c$ such that $\|W_n\|_{H^1(\Gamma)} = 1$ and $\langle L_- W_n, W_n\rangle_{L^2(\Gamma)} \to 0$, see also Lemma 5.2.3 in [45]. $\square$

2.4.2 The operator L_+

We are interested in the discrete spectrum of the Hessian operator L_+ given by (2.4.3) with $\omega = 1$. Below we show the reduction of the spectral problem for L_+ on the star graph Γ to the eigenvalue problem for scalar Schrödinger equations.

By using the representation (2.3.6), the general form of components for the spectral problem $L_+ U = \lambda U$ on Γ is given by the following second-order differential equation

$$-u''(x) + u(x) - (2p+1)(p+1)\operatorname{sech}^2(p(x+a_0))u(x) = \lambda u(x), \quad x \in (0,\infty), \tag{2.4.12}$$

where a_0 represents either $+a$ or $-a$ shift in (2.3.6) depending on the edge. By the Sobolev's embedding of $H^2(\mathbb{R}^+)$ into $C^1(\mathbb{R}^+)$, we consider only the exponentially decaying solutions to (2.4.12). By means of the substitution $u(x) = v(x+a_0)$ for $x \in (0,\infty)$, exponentially decaying solutions u to the equation (2.4.12) are equivalent to exponentially decaying solutions v of the second-order differential equation

$$-v''(x) + v(x) - (2p+1)(p+1)\operatorname{sech}^2(px)v(x) = \lambda v(x), \quad x \in (a_0,\infty). \tag{2.4.13}$$

The following lemmas extend some well-known results on the scalar Schrödinger equation (2.4.13).

Lemma 2.19. *For every $\lambda < 1$, there exists a unique solution $v \in C^1(\mathbb{R})$ to equation (2.4.13) such that*

$$\lim_{x\to+\infty} v(x)e^{\sqrt{1-\lambda}x} = 1. \tag{2.4.14}$$

Moreover, for any fixed $x_0 \in \mathbb{R}$, $v(x_0)$ is a C^1 function of λ for $\lambda < 1$ such that $\frac{v'(x_0)}{v(x_0)} \to -\infty$ as $\lambda \to -\infty$. The other linearly independent solution to equation (2.4.13) diverges as $x \to +\infty$.

Proof. The proof is based on the reformulation of the boundary–value problem (2.4.13)–(2.4.14) as Volterra's integral equation. By means of Green's function, the solution to (2.4.13)–(2.4.14) can be found from the inhomogeneous integral equation

$$v(x) = e^{-\sqrt{1-\lambda}x} - \frac{(2p+1)(p+1)}{\sqrt{1-\lambda}} \int_x^\infty \sinh(\sqrt{1-\lambda}(x-y))\,\mathrm{sech}^2(py)v(y)\,\mathrm{d}\,y. \tag{2.4.15}$$

Setting $w(x;\lambda) = v(x)e^{\sqrt{1-\lambda}x}$ yields the following Volterra's integral equation with a bounded kernel:

$$w(x;\lambda) = 1 + \frac{(2p+1)(p+1)}{2\sqrt{1-\lambda}} \int_x^\infty (1 - e^{-2\sqrt{1-\lambda}(y-x)})\,\mathrm{sech}^2(py)w(y;\lambda)\,\mathrm{d}\,y. \tag{2.4.16}$$

By standard Neumann series, there exists a unique solution $w(\cdot;\lambda) \in C^1(x_0,\infty)$ satisfying $\lim_{x\to\infty} w(x;\lambda) = 1$ for every $\lambda < 1$ and sufficiently large $x_0 \gg 1$. By the ODE theory, this solution is extended globally as a solution $w(\cdot;\lambda) \in C^1(\mathbb{R})$ of the integral equation (2.4.16). This construction yields a solution $v \in C^1(\mathbb{R})$ to the differential equation (2.4.13) with the exponential decay as $x \to +\infty$ given by (2.4.14). Since the Volterra's integral equation (2.4.15) depends analytically on λ for $\lambda < 1$, then $v(x_0)$ is (at least) C^1 function of $\lambda < 1$ for any fixed $x_0 \in \mathbb{R}$. Thanks to the x-independent and nonzero Wronskian determinant between two linearly independent solutions to the second-order equation (2.4.13), the other linearly independent solution diverges exponentially as $x \to +\infty$.

It remains to prove that $\frac{v'(x_0)}{v(x_0)} \to -\infty$ as $\lambda \to -\infty$ for any fixed $x_0 \in \mathbb{R}$. Using the setting $w(x;\lambda) = v(x)e^{\sqrt{1-\lambda}x}$, we get

$$\frac{v'(x_0)}{v(x_0)} = -\sqrt{1-\lambda} + \frac{w'(x_0;\lambda)}{w(x_0;\lambda)}. \tag{2.4.17}$$

Since $w(\cdot,\lambda) \in C^1(\mathbb{R})$ and $\lim_{x\to\infty} w(x;\lambda) = 1$, we get $w(\cdot,\lambda) \in L^\infty[x_0,\infty)$. The construction (2.4.16) yields that $\|w\|_{L^\infty[x_0,\infty)} \leq 2$ for large enough negative λ, and so, as $\lambda \to -\infty$, (2.4.16) implies

$$|w(x_0;\lambda) - 1| \leq \frac{C_p}{\sqrt{1-\lambda}} \|w\|_{L^\infty[x_0,\infty)} \leq \frac{2C_p}{\sqrt{1-\lambda}} \to 0,$$

where C_p is constant which depends on p only. Therefore,

$$w(x_0;\lambda) \to 1 \quad \text{as} \quad \lambda \to -\infty. \tag{2.4.18}$$

Differentiating the equation (2.4.16) in x, we get

$$w'(x;\lambda) = -(2p+1)(p+1) \int_x^\infty e^{-2\sqrt{1-\lambda}(y-x)} \operatorname{sech}^2(py) w(y;\lambda) \,\mathrm{d}\, y.$$

Since the integrand in the latter expression is bounded for $\lambda < 1$, for $\lambda \to -\infty$ we get

$$|w'(x_0;\lambda)| \leq \hat{C}_p \|w\|_{L^\infty[x_0,\infty)} \leq 2\hat{C}_p, \tag{2.4.19}$$

where $\hat{C}_p$ is constant which depends on p only.

Finally, by using the bounds in (2.4.18) and (2.4.19), the expression (2.4.17) implies that $\frac{v'(x_0)}{v(x_0)} \to -\infty$ as $\lambda \to -\infty$. □

Lemma 2.20. *Let v be the solution defined in Lemma 2.19. If $v(0) = 0$ (resp. $v'(0) = 0$) for some $\lambda_0 < 1$, then the corresponding eigenfunction v to the Schrödinger equation (2.4.13) is an odd (resp. even) function on $\mathbb{R}$, whereas λ_0 is an eigenvalue of the associated Schrödinger operator defined in $L^2(\mathbb{R})$. There exists exactly one simple eigenvalue $\lambda_0 < 0$ corresponding to $v'(0) = 0$ and a simple eigenvalue $\lambda_0 = 0$ corresponding to $v(0) = 0$, all other possible points λ_0 are located in $(0, 1)$ bounded away from zero.*

Proof. Extension of v to an eigenfunction of the associated Schrödinger operator defined in $L^2(\mathbb{R})$ follows by the reversibility of the Schrödinger equation (2.4.13) with respect to the transformation $x \mapsto -x$. The count of eigenvalues follows by Sturm's theorem since the odd eigenfunction for the eigenvalue $\lambda_0 = 0$,

$$\phi'(x) = -\operatorname{sech}^{\frac{1}{p}}(px)\tanh(px) \tag{2.4.20}$$

has one zero on the infinite line. By Sturm's theorem, $\lambda_0 = 0$ is the second eigenvalue of the Schrödinger equation (2.4.13) with exactly one simple negative eigenvalue $\lambda_0 < 0$ that corresponds to an even eigenfunction. □

Remark 2.21. *For $p = 1$, the solution v in Lemma 2.19 is available in the closed analytic form:*

$$v(x) = e^{-\sqrt{1-\lambda}x} \frac{3 - \lambda + 3\sqrt{1-\lambda}\tanh x - 3\operatorname{sech}^2 x}{3 - \lambda + 3\sqrt{1-\lambda}}.$$

In this case, the eigenvalues and eigenfunctions in Lemma 2.20 are given by

$$\lambda = -3: \qquad v(x) = \tfrac{1}{4}\operatorname{sech}^2 x,$$
$$\lambda = 0: \qquad v(x) = \tfrac{1}{2}\tanh x \operatorname{sech} x.$$

No other eigenvalues of the associated Schrödinger operator on $L^2(\mathbb{R})$ *exist in* $(-\infty, 1)$*.*

Lemma 2.22. *Let* $v = v(x;\lambda)$ *be the solution defined by Lemma 2.19. Assume that* $v(x;\lambda_1)$ *has a simple zero at* $x = x_1 \in \mathbb{R}$ *for some* $\lambda_1 \in (-\infty, 1)$*. Then, there exists a unique* C^1 *function* $\lambda \mapsto x_*(\lambda)$ *for* λ *near* λ_1 *such that* $v(x;\lambda)$ *has a simple zero at* $x = x_*(\lambda)$ *with* $x_*(\lambda_1) = x_1$ *and* $x_*'(\lambda_1) > 0$*.*

Proof. By Lemma 2.19, v is a C^1 function of x and λ for every $x \in \mathbb{R}$ and $\lambda \in (-\infty, 1)$. Since x_1 is a simple zero of $v(x;\lambda_1)$, we have $\partial_x v(x_1;\lambda_1) \neq 0$. By the implicit function theorem, there exists a unique C^1 function $\lambda \mapsto x_*(\lambda)$ for λ near λ_1 such that $v(x;\lambda)$ has a simple zero at $x = x_*(\lambda)$ with $x_*(\lambda_1) = x_1$. It remains to show that $x_*'(\lambda_1) > 0$.

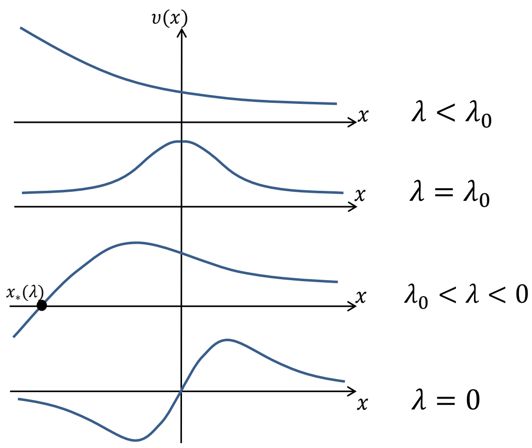

FIGURE 2.4: Profiles of the solution v in Lemma 2.19 for different values of λ.

Differentiating $v(x_*(\lambda);\lambda) = 0$ in λ at $\lambda = \lambda_1$, we obtain

$$\partial_x v(x_1;\lambda_1)x_*'(\lambda_1) + \partial_\lambda v(x_1;\lambda_1) = 0. \tag{2.4.21}$$

Let us denote $\tilde{v}(x) = \partial_\lambda v(x;\lambda_1)$. Differentiating equation (2.4.13) in λ yields the inhomogeneous differential equation for $\tilde{v}$:

$$-\tilde{v}''(x) + \tilde{v}(x) - (2p+1)(p+1)\operatorname{sech}^2(px)\tilde{v}(x) = \lambda_1\tilde{v}(x) + v(x;\lambda_1), \quad x \in (a,\infty). \tag{2.4.22}$$

By the same method based on the Volterra's integral equation as in Lemma 2.19, the function $\tilde{v}$ is C^1 in x and decays to zero as $x \to \infty$. Therefore, by multiplying equation (2.4.22) by $v(x;\lambda_1)$, integrating by parts on $[x_1, \infty)$, and using equation (2.4.13), we obtain

$$-\partial_x v(x_1;\lambda_1)\tilde{v}(x_1) = \int_{x_1}^{\infty} v(x;\lambda_1)^2 dx, \tag{2.4.23}$$

where we have used $v(x_1;\lambda_1) = 0$ as well as the decay of $v(x;\lambda_1)$, $\partial_x v(x;\lambda_1)$, $\tilde{v}(x)$, and $\tilde{v}'(x)$ to zero as $x \to \infty$. Combining (2.4.21) and (2.4.23) yields

$$(\partial_x v(x_1;\lambda_1))^2 x_*'(\lambda_1) = \int_{x_1}^{\infty} v(x;\lambda_1)^2 dx > 0, \tag{2.4.24}$$

so that $x_*'(\lambda_1) > 0$ follows from the fact that $\partial_x v(x_1;\lambda_1) \neq 0$. □

Remark 2.23. *We can obtain same results for a general $\omega > 0$ by using the scaling transformation (2.3.2).*

The results of Lemmas 2.19, 2.20, and 2.22 are illustrated on Figure 2.4 which shows profiles of the solution v satisfying the limit (2.4.14) for four cases of λ in $(-\infty, 0]$. The even eigenfunction for $\lambda_0 < 0$ and the odd eigenfunction for $\lambda = 0$ correspond to the solutions of the Schrödinger equation defined in $L^2(\mathbb{R})$. The only zero $x_*(\lambda)$ of v appears from negative infinity at $\lambda = \lambda_0$ and it is a monotonically increasing function of λ in $(\lambda_0, 0)$ such that $x_*(0) = 0$.

Chapter 3

Nonlinear Instability of Half-Solitons on Star Graphs

This chapter is devoted to the study of nonlinear stability of the half-soliton state Φ defined in (2.3.4) with $\alpha = 1$. According to Lemma 2.13 and Remark 2.14, the stationary NLS equation (2.3.1) on the star graph Γ for every N admits the half-soliton state.

The analysis of variational properties of the half-soliton state on star graphs was initiated in [2]. Namely, it was shown that the half-soliton state is a saddle point of the constrained energy functional associated to the cubic NLS equation on Γ with $N = 3$ edges. The saddle point geometry was not related to the instability of the half-soliton state in the time evolution of the NLS, and was obtained by considering two constrained families of states on Γ such that the half-soliton state minimizes the energy along one family, but maximizes along the other.

The main result of this section is to provide a dynamical characterization of the result in [2] for the NLS with the power nonlinearity and in the case of an arbitrary star graph. By using dynamical system methods (in particular, normal forms), we will verify that the half-soliton state is the saddle point of the constrained NLS energy on the star graph and moreover it is dynamically unstable due to the slow growth of perturbations. This nonlinear instability is likely to result in the destruction of the half-soliton state pinned to the vertex and the formation of a solitary wave escaping to infinity along one edge of the star graph.

For every $p > 0$, we define the orbital stability and instability of the half-soliton state Φ with respect to its orbit $\{e^{i\theta}\Phi : \theta \in \mathbb{R}\}$ as the following:

Definition 3.1. *The stationary state Φ is* **orbitally stable** *if for every $\epsilon > 0$ there is $\delta > 0$, such that for every $\Psi_0 \in H^1_\Gamma$ with $\|\Psi_0 - \Phi\|_{H^1(\Gamma)} < \delta$, the unique global solution $\Psi(t) \in C(\mathbb{R}, H^1_\Gamma) \cap C^1(\mathbb{R}, H^{-1}_\Gamma)$ to the NLS equation (3.1.1) starting with the initial datum $\Psi(0) = \Psi_0$ satisfies*

$$\inf_{\theta\in\mathbb{R}} \|e^{-i\theta}\Psi(t) - \Phi\|_{H^1(\Gamma)} < \epsilon \quad \text{for all} \quad t > 0.$$

Otherwise, it is **orbitally unstable**.

3.1 Main results

We consider a star graph Γ with $N \geq 3$ edges, and set $\alpha = 1$ in the boundary conditions (2.1.1) and (2.1.2). Then, the NLS equation (2.2.1) is

$$i\frac{\partial \Psi}{\partial t} = -\Delta\Psi - (p+1)|\Psi|^{2p}\Psi, \quad x \in \Gamma, \quad t \in \mathbb{R}, \tag{3.1.1}$$

and the half-soliton state (2.3.4) solving the stationary NLS equation (2.3.3) with $\omega = 1$ is given by

$$\Phi(x) = \phi(x)(1,1,\dots,1)^T, \quad \text{with } \phi(x) = \operatorname{sech}^{\frac{1}{p}}(px). \tag{3.1.2}$$

Our main results are given as follows. Thanks to the scaling transformation, we set $\omega = 1$ and use the notation Λ for $\Lambda_{\omega=1}$.

Theorem 3.2. *Let $\Lambda''(\Phi)$ be the Hessian operator for the second variation of $\Lambda(\Psi)$ at $\Psi = \Phi$ in H^1_Γ. For every $p \in (0,2)$, it is true that $\langle \Lambda''(\Phi)V, V\rangle_{L^2(\Gamma)} \geq 0$ for every $V \in H^1_\Gamma \cap L^2_c$, where L^2_c is defined in (2.4.7) as*

$$L^2_c := \left\{ V \in L^2(\Gamma) : \quad \langle V, \Phi\rangle_{L^2(\Gamma)} = 0 \right\}.$$

Moreover, $\langle \Lambda''(\Phi)V, V\rangle_{L^2(\Gamma)} = 0$ if and only if $V \in H^1_\Gamma \cap L^2_c$ belongs to a $(N-1)$-dimensional subspace $X_c := \operatorname{span}\{U^{(1)}, U^{(2)}, \dots, U^{(N-1)}\} \subset L^2_c$. Consequently, $V = 0$ is a degenerate minimizer of $\langle \Lambda''(\Phi)V, V\rangle_{L^2(\Gamma)}$ in $H^1_\Gamma \cap L^2_c$.

Remark 3.3. *If $p = 2$, then $\langle \Lambda''(\Phi)V, V\rangle_{L^2(\Gamma)} = 0$ if and only if $V \in H^1_\Gamma \cap L^2_c$ belongs to a N-dimensional subspace of L^2_c with an additional degeneracy. For $p > 2$, the second variation is not positive in $H^1_\Gamma \cap L^2_c$.*

Theorem 3.4. *Let $X_c = \operatorname{span}\{U^{(1)}, U^{(2)}, \dots, U^{(N-1)}\} \subset L^2_c$ be defined in Theorem 3.2. For every $p \in \left[\frac{1}{2}, 2\right)$, there exists $\delta > 0$ such that for every $c = (c_1, c_2, \dots, c_{N-1})^T \in \mathbb{R}^{N-1}$ satisfying $\|c\| \leq \delta$, there exists a unique minimizer of the variational problem*

$$M(c) := \inf_{U^\perp \in H^1_\Gamma \cap L^2_c \cap [X_c]^\perp} \left[\Lambda(\Phi + c_1 U^{(1)} + \dots + c_{N-1}U^{(N-1)} + U^\perp) - \Lambda(\Phi) \right] \tag{3.1.3}$$

such that $\|U^\perp\|_{H^1(\Gamma)} \leq A\|c\|^2$ for a c-independent constant $A > 0$. Moreover, $M(c)$ is sign-indefinite in c. Consequently, Φ is a nonlinear saddle point of Λ in H^1_Γ with respect to perturbations in $H^1_\Gamma \cap L^2_c$.

Remark 3.5. *The restriction $p \geq \frac{1}{2}$ is used in order to expand $\Lambda(\Phi + U)$ up to the cubic terms with respect to the perturbation $U \in H^1_\Gamma \cap L^2_c$ and then to pass to normal forms. If $p = 2$, Φ is still a nonlinear saddle point of Λ in $H^1_\Gamma \cap L^2_c$ but the proof needs to be modified by the fact that X_c is N-dimensional. If $p > 2$, it follows already from the second derivative test that Φ is a saddle point of Λ in $H^1_\Gamma \cap L^2_c$.*

Theorem 3.6. *For every $p \in \left[\frac{1}{2}, 2\right)$, there exists $\epsilon > 0$ such that for every $\delta > 0$ (sufficiently small) there exists $V \in H^1_\Gamma$ with $\|V\|_{H^1_\Gamma} \leq \delta$ such that the unique global solution $\Psi(t) \in C(\mathbb{R}, H^1_\Gamma) \cap C^1(\mathbb{R}, H^{-1}_\Gamma)$ to the NLS equation (3.1.1) starting with the initial datum $\Psi(0) = \Phi + V$ satisfies*

$$\inf_{\theta \in \mathbb{R}} \|e^{-i\theta}\Psi(t_0) - \Phi\|_{H^1(\Gamma)} > \epsilon \quad \text{for some} \;\; t_0 > 0. \tag{3.1.4}$$

Consequently, the orbit $\{\Phi e^{i\theta}\}_{\theta\in\mathbb{R}}$ is unstable in the time evolution of the NLS equation (3.1.1) in H^1_Γ.

Remark 3.7. *If $p = 2$, the instability claim of Theorem 3.6 follows from the same analysis as in the case of the NLS equation on the real line [22, 57]. If $p > 2$, the instability claim of Theorem 3.6 follows from the spectral instability [37].*

3.2 Degeneracy of the second variation

This section is devoted to the proof of Theorem 3.2.

It follows from the expansion of the action functional $\Lambda(\Psi)$ with $\Psi = \Phi + V$ around Φ, that the second variation $\Lambda''(\Phi)$ satisfies

$$\frac{1}{2}\langle \Lambda''(\Phi)V, V\rangle_{L^2(\Gamma)} = \langle L_+U, U\rangle_{L^2(\Gamma)} + \langle L_-W, W\rangle_{L^2(\Gamma)} \quad \text{with} \;\; V = U + iW, \tag{3.2.1}$$

where $U, W \in H^1_\Gamma$ are real-valued. In the strong formulation, the operators L_+ and L_- are equivalent to the Hessian operators in (2.4.3) and (2.4.4), respectively, with $\omega = 1$ and $\alpha = 1$:

$$\begin{aligned} L_+ &= -\Delta + 1 - (2p+1)(p+1)\Phi^{2p} : H^2_\Gamma \subset L^2(\Gamma) \to L^2(\Gamma), \\ L_- &= -\Delta + 1 - (p+1)\Phi^{2p} : \qquad\qquad H^2_\Gamma \subset L^2(\Gamma) \to L^2(\Gamma). \end{aligned}$$

Recall that the continuous spectrum of $L_\pm$ is given in (2.4.5), and the point spectrum is located in $(-\infty, 1)$. Moreover, by Lemma 2.18, the operator L_- is coercive in the subspace $H^1_\Gamma \cap L^2_c$. Therefore, we are only concerned with the eigenvalues of L_+.

By using Lemmas 2.19 and 2.20, we can now characterize $\sigma_p(L_+)$ in $(-\infty, 1)$. The following result shows that $\sigma_p(L_+)$ includes a simple negative eigenvalue and a zero eigenvalue of multiplicity $N - 1$.

Lemma 3.8. *Let u be a solution of Lemma 2.19 for $\lambda \in (-\infty, 1)$. Then, $\lambda_0 \in (-\infty, 1)$ is an eigenvalue of $\sigma_p(L_+)$ if and only if either $u(0) = 0$ or $u'(0) = 0$ (both $u(0)$ and $u'(0)$ cannot vanish simultaneously). Moreover, λ_0 in $\sigma_p(L_+)$ has multiplicity $N - 1$ if $u(0) = 0$ and multiplicity 1 if $u'(0) = 0$.*

Proof. Let $\lambda_0 \in (-\infty, 1)$ be an eigenvalue of $\sigma_p(L_+)$ and denote the eigenvector by $U \in H^2_\Gamma$. Since $U(x)$ and $U'(x)$ decay to zero as $x \to +\infty$, by Sobolev's embedding of $H^2(\mathbb{R}^+)$ to the space $C^1(\mathbb{R}^+)$, we can parameterize $U \in H^2_\Gamma$ by using u from Lemma 2.19 as follows

$$U(x) = u(x) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix},$$

where $(c_1, c_2, \dots, c_N)$ are some coefficients. By using the boundary conditions in the definition of H^2_Γ in (2.1.2), we obtain a homogeneous linear system on the coefficients:

$$c_1 u(0) = c_2 u(0) = \cdots = c_N u(0), \quad c_1 u'(0) + c_2 u'(0) + \cdots + c_N u'(0) = 0. \tag{3.2.2}$$

The determinant of the associated matrix is

$$\Delta = [u(0)]^{N-1} u'(0) \begin{vmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{vmatrix} = N[u(0)]^{N-1} u'(0). \tag{3.2.3}$$

Therefore, $U \neq 0$ is an eigenvector for an eigenvalue $\lambda_0 \in (-\infty, 1)$ if and only if $\Delta = 0$, which is only possible in (3.2.3) if either $u(0) = 0$ or $u'(0) = 0$. Moreover, multiplicity of $u(0)$ and $u'(0)$ in Δ coincides with the multiplicity of the eigenvalue λ_0 because it gives the number of linearly independent solutions of the homogeneous linear system (3.2.2). The assertion of the lemma is proven. □

Corollary 3.9. *There exists exactly one simple negative eigenvalue $\lambda_0 < 0$ in $\sigma_p(L_+)$ and a zero eigenvalue $\lambda_0 = 0$ in $\sigma_p(L_+)$ of multiplicity $N - 1$, all other possible eigenvalues of $\sigma_p(L_+)$ in $(0, 1)$ are bounded away from zero.*

Proof. The result follows from Lemmas 2.20 and 3.8. □

Remark 3.10. *For the simple eigenvalue $\lambda_0 < 0$ in $\sigma_p(L_+)$, the corresponding eigenvector is*

$$U = u(x) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

where $u(x) > 0$ for every $x \in \mathbb{R}^+$ with $u'(0) = 0$. For the eigenvalue $\lambda_0 = 0$ of multiplicity $N - 1$ in $\sigma_p(L_+)$, the invariant subspace of L_+ can be spanned by an orthogonal basis of eigenvectors $\{U^{(1)}, U^{(2)}, \dots, U^{(N-1)}\}$. The orthogonal basis of eigenvectors can *be constructed by induction as follows:*

$$N = 3: \qquad U^{(1)} = \phi'(x)\begin{pmatrix}1\\-1\\0\end{pmatrix}, \quad U^{(2)} = \phi'(x)\begin{pmatrix}1\\1\\-2\end{pmatrix},$$

$$N = 4: \qquad U^{(1)} = \phi'(x)\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \quad U^{(2)} = \phi'(x)\begin{pmatrix}1\\1\\-2\\0\end{pmatrix}, \quad U^{(3)} = \phi'(x)\begin{pmatrix}1\\1\\1\\-3\end{pmatrix},$$

and so on.

The following result shows that the operator L_+ is positive in the subspace L^2_c associated with a scalar constraint in (2.4.7), provided the nonlinearity power p is in $(0, 2)$, and coercive on a subspace of L^2_c orthogonal to $\ker(L_+)$.

Lemma 3.11. *For every $p \in (0, 2)$, $\langle L_+U, U\rangle_{L^2(\Gamma)} \geq 0$ for every $U \in H^1_\Gamma \cap L^2_c$, where L^2_c is given by (2.4.7). Moreover $\langle L_+U, U\rangle_{L^2(\Gamma)} = 0$ if and only if $U \in H^1_\Gamma \cap L^2_c$ belongs to the $(N-1)$-dimensional subspace $X_c = \mathrm{span}\{U^{(1)}, U^{(2)}, \dots, U^{(N-1)}\} \subset L^2_c$ in the kernel of L_+. Consequently, there exists $C_p > 0$ such that*

$$\langle L_+U, U\rangle_{L^2(\Gamma)} \geq C_p\|U\|^2_{H^1(\Gamma)} \quad \text{for every } \ U \in H^1_\Gamma \cap L^2_c \cap [X_c]^\perp. \tag{3.2.4}$$

Proof. Since $\sigma_c(L_+) = \sigma(-\Delta + 1) = [1, \infty)$ by (2.4.5), the eigenvalues of $\sigma_p(L_+)$ at $\lambda_0 < 0$ and $\lambda = 0$ given by Corollary 3.9 are isolated. Since $\langle U^{(k)}, \Phi\rangle_{L^2(\Gamma)} = 0$ for every $1 \leq k \leq N - 1$, $L_+^{-1}\Phi$ exists in $L^2(\Gamma)$ and is in fact given by $L_+^{-1}\Phi = -\partial_\omega\Phi_\omega|_{\omega=1}$ up to an addition of an arbitrary element in $\ker(L_+)$. By the well-known result (see Theorem 3.3 in [37]), $L_+|_{L^2_c}$ (that is, L_+ restricted on subspace L^2_c) is nonnegative if and only if

$$0 \geq \langle L_+^{-1}\Phi, \Phi\rangle_{L^2(\Gamma)} = -\langle \partial_\omega\Phi_\omega|_{\omega=1}, \Phi\rangle_{L^2(\Gamma)} = -\frac{1}{2}\frac{d}{d\omega}\|\Phi_\omega\|^2_{L^2(\Gamma)}\bigg|_{\omega=1}. \tag{3.2.5}$$

Moreover, $\ker(L_+|_{L^2_c}) = \ker(L_+)$ if $\langle L_+^{-1}\Phi, \Phi\rangle_{L^2(\Gamma)} \neq 0$. By the direct computation, we obtain

$$\|\Phi_\omega\|^2_{L^2(\Gamma)} = N\omega^{\frac{1}{p}-\frac{1}{2}}\int_0^\infty \phi(z)^2 dz$$

so that

$$\frac{d}{d\omega}\|\Phi_\omega\|^2_{L^2(\Gamma)} = N\left(\frac{1}{p} - \frac{1}{2}\right)\omega^{\frac{1}{p}-\frac{3}{2}}\int_0^\infty \phi(z)^2 dz, \tag{3.2.6}$$

so that $L_+|_{L^2_c} \geq 0$ if $p \in (0, 2]$ and $\ker(L_+|_{L^2_c}) = \ker(L_+)$ if $p \in (0, 2)$. This argument gives the first two assertions of the lemma. The coercivity bound (3.2.4) follows from the spectral theorem in L^2_c and Gårding inequality. □

Proof of Theorem 3.2. The result of Theorem 3.2 follows by Lemmas 2.18 and 3.11 applied to (3.2.1). ■

3.3 Half-solitons as saddle points of $\Lambda(\Psi)$

To prove Theorem 3.4, it is sufficient to work with real-valued perturbations $U \in H^1_\Gamma \cap L^2_c$ to the critical point $\Phi \in H^1_\Gamma$ of the action functional Λ. Assuming $p \geq \frac{1}{2}$, we substitute $\Psi = \Phi + U$ with real-valued $U \in H^1_\Gamma$ into $\Lambda(\Psi)$ and expand in U to obtain

$$\Lambda(\Phi+U) = \Lambda(\Phi) + \langle L_+ U, U\rangle_{L^2(\Gamma)} - \frac{2}{3}p(p+1)(2p+1)\langle \Phi^{2p-1}U^2, U\rangle_{L^2(\Gamma)} + S(U), \quad (3.3.1)$$

where

$$S(U) = \begin{cases} \mathrm{o}(\|U\|^3_{H^1(\Gamma)}), & p \in \left(\frac{1}{2}, 1\right), \\ \mathrm{O}(\|U\|^4_{H^1(\Gamma)}), & p \geq 1. \end{cases}$$

Compared to the expansion (2.4.2), we have set $W = 0$ and have expanded the cubic term explicitly, under the additional assumption $p \geq \frac{1}{2}$. In what follows, we inspect convexity of $\Lambda(\Phi + U)$ with respect to the small perturbation $U \in H^1_\Gamma \cap L^2_c$.

The quadratic form $\langle L_+ U, U\rangle_{L^2(\Gamma)}$ is associated with the same operator L_+ given by (2.4.3). By Lemma 3.11, $\ker(L_+) \equiv X_c = \mathrm{span}\{U^{(1)}, U^{(2)}, \dots, U^{(N-1)}\}$ for every $p > 0$, where the orthogonal vectors $\{U^{(1)}, U^{(2)}, \dots, U^{(N-1)}\}$ are constructed inductively in Remark 3.10. Furthermore, by Lemma 3.11, if $U \in H^1_\Gamma \cap L^2_c$, that is, if U satisfies $\langle U, \Phi\rangle_{L^2(\Gamma)} = 0$, then the quadratic form $\langle L_+ U, U\rangle_{L^2(\Gamma)}$ is positive for $p \in (0, 2)$, whereas if $U \in H^1_\Gamma \cap L^2_c \cap [X_c]^\perp$, the quadratic form is coercive. Hence, we use the orthogonal decomposition for $U \in H^1_\Gamma \cap L^2_c$:

$$U = c_1 U^{(1)} + c_2 U^{(2)} + \dots + c_{N-1} U^{(N-1)} + U^\perp, \quad (3.3.2)$$

where $U^\perp \in H^1_\Gamma \cap L^2_c \cap [X_c]^\perp$ satisfies $\langle U^\perp, U^{(j)}\rangle_{L^2(\Gamma)} = 0$ for every j and the coefficients $(c_1, c_2, \dots, c_{N-1})$ are found uniquely by

$$c_j = \frac{\langle U, U^{(j)}\rangle_{L^2(\Gamma)}}{\|U^{(j)}\|^2_{L^2(\Gamma)}}, \quad \text{for every} \quad j.$$

The following result shows how to define a unique mapping from $c = (c_1, c_2, \dots, c_{N-1})^t \in \mathbb{R}^{N-1}$ to $U^\perp \in H^1_\Gamma \cap L^2_c \cap [X_c]^\perp$ for small c.

Lemma 3.12. *For every $p \in \left[\frac{1}{2}, 2\right)$, there exists $\delta > 0$ and $A > 0$ such that for every $c \in \mathbb{R}^{N-1}$ satisfying $\|c\| \leq \delta$, there exists a unique minimizer $U^\perp \in H^1_\Gamma \cap L^2_c \cap [X_c]^\perp$ of the variational problem*

$$\inf_{U^\perp \in H^1_\Gamma \cap L^2_c \cap [X_c]^\perp} \left[\Lambda(\Phi + c_1 U^{(1)} + c_2 U^{(2)} + \dots + c_{N-1} U^{(N-1)} + U^\perp) - \Lambda(\Phi)\right]. \quad (3.3.3)$$

satisfying

$$\|U^\perp\|_{H^1(\Gamma)} \leq A\|c\|^2. \quad (3.3.4)$$

Proof. First, we find the critical point of $\Lambda(\Phi+U)$ with respect to $U^\perp \in H^1_\Gamma \cap L^2_c \cap [X_c]^\perp$ for a given small $c \in \mathbb{R}^{N-1}$. Therefore, we set up the Euler–Lagrange equation in the form $F(U^\perp, c) = 0$, where

$$F(U^\perp, c) : X \times \mathbb{R}^{N-1} \mapsto Y, \quad X := H^1_\Gamma \cap L^2_c \cap [X_c]^\perp, \quad Y := H^{-1}_\Gamma \cap L^2_c \cap [X_c]^\perp \tag{3.3.5}$$

is given explicitly by

$$F(U^\perp, c) := L_+ U^\perp - p(p+1)(2p+1)\Pi_c \Phi^{2p-1}\left(\sum_{j=1}^{N-1} c_j U^{(j)} + U^\perp\right)^2 - \Pi_c R\left(\sum_{j=1}^{N-1} c_j U^{(j)} + U^\perp\right),$$

where $\Pi_c : L^2(\Gamma) \mapsto L^2_c \cap [X_c]^\perp$ is the orthogonal projection operator and $R(U)$ satisfies

$$\|R(U)\|_{H^1(\Gamma)} = \begin{cases} \mathrm{o}(\|U\|^2_{H^1(\Gamma)}), & p \in \left(\frac{1}{2}, 1\right), \\ \mathrm{O}(\|U\|^3_{H^1(\Gamma)}), & p \geq 1. \end{cases}$$

Operator function F satisfies the conditions of the implicit function theorem:

(i) F is a C^2 map from $X \times \mathbb{R}^{N-1}$ to Y;

(ii) $F(0,0) = 0$;

(iii) $D_{U^\perp} F(0,0) = \Pi_c L_+ \Pi_c : X \mapsto Y$ has a bounded inverse from Y to X.

By the implicit function theorem (see Theorem 4.E in [80]), there are $r > 0$ and $\delta > 0$ such that for each $c \in \mathbb{R}^{N-1}$ with $\|c\| \leq \delta$ there exists a unique solution $U^\perp \in X$ of the operator equation $F(U^\perp, c) = 0$ with $\|U^\perp\|_{H^1(\Gamma)} \leq r$ such that the map

$$\mathbb{R}^{N-1} \ni c \to U^\perp(c) \in X \tag{3.3.6}$$

is C^2 near $c = 0$ and $U^\perp(0) = 0$. Since $D_{U^\perp} F(0,0) = \Pi_c L_+ \Pi_c : X \mapsto Y$ is strictly positive, the associated quadratic form is coercive according to the bound (3.2.4), hence the critical point $U^\perp = U^\perp(c)$ is a unique infimum of the variational problem (3.3.3) near $c = 0$.

It remains to prove the bound (3.3.4). To show this, we note that

$$F(0, c) = -p(p+1)(2p+1)\Pi_c \Phi^{2p-1}\left(\sum_{j=1}^{N-1} c_j U^{(j)}\right)^2 - \Pi_c R\left(\sum_{j=1}^{N-1} c_j U^{(j)}\right)$$

satisfies $\|F(0,c)\|_{L(\Gamma)} \leq \tilde{A}\|c\|^2$ for a c-independent constant $\tilde{A} > 0$. Since F is a C^2 map from $X \times \mathbb{R}^{N-1}$ to Y and $D_c F(0,0) = 0$, we have $D_c U^\perp(0) = 0$, so that the C^2 map (3.3.6) satisfies the bound (3.3.4). □

Proof of Theorem 3.4. Let us denote

$$M(c) := \inf_{U^\perp \in H^1_\Gamma \cap L^2_c \cap [X_c]^\perp} \left[\Lambda(\Phi + c_1 U^{(1)} + \cdots + c_{N-1} U^{(N-1)} + U^\perp) - \Lambda(\Phi)\right], \tag{3.3.7}$$

where the infimum is achieved by Lemma 3.12 for sufficiently small $c \in \mathbb{R}^{N-1}$. Thanks to the representation (3.3.1) and the bound (3.3.4), we obtain $M(c) = M_0(c) + \widetilde{M}(c)$, where

$$M_0(c) := -\frac{2}{3}p(p+1)(2p+1)\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1} c_i c_j c_k \langle \Phi^{2p-1}U^{(i)}U^{(j)}, U^{(k)}\rangle_{L^2(\Gamma)} \tag{3.3.8}$$

and

$$\tilde{M}(c) = \begin{cases} \mathrm{o}(\|c\|^3), & p \in \left(\frac{1}{2}, 1\right), \\ \mathrm{O}(\|c\|^4), & p \geq 1. \end{cases}$$

In order to show that $M_0(c)$ is sign-indefinite near $c = 0$, it is sufficient to show that at least one diagonal cubic coefficient in $M_0(c)$ is nonzero. Since

$$\int_0^{+\infty} \phi^{2p-1}(\phi')^3 dx = -\int_0^{+\infty} \operatorname{sech}^{\frac{2p+2}{p}}(px)\tanh^3(px)dx = -\frac{p}{2(p+1)(2p+1)},$$

we obtain

$$\langle \Phi^{2p-1}U^{(j)}U^{(j)}, U^{(j)}\rangle_{L^2(\Gamma)} = \frac{pj(j^2-1)}{2(p+1)(2p+1)} \neq 0, \quad j \geq 2, \tag{3.3.9}$$

where the algorithmic construction of the orthogonal vectors $\{U^{(1)}, U^{(2)}, \dots, U^{(N-1)}\}$ in Remark 3.10 has been used. Since the diagonal coefficients in front of the cubic terms $c_2^3, c_3^3, \dots, c_{N-1}^3$ in $M_0(c)$ are nonzero, $M_0(c)$ and hence $M(c)$ is sign-indefinite near $c = 0$. ■

Remark 3.13. *We give explicit expressions for the function $M_0(c)$:*

$$\begin{aligned} N = 3: \quad & M_0(c) = 2p^2(c_1^2 - c_2^2)c_2, \\ N = 4: \quad & M_0(c) = 2p^2(c_1^2 c_2 + c_1^2 c_3 - c_2^3 + 3c_2^2 c_3 - 4c_3^3), \end{aligned}$$

and so on. Note that the diagonal coefficients in front of c_2^3 and c_3^3 are nonzero, in agreement with (3.3.8) and (3.3.9).

3.4 Nonlinear instability of half-solitons

The half-soliton state Φ is a degenerate saddle point of the constrained action functional Λ. We develop the proof of nonlinear instability of Φ by using the energy method. The steps in the proof of Theorem 3.6 are as follows.

First, we use the gauge symmetry and project a unique global solution to the NLS equation (2.2.1) with $p \in (0, 2)$ in H^1_Γ to the modulated stationary state $\{e^{i\theta}\Phi_\omega\}_{\theta,\omega}$ with ω near $\omega_0 = 1$ and the symplectically orthogonal remainder term V. Second, we project the remainder term V into the $2(N-1)$-dimensional subspace associated with the $(N-1)$-dimensional subspace X_c defined in Theorem 3.4 and the symplectically orthogonal complement $V^\perp$. Third, we define a truncated Hamiltonian system of $(N-1)$ degrees of freedom for the coefficients of the projection on X_c. Fourth, we use the energy conservation to control globally the time evolution of ω and $V^\perp$ in terms of the initial conditions and the reduced energy for the finite-dimensional Hamiltonian system. Finally, we transfer the instability of the zero equilibrium in the finite-dimensional system to the instability result (3.1.4) for the NLS equation (2.2.1).

3.4.1 Modulated stationary states

We start with the standard result, which holds if $\langle \Phi_\omega, \partial_\omega \Phi_\omega \rangle_{L^2(\Gamma)} \neq 0$.

Lemma 3.14. *For every $p \in (0,2)$, there exists $\delta_0 > 0$ such that for every $\Psi \in H^1_\Gamma$ satisfying*

$$\delta := \inf_{\theta \in \mathbb{R}} \|e^{-i\theta}\Psi - \Phi\|_{H^1(\Gamma)} \leq \delta_0, \tag{3.4.1}$$

there exists a unique choice for real-valued (θ, ω) and real-valued $U, W \in H^1_\Gamma$ in the orthogonal decomposition

$$\Psi = e^{i\theta} \left[\Phi_\omega + U + iW\right], \quad \langle U, \Phi_\omega \rangle_{L^2(\Gamma)} = \langle W, \partial_\omega \Phi_\omega \rangle_{L^2(\Gamma)} = 0, \tag{3.4.2}$$

satisfying the estimate

$$|\omega - 1| + \|U + iW\|_{H^1(\Gamma)} \leq C\delta, \tag{3.4.3}$$

for some positive constant $C > 0$.

Proof. Let us define the following vector function $G(\theta, \omega; \Psi) : \mathbb{R}^2 \times H^1_\Gamma \mapsto \mathbb{R}^2$ given by

$$G(\theta, \omega; \Psi) := \begin{bmatrix} \langle \mathrm{Re}(e^{-i\theta}\Psi - \Phi_\omega), \Phi_\omega \rangle_{L^2(\Gamma)} \\ \langle \mathrm{Im}(e^{-i\theta}\Psi - \Phi_\omega), \partial_\omega \Phi_\omega \rangle_{L^2(\Gamma)} \end{bmatrix},$$

the zeros of which represent the orthogonal constraints in (3.4.2).

Let θ_0 be the argument in $\inf_{\theta \in \mathbb{R}} \|e^{-i\theta}\Psi - \Phi\|_{H^1(\Gamma)}$ for a given $\Psi \in H^1_\Gamma$ satisfying (3.4.1). Since the map $\mathbb{R} \ni \omega \mapsto \Phi_\omega \in L^2(\Gamma)$ is smooth, the vector function G is a C^∞ map from $\mathbb{R}^2 \times H^1_\Gamma$ to $\mathbb{R}^2$. Thanks to the bound (3.4.1), there exists a δ-independent constant $C > 0$ such that $|G(\theta_0, 1; \Psi)| \leq C\delta$. Also we obtain that the matrix $\mathcal{D} := D_{(\theta,\omega)} G(\theta_0, 1; \Psi)$ is given by

$$\begin{aligned} \mathcal{D} &= -\begin{bmatrix} 0 & \langle \Phi, \partial_\omega \Phi_\omega|_{\omega=1} \rangle_{L^2(\Gamma)} \\ \langle \Phi, \partial_\omega \Phi_\omega|_{\omega=1} \rangle_{L^2(\Gamma)} & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} \langle \mathrm{Im}(e^{-i\theta_0}\Psi - \Phi), \Phi \rangle_{L^2(\Gamma)} & \langle \mathrm{Re}(e^{-i\theta_0}\Psi - \Phi), \partial_\omega \Phi_\omega|_{\omega=1} \rangle_{L^2(\Gamma)} \\ -\langle \mathrm{Re}(e^{-i\theta_0}\Psi - \Phi), \partial_\omega \Phi_\omega|_{\omega=1} \rangle_{L^2(\Gamma)} & \langle \mathrm{Im}(e^{-i\theta_0}\Psi - \Phi), \partial^2_\omega \Phi_\omega|_{\omega=1} \rangle_{L^2(\Gamma)} \end{bmatrix}, \end{aligned}$$

where $\langle \Phi, \partial_\omega \Phi_\omega|_{\omega=1} \rangle_{L^2(\Gamma)} \neq 0$ if $p \in (0,2)$ and the second matrix is bounded by $C\delta$ with a δ-independent constant $C > 0$. Because the first matrix is invertible if $p \in (0,2)$ and

δ is small, we conclude that there is $\delta_0 > 0$ such that $D_{(\theta,\omega)}G(\theta_0, 1; \Psi) : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible with the $\mathcal{O}(1)$ bound on the inverse matrix for every $\delta \in (0, \delta_0)$. By the local inverse mapping theorem (see Theorem 4.F in [80]), for any $\Psi \in H^1_\Gamma$ satisfying (3.4.1), there exists a unique solution $(\theta, \omega) \in \mathbb{R}^2$ of the vector equation $G(\theta, \omega; \Psi) = 0$ such that $|\theta - \theta_0| + |\omega - 1| \leq C\delta$ with a δ-independent constant $C > 0$. Thus, the bound (3.4.3) is satisfied for ω.

By using the definition of (U, W) in the decomposition (3.4.2) and the triangle inequality for (θ, ω) near $(\theta_0, 1)$, it is then straightforward to show that (U, W) are uniquely defined in H^1_Γ and satisfy the bounds in (3.4.3). □

By global well-posedness theory, see Proposition 2.6, if $\Psi_0 \in H^1_\Gamma$, then there exists a unique solution $\Psi(t) \in C(\mathbb{R}, H^1_\Gamma) \cap C^1(\mathbb{R}, H^{-1}_\Gamma)$ to the NLS equation (2.2.1) with $p \in (0, 2)$ such that $\Psi(0) = \Psi_0$. For every $\delta > 0$ (sufficiently small), we set

$$\Psi_0 = \Phi + U_0 + iW_0, \quad \|U_0 + iW_0\|_{H^1(\Gamma)} \leq \delta, \tag{3.4.4}$$

such that

$$\langle U_0, \Phi \rangle_{L^2(\Gamma)} = 0, \quad \langle W_0, \partial_\omega \Phi_\omega|_{\omega=1} \rangle_{L^2(\Gamma)} = 0. \tag{3.4.5}$$

Thus, in the initial decomposition (3.4.2), we choose $\theta_0 = 0$ and $\omega_0 = 1$ at $t = 0$.

Remark 3.15. *Compared to the statement of Theorem 3.6, the initial datum* $V := \Psi(0) - \Phi = U_0 + iW_0 \in H^1_\Gamma$ *is required to satisfy the constraints (3.4.5). A more general unstable solution can be constructed by choosing different initial values for* (θ_0, ω_0) *in the decomposition (3.4.2).*

Let us assume that $\Psi(t)$ satisfies a priori bound

$$\inf_{\theta \in \mathbb{R}} \|e^{-i\theta}\Psi(t) - \Phi\|_{H^1(\Gamma)} \leq \epsilon, \quad t \in [0, t_0], \tag{3.4.6}$$

for some $t_0 > 0$ and $\epsilon > 0$. This assumption is true at least for small $t_0 > 0$ by the continuity of the global solution $\Psi(t)$. Fix $\epsilon = \delta_0$ defined by Lemma 3.14. As long as a priori assumption (3.4.6) is satisfied, Lemma 3.14 yields that the unique solution $\Psi(t)$ to the NLS equation (2.2.1) can be represented as

$$\Psi(t) = e^{i\theta(t)} \left[\Phi_{\omega(t)} + U(t) + iW(t) \right], \tag{3.4.7}$$

with

$$\langle U(t), \Phi_{\omega(t)} \rangle_{L^2(\Gamma)} = \langle W(t), \partial_\omega \Phi_\omega|_{\omega=\omega(t)} \rangle_{L^2(\Gamma)} = 0. \tag{3.4.8}$$

Since $\Psi(t) \in C(\mathbb{R}, H^1_\Gamma) \cap C^1(\mathbb{R}, H^{-1}_\Gamma)$ and the map $\mathbb{R} \ni \omega \mapsto \Phi_\omega \in H^1_\Gamma$ is smooth, we obtain $(\theta(t), \omega(t)) \in C^1([0, t_0], \mathbb{R}^2)$, hence $U(t), W(t) \in C([0, t_0], H^1_\Gamma) \cap C^1([0, t_0], H^{-1}_\Gamma)$. The proof of Theorem 3.6 is achieved if we can show that there exists $t_0 > 0$ such that the bound (3.4.6) is true for $t \in [0, t_0]$ but fails as $t > t_0$.

Substituting (3.4.7) into the NLS equation (2.2.1) yields the time evolution system for the remainder terms:

$$\frac{d}{dt}\begin{pmatrix} U \\ W \end{pmatrix} = \begin{pmatrix} 0 & L_-(\omega) \\ -L_+(\omega) & 0 \end{pmatrix}\begin{pmatrix} U \\ W \end{pmatrix} + (\dot\theta-\omega)\begin{pmatrix} W \\ -(\Phi_\omega + U) \end{pmatrix} - \dot\omega\begin{pmatrix} \partial_\omega\Phi_\omega \\ 0 \end{pmatrix} + \begin{pmatrix} -R_U \\ R_W \end{pmatrix}, \quad (3.4.9)$$

where $L_+(\omega)$ and $L_-(\omega)$ are given by (2.4.3)–(2.4.4) with $\alpha = 1$, and

$$R_U = (p+1)\left[\left((\Phi_\omega+U)^2+W^2\right)^p - \Phi_\omega^{2p}\right]W, \quad (3.4.10)$$

$$R_W = (p+1)\left[\left((\Phi_\omega+U)^2+W^2\right)^p(\Phi_\omega+U) - \Phi_\omega^{2p}(\Phi_\omega+U) - 2p\Phi_\omega^{2p}U\right]. \quad (3.4.11)$$

By using the symplectically orthogonal conditions (3.4.8) in the decomposition (3.4.7), we obtain from system (3.4.9) the modulation equations for parameters (θ,ω):

$$\begin{pmatrix} \langle\Phi_\omega, W\rangle_{L^2(\Gamma)} & -\langle\partial_\omega\Phi_\omega, \Phi_\omega - U\rangle_{L^2(\Gamma)} \\ \langle\partial_\omega\Phi_\omega, \Phi_\omega + U\rangle_{L^2(\Gamma)} & -\langle\partial_\omega^2\Phi_\omega, W\rangle_{L^2(\Gamma)} \end{pmatrix}\begin{pmatrix} \dot\theta-\omega \\ \dot\omega \end{pmatrix} = \begin{pmatrix} \langle\Phi_\omega, R_U\rangle_{L^2(\Gamma)} \\ \langle\partial_\omega\Phi_\omega, R_W\rangle_{L^2(\Gamma)} \end{pmatrix}. \quad (3.4.12)$$

The modulation equations (3.4.12) and the time-evolution system (3.4.9) have been studied in many contexts involving dynamics of solitary waves [24, 52, 62, 74]. In the context of orbital instability of the half-soliton states, we are able to avoid detailed analysis of system (3.4.9) and (3.4.12) by using conservation of the energy E and mass Q defined by (2.2.4). The following result provide some estimates on the derivatives of the modulation parameters θ and ω.

Lemma 3.16. *Assume that $\omega \in \mathbb{R}$ and $U, W \in H^1_\Gamma$ satisfy*

$$|\omega - 1| + \|U + iW\|_{H^1(\Gamma)} \leq \epsilon \quad (3.4.13)$$

for sufficiently small $\epsilon > 0$. For every $p \in \left[\frac{1}{2}, 2\right)$, there exists an ϵ-independent constant $A > 0$ such that

$$|\dot\theta - \omega| \leq A\left(\|U\|^2_{H^1(\Gamma)} + \|W\|^2_{H^1(\Gamma)}\right), \quad |\dot\omega| \leq A\|U\|_{H^1(\Gamma)}\|W\|_{H^1(\Gamma)}. \quad (3.4.14)$$

Proof. Since $\langle\Phi_\omega, \partial_\omega\Phi_\omega\rangle_{L^2(\Gamma)} \neq 0$ for $p \neq 2$ and under assumption (3.4.13), the coefficient matrix of system (3.4.12) is invertible with the $\mathcal{O}(1)$ bound on the inverse matrix for sufficiently small $\epsilon > 0$. For every $p \geq \frac{1}{2}$, the Taylor expansion of the nonlinear functions R_U and R_W in (3.4.10) and (3.4.11) yield

$$R_U = 2p(p+1)\Phi_\omega^{2p-1}UW + \tilde R_U \quad (3.4.15)$$

and

$$R_W = p(p+1)\Phi_\omega^{2p-1}\left[(2p+1)U^2 + W^2\right] + \tilde R_W, \quad (3.4.16)$$

where $\tilde{R}_U$ and $\tilde{R}_W$ satisfies

$$\|\tilde{R}_U\|_{H^1(\Gamma)} + \|\tilde{R}_W\|_{H^1(\Gamma)} = \left\{ \begin{array}{ll} \mathrm{o}(\|U+iW\|^2_{H^1(\Gamma)}), & p \in \left(\frac{1}{2}, 1\right), \\ \mathrm{O}(\|U+iW\|^3_{H^1(\Gamma)}), & p \geq 1. \end{array} \right.$$

The leading-order terms in (3.4.15)–(3.4.16) and the Banach algebra property of $H^1(\Gamma)$ yield the bound (3.4.14). $\square$

3.4.2 Symplectic projections to the neutral modes

Let us recall the orthogonal basis of eigenvectors constructed in Remark 3.10. We denote the corresponding invariant subspace by

$$X_c := \mathrm{span}\{U^{(1)}, U^{(2)}, \dots, U^{(N-1)}\}.$$

For each vector $U^{(j)}$ with $1 \leq j \leq N-1$, we construct the generalized vector $W^{(j)}$ from solutions of the linear system $L_- W^{(j)} = U^{(j)}$, which exists uniquely in L^2_c thanks to the fact that $U^{(j)} \in L^2_c$ in (2.4.7) and $\ker(L_-) = \mathrm{span}\{\Phi\}$. Let us denote the corresponding invariant subspace by $X_c^* := \mathrm{span}\{W^{(1)}, W^{(2)}, \dots, W^{(N-1)}\}$.

Lemma 3.17. *Basis vectors in X_c and X_c^* are symplectically orthogonal in the sense*

$$\langle U^{(j)}, W^{(k)} \rangle_{L^2(\Gamma)} = 0, \quad j \neq k \quad \text{and} \quad \langle U^{(j)}, W^{(j)} \rangle_{L^2(\Gamma)} > 0. \tag{3.4.17}$$

Moreover, basis vectors are also orthogonal to each other.

Proof. Let us represent $U^{(j)}$ by

$$U^{(j)}(x) = \phi'(x) e_j,$$

where $e_j \in \mathbb{R}^N$ is x-independent and $\phi(x) = \mathrm{sech}^{\frac{1}{p}}(px)$. Then $W^{(j)}$ can be represented by the explicit expression

$$W^{(j)}(x) = -\frac{1}{2} x \phi(x) e_j.$$

The orthogonality of the set $\{e_1, e_2, \dots, e_{N-1}\}$ implies the orthogonality of the set $\{W^{(1)}, W^{(2)}, \dots, W^{(N-1)}\}$, and its orthogonality with respect to the set $\{U^{(1)}, U^{(2)}, \dots, U^{(N-1)}\}$. Normalization is computed from

$$\langle U^{(j)}, W^{(j)} \rangle_{L^2(\Gamma)} = \frac{1}{4} \|\phi\|^2_{L^2(\mathbb{R}_+)} \|e_j\|^2 \tag{3.4.18}$$

such that $\langle U^{(j)}, W^{(j)} \rangle_{L^2(\Gamma)} > 0$ for each j. Thus, (3.4.17) is proved. $\square$

Although the coercivity of L_+ was only proved with respect to the bases in X_c, see Lemma 3.11, the result can now be transferred to the symplectically dual basis.

Lemma 3.18. *For every $p \in (0,2)$, there exists $C_p > 0$ such that*

$$\langle L_+ U, U \rangle_{L^2(\Gamma)} \geq C_p \|U\|^2_{H^1(\Gamma)} \quad \text{for every} \;\; U \in H^1_\Gamma \cap L^2_c \cap [X_c^*]^\perp. \tag{3.4.19}$$

Proof. It follows from Lemma 3.11 that $\langle L_+ U, U \rangle_{L^2(\Gamma)} \geq 0$ for $p \in (0,2)$ if $U \in H^1_\Gamma \cap L^2_c$. Moreover, $\langle L_+ U, U \rangle_{L^2(\Gamma)} = 0$ if and only if $U \in X_c$. Thanks to the orthogonality and positivity of diagonal terms in the symplectically dual bases in X_c and X_c^*, see (3.4.17), the coercivity bound (3.4.19) follows from the bound (3.2.4) by the standard variational analysis. □

Similarly, the coercivity of L_- was proved with respect to the constraint in L^2_c, see Lemma 2.18. The following lemma transfers the result to the symplectically dual constraint.

Lemma 3.19. *For every $p \in (0,2)$, there exists $C_p > 0$ such that*

$$\langle L_- W, W \rangle_{L^2(\Gamma)} \geq C \|W\|^2_{H^1(\Gamma)} \quad \text{for every} \;\; W \in H^1_\Gamma \cap (L^2_c)^*, \tag{3.4.20}$$

where $(L^2_c)^* = \{W \in L^2(\Gamma) : \;\; \langle W, \partial_\omega \Phi_\omega |_{\omega=1} \rangle_{L^2(\Gamma)} = 0\}$.

Proof. It follows from Lemma 2.18 that $\langle L_- W, W \rangle_{L^2(\Gamma)} \geq 0$ if $W \in H^1_\Gamma$. Moreover, $\langle L_- W, W \rangle_{L^2(\Gamma)} = 0$ if and only if $W \in \text{span}(\Phi)$. Due to the positivity of the expression $\langle \partial_\omega \Phi_\omega |_{\omega=1}, \Phi \rangle_{L^2(\Gamma)} > 0$ in (3.2.5) and (3.2.6) for $p \in (0,2)$, the coercivity bound (3.4.20) follows from the bound (2.4.6) by the standard variational analysis. □

Remark 3.20. *By using the scaling transformation (2.3.2), we can continue the basis vectors for $\omega \neq 1$. For notational convenience, ω is added as a subscript if the expressions are continued with respect to ω.*

Recall the symplectically orthogonal decomposition of the unique solution $\Psi(t)$ to the NLS equation (2.2.1) in the form (3.4.7)–(3.4.8). Let us further decompose the remainder terms $U(t)$ and $W(t)$ in (3.4.7) over the orthogonal bases in X_c and X_c^*, which are also symplectically orthogonal to each other by Lemma 3.17. More precisely, since $\omega(t)$ changes we set

$$U(t) = \sum_{j=1}^{N-1} c_j(t) U^{(j)}_{\omega(t)} + U^\perp(t), \quad W(t) = \sum_{j=1}^{N-1} b_j(t) W^{(j)}_{\omega(t)} + W^\perp(t), \tag{3.4.21}$$

and require

$$\langle U^\perp(t), W^{(j)}_{\omega(t)} \rangle_{L^2(\Gamma)} = \langle W^\perp(t), U^{(j)}_{\omega(t)} \rangle_{L^2(\Gamma)} = 0, \quad 1 \leq j \leq N-1. \tag{3.4.22}$$

Since $\{\langle U^{(j)}_\omega, W^{(k)}_\omega \rangle_{L^2(\Gamma)}\}_{1 \leq j,k \leq N-1}$ is a positive diagonal matrix by the conditions (3.4.17), the projections $c = (c_1, c_2, \dots, c_{N-1}) \in \mathbb{R}^{N-1}$ and $b = (b_1, b_2, \dots, b_{N-1}) \in \mathbb{R}^{N-1}$ in (3.4.21) are uniquely determined by U and W and so are the remainder terms $U^\perp$ and

$W^\perp$. Because $\omega(t) \in C^1([0,t_0],\mathbb{R})$ and $U(t), W(t) \in C([0,t_0], H^1_\Gamma) \cap C^1([0,t_0], H^{-1}_\Gamma)$, we have $c(t), b(t) \in C^1([0,t_0],\mathbb{R}^{N-1})$ and $U^\perp(t), W^\perp(t) \in C([0,t_0], H^1_\Gamma) \cap C^1([0,t_0], H^{-1}_\Gamma)$.

When the decomposition (3.4.21) is substituted to the time evolution problem (3.4.9), we obtain

$$\begin{aligned} \frac{dU^\perp}{dt} + \sum_{j=1}^{N-1} \left[\frac{dc_j}{dt} - b_j\right] U_\omega^{(j)} &= L_-(\omega) W^\perp + (\dot\theta - \omega) W \\ &\quad -\dot\omega \left[\partial_\omega \Phi_\omega + \sum_{j=1}^{N-1} c_j(t) \partial_\omega U_\omega^{(j)}\right] - R_U \end{aligned} \tag{3.4.23}$$

and

$$\frac{dW^\perp}{dt} + \sum_{j=1}^{N-1} \frac{db_j}{dt} W_\omega^{(j)} = -L_+(\omega) U^\perp - (\dot\theta - \omega)\left[\Phi_\omega + U\right] - \dot\omega \sum_{j=1}^{N-1} b_j(t) \partial_\omega W_\omega^{(j)} + R_W, \tag{3.4.24}$$

where R_U and R_W are rewritten from (3.4.10) and (3.4.11) after U and W are expressed by (3.4.21).

By using symplectically orthogonal projections (3.4.22), we obtain from (3.4.23) and (3.4.24) a system of differential equations for the amplitudes (c_j, b_j) for every $1 \le j \le N-1$:

$$\langle W_\omega^{(j)}, U_\omega^{(j)} \rangle_{L^2(\Gamma)} \left[\frac{dc_j}{dt} - b_j\right] = R_c^{(j)}, \quad \langle W_\omega^{(j)}, U_\omega^{(j)} \rangle_{L^2(\Gamma)} \frac{db_j}{dt} = R_b^{(j)}, \tag{3.4.25}$$

where

$$\begin{aligned} R_c^{(j)} &= \dot\omega \left[\langle \partial_\omega W_\omega^{(j)}, U^\perp \rangle_{L^2(\Gamma)} - \sum_{i=1}^{N-1} c_i \langle W_\omega^{(j)}, \partial_\omega U_\omega^{(i)} \rangle_{L^2(\Gamma)}\right] \\ &\quad + (\dot\theta - \omega) \langle W_\omega^{(j)}, W \rangle_{L^2(\Gamma)} - \langle W_\omega^{(j)}, R_U \rangle_{L^2(\Gamma)}, \\ R_b^{(j)} &= \dot\omega \left[\langle \partial_\omega U_\omega^{(j)}, W^\perp \rangle_{L^2(\Gamma)} - \sum_{i=1}^{N-1} b_i \langle U_\omega^{(j)}, \partial_\omega W_\omega^{(i)} \rangle_{L^2(\Gamma)}\right] \\ &\quad - (\dot\theta - \omega) \langle U_\omega^{(j)}, U \rangle_{L^2(\Gamma)} + \langle U_\omega^{(j)}, R_W \rangle_{L^2(\Gamma)}, \end{aligned}$$

and we have used the orthogonality conditions:

$$\langle U_\omega^{(j)}, \Phi_\omega \rangle_{L^2(\Gamma)} = \langle W_\omega^{(j)}, \partial_\omega \Phi_\omega \rangle_{L^2(\Gamma)} = 0, \quad 1 \le j \le N-1.$$

The terms $\dot\omega$ and $\dot\theta - \omega$ can be expressed from the system (3.4.12), where U and W are again expressed by (3.4.22).

3.4.3 Truncated Hamiltonian system

The truncated Hamiltonian system of $(N-1)$ degrees of freedom follows from the formal truncation of system (3.4.25) with $\omega = 1$ at the leading order:

$$\begin{cases} \dot{\gamma}_j = \beta_j, \\ \langle W^{(j)}, U^{(j)} \rangle_{L^2(\Gamma)} \dot{\beta}_j = p(p+1)(2p+1) \sum_{k=1}^{N-1} \sum_{n=1}^{N-1} \langle \Phi^{2p-1} U^{(k)} U^{(n)}, U^{(j)} \rangle_{L^2(\Gamma)} \gamma_k \gamma_n. \end{cases} \tag{3.4.26}$$

By using the function $M_0(\gamma)$ given by (3.3.8), we can write the truncated system (3.4.26) in the Hamiltonian form

$$\begin{cases} 2\langle W^{(j)}, U^{(j)} \rangle_{L^2(\Gamma)} \dot{\gamma}_j = \partial_{\beta_j} H_0(\gamma, \beta), \\ 2\langle W^{(j)}, U^{(j)} \rangle_{L^2(\Gamma)} \dot{\beta}_j = -\partial_{\gamma_j} H_0(\gamma, \beta), \end{cases} \tag{3.4.27}$$

which is generated by the Hamiltonian

$$H_0(\gamma, \beta) := \sum_{j=1}^{N-1} \langle W^{(j)}, U^{(j)} \rangle_{L^2(\Gamma)} \beta_j^2 + M_0(\gamma). \tag{3.4.28}$$

The reduced Hamiltonian H_0 arises naturally in the expansion of the action functional Λ. The following result implies nonlinear instability of the zero equilibrium point in the finite-dimensional Hamiltonian system (3.4.27)–(3.4.28).

Lemma 3.21. *There exists $\epsilon > 0$ such that for every $\delta > 0$ (sufficiently small), there is an initial point $(\gamma(0), \beta(0))$ with $\|\gamma(0)\| + \|\beta(0)\| \leq \delta$ such that the unique solution of the finite-dimensional system (3.4.26) satisfies $\|\gamma(t_0)\| > \epsilon$ for some $t_0 = \mathcal{O}(\epsilon^{-1/2})$.*

Proof. We claim that $\gamma_1 = \gamma_2 = \cdots = \gamma_{N-2} = 0$ is an invariant reduction of system (3.4.26). In order to show this, we compute coefficients of the function $M_0(\gamma)$ in (3.3.8) that contains $\gamma_i \gamma_j \gamma_{N-1}$ for $i, j \neq N-1$:

$$\langle \Phi^{2p-1} U^{(i)} U^{(j)}, U^{(N-1)} \rangle_{L^2(\Gamma)} = \langle e_i, e_j \rangle \int_0^\infty \phi^{2p-1} (\phi')^3 dx$$

Since $\langle e_i, e_j \rangle = 0$ for every $i \neq j$, the function $M_0(\gamma)$ depends on γ_{N-1} only in the terms $\gamma_1^2 \gamma_{N-1}$, $\gamma_2^2 \gamma_{N-1}$, ..., $\gamma_{N-2}^2 \gamma_{N-1}$, as well as γ_{N-1}^3. Therefore, $\gamma_1 = \gamma_2 = \cdots = \gamma_{N-2} = 0$ is an invariant solution of the first $(N-2)$ equations of system (3.4.26). The last equation yields the following second-order differential equation for γ_{N-1}:

$$C \ddot{\gamma}_{N-1} = p(p+1)(2p+1) \langle \Phi^{2p-1} U^{(N-1)} U^{(N-1)}, U^{(N-1)} \rangle_{L^2(\Gamma)} \gamma_{N-1}^2, \tag{3.4.29}$$

where the coefficient $C = \langle W^{(N-1)}, U^{(N-1)} \rangle_{L^2(\Gamma)}$ is nonzero thanks to (3.3.9) and (3.4.18). Since the zero equilibrium is unstable in the scalar equation (3.4.29), it is then unstable in system (3.4.26). If $\gamma(t) = \mathcal{O}(\epsilon)$ for $t \in [0, t_0]$, then $\epsilon^2 t_0^2 = \mathcal{O}(\epsilon)$, hence the nonlinear instability develops at the time span $[0, t_0]$ with $t_0 = \mathcal{O}(\epsilon^{-1/2})$. □

Remark 3.22. *For $N = 3$, we have $M_0(\gamma) = 2p^2(\gamma_1^2 - \gamma_2^2)\gamma_2$. Computing the normalization conditions (3.4.18), we obtain the following finite-dimensional system of degree two:*

$$\left\{ \begin{array}{l} \|\phi\|^2_{L^2(\mathbb{R}_+)}\ddot{\gamma}_1 = -4p^2\gamma_1\gamma_2, \\ 3\|\phi\|^2_{L^2(\mathbb{R}_+)}\ddot{\gamma}_2 = -2p^2(\gamma_1^2 - 3\gamma_2^2). \end{array} \right. \tag{3.4.30}$$

For $N = 4$, we have $M_0(\gamma) = 2p^2(\gamma_1^2\gamma_2 + \gamma_1^2\gamma_3 - \gamma_2^3 + 3\gamma_2^2\gamma_3 - 4\gamma_3^3)$. Computing the normalization conditions (3.4.18), we obtain the following finite-dimensional system of degree three:

$$\left\{ \begin{array}{l} \|\phi\|^2_{L^2(\mathbb{R}_+)}\ddot{\gamma}_1 = -4p^2\gamma_1(\gamma_2 + \gamma_3), \\ 3\|\phi\|^2_{L^2(\mathbb{R}_+)}\ddot{\gamma}_2 = -2p^2(\gamma_1^2 - 3\gamma_2^2 + 6\gamma_2\gamma_3), \\ 3\|\phi\|^2_{L^2(\mathbb{R}_+)}\ddot{\gamma}_3 = -p^2(\gamma_1^2 + 3\gamma_2^2 - 12\gamma_3^2). \end{array} \right. \tag{3.4.31}$$

Remark 3.23. *For $N = 3$, the zero point $(\gamma_1, \gamma_2) = (0, 0)$ is the only equilibrium point of system (3.4.30). For $N = 4$, the zero point $(\gamma_1, \gamma_2, \gamma_3) = (0, 0, 0)$ is located at the intersections of three lines of equilibria of system (3.4.31): $\gamma_1 = 0$, $\gamma_2 = 2\gamma_3$; $\gamma_1 = 3\gamma_3$, $\gamma_2 = -\gamma_3$; and $\gamma_1 = -3\gamma_3$, $\gamma_2 = -\gamma_3$. The lines of equilibria correspond to the shifted states in Lemma 2.9 studied in Chapter 4.*

3.4.4 Expansion of the action functional $\Lambda(\Psi)$

Recall the action functional $\Lambda(\Psi) = E(\Psi) + Q(\Psi)$, for which Φ is a critical point. By using the scaling transformation (2.3.2), we continue the action functional for $\omega \neq 1$ and define the following function:

$$\begin{aligned} \Delta(t) \;:=\; & E(\Phi_{\omega(t)} + U(t) + iW(t)) - E(\Phi) \\ & + \omega(t)\left[Q(\Phi_{\omega(t)} + U(t) + iW(t)) - Q(\Phi)\right]. \end{aligned} \tag{3.4.32}$$

As long as a priori bound (3.4.6) is satisfied, one can expand Δ by using the primary decomposition (3.4.7) as follows:

$$\Delta = D(\omega) + \langle L_+(\omega)U, U\rangle_{L^2(\Gamma)} + \langle L_-(\omega)W, W\rangle_{L^2(\Gamma)} + N_\omega(U, W), \tag{3.4.33}$$

where the dependence of all quantities on t is ignored, $D(\omega)$ is defined by

$$D(\omega) := E(\Phi_\omega) - E(\Phi) + \omega\left[Q(\Phi_\omega) - Q(\Phi)\right],$$

and

$$N_\omega(U, W) = \left\{ \begin{array}{ll} \mathrm{o}(\|U + iW\|^2_{H^1(\Gamma)}), & p \in \left(0, \frac{1}{2}\right), \\ \mathrm{O}(\|U + iW\|^3_{H^1(\Gamma)}), & p \geq \frac{1}{2}, \end{array} \right.$$

is a continuation of $N(U, W)$ defined by (2.4.2) with respect to ω.

Since $D'(\omega) = Q(\Phi_\omega) - Q(\Phi)$ thanks to the variational characterization of Φ_ω, we have $D(1) = D'(1) = 0$, and

$$D(\omega) = (\omega - 1)^2 \langle \Phi, \partial_\omega \Phi_\omega|_{\omega=1} \rangle_{L^2(\Gamma)} + \tilde{D}(\omega), \tag{3.4.34}$$

where $\tilde{D}(\omega) = \mathcal{O}(|\omega - 1|^3)$. Thanks to conservation of the energy E and mass Q defined by (2.2.4) and to the phase invariance in the NLS, we represent $\Delta(t)$ in terms of the initial data $\omega(0) = \omega_0 = 1$, $U(0) = U_0$, and $W(0) = W_0$ as follows:

$$\Delta(t) = \Delta_0 + (\omega(t) - 1) \left[Q(\Phi + U_0 + iW_0) - Q(\Phi) \right], \tag{3.4.35}$$

where

$$\Delta_0 := E(\Phi + U_0 + iW_0) - E(\Phi) + Q(\Phi + U_0 + iW_0) - Q(\Phi) \tag{3.4.36}$$

is a constant of motion.

Let us now consider the secondary decomposition (3.4.21)–(3.4.22). If the solution given by (3.4.7) and (3.4.21) satisfies a priori bound (3.4.6) for some $t_0 > 0$ and $\epsilon > 0$, then the coefficients of the secondary decomposition (3.4.21) are required to satisfy the bound

$$|\omega(t) - 1| + \|c(t)\| + \|b(t)\| + \|U^\perp(t) + iW^\perp(t)\|_{H^1(\Gamma)} \leq A\epsilon, \quad t \in [0, t_0], \tag{3.4.37}$$

for an ϵ-independent constant $A > 0$. We substitute the secondary decomposition (3.4.21)–(3.4.22) into the representation (3.4.33) and estimate the corresponding expansion.

Lemma 3.24. *Assume that $\omega \in \mathbb{R}$, $c, b \in \mathbb{R}^{N-1}$, and $U^\perp, W^\perp \in H^1_\Gamma$ satisfy the bound (3.4.37) for sufficiently small $\epsilon > 0$. For every $p \geq \frac{1}{2}$, there exists an ϵ-independent constant $A > 0$ such that the representation (3.4.33) is expanded as follows:*

$$\begin{aligned} \Delta \;=\; & D(\omega) + \langle L_+(\omega) U^\perp, U^\perp \rangle_{L^2(\Gamma)} + \langle L_-(\omega) W^\perp, W^\perp \rangle_{L^2(\Gamma)} \\ & + \sum_{j=1}^{N-1} \langle W^{(j)}, U^{(j)} \rangle_{L^2(\Gamma)} b_j^2 + M_0(c) + \widetilde{\Delta}(\omega, c, b, U^\perp, W^\perp), \end{aligned} \tag{3.4.38}$$

with

$$\begin{aligned} |\widetilde{\Delta}(\omega, c, b, U^\perp, W^\perp)| \leq A \Big(& \mu(\|c\|) + \|c\|^2 \|U^\perp\|_{H^1(\Gamma)} + \|U^\perp\|^3_{H^1(\Gamma)} + |\omega - 1| \|b\|^2 \\ & + \|c\| \|b\|^2 + \|c\| \|W^\perp\|^2_{H^1(\Gamma)} + \|b\|^2 \|U^\perp\|_{H^1(\Gamma)} + \|U^\perp\|_{H^1(\Gamma)} \|W^\perp\|^2_{H^1(\Gamma)} \Big), \end{aligned} \tag{3.4.39}$$

where $M_0(c)$ is given by (3.3.8) and

$$\mu(\|c\|) = \begin{cases} \mathrm{o}(\|c\|^3), & p \in \left(\frac{1}{2}, 1\right), \\ \mathrm{O}(\|c\|^4), & p \geq 1. \end{cases} \tag{3.4.40}$$

Proof. For every $p \geq \frac{1}{2}$, Taylor expansion of $N_\omega(U,W)$ yields

$$N_\omega(U,W) = -\frac{2}{3}p(p+1)(2p+1)\langle \Phi^{2p-1}U^2, U\rangle_{L^2(\Gamma)} - 2p(p+1)\langle \Phi^{2p-1}W^2, U\rangle_{L^2(\Gamma)} + S_\omega(U,W),$$

where

$$S_\omega(U,W) = \begin{cases} \mathrm{o}(\|U+iW\|^3_{H^1(\Gamma)}), & p \in \left(\frac{1}{2},1\right), \\ \mathrm{O}(\|U+iW\|^4_{H^1(\Gamma)}), & p \geq 1. \end{cases}$$

is a continuation of $S(U,W)$ defined by (3.3.1) with respect to ω. The expansion (3.4.38) holds by substituting of (3.4.21) into (3.4.33) and estimating the remainder terms thanks to Banach algebra property of $H^1(\Gamma)$ and the assumption (3.4.37). Only the end-point bounds are incorporated into the estimate (3.4.39). □

We bring (3.4.35) and (3.4.38) together as follows:

$$\begin{aligned} \Delta_0 - H_0(c,b) \;=\; & D(\omega) - (\omega-1)\left[Q(\Phi+U_0+iW_0) - Q(\Phi)\right] \\ & + \langle L_+(\omega)U^\perp, U^\perp\rangle_{L^2(\Gamma)} + \langle L_-(\omega)W^\perp, W^\perp\rangle_{L^2(\Gamma)} + \widetilde{\Delta}(\omega,c,b,U^\perp,W^\perp), \end{aligned} \tag{3.4.41}$$

where $H_0(c,b)$ is given by (3.4.28). Recall that the energy $E(\Psi)$ and mass $Q(\Psi)$ are bounded in H^1_Γ, whereas Φ is a critical point of E under fixed Q. Thanks to the bound (3.4.4) on the initial data, the orthogonality (3.4.5), and the representation (3.4.36), there is an δ-independent constant $A > 0$ such that

$$|\Delta_0| + |Q(\Phi+U_0+iW_0) - Q(\Phi)| \leq A\delta^2. \tag{3.4.42}$$

Thanks to the representations (3.3.8) and (3.4.28), there is a generic constant $A > 0$ such that

$$|H_0(c,b)| \leq A\left(\|c\|^3 + \|b\|^2\right). \tag{3.4.43}$$

The value of ω near $\omega_0 = 1$ and the remainder terms $U^\perp, W^\perp$ in the $H^1(\Gamma)$ norm can be controlled in the time evolution of the NLS equation (2.2.1) by using the energy expansion (3.4.41). The following lemma presents this result.

Lemma 3.25. *Consider a solution to the NLS with $p \geq \frac{1}{2}$ given by (3.4.7) and (3.4.21) with $\omega(t) \in C^1([0,t_0],\mathbb{R})$, $c(t), b(t) \in C^1([0,t_0],\mathbb{R}^{N-1})$, and $U^\perp(t), W^\perp(t) \in C([0,t_0],H^1_\Gamma)$ satisfying the bound (3.4.37) for sufficiently small $\epsilon > 0$. Then, there exists an ϵ-independent constant $A > 0$ such that for every $t \in [0,t_0]$,*

$$|\omega-1|^2 + \|U^\perp + iW^\perp\|^2_{H^1(\Gamma)} \leq A\left[\delta^2 + |H_0(c,b)| + \mu(\|c\|) + \|c\|\|b\|^2 + \|b\|^3\right], \tag{3.4.44}$$

where $\mu(\|c\|)$ is the same as in (3.4.40).

Proof. The bound on $|\omega-1|^2$ follows from (3.4.34), (3.4.39), (3.4.41), and (3.4.42) thanks to the positivity of $D''(1) = 2\langle \Phi, \partial_\omega\Phi_\omega|_{\omega=1}\rangle_{L^2(\Gamma)}$. The bounds on $\|U^\perp\|^2_{H^1(\Gamma)}$ and

$\|W^\perp\|^2_{H^1(\Gamma)}$ follow from (3.4.39), (3.4.41), and (3.4.42) thanks to the coercivity of $L_+(\omega)$ and $L_-(\omega)$ in Lemmas 3.18 and 3.19. □

3.4.5 Closing the energy estimates

By Lemma 3.21, there exists a trajectory of the finite-dimensional system (3.4.26) near the zero equilibrium which leaves the ϵ-neighborhood of the zero equilibrium. This nonlinear instability developes over the time span $[0, t_0]$ with $t_0 = \mathcal{O}(\epsilon^{-1/2})$. The second equation of system (3.4.26) shows that if $\gamma(t) = \mathcal{O}(\epsilon)$ for $t \in [0, t_0]$ and $t_0 = \mathcal{O}(\epsilon^{-1/2})$, then $\beta(t) = \mathcal{O}(\epsilon^{3/2})$ for $t \in [0, t_0]$. It is also clear that the scaling is consistent with the first equation of system (3.4.26). The scaling suggests to consider the following region in the phase space $\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$:

$$\|c(t)\| \le A\epsilon, \quad \|b(t)\| \le A\epsilon^{3/2}, \quad t \in [0, t_0], \quad t_0 \le A\epsilon^{-1/2}, \tag{3.4.45}$$

for an ϵ-independent constant $A > 0$. The region in (3.4.45) satisfies a priori assumption (3.4.37) for c and b. The following result shows that a trajectory of the full system (3.4.25) follows closely to the trajectory of the finite-dimensional system (3.4.26) in the region (3.4.45).

Lemma 3.26. *Consider a solution $\gamma(t), \beta(t) \in C^1([0, t_0], \mathbb{R}^{N-1})$ to the finite-dimensional system (3.4.26) in the region (3.4.45) with sufficiently small $\epsilon > 0$. Then, a solution $c(t), b(t) \in C^1([0, t_0], \mathbb{R}^{N-1})$ to system (3.4.25) remains in the region (3.4.45) and there exist an ϵ-independent constant $A > 0$ such that*

$$\|c(t) - \gamma(t)\| \le A\nu(\epsilon), \quad \|b(t) - \beta(t)\| \le A\epsilon^{1/2}\nu(\epsilon), \quad t \in [0, t_0], \tag{3.4.46}$$

where

$$\nu(\epsilon) = \left\{ \begin{array}{ll} \mathrm{o}(\epsilon), & p \in \left(\frac{1}{2}, 1\right), \\ \mathrm{O}(\epsilon^{3/2}), & p \ge 1. \end{array} \right. \tag{3.4.47}$$

Proof. By the bounds (3.4.43) and (3.4.44), as well as a priori assumption (3.4.45), there exists an (δ, ϵ)-independent constant $A > 0$ such that

$$|\omega(t) - 1| + \|U^\perp(t) + iW^\perp(t)\|_{H^1(\Gamma)} \le A\left(\delta + \epsilon^{3/2}\right), \quad t \in [0, t_0]. \tag{3.4.48}$$

It makes sense to define $\delta = \mathcal{O}(\epsilon^{3/2})$ in the bound (3.4.4) on the initial data, which we will adopt here. By using the decomposition (3.4.21) and the bounds (3.4.45) and (3.4.48) in (3.4.14), we get

$$|\dot{\theta} - \omega| \le A\epsilon^2, \quad |\dot{\omega}| \le A\epsilon^{5/2}, \tag{3.4.49}$$

for an ϵ-independent constant $A > 0$. By subtracting the first equation of system (3.4.26) from the first equation of system (3.4.25), we obtain

$$\dot{c}_j - \dot{\gamma}_j = b_j - \beta_j + [F(c, b)]_j, \tag{3.4.50}$$

where the vector $F(c, b) \in \mathbb{R}^{N-1}$ satisfies the estimate

$$\|F(c, b)\| \leq A\epsilon^{5/2}, \tag{3.4.51}$$

thanks to (3.4.15), (3.4.21), (3.4.48), and (3.4.49). By subtracting the second equation of system (3.4.26) from the second equation of system (3.4.25), we obtain

$$\begin{aligned} \dot{b}_j - \dot{\beta}_j &= [G(c, b)]_j \\ &+ p(p+1)(2p+1) \sum_{k=1}^{N-1} \sum_{n=1}^{N-1} \frac{\langle \Phi^{2p-1} U^{(k)} U^{(n)}, U^{(j)} \rangle_{L^2(\Gamma)}}{\langle W^{(j)}, U^{(j)} \rangle_{L^2(\Gamma)}} (c_k c_n - \gamma_k \gamma_n) \end{aligned} \tag{3.4.52}$$

where the vector $G(c, b) \in \mathbb{R}^{N-1}$ satisfies the estimate

$$\|G(c, b)\| \leq A\epsilon \nu(\epsilon), \tag{3.4.53}$$

thanks to (3.4.16), (3.4.21), (3.4.48), and (3.4.49), where $\nu(\epsilon)$ is given by (3.4.47).

Let us assume than $\gamma(0) = c(0)$ and $\beta(0) = \beta(0)$. Integrating equations (3.4.50) and (3.4.52) over $t \in [0, t_0]$ with $t_0 \leq A\epsilon^{-1/2}$ in the region (3.4.45), we obtain

$$\|c(t) - \gamma(t)\| \leq \int_0^t \|b(t') - \beta(t')\| dt' + A\epsilon^2 \tag{3.4.54}$$

and

$$\|b(t) - \beta(t)\| \leq A\epsilon \int_0^t \|c(t') - \gamma(t')\| dt' + A\epsilon^{1/2} \nu(\epsilon), \tag{3.4.55}$$

for a generic ϵ-independent constant $A > 0$. Gronwall's inequality for

$$\|b(t) - \beta(t)\| + A\epsilon^{1/2} \|c(t) - \gamma(t)\|$$

yields (3.4.46). □

Proof of Theorem 3.6. Let us consider the unstable solution (γ, β) to the finite-dimensional system (3.4.26) according to Lemma 3.21. This solution belongs to the region (3.4.45). By Lemma 3.26, the correction terms satisfy (3.4.46), hence the solution (c, b) to system (3.4.25) also satisfies the bound (3.4.45) over the time span $[0, t_0]$ with $t_0 = \mathcal{O}(\epsilon^{-1/2})$.

By Lemma 3.25 and the elementary continuation argument, the components ω, $U^\perp$, and $W^\perp$ satisfy the bound (3.4.48) with $\delta = \mathcal{O}(\epsilon^{3/2})$, so that the solution to the NLS equation (2.2.1) given by (3.4.7) and (3.4.21) satisfies the bound (3.4.6) for $t \in [0, t_0]$.

Finally, the solution γ to the finite-dimensional system (3.4.26) grows in time and reaches the boundary in the region (3.4.45) by Lemma 3.21. The same is true for the full solution to the NLS equation (3.4.21) thanks to the bounds (3.4.46) and (3.4.48). Hence, the solution starting with the initial data satisfying the bound (3.4.4) with $\delta = \mathcal{O}(\epsilon^{3/2})$ reaches and crosses the boundary in (3.1.4) for some $t_0 = \mathcal{O}(\epsilon^{-1/2})$. ■

Chapter 4

Spectral stability of shifted states on star graphs

In this chapter we explore the NLS equation on star graphs with the generalized Kirchhoff boundary conditions in (2.1.1)–(2.1.2) satisfying the constraint (2.3.5). We study the stability of *shifted states*, existence of which was obtained in Lemma 2.9.

In the variational characterization of NLS stationary states on star graphs, the shifted states with $\alpha = 1$ were mentioned in Remarks 5.3 and 5.4 in [4], where it was conjectured that all shifted states are saddle points of the action functional and are thus unstable if the even number of edges in the star graph exceeds two.

Here we will prove this conjecture with an explicit count of the Morse index for the shifted states. By extending the Sturm theory to Schrödinger operators on the star graph, we give a very precise characterization of the negative and zero eigenvalues of the linearized Schrödinger operators, avoiding the theory of deficiency indices for star graphs with point interactions [60]. As a result of our analysis, we prove that these shifted states are saddle points of energy subject to fixed mass, which are spectrally unstable under the NLS flow. In comparison, the half-soliton states are degenerate saddle points of energy and they are spectrally stable but nonlinearly unstable under the NLS flow.

We also show that the shifted states with $\alpha \neq 1$ satisfy the reduction of the NLS equation on the star graph to the homogeneous NLS equation on the infinite line. Nevertheless, with one exception, for $\alpha \neq 1$ the shifted states are spectrally unstable under the NLS flow due to perturbations that break this reduction.

4.1 Main results

Let Γ be the star graph with $N \geq 2$ infinite edges. Consider the scaled ($\omega = 1$) stationary NLS equation (2.3.3) with the domain H^2_Γ as in (2.1.2), and assume that the coefficients $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{R}^N_+$ in (2.1.2) satisfy the constraint (2.3.5) for some K. Let Φ denote the shifted state solution given in Lemma 2.9.

Let $\Psi = \Phi + U + iW$ with real valued $U, W \in H^2_\Gamma$ be the complex perturbation of Φ. Substituting Ψ into the NLS equation (2.2.1), we obtain the linearized time evolution system for the perturbation terms as in

$$\frac{d}{dt}\left[\begin{array}{c} U \\ W \end{array}\right] = \left[\begin{array}{cc} 0 & L_- \\ -L_+ & 0 \end{array}\right]\left[\begin{array}{c} U \\ W \end{array}\right], \tag{4.1.1}$$

where the operators L_+ and L_- are equivalent to the Hessian operators in (2.4.3) and (2.4.4), respectively, with $\omega = 1$:

$$\begin{array}{rcll} L_+ & = & -\Delta + 1 - (2p+1)(p+1)\alpha^{2p}\Phi^{2p} : & H^2_\Gamma \subset L^2(\Gamma) \to L^2(\Gamma), \\ L_- & = & -\Delta + 1 - (p+1)\alpha^{2p}\Phi^{2p} : & H^2_\Gamma \subset L^2(\Gamma) \to L^2(\Gamma). \end{array}$$

The spectral stability of the shifted state Φ is related to the spectral problem associated with the linearized system (4.1.1), and we define it as follows:

Definition 4.1. *The shifted state Φ is* **spectrally stable** *if the spectrum of the spectral problem*

$$\lambda \left[\begin{array}{c} U \\ W \end{array}\right] = \left[\begin{array}{cc} 0 & L_- \\ -L_+ & 0 \end{array}\right]\left[\begin{array}{c} U \\ W \end{array}\right] \tag{4.1.2}$$

in $L^2(\Gamma)$ satisifes $\lambda \in i\mathbb{R}$. If there exists λ with $\mathrm{Re}(\lambda) > 0$, then Φ is **spectrally unstable**.

Existence of unstable eigenvalues λ of the spectral problem (4.1.2) with $\mathrm{Re}(\lambda) > 0$ depends on the number of negative eigenvalues of the operators $L_\pm$. It is known from [25], [36], [40] that the spectral problem (4.1.2) admits a positive real eigenvalue if the Morse indices of the operator L_- and constrained operator L_+ differ by one. The following theorem gives the explicit count of the Morse indices.

Theorem 4.2. *Let Φ be a shifted state given by Lemma 2.9 with $a \neq 0$. Then $\sigma_p(L_-) \geq 0$ and 0 is a simple eigenvalue of L_-, whereas the non-positive part of $\sigma_p(L_+)$ consists of a simple eigenvalue $\lambda_0 < 0$, another eigenvalue $\lambda_1 \in (\lambda_0, 0)$ of multiplicity $K-1$ for $a < 0$ and $N-K-1$ for $a > 0$, and a simple zero eigenvalue. The rest of $\sigma_p(L_-)$ and $\sigma_p(L_+)$ is strictly positive and is bounded away from zero.*

Remark 4.3. *If $a = 0$, it was established in Corollary 3.9 that the non-positive part of $\sigma_p(L_+)$ for the half-solitons (2.3.4) consists of a simple eigenvalue $\lambda_0 < 0$ and a zero eigenvalue of multiplicity $N-1$.*

By using the Theorem 1.2 in [36] (see also [40]), we can deduce spectral instability of the shifted states from Theorem 4.2.

Corollary 4.4. *If $1 < K < N-1$ in Lemma 2.9, the shifted states with $a \neq 0$ are spectrally unstable in the time evolution of the NLS equation (2.2.1), in particular, there exists real positive eigenvalues λ in the spectral stability problem (4.1.2). To be precise, for $p \in (0,2)$, there exist $K-1$ real positive eigenvalues λ of the spectral stability problem (4.1.2) for $a < 0$ and $N-K-1$ real positive eigenvalues λ for $a > 0$.*

Remark 4.5. *The result of Theorem 4.2 and Corollary 4.4 in case of $\alpha = 1$ and even N agrees with the qualitative picture described in Remark 5.3 in [4] and proves the conjecture formulated in Remark 5.4 in [4] that all shifted states (2.3.10) given by Lemma 2.13 are unstable for all even $N \geq 4$. In case $N = 2$, we have $K = 1$, and Theorem 4.2 implies that the shifted states are spectrally stable. In fact, since $\alpha_1 = \alpha_2$ by the constraint (2.3.5), the spectrally stable shifted states are also orbitally stable because the NLS equation on the star graph with $N = 2$ becomes equivalent to the NLS equation on the real line.*

Remark 4.6. *By construction in Lemma 2.9, the shifted states with $K = 1$ and $a > 0$ are equivalent to the shifted states with $K = N - 1$ and $a < 0$. In such case, by Theorem 4.2 and Theorem 1.2 in [36], the shifted state is spectrally stable. However, compared to Remark 4.5, the orbital instability is more challenging problem since the orbit of the shifted state has to be two-parametric due to phase rotation and translation in space, whereas the star graph Γ with $N \geq 3$ edges is not equivalent to the real line $\mathbb{R}$. We prove the orbital (nonlinear) instability of the spectrally stable shifted states in Chapter 5 analyzing symmetry breaking perturbations.*

4.2 The count of the Morse index

This section is devoted to the proof of Theorem 4.2.

Since the discrete spectrum of the operator L_- is described by Lemma 2.18 and $\sigma_c(L_\pm)$ is given in (2.4.5), we are mainly concerned by the eigenvalues of L_+ in $(-\infty, 1)$. By using the results of Lemmas 2.19, 2.20, and 2.22, we determine the sufficient conditions for the presence of eigenvalues in $\sigma_p(L_+)$.

Lemma 4.7. *Let $a \in \mathbb{R}$ be arbitrary, and v be the solution defined by Lemma 2.19. Then, $\lambda_* \in (-\infty, 1)$ is an eigenvalue of $\sigma_p(L_+)$ if and only if one of the following equations holds:*

(a) $v(a) = 0$,

(b) $v(-a) = 0$,

(c) $v(-a)v'(a) + v(a)v'(-a) = 0$.

Moreover, $\lambda_* \in \sigma_p(L_+)$ has multiplicity $K - 1$ in the case (a), $N - K - 1$ in the case (b), and is simple in the case (c). If v satisfies several cases, then multiplicity of λ_* is the sum of the multiplicities in each case.

Proof. Let $U \in H^2_\Gamma$ be the eigenvector of the operator L_+ for the eigenvalue $\lambda_* \in \sigma_p(L_+)$. By Sobolev embedding of $H^2(\mathbb{R}^+)$ into $C^1(\mathbb{R}^+)$, both $U(x)$ and $U'(x)$ decay to zero as $x \to +\infty$. By using the representation (2.3.6) and the transformation of (2.4.12) to

(2.4.13), we can write $U = (u_1, \dots, u_N)^T$ in the form

$$u_j(x) = \begin{cases} c_j v(x+a), & j = 1, \dots, K, \\ c_j v(x-a), & j = K+1, \dots, N, \end{cases}$$

where $(c_1, c_2, \dots, c_N)$ are coefficients and v is the solution defined in Lemma 2.19. The boundary conditions for $U \in H^2_\Gamma$ in (2.1.1) and (2.1.2) imply the homogeneous linear system on the coefficients on $(c_1, c_2, \dots, c_N)$:

$$c_1\alpha_1 v(a) = \dots = c_K\alpha_K v(a) = c_{K+1}\alpha_{K+1}v(-a) = \dots = c_N\alpha_N v(-a) \tag{4.2.1}$$

and

$$\sum_{j=1}^{K} c_j\alpha_j^{-1}v'(a) + \sum_{j=K+1}^{N} c_j\alpha_j^{-1}v'(-a) = 0. \tag{4.2.2}$$

The associated matrix is

$$\begin{pmatrix} \alpha_1 v(a) & -\alpha_2 v(a) & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \alpha_1 v(a) & 0 & -\alpha_3 v(a) & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_1 v(a) & 0 & 0 & \dots & -\alpha_K v(a) & 0 & \dots & 0 & 0 \\ \alpha_1 v(a) & 0 & 0 & \dots & 0 & -\alpha_{K+1}v(-a) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_1 v(a) & 0 & 0 & \dots & 0 & 0 & \dots & 0 & -\alpha_N v(-a) \\ b_1 & b_2 & b_3 & \dots & b_K & b_{K+1} & \dots & b_{N-1} & b_N \end{pmatrix}$$

where

$$b_j = \begin{cases} \alpha_j^{-1}v'(a), & 1 \le j \le K \\ \alpha_j^{-1}v'(-a), & K+1 \le j \le N. \end{cases}$$

In order to calculate the determinant of the associate matrix, we perform elementary column operations and obtain a lower triangular matrix. Let the associate matrix be of the form $[A_1^0 A_2^0 \dots A_N^0]$, where A_j^0 represents the j-th column of the matrix in the beginning of the algorithm. We perform the following elementary column operation:

$$[A_1^0\ A_2^0\ A_3^0\ \dots\ A_N^0] \longrightarrow [A_1^1\ A_2^1\ A_3^1\ \dots\ A_N^1] := [A_1^0\ A_2^0 + \alpha_1^{-1}\alpha_2 A_1^0\ A_3^0\ \dots\ A_N^0],$$

then

$$[A_1^1\ A_2^1\ A_3^1\ \dots\ A_N^1] \longrightarrow [A_1^2\ A_2^2\ A_3^2\ \dots\ A_N^2] := [A_1^1\ A_2^1\ A_3^1 + \alpha_2^{-1}\alpha_3 A_2^1\ \dots\ A_N^1],$$

and so on, until the K-th step. At the K-th step, we need to take into account the change of $v(a)$ to $v(-a)$ in the $(K+1)$-th column, hence the K-th step involves

$$A_{K+1}^{K-1} \longrightarrow A_{K+1}^{K} := A_{K+1}^{K-1} + \frac{\alpha_{K+1} v(-a)}{\alpha_K v(a)} A_K^{K-1}.$$

At the $(K+1)$-th and subsequent steps, no further changes of $v(-a)$ occurs, so that we apply the same rule as the one before the K-th step in all subsequent transformations. Finally, after $(N-1)$ transformations, we obtain a lower triangular matrix in the form:

$$\begin{pmatrix}
\alpha_1 v(a) & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
\alpha_1 v(a) & \alpha_2 v(a) & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
\alpha_1 v(a) & \alpha_2 v(a) & \alpha_3 v(a) & \dots & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_1 v(a) & \alpha_2 v(a) & \alpha_3 v(a) & \dots & \alpha_K v(a) & 0 & \dots & 0 & 0 \\
\alpha_1 v(a) & \alpha_2 v(a) & \alpha_3 v(a) & \dots & \alpha_K v(a) & \alpha_{K+1} v(-a) & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_1 v(a) & \alpha_2 v(a) & \alpha_3 v(a) & \dots & \alpha_K v(a) & \alpha_{K+1} v(-a) & \dots & \alpha_{N-1} v(-a) & 0 \\
B_1 & B_2 & B_3 & \dots & B_K & B_{K+1} & \dots & B_{N-1} & B_N
\end{pmatrix}$$

where $\{B_j\}_{j=1}^N$ are some numerical coefficients, in particular, $B_1 = \alpha_1^{-1} v'(a)$ and

$$B_N = \frac{\alpha_N}{v(a)} \left[\sum_{j=1}^{K} \alpha_j^{-2} v'(a) v(-a) + \sum_{j=K+1}^{N} \alpha_j^{-2} v'(-a) v(a) \right].$$

Under the constraint (2.3.5), the determinant of the lower triangular matrix is evaluated in the form:

$$\Delta = \left(\prod_{j=1}^{N} \alpha_j \right) \left(\sum_{j=1}^{K} \alpha_j^{-2} \right) v(a)^{K-1} v(-a)^{N-K-1} \left[v(-a) v'(a) + v(a) v'(-a) \right].$$

Therefore, $U \neq 0$ is the eigenvector of L_+ for the eigenvalue $\lambda_* \in (-\infty, 1)$ if and only if $\Delta = 0$, or equivalently, if either $v(a) = 0$ or $v(-a) = 0$ or $v(-a)v'(a) + v(a)v'(-a) = 0$.

In the case of $v(a) = 0$ and $v(-a) \neq 0$, it follows from the linear system (6.3.4) that $c_j = 0$ for all $K+1 \leq j \leq N$ and $c_j \in \mathbb{R}$ are arbitrary for all $1 \leq j \leq K$. The linear equation (6.3.5) implies that $\sum_{j=1}^{K} c_j \alpha_j^{-1} = 0$, since $v'(a) \neq 0$ when $v(a) = 0$. Therefore, the eigenvalue λ_* has a multiplicity $K-1$.

Similarly, the eigenvalue λ_* has a multiplicity $N-K-1$ if $v(a) \neq 0$ and $v(-a) = 0$.

In the case $v(-a)v'(a) + v(a)v'(-a) = 0$ but $v(a) \neq 0$ and $v(-a) \neq 0$, the linear system (6.3.4) implies that all coefficients are related to one coefficient. The linear equation (6.3.5) is then satisfied due to the constraint (2.3.5), hence λ_* is a simple eigenvalue.

If several cases are satisfied simultaneously, then it follows from the linear system (6.3.4) and (6.3.5) that multiplicity of λ_* is equal to the sum of the multiplicities for each of the cases. $\square$

Proof of Theorem 4.2. The result on $\sigma_p(L_-)$ is proved in Lemma 2.18. The construction of $\sigma_p(L_+)$ follows from Lemma 4.7.

The condition (c) in Lemma 4.7 is satisfied if the solution v in Lemma 2.19 is either odd or even function of a. For the simple eigenvalue $\lambda_0 < 0$ in Lemma 2.20, the eigenfunction is even and positive. Hence, $v(a) \neq 0$ and $v(-a) \neq 0$, so that λ_0 is a simple eigenvalue in $\sigma_p(L_+)$ by the case (c) in Lemma 4.7. The corresponding eigenvector $U \in H^2_\Gamma$ is strictly positive definite on Γ.

For the simple zero eigenvalue in Lemma 2.20, the eigenfunction (2.4.20) is odd and positive on $(-\infty, 0)$. Since $v(a) \neq 0$ and $v(-a) \neq 0$ if $a \neq 0$. then 0 is a simple eigenvalue in $\sigma_+(L_+)$ by the case (c) in Lemma 4.7. The corresponding eigenvector $U \in H^2_\Gamma$ can be represented in the form:

$$U(x) = \begin{cases} \alpha_j^{-1} \phi'(x+a), & j = 1, \dots, K \\ -\alpha_j^{-1} \phi'(x-a), & j = K+1, \dots, N \end{cases}. \tag{4.2.3}$$

which represent the translation of the shifted state (2.3.6) with respect to parameter a.

No other values of λ exists in $(-\infty, \lambda_2)$ such that the condition (c) in Lemma 4.7 is satisfied, where $\lambda_2 > 0$ is either the positive eigenvalue of the scalar Schrödinger equation (2.4.13) or the bottom of $\sigma_c(L_+)$ at $\lambda_2 = 1$.

If $a > 0$, then we claim that $v(a) > 0$ for every $\lambda \in (-\infty, 0]$. Indeed, by Lemma 2.22, simple zeros of v are monotonically increasing functions of λ, whereas no multiple zeros of v may exist for a nonzero solution of the second-order differential equation. Since the only zero of v bifurcates from $x = -\infty$ at $\lambda = \lambda_0 < 0$ and reaches $x = 0$ at $\lambda = 0$, $v(x)$ remains positive for every $x > 0$ for $\lambda \in (-\infty, 0]$. Hence the condition (a) in Lemma 4.7 is not satisfied for every $\lambda \in (-\infty, 0]$.

We now consider vanishing of $v(-a)$ for $a > 0$ for the condition (b) in Lemma 4.7. By the same continuation argument from Lemma 2.22, there exists exactly one $\lambda_1 \in (\lambda_0, 0)$ such that $v(-a) = 0$ for any given $a > 0$. Since $v'(-a) \neq 0$ and $v(a) \neq 0$, λ_1 is an eigenvalue of $\sigma_p(L_+)$ of multiplicity $N - K - 1$.

For $a < 0$, the roles of cases (a) and (b) are swapped. The condition (b) is never satisfied, while the condition (a) is satisfied for exactly one $\lambda_1 \in (\lambda_0, 0)$, which becomes an eigenvalue of $\sigma_p(L_+)$ of multiplicity $K - 1$. The assertion of Theorem 4.2 is proved. $\blacksquare$

Remark 4.8. *By Remark 2.21, the solution v in Lemma 2.19 for $p = 1$ is available in the closed analytic form, and has a simple negative eigenvalue $\lambda_0 = -3$ with the corresponding eigenvector $v(x) = \frac{1}{4}\operatorname{sech}^2 x$ and a simple zero eigenvalue corresponding to $v(x) = \frac{1}{2}\tanh x \operatorname{sech} x$. If $a \neq 0$, the negative eigenvalue $\lambda_1 \in (\lambda_0, 0)$ in the proof of*

Theorem 4.2 is given by the root of the following equation

$$3 - \lambda - 3\sqrt{1-\lambda}\tanh|a| - 3\mathrm{sech}^2(a) = 0,$$

or explicitly, by

$$\lambda_1 = -\frac{3}{2}\tanh|a|\left[\tanh|a| + \sqrt{1 + 3\,\mathrm{sech}^2(a)}\right],$$

We note that $\lambda_1 \to 0$ *when* $a \to 0$ *and* $\lambda_1 \to \lambda_0 = -3$ *when* $|a| \to \infty$.

4.3 Morse index = Sturm index

Definition 4.9. *Let the first nonnegative discrete eigenvalue* λ *of the operator* L_+ *be simple. Then, we define the* **Sturm index** *of the operator* L_+ *as the number of nodes of the eigenfunction corresponding to the eigenvalue* λ.

One of the consequences of the well-known Sturm comparison theorem (see e.g. [23, 76]) is that the Morse index of Schrödinger operators on the real line is equal to the number of eigenfunction nodes correponding to its first nonnegative eigenvalue. In the case of the NLS equation with a power-type nonlinearity on the real line, the equivalence was used, for example, in the works of Weinstein [78, 79]. Theorem 4.2 confirms the equality for the operator L_+ by considering the star graph Γ with $N = 2$ edges. In this case, we have $K = 1$ and $\alpha_1 = \alpha_2$ by (2.3.5), and the NLS equation on the star graph with $N = 2$ becomes equivalent to the NLS equation $\mathbb{R}$. As a result, applying Theorem 4.2 with an arbitrary $a \neq 0$, the Morse index of L_+ is equal to 1, which coincides with the number of nodes of the eigenfunction (4.2.3) corresponding to the first nonnegative eigenvalue of L_+, namely $\lambda = 0$.

Considering the operators L_- and L_+ on the star graph with even N edges, we can observe the relation of the structure of the shifted states on the graph model with the case on the real line. If N is even, the graph can be considered as a set of $K = N/2$ copies of the real line and the shifted state can be interpreted as $K = N/2$ identical solitary waves on each real line translated by the shift parameter $a \in \mathbb{R}$. Since L_- is positive at each solitary wave with a simple zero eigenvalue and L_+ has a simple negative and a simple zero eigenvalue at each solitary wave for $a \neq 0$, the number of zeros for the eigenfunction (4.2.3) gives $N/2$ negative eigenvalues (with the account of their multiplicity), in agreement with the statement of Theorem 4.2.

For a general case of a bounded star graph with an arbitrary number of edges $N \geq 3$, the equality of the Morse index of Schrödinger operators with the number of eigenfunction nodes corresponding to its first nonnegative eigenvalue was obtained in [14, 69], because the Betti number of the star graph is zero. Although the previous result was proven only for compact star graphs, the result of Theorem 4.2 confirms the equality for the operator L_+ on the unbounded star graph also. By the count of Theorem 4.2, the Morse index of L_+ is K if $a < 0$ and $N - K$ if $a > 0$. On the other hand, the number of zeros for the eigenfunction (4.2.3) corresponding to the eigenvalue $\lambda = 0$ is K if $a < 0$ and $N - K$ if $a > 0$. Hence, the two indices are equal to each other for $a \neq 0$.

4.4 Homogenization of the star graph

The translational symmetry of the infinite line $\mathbb{R}$ is broken in the star graph Γ due to the vertex at $x = 0$. As a result, a momentum functional is not generally conserved under the NLS flow. However, we will show here that if the coefficients $(\alpha_1, \alpha_2, \dots, \alpha_N)$ satisfy the constraint (2.3.5), then there exist solutions to the NLS equation (2.2.1), for which the following momentum functional is conserved:

$$P(\Psi) := \sum_{j=1}^{N} (-1)^{m_j} \int_{\mathbb{R}^+} \mathrm{Im}\left(\psi_j' \overline{\psi}_j\right) dx, \tag{4.4.1}$$

where the N-tuple $(m_1, m_2, \dots, m_N)$ is given by (2.3.8). The following lemma yields the momentum balance equation.

Lemma 4.10. *For every $p > 0$ and every $(\alpha_1, \alpha_2, \dots, \alpha_N)$ satisfying the constraint (2.3.5), the local solution (2.2.3) in Lemma 2.3 satisfies the following momentum balance equation:*

$$\frac{d}{dt} P(\Psi) = \sum_{j=1}^{N} (-1)^{m_j} |\psi_j'(0)|^2. \tag{4.4.2}$$

for all $t \in (-t_0, t_0)$, where P is given by (4.4.1).

Proof. If $p \geq 1$, we can consider the smooth solutions (2.2.7) to the NLS equation (2.2.1) in H^3_Γ and compute the following momentum balance equation for P in (4.4.1):

$$\frac{d}{dt} P(\Psi) = \sum_{j=1}^{N} (-1)^{m_j} \int_{\mathbb{R}^+} \mathrm{Im}\left(\psi_j' \partial_t \overline{\psi}_j + \overline{\psi}_j \partial_t \psi_j'\right) dx. \tag{4.4.3}$$

By substituting the NLS equation (2.2.1) into (4.4.3), we integrate by parts and obtain:

$$\begin{aligned} \frac{d}{dt} P(\Psi) &= \sum_{j=1}^{N} (-1)^{m_j} \int_{\mathbb{R}^+} \mathrm{Re}\left(\overline{\psi}_j \psi_j''' - \psi_j' \overline{\psi}_j'' + p\alpha_j^{2p} (|\psi_j|^{2p+2})'\right) dx \\ &= \sum_{j=1}^{N} (-1)^{m_j} \left(-\mathrm{Re}[\overline{\psi}_j(0) \psi_j''(0)] + |\psi_j'(0)|^2 - p\alpha_j^{2p} |\psi_j(0)|^{2p+2}\right), \end{aligned}$$

where the decay of $\Psi(x)$, $\Psi'(x)$, and $\Psi''(x)$ to zero at infinity has been used for the solution in H^3_Γ. Applying the boundary conditions in (2.1.1) and (2.2.6), the constraint (2.3.5), and the choice of values of m_j in (2.3.8) yields the momentum balance equation in the form (4.4.2).

Although our derivation was restricted to the case $p \geq 1$ and to solutions in H^3_Γ, the proof can be extended to the local solution (2.2.3) for all values of $p > 0$ by using standard approximation techniques [20]. $\square$

The momentum $P(\Psi)$ is conserved in t if the boundary conditions for derivatives satisfy the additional constraints:

$$(-1)^{m_1}\alpha_1\psi_1'(0) = (-1)^{m_2}\alpha_2\psi_2'(0) = \cdots = (-1)^{m_N}\alpha_N\psi_N'(0), \tag{4.4.4}$$

which are compatible with the boundary conditions in (2.1.2) under the constraint (2.3.5). Indeed, equation (4.4.2) with the constraint (4.4.4) yields:

$$\frac{d}{dt}P(\Psi) = (-1)^{m_1}\alpha_1^2|\psi_1'(0)|^2\left(\sum_{j=1}^{K}\frac{1}{\alpha_j^2} - \sum_{j=K+1}^{N}\frac{1}{\alpha_j^2}\right) = 0,$$

hence $P(\Psi)$ is conserved in t.

In order to make sure that the constraint (4.4.4) is satisfied for every t, we observe the following reduction of the NLS equation (2.2.1) on the star graph Γ to the homogeneous NLS equation on the infinite line $\mathbb{R}$.

Lemma 4.11. *Under the constraint (2.3.5), there exist solutions of the NLS equation (2.2.1) on the graph Γ which satisfy the the following homogeneous NLS equation on the infinite line:*

$$iU_t + U_{xx} + (p+1)\,|U|^{2p}\,U = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{4.4.5}$$

where $U = U(t,x) \in \mathbb{C}$.

Proof. The class of suitable solutions Ψ to the NLS equation (2.2.1) on the star graph Γ must satisfy the following reduction:

$$\begin{cases} \alpha_1\psi_1(t,x) = \cdots = \alpha_K\psi_K(t,x), \\ \alpha_{K+1}\psi_{K+1}(t,x) = \cdots = \alpha_N\psi_N(t,x), \end{cases} \quad x \in \mathbb{R}^+, \quad t \in \mathbb{R}, \tag{4.4.6}$$

subject to the boundary conditions at the vertex point $x = 0$:

$$\alpha_K\psi_K(t,0) = \alpha_{K+1}\psi_{K+1}(t,0), \quad \alpha_K\partial_x\psi_K(t,0) = -\alpha_{K+1}\partial_x\psi_{K+1}(t,0). \tag{4.4.7}$$

Note that the boundary conditions (4.4.6) and (4.4.7) are compatible with the generalized Kirchhoff boundary conditions in (2.1.2) under the constraint (2.3.5). Thanks to the reduction (4.4.6), the following function can be defined on the infinite line:

$$U(t,x) := \begin{cases} \alpha_j\psi_j(t,-x), & 1 \leq j \leq K, & x \in \mathbb{R}^-, \\ \alpha_j\psi_j(t,x), & K+1 \leq j \leq N, & x \in \mathbb{R}^+. \end{cases} \tag{4.4.8}$$

Thanks to the boundary conditions (4.4.7), U is a C^1 function across $x = 0$. Substitution (4.4.8) into the NLS equation (2.2.1) on the graph Γ yields the homogeneous NLS equation (4.4.5), where the point $x = 0$ is a regular point on the infinite line $\mathbb{R}$. □

Remark 4.12. *The shifted state (2.3.6) corresponds to the NLS soliton in the homogeneous NLS equation (4.4.5), which is translational invariant along the line $\mathbb{R}$. The eigenvalue count of Theorem 4.2 and the instability result of Corollary 4.4 are related to the symmetry-breaking perturbations, which do not satisfy the reduction (4.4.6). These perturbations satisfy the NLS equation (2.2.1) on the graph Γ but do not satisfy the homogeneous NLS equation (4.4.5) on the line $\mathbb{R}$. Such symmetry-breaking perturbations were not considered in the numerical experiments in [66, 72, 73].*

4.5 Variational characterization of the shifted states

Here we give a simple argument suggesting that the spectrally stable shifted states with $K = 1$ and $a < 0$ in Remark 4.6 are nonlinearly unstable under the NLS flow. This involves the variational characterization of the shifted states in the graph Γ as critical points of energy under the fixed mass, where the mass and energy are defined by (2.2.4).

The mass and energy are computed at the shifted states (2.3.6) as follows:

$$Q(\Phi) = \left(\sum_{j=1}^{K} \alpha_j^{-2} \right) \|\phi\|^2_{L^2(\mathbb{R})} \tag{4.5.1}$$

and

$$E(\Phi) = \left(\sum_{j=1}^{K} \alpha_j^{-2} \right) \left(\|\phi'\|^2_{L^2(\mathbb{R})} - \|\phi\|^{2p+2}_{L^{2p+2}(\mathbb{R})} \right), \tag{4.5.2}$$

where the constraint (2.3.5) has been used. In the case $K = 1$, the mass and energy at the shifted states is the same as the mass and energy of a free solitary wave escaping to infinity along the only incoming edge. This property signals out that the infimum of energy is not achieved, as is discussed in [7].

Furthermore, the constraint (2.3.5) implies that $\alpha_2, \dots, \alpha_N > \alpha_1$ (if $N \geq 3$). Pick the j-th outgoing edge for $2 \leq j \leq N$ and fix the mass at the level $\mu > 0$. Then, it is well-known [7] that the energy of a free solitary wave escaping to infinity along the j-th outgoing edge is given by

$$E_j = -C_p \alpha_j^{\frac{4p}{2-p}} \mu^{\frac{p+2}{2-p}} < -C_p \alpha_1^{\frac{4p}{2-p}} \mu^{\frac{p+2}{2-p}} = E(\Phi), \tag{4.5.3}$$

where $p \in (0, 2)$ and C_p is a universal constant that only depends on p. Thus, a free solitary wave escaping the graph Γ along any outgoing edge has a lower energy level at fixed mass compared to the shifted state. This suggests that any shifted state is energetically unstable.

Let us now give a simple argument suggesting nonlinear instability of the shifted states (2.3.6) with $K = 1$ and $a < 0$ under the NLS flow. If $K = 1$, it follows from the momentum balance equation (4.4.2) in Lemma 4.10 that the momentum $P(\Psi)$ defined by (4.4.1) is an increasing function of time if $m_1 = 1$ and a decreasing function of time if $m_1 = 0$, for the two choices in (2.3.8). Indeed, we obtain the following chain of transformations by using the boundary conditions in (2.1.2) and the constraint (2.3.5):

$$\begin{aligned} \frac{d}{dt}P(\Psi) &= (-1)^{m_1+1}\left[\sum_{j=2}^{N}|\psi_j'(0)|^2 - \sum_{j=2}^{N}\sum_{i=2}^{N}\frac{\alpha_1^2}{\alpha_j\alpha_i}\psi_j'(0)\overline{\psi_i}'(0)\right] \\ &= (-1)^{m_1+1}\left(\sum_{j=2}^{N}\sum_{\substack{i=2\\ i\neq j}}^{N}\frac{\alpha_1^2}{\alpha_i^2}|\psi_j'(0)|^2 - \sum_{j=2}^{N}\sum_{\substack{i=2\\ i\neq j}}^{N}\frac{\alpha_1^2}{\alpha_j\alpha_i}\psi_j'(0)\overline{\psi_i}'(0)\right) \\ &= \frac{1}{2}(-1)^{m_1+1}\sum_{j=2}^{N}\sum_{\substack{i=2\\ i\neq j}}^{N}\frac{\alpha_1^2}{\alpha_j^2\alpha_i^2}\left|\alpha_j\psi_j'(0) - \alpha_i\psi_i'(0)\right|^2 \end{aligned} \tag{4.5.4}$$

Hence $\frac{d}{dt}P(\Psi) \geq 0$ if $m_1 = 1$ and $\frac{d}{dt}P(\Psi) \leq 0$ if $m_1 = 0$.

Since the shifted states (2.3.6) satisfies $P(\Phi) = 0$, monotonicity of the momentum $P(\Psi)$ in time t immediately implies nonlinear instability of the shifted states (2.3.6) with $a < 0$ under the NLS flow, despite that these shifted states are spectrally stable. Indeed, if $a < 0$ (or $m_1 = 1$), the value of the momentum $P(\Psi)$ is monotonically increasing in time as soon as the right-hand side of (4.5.4) is nonzero. Therefore, if $P(\Psi)$ is initially near zero, which is the value of $P(\Phi)$ for every shifted state (2.3.6) with $a \in \mathbb{R}$, then $P(\Psi)$ grows far away from the zero value. By this simple argument, we expect that the branch of shifted states (2.3.6) with $K = 1$ and $a < 0$ is nonlinearly unstable under the NLS flow. We give a rigorous proof of this observation in Chapter 5.

Chapter 5

Drift of spectrally stable shifted states

In Chapter 4, we proved that the shifted states on star graphs with exactly $K = 1$ incoming edge and $N - 1$ outgoing edges are spectrally stable if their monotonic tails are located on the outgoing edges, see Figure 5.1. These shifted states are constrained minimizers of the energy and the only degeneracies of the second variation of energy are due to phase rotation and the spatial translation of the shifted state along the star graph. However, in contrast to the well-known result, that standing waves in the NLS equation on the real line with symmetry-related degeneracies are orbitally stable, we show in this paper that *the shifted states are orbitally unstable in the NLS equation on the star graph.*

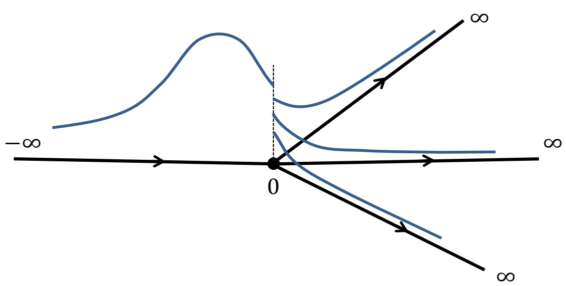

FIGURE 5.1: A shifted state on a star graph with one incoming and three outgoing edges. The shifted state has symmetric monotonic tails on the outgoing edges and a non-monotonic tail on the incoming edge.

The instability is related to the following observation. The shifted state is symmetric with respect to the exchange of components on the outgoing edges. If the initial perturbation to the shifted state preserves this symmetry, then the NLS on the balanced star graph can be reduced to the NLS equation on a line, and the solution of the time evolution problem has translational symmetry. Perturbations that lack this exchange symmetry also break the translational symmetry and the solution fails to conserve momentum. Moreover, the value of the momentum functional increases monotonically in the time flow of the NLS equation and this monotone increase results in the irreversible drift of the shifted state along the incoming edge towards the outgoing edges of the balanced star graph. When the center of mass of the shifted state reaches the vertex, the shifted state becomes a saddle point of energy under the fixed mass. At this point in time, orbital instability of the shifted state develops as a result of the saddle point geometry similar to the instability studied in Chapter 3.

The main novelty of this result is that *degeneracy of the positive second variation of energy may lead to orbital instability of constrained minimizers if this degeneracy is not related to the symmetry of the Hamiltonian PDE*. The orbital instability appears due to irreversible drift of shifted states from a spectrally stable state towards the orbitally unstable states.

5.1 Main results

For simplicity, consider the cubic ($p = 1$) NLS equation (2.2.1) on the star graph Γ with $N \geq 3$ edges given by

$$i\frac{\partial \Psi}{\partial t} = -\Delta\Psi - 2\alpha^2|\Psi|^2\Psi, \tag{5.1.1}$$

In the construction of the graph Γ, we change the parametrization of half-lines, so that one edge represents an incoming bond and the remaining $N-1$ edges represent outgoing bonds. We place the vertex at the origin and parameterize the incoming edge by $\mathbb{R}^-$ and the $N-1$ outgoing edges by $\mathbb{R}^+$. We assume the coefficients $(\alpha_1, \alpha_2, \dots, \alpha_N)$ in (2.1.1)–(2.1.2) satisfy the constraint (2.3.5) with $K = 1$, and we set the incoming edge to be the edge, where the NLS equation is given by (2.2.2) with $j = 1$. The star graph Γ with one incoming and three outgoing edges is illustrated on Fig. 5.1.

The new parametrization changes the structures of spaces and the boundary conditions compared to the definitions in Section 2.1. That is, we re-define the Hilbert space $L^2(\Gamma)$ as

$$L^2(\Gamma) = L^2(\mathbb{R}^-) \oplus \underbrace{L^2(\mathbb{R}^+) \oplus \cdots \oplus L^2(\mathbb{R}^+)}_{\text{(N-1) elements}}, \tag{5.1.2}$$

and Sobolev spaces $H^k(\Gamma)$ as

$$H^k(\Gamma) = H^k(\mathbb{R}^-) \oplus \underbrace{H^k(\mathbb{R}^+) \oplus \cdots \oplus H^k(\mathbb{R}^+)}_{\text{(N-1) elements}}$$

for $k = 1, 2$. The generalized Kirchhoff boundary conditions in (2.1.1)–(2.1.2) become equivalent to

$$H^1_\Gamma := \{\Psi \in H^1(\Gamma) : \quad \alpha_1\psi_1(0) = \alpha_2\psi_2(0) = \cdots = \alpha_N\psi_N(0)\} \tag{5.1.3}$$

and

$$H^2_\Gamma := \left\{\Psi \in H^2(\Gamma) \cap H^1_\Gamma : \quad \alpha_1^{-1}\psi_1'(0) = \sum_{j=2}^{N} \alpha_j^{-1}\psi_j'(0)\right\}. \tag{5.1.4}$$

where derivatives are defined as $\lim_{x\to 0^-}$ for the incoming edge and $\lim_{x\to 0^+}$ for the $(N-1)$ outgoing edges.

Under such configuration, we can re-write the shifted states $\Phi(x;a)$ with $\omega = 1$ given in Lemma 2.9 as

$$\phi_j(x;a) = \alpha_j^{-1}\phi(x+a), \quad 1 \leq j \leq n, \tag{5.1.5}$$

with $\phi(x) = \operatorname{sech}(x)$. Here, we also replace the parameter $a \in \mathbb{R}$ in Lemma 2.9 by $-a \in \mathbb{R}$ for convenience.

Remark 5.1. *As a result of re-parametrization of Γ, $\phi_1(x)$ in (5.1.5) is defined for $x \in \mathbb{R}^-$, while all other $\phi_j(x)$ are defined for $x \in \mathbb{R}^+$.*

The shifted state (5.1.5) with $K = 1$ satisfies the symmetry (4.4.6). Under the symmetry, the free parameter a in the family of shifted states (5.1.5) is related to the translational symmetry of the NLS equation (5.1.1) in x. However, the translational symmetry is broken for the NLS equation (5.1.1) on the star graph Γ due to the vertex at $x = 0$. As a result, the momentum functional $P(\Psi)$

$$P(\Psi) := \operatorname{Im}\langle\Psi', \Psi\rangle_{L^2(\Gamma)} = \int_{\mathbb{R}^-} \operatorname{Im}\left(\psi_1'\overline{\psi}_1\right) dx + \sum_{j=2}^{N} \int_{\mathbb{R}^+} \operatorname{Im}\left(\psi_j'\overline{\psi}_j\right) dx \tag{5.1.6}$$

is no longer constant under the time flow of (5.1.1). Note that (5.1.6) is obtained from (4.4.1) after the parametrization (5.1.2) applied. By (4.5.4), for every strong solution $\Psi \in C(\mathbb{R}, H^2_\Gamma) \cap C^1(\mathbb{R}, L^2(\Gamma))$ to the NLS equation (5.1.1) the map $t \mapsto P(\Psi)$ is monotonically increasing, thanks to the following inequality:

$$\frac{d}{dt}P(\Psi) = \frac{1}{2}\sum_{j=2}^{N}\sum_{\substack{i=2\\ i\neq j}}^{N} \frac{\alpha_1^2}{\alpha_j^2\alpha_i^2}\left|\alpha_j\psi_j'(0) - \alpha_i\psi_i'(0)\right|^2 \geq 0. \tag{5.1.7}$$

If the strong solution Ψ satisfies the symmetry (4.4.6) with $K = 1$, then $P(\Psi)$ is conserved in time.

In Section 4.5 we presented simple arguments on nonlinear instability of the shifted state $\Phi(\cdot;a)$ given in (5.1.5), which we rigorously prove below. First, we give the following definition of nonlinear instability of a shifted state $\Phi(\cdot;a)$.

Definition 5.2. *Fix $a \in \mathbb{R}$. The shifted state $\Phi(\cdot;a)$ is said to be* **nonlinearly unstable** *in H^1_Γ if there exists $\epsilon > 0$ such that for every $\delta > 0$ there exists an initial datum $\Psi_0 \in H^1_\Gamma$ satisfying*

$$\|\Psi_0 - \Phi(\cdot;a)\|_{H^1(\Gamma)} \leq \delta$$

and $T > 0$ such that the unique global solution $\Psi \in C(\mathbb{R}, H^1_\Gamma) \cap C^1(\mathbb{R}, H^{-1}_\Gamma)$ to the NLS equation (5.1.1) with $\Psi(0,\cdot) = \Psi_0$ satisfies

$$\inf_{\theta\in\mathbb{R}} \|\Psi(T,\cdot) - e^{i\theta}\Phi(\cdot;a)\|_{H^1(\Gamma)} > \epsilon.$$

Our first main result shows that the monotone increase of the map $t \mapsto P(\Psi)$ as in (5.1.7) leads to a drift along the family of shifted states (5.1.5) in which the parameter a decreases monotonically in t towards $a = 0$. This drift induces nonlinear instability of the spectrally stable shifted states in Lemma 2.9 with $a > 0$ according to Definition 5.2. The following theorem formulates the result.

Theorem 5.3. *Fix $a_0 > 0$. For every $\mathfrak{a} \in (0, a_0)$ there exists $\epsilon_0 > 0$ (sufficiently small) such that for every $\epsilon \in (0, \epsilon_0)$, there exists $\delta > 0$ and $T > 0$ such that for every initial datum $\Psi_0 \in H^1_\Gamma$ satisfying*

$$\|\Psi_0 - \Phi(\cdot; a_0)\|_{H^1(\Gamma)} \leq \delta \tag{5.1.8}$$

and $P(\Psi_0) \geq C_0\delta$ for some independent constant $C_0 > 0$, the unique global solution $\Psi \in C(\mathbb{R}, H^1_\Gamma) \cap C^1(\mathbb{R}, H^{-1}_\Gamma)$ to the NLS equation (5.1.1) with $\Psi(0, \cdot) = \Psi_0$ satisfies

$$\inf_{\theta\in\mathbb{R}} \|\Psi(t, \cdot) - e^{i\theta}\Phi(\cdot; a(t))\|_{H^1(\Gamma)} \leq \epsilon, \quad t \in [0, T], \tag{5.1.9}$$

where $a \in C^1([0, T])$ is a strictly decreasing function such that $\lim_{t\to T} a(t) = \mathfrak{a}$.

By Theorem 5.3, the shifted state (5.1.5) with $a > 0$ drifts towards the half-soliton state with $a = 0$. The half-soliton state is more degenerate than the shifted state with $a > 0$ because the zero eigenvalue of the linearized operator to the stationary NLS equation (2.3.3) is simple for $a > 0$ and has multiplicity $N-1$ for $a = 0$. Moreover, while the shifted state $\Phi(\cdot; a)$ with $a > 0$ is a degenerate minimizer of the action functional $\Lambda_{\omega=1}(\Psi) = E(\Psi) + Q(\Psi)$, the half-soliton state $\Phi_0 := \Phi(\cdot; a = 0)$ is a degenerate saddle point of the same action functional. The following theorem shows the nonlinear instability of the half-soliton state according to Definition 5.2. This nonlinear instability is related to the saddle point geometry of the critical point Φ_0.

Theorem 5.4. *Denote $\Phi_0 := \Phi(\cdot; a = 0)$. There exists $\epsilon > 0$ such that for every small $\delta > 0$ there exists $V \in H^1_\Gamma$ with $\|V\|_{H^1_\Gamma} \leq \delta$ such that the unique global solution $\Psi \in C(\mathbb{R}, H^1_\Gamma) \cap C^1(\mathbb{R}, H^{-1}_\Gamma)$ to the NLS equation (5.1.1) with the initial datum $\Psi(0, \cdot) = \Phi + V$ satisfies*

$$\inf_{\theta\in\mathbb{R}} \|e^{-i\theta}\Psi(T, \cdot) - \Phi\|_{H^1(\Gamma)} > \epsilon \quad \text{for some } T > 0. \tag{5.1.10}$$

Remark 5.5. *The result of Theorem 5.4 is very similar to the instability result in Theorem 3.6.*

5.2 Linear estimates

Recall that the scaling transformation (2.3.2) converts the normalized shifted states Φ in (5.1.5) to the ω-dependent family Φ_ω of the shifted states. We note the following elementary computations:

$$D_1(\omega) = -\langle \Phi_\omega(\cdot;a), \partial_\omega \Phi_\omega(\cdot;a)\rangle_{L^2(\Gamma)} = -\frac{1}{2}\frac{d}{d\omega}\|\Phi_\omega\|^2_{L^2(\Gamma)} = -\frac{1}{2\alpha_1^2\omega^{\frac{1}{2}}} \tag{5.2.1}$$

and

$$D_2(\omega) = -\langle \Phi'_\omega(\cdot;a), (\cdot + a)\Phi_\omega(\cdot;a)\rangle_{L^2(\Gamma)} = \frac{1}{2}\|\Phi_\omega\|^2_{L^2(\Gamma)} = \frac{\omega^{\frac{1}{2}}}{\alpha_1^2}. \tag{5.2.2}$$

We discuss separately the linearization of the shifted state with $a \neq 0$ and the half-soliton state with $a = 0$.

5.2.1 Linearization at the shifted state with $a \neq 0$

For every standing wave solution $\Phi_\omega(\cdot;a)$ we recall two self-adjoint linear operators $L_\pm(\omega,a) : H^2_\Gamma \subset L^2(\Gamma) \to L^2(\Gamma)$ given in (2.4.3)–(2.4.4):

$$\begin{cases} L_-(\omega,a) = -\Delta + \omega - 2\alpha^2\Phi_\omega(\cdot;a)^2, \\ L_+(\omega,a) = -\Delta + \omega - 6\alpha^2\Phi_\omega(\cdot;a)^2. \end{cases}$$

We recall the spectral properties of these operators for general $\omega > 0$.

The continuous spectrum is strictly positive thanks to the fast exponential decay of $\Phi_\omega(x;a)$ to zero as $|x| \to \infty$ and Weyl's Theorem:

$$\sigma_c(L_\pm(\omega,a)) = [\omega,\infty)$$

with $\omega > 0$. The discrete spectrum $\sigma_p(L_\pm(\omega,a)) \subset (-\infty,\omega)$ includes finitely many negative, zero, and positive eigenvalues of finite multiplicities.

Eigenvalues of $\sigma_p(L_+(\omega,a)) \subset (-\infty,\omega)$ are known explicitly, see Remarks 2.21 and 4.8. For $\omega = 1$, these eigenvalues are given by:

- a simple negative eigenvalue $\lambda_0 = -3$;
- a zero eigenvalue $\lambda = 0$ which is simple when $a \neq 0$;
- the additional eigenvalue $\lambda = \lambda_1(a)$ of multiplicity $N-2$ given by

$$\lambda_1(a) = -\frac{3}{2}\tanh(a)\left[\tanh(a) - \sqrt{1 + 3\,\mathrm{sech}(a)}\right]. \tag{5.2.3}$$

It is negative for $a < 0$, zero for $a = 0$, and positive for $a \in (0, a_*)$, where $a_* = \tanh^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.66$. The eigenvalue merges into the continuous spectrum as $a \nearrow a_*$.

The spectrum of $L_+(\omega,a)$ for $\omega = 1$ is illustrated in Fig. 5.2.

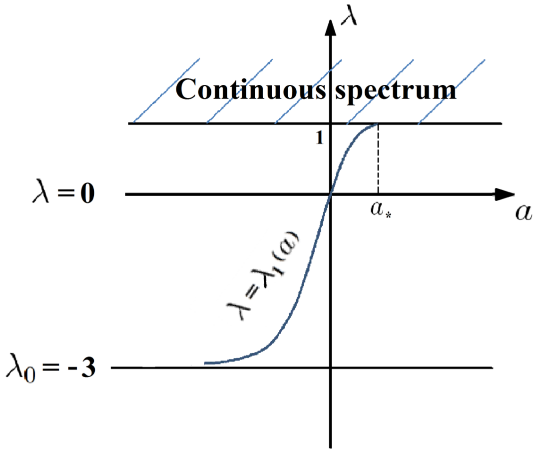

FIGURE 5.2: The spectrum of $L_+(\omega, a)$ for $\omega = 1$. The continuous spectrum is $[1, \infty)$, while the discrete spectrum is given by the eigenvalues $\lambda = 0$, $\lambda = -3$, and $\lambda = \lambda_1(a)$ in (5.2.3).

Eigenvalues of $\sigma_p(L_-(\omega, a)) \subset (-\infty, \omega)$ are non-negative and the zero eigenvalue is simple, see Lemma 2.18. If $a \neq 0$, the zero eigenvalues of $L_+(\omega, a)$ and $L_-(\omega, a)$ are each simple with the eigenvectors given by

$$L_+(\omega, a)\Phi_\omega'(\cdot; a) = 0, \quad L_-(\omega, a)\Phi_\omega(\cdot; a) = 0. \tag{5.2.4}$$

The eigenvectors in (5.2.4) induce the generalized eigenvectors in

$$L_+(\omega, a)\partial_\omega\Phi_\omega(\cdot; a) = -\Phi_\omega(\cdot; a), \quad L_-(\omega, a)(\cdot + a)\Phi_\omega(\cdot; a) = -2\Phi_\omega'(\cdot; a). \tag{5.2.5}$$

The following lemma gives coercivity of the quadratic forms associated with the operators $L_+(\omega, a)$ and $L_-(\omega, a)$ for $a > 0$.

Lemma 5.6. *For every $\omega > 0$ and $a > 0$, there exists a positive constant $C(\omega, a)$ such that*

$$\langle L_+(\omega, a)U, U\rangle_{L^2(\Gamma)} + \langle L_-(\omega, a)W, W\rangle_{L^2(\Gamma)} \geq C(\omega, a)\|U + iW\|^2_{H^1(\Gamma)} \tag{5.2.6}$$

if U and W satisfy the orthogonality conditions

$$\begin{cases} \langle W, \partial_\omega\Phi_\omega(\cdot; a)\rangle_{L^2(\Gamma)} = 0, \\ \langle U, \Phi_\omega(\cdot; a)\rangle_{L^2(\Gamma)} = 0, \\ \langle U, (\cdot + a)\Phi_\omega(\cdot; a)\rangle_{L^2(\Gamma)} = 0, \end{cases} \tag{5.2.7}$$

Proof. The first orthogonality condition in (5.2.7) shifts the lowest (zero) eigenvalue of $L_-(\omega, a)$ to a positive eigenvalue thanks to the condition (5.2.1) (see Lemma 3.19) and yields by Gårding's inequality the following coercivity bound

$$\langle L_-(\omega, a)W, W\rangle_{L^2(\Gamma)} \geq C(\omega)\|W\|^2_{H^1(\Gamma)}$$

independently of a. The second orthogonality condition in (5.2.7) shifts the lowest (negative) eigenvalue of $L_+(\omega, a)$ to a positive eigenvalue thanks to the same condition (5.2.1) (see Lemma 3.11) and yields

$$\langle L_+(\omega, a)U, U\rangle_{L^2(\Gamma)} \geq 0$$

with $\langle L_+(\omega, a)U, U\rangle_{L^2(\Gamma)} = 0$ if and only if U is proportional to $\Phi'_\omega(\cdot; a)$. The zero eigenvalue of $L_+(\omega, a)$ is preserved by the constraint since

$$\langle \Phi_\omega(\cdot; a), \Phi'_\omega(\cdot; a)\rangle_{L^2(\Gamma)} = 0.$$

Finally, the third orthogonality condition in (5.2.7) shifts the zero eigenvalue of $L_+(\omega, a)$ to a positive eigenvalue thanks to the condition (5.2.2). By Gårding's inequality, this yields the coercivity bound

$$\langle L_+(\omega, a)U, U\rangle_{L^2(\Gamma)} \geq C(\omega, a)\|U\|^2_{H^1(\Gamma)},$$

where $C(\omega, a)$ depends on a because the gap between the zero eigenvalue and the rest of the positive spectrum in $\sigma_p(L_+(\omega, a))$ exists for $a > 0$ but vanishes as $a \to 0$. $\square$

Remark 5.7. *For every $\omega > 0$, the positive constant $C(\omega, a)$ in (5.2.6) satisfies*

$$C(\omega, a) \to 0 \quad \text{as} \quad a \searrow 0.$$

This is because the zero eigenvalue in $\sigma_p(L_+(\omega, a = 0))$ has multiplicity $(N - 1)$ and the $(N-2)$ eigenvectors of $L_+(\omega, a = 0)$ satisfy the last two orthogonality conditions (5.2.7) as is seen from the proof of Lemma 5.11.

Remark 5.8. *For $a < 0$, the result of Lemma 5.6 is false because $\sigma_p(L_+(\omega, a))$ includes another negative eigenvalue as is seen from Fig. 5.2.*

Remark 5.9. *The orthogonality conditions in (5.2.7) are typically referred to as the symplectic orthogonality conditions, because they express orthogonality of the residual terms U and W for the real and imaginary parts of the perturbation to $\Phi_\omega(\cdot; a)$ to the eigenvectors and generalized eigenvectors of the spectral stability problem expressed by $L_+(\omega, a)$ and $L_-(\omega, a)$ and the symplectic structure of the NLS equation. Compared to the classical approach of four orthogonality conditions and four parameters of modulated states [78], we do not use the orthogonality condition $\langle W, \Phi'_\omega(\cdot; a)\rangle_{L^2(\Gamma)} = 0$ and work with three parameters for modulations of the orbit $\{e^{i\theta}\Phi_\omega(\cdot; a)\}_{\theta\in\mathbb{R}, a\in\mathbb{R}^+, \omega\in\mathbb{R}^+}$. The reason for this is that the coercivity (5.2.6) is already obtained under the three orthogonality conditions (5.2.7) and that it is difficult to control the fourth parameter, corresponding to the velocity of the shifted state, with the energy method.*

5.2.2 Linearization at the half-soliton state

For $a = 0$, we denote operators $L_\pm(\omega) \equiv L_\pm(\omega, a = 0)$. The kernel of the operator $L_+(\omega)$ is spanned by an orthogonal basis consisting of $N-1$ eigenvectors, which we denote by $\{U_\omega^{(1)}, U_\omega^{(2)}, \cdots, U_\omega^{(N-1)}\}$. The following lemma specifies an explicit construction of these basis eigenvectors.

Lemma 5.10. *There exists an orthogonal basis $\{U_\omega^{(1)}, U_\omega^{(2)}, \cdots, U_\omega^{(N-1)}\}$ of the kernel of $L_+(\omega)$ satisfying the orthogonality condition*

$$\langle U, \Phi_\omega \rangle_{L^2(\Gamma)} = 0. \tag{5.2.8}$$

The eigenvectors can be represented in the following way: for $j = 1$,

$$U_\omega^{(1)} := (\alpha_1^{-1}\phi'_\omega, \alpha_2^{-1}\phi'_\omega, \dots, \alpha_N^{-1}\phi'_\omega), \tag{5.2.9}$$

and for $j = 2, \dots, N-1$,

$$U_\omega^{(j)} := (\underbrace{0, \dots, 0}_{\text{(j-1) elements}}, r_j\phi'_\omega, \alpha_{j+1}^{-1}\phi'_\omega, \dots, \alpha_N^{-1}\phi'_\omega), \quad r_j = -\left(\sum_{i=j+1}^{N} \frac{1}{\alpha_i^2}\right)\alpha_j, \tag{5.2.10}$$

where $\phi_\omega(x) = \omega^{\frac{1}{2}}\operatorname{sech}(\omega^{\frac{1}{2}}x)$, $x \in \mathbb{R}$.

Proof. Let $U = (u_1, u_2, \dots, u_N) \in H^2_\Gamma$ be an eigenvector for the zero eigenvalue of the operator $L_+(\omega)$. Each component of the eigenvalue problem $L_+(\omega)U = 0$ satisfies

$$-u_j''(x) + \omega u_j(x) - 6\omega\operatorname{sech}^2(\sqrt{\omega}x)u_j(x) = 0, \tag{5.2.11}$$

where $x \in \mathbb{R}^-$ on the first edge and $x \in \mathbb{R}^+$ on the remaining edges. Since $H^2(\mathbb{R}^\pm)$ are continuously embedded into $C^1(\mathbb{R}^\pm)$, if $U \in H^2(\Gamma)$, then both $u_j(x)$ and $u_j'(x)$ decay to zero as $|x| \to \infty$. Such solutions to the differential equations (5.2.11) are given uniquely by $u_j(x) = a_j\phi'_\omega(x)$ up to multiplication by a constant a_j. Therefore, the eigenvector U is given by

$$U = (a_1\phi'_\omega, a_2\phi'_\omega, \dots, a_N\phi'_\omega). \tag{5.2.12}$$

The eigenvector $U \in H^2_\Gamma$ must satisfy the boundary conditions in (5.1.4). The continuity conditions hold since $\phi'_\omega(0) = 0$, whereas the Kirchhoff condition implies

$$\frac{a_1}{\alpha_1} = \sum_{j=2}^{N} \frac{a_j}{\alpha_j}. \tag{5.2.13}$$

Since the equation (5.2.13) relates N unknowns, the space of solutions for $(a_1, a_2, \dots, a_N)$ is $(N-1)$-dimensional and the kernel of the operator $L_+(\omega)$ is $(N-1)$-dimensional. Let $\{U_\omega^{(1)}, U_\omega^{(2)}, \dots, U_\omega^{(N-1)}\}$ be an orthogonal basis of the kernel, which can be constructed from any set of basis vectors by applying the Gram-Schmidt orthogonalization process.

Direct computations show that if U is given by (5.2.12), then

$$\langle U, \Phi_\omega \rangle_{L^2(\Gamma)} = \left(\sum_{j=2}^{N} \frac{a_j}{\alpha_j} - \frac{a_1}{\alpha_1} \right) \langle \phi'_\omega, \phi_\omega \rangle_{L^2(\mathbb{R}^+)},$$

which means that the condition (5.2.13) is equivalent to $\langle U, \Phi_\omega \rangle_{L^2(\Gamma)} = 0$. Therefore, all elements in the orthogonal basis satisfy the orthogonality condition (5.2.8).

It remains to prove that the orthogonal basis can be characterized in the form given in (5.2.9)–(5.2.10). From the constraint (2.3.5), we can take $a_j = \alpha_j^{-1}$ for all j in (5.2.13) to set the first eigenvector $U_\omega^{(1)}$ to be defined by (5.2.9). The last eigenvector $U_\omega^{(N-1)}$ can be defined by

$$U_\omega^{(N-1)} := (0, \dots, 0, r_{N-1}\phi'_\omega, \alpha_N^{-1}\phi'_\omega), \tag{5.2.14}$$

where r_{N-1} is defined to satisfy the orthogonality condition $\langle U_\omega^{(1)}, U_\omega^{(N-1)} \rangle_{L^2(\Gamma)} = 0$ and the condition (5.2.13). In fact, both conditions are equivalent since the first $(N-2)$ entries of $U_\omega^{(N-1)}$ are zero and

$$\langle U_\omega^{(1)}, U_\omega^{(N-1)} \rangle_{L^2(\Gamma)} = \|\phi'_\omega\|^2_{L^2(\mathbb{R}^+)} \left(\frac{r_{N-1}}{\alpha_{N-1}} + \frac{1}{\alpha_N^2} \right)$$

with $\|\phi'_\omega\|^2_{L^2(\mathbb{R}^+)} \neq 0$. Hence r_{N-1} is defined by

$$r_{N-1} = -\frac{\alpha_{N-1}}{\alpha_N^2}.$$

The remaining eigenvectors $U_\omega^{(j)}$ in (5.2.10) are constructed recursively from $j = N-2$ to $j = 2$. By direct computations the orthogonality condition $\langle U_\omega^{(1)}, U_\omega^{(j)} \rangle_{L^2(\Gamma)} = 0$ is equivalent to the constraint (5.2.13). Moreover, all the eigenvectors are mutually orthogonal thanks to the recursive construction of $U_\omega^{(j)}$, $\square$

We denote the kernel of $L_+(\omega)$ by

$$X_\omega := \text{span}\{U_\omega^{(1)}, U_\omega^{(2)}, \cdots, U_\omega^{(N-1)}\}. \tag{5.2.15}$$

For each $j = 1, 2, \dots, N-1$, we construct the generalized eigenvector $W_\omega^{(j)} \in H^2_\Gamma$ by solving

$$L_-(\omega) W_\omega^{(j)} = U_\omega^{(j)},$$

which exists thanks to the orthogonality condition (5.2.8) since Φ_ω spans the kernel of $L_-(\omega)$. Explicitly, representing $U_\omega^{(j)}$ from (5.2.9)–(5.2.10) by

$$U_\omega^{(j)} = \phi'_\omega e_j \tag{5.2.16}$$

with some x-independent vectors $e_j \in \mathbb{R}^N$, we get for the same vectors e_j

$$W_\omega^{(j)} = \chi_\omega e_j, \tag{5.2.17}$$

where $\chi_\omega(x) = -\frac{1}{2}x\phi_\omega(x)$, $x \in \mathbb{R}$. We denote the generalized kernel of $L_-(\omega)$ by

$$X_\omega^* := \text{span}\{W_\omega^{(1)}, W_\omega^{(2)}, \cdots, W_\omega^{(N-1)}\}, \tag{5.2.18}$$

The following lemma gives coercivity of the quadratic forms associated with the operators $L_+(\omega)$ and $L_-(\omega)$.

Lemma 5.11. *For every $\omega > 0$, there exists a positive constant $C(\omega)$ such that*

$$\langle L_+(\omega)U, U\rangle_{L^2(\Gamma)} + \langle L_-(\omega)W, W\rangle_{L^2(\Gamma)} \geq C(\omega)\|U + iW\|^2_{H^1(\Gamma)} \tag{5.2.19}$$

if $U \in X_\omega^*$ and $W \in X_\omega$ satisfying the additional orthogonality conditions

$$\begin{cases} \langle W, \partial_\omega \Phi_\omega\rangle_{L^2(\Gamma)} = 0, \\ \langle U, \Phi_\omega\rangle_{L^2(\Gamma)} = 0. \end{cases} \tag{5.2.20}$$

Proof. We claim that basis vectors in X_ω and X_ω^* satisfy the following orthogonality conditions:

- $\{\langle U_\omega^{(j)}, U_\omega^{(k)}\rangle_{L^2(\Gamma)}\}_{1\leq j,k\leq N-1}$ is a positive diagonal matrix;
- $\{\langle W_\omega^{(j)}, W_\omega^{(k)}\rangle_{L^2(\Gamma)}\}_{1\leq j,k\leq N-1}$ is a positive diagonal matrix;
- $\{\langle U_\omega^{(j)}, W_\omega^{(k)}\rangle_{L^2(\Gamma)}\}_{1\leq j,k\leq N-1}$ is a positive diagonal matrix.

Indeed, the orthogonality of $\{U_\omega^{(1)}, \dots, U_\omega^{(N-1)}\}$ is established by Lemma 5.10. Therefore, the vectors $\{e_1, \dots, e_{N-1}\}$ in (5.2.16) are orthogonal in $\mathbb{R}^{N-1}$.

The orthogonality of $\{W_\omega^{(1)}, \dots, W_\omega^{(N-1)}\}$ follows by the representation (5.2.17) due to the orthogonality of the vectors $\{e_1, \dots, e_{N-1}\}$ in $\mathbb{R}^{N-1}$. The sets $\{U_\omega^{(1)}, \dots, U_\omega^{(N-1)}\}$ and $\{W_\omega^{(1)}, \dots, W_\omega^{(N-1)}\}$ are mutually orthogonal by the same reason. Finally, we have for every $j = 1, \dots, N$

$$\langle U_\omega^{(j)}, W_\omega^{(j)}\rangle_{L^2(\Gamma)} = \frac{\alpha_j^2}{4}\left(\sum_{i=j}^{N}\frac{1}{\alpha_i^2}\right)\left(\sum_{i=j+1}^{N}\frac{1}{\alpha_i^2}\right)\|\phi_\omega\|^2_{L^2(\mathbb{R}^+)}(>0). \tag{5.2.21}$$

The rest of the proof is similar to the proof of Lemma 5.6 with the only difference being that the third orthogonality condition (5.2.7) is replaced by the $(N-1)$ orthogonality conditions in $U \in X_\omega^*$. The constraint $U \in X_\omega^*$ provide the shift of the zero eigenvalue of $L_+(\omega)$ of algebraic multiplicity $(N-1)$ to positive eigenvalues thanks to the condition that $\{\langle U_\omega^{(j)}, W_\omega^{(k)}\rangle_{L^2(\Gamma)}\}_{1\leq j,k\leq N-1}$ is a positive diagonal matrix. $\square$

5.3 Drift of the shifted states with $a > 0$

The proof of Theorem 5.3 is divided into several steps. First, we decompose the unique global solution Ψ to the NLS equation (5.1.1) into the modulated stationary state $\{e^{i\theta}\Phi_\omega(\cdot; a)\}_{\theta\in\mathbb{R},a\in\mathbb{R},\omega\in\mathbb{R}^+}$ and the symplectically orthogonal remainder terms. Second, we estimate the rate of change of the modulation parameter $a(t)$ in time t and show that $a'(t) < 0$ for $t > 0$. Third, we use energy estimates to control the time evolution of the modulation parameter $\omega(t)$ and the remainder terms. Although the decomposition works for any $a(t)$, we only consider $a(t) > 0$ in order to use the coercivity bound in Lemma 5.6.

5.3.1 Symplectically orthogonal decomposition

Any point in H^1_Γ close to an orbit $\{e^{i\theta}\Phi(\cdot;a_0)\}_{\theta\in\mathbb{R}}$ for some $a_0 \in \mathbb{R}$ can be represented by a superposition of a point on the family $\{e^{i\theta}\Phi_\omega(\cdot;a)\}_{\theta\in\mathbb{R},a\in\mathbb{R},\omega\in\mathbb{R}^+}$ and a symplectically orthogonal remainder term. Here and in what follows, we denote $\Phi \equiv \Phi_{\omega=1}$. The following lemma provides details of this symplectically orthogonal decomposition.

Lemma 5.12. *Fix $a_0 \in \mathbb{R}$. There exists some $\delta_0 > 0$ such that for every $\Psi \in H^1_\Gamma$ satisfying*

$$\delta := \inf_{\theta\in\mathbb{R}} \|\Psi - e^{i\theta}\Phi(\cdot;a_0)\|_{H^1(\Gamma)} \leq \delta_0, \tag{5.3.1}$$

there exists a unique choice for real-valued $(\theta,\omega,a) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}$ and real-valued $(U,W) \in H^1_\Gamma \times H^1_\Gamma$ in the decomposition

$$\Psi(x) = e^{i\theta}\left[\Phi_\omega(x;a) + U(x) + iW(x)\right], \tag{5.3.2}$$

subject to the orthogonality conditions

$$\begin{cases} \langle W, \partial_\omega\Phi_\omega(\cdot;a)\rangle_{L^2(\Gamma)} = 0, \\ \langle U, \Phi_\omega(\cdot;a)\rangle_{L^2(\Gamma)} = 0, \\ \langle U, (\cdot + a)\Phi_\omega(\cdot;a)\rangle_{L^2(\Gamma)} = 0, \end{cases} \tag{5.3.3}$$

where ω, a, and (U,W) satisfy the estimate

$$|\omega - 1| + |a - a_0| + \|U + iW\|_{H^1(\Gamma)} \leq C\delta, \tag{5.3.4}$$

for some δ-independent constant $C > 0$. Moreover, the map from $\Psi \in H^1_\Gamma$ to $(\theta,\omega,a) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}$ and $(U,W) \in H^1_\Gamma\times H^1_\Gamma$ is C^ω.

Proof. Define the following vector function $G(\theta,\omega,a;\Psi) : \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}\times H^1_\Gamma \mapsto \mathbb{R}^3$ given by

$$G(\theta,\omega,a;\Psi) := \begin{bmatrix} \langle \mathrm{Im}(\Psi - e^{i\theta}\Phi_\omega(\cdot;a)), \partial_\omega\Phi_\omega(\cdot;a)\rangle_{L^2(\Gamma)} \\ \langle \mathrm{Re}(\Psi - e^{i\theta}\Phi_\omega(\cdot;a)), \Phi_\omega(\cdot;a)\rangle_{L^2(\Gamma)} \\ \langle \mathrm{Re}(\Psi - e^{i\theta}\Phi_\omega(\cdot;a)), (\cdot + a)\Phi_\omega(\cdot;a)\rangle_{L^2(\Gamma)} \end{bmatrix},$$

the zeros of which represent the orthogonality constraints in (5.3.3).

Let θ_0 be the argument of $\inf_{\theta\in\mathbb{R}} \|\Psi - e^{i\theta}\Phi(\cdot;a_0)\|_{H^1(\Gamma)}$ for a given $\Psi \in H^1_\Gamma$. The vector function $G(\theta,\omega,a;\Psi)$ is a C^ω map from $\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}\times H^1_\Gamma$ to $\mathbb{R}^3$ since the map $\mathbb{R}^+\times\mathbb{R} \ni (\omega,a) \mapsto \Phi_\omega(\cdot;a) \in L^2(\Gamma)$ is C^ω in both variables. Moreover, if $\Psi \in H^1_\Gamma$

satisfies (5.3.1), then

$$\|G(\theta_0, 1, a_0; \Psi)\|_{\mathbb{R}^3} \leq C\delta \tag{5.3.5}$$

for a δ-independent constant $C > 0$. Also we have

$$D_{(\theta,\omega,a)} G(\theta_0, 1, a_0; \Psi) = D + B,$$

where $D = \mathrm{diag}(d_1, d_1, d_2)$ with entries $d_1 \equiv D_1(\omega = 1)$ and $d_2 \equiv D_2(\omega = 1)$ given by (5.2.1) and (5.2.2), whereas B is a matrix satisfying the estimate $\|B\|_{\mathbb{M}_{3\times 3}} \leq C\delta$ for a δ-independent constant $C > 0$. Since $d_1, d_2 \neq 0$, the matrix D is invertible and there exists $\delta_0 > 0$ such that the Jacobian $D_{(\theta,\omega,a)} G(\theta_0, 1, a_0; \Psi)$ is invertible for every $\delta \in (0, \delta_0)$ with the bound

$$\|[D_{(\theta,\omega,a)} G(\theta_0, 1, a_0; \Psi)]^{-1}\|_{\mathbb{M}_{3\times 3}} \leq C \tag{5.3.6}$$

for a δ-independent constant $C > 0$. By the local inverse mapping theorem, for the given $\Psi \in H^1_\Gamma$ satisfying (5.3.1), the equation $G(\theta, \omega, a; \Psi) = 0$ has a unique solution $(\theta, \omega, a) \in \mathbb{R}^3$ in a neighborhood of the point $(\theta_0, 1, a_0)$. Since $G(\theta, \omega, a; \Psi)$ is C^ω with respect to its arguments, the solution $(\theta, \omega, a) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$ is C^ω with respect to $\Psi \in H^1_\Gamma$. The Taylor expansion of $G(\theta, \omega, a; \Psi) = 0$ around $(\theta_0, 1, a_0)$,

$$G(\theta_0, 1, a_0; \Psi) + D_{(\theta,\omega,a)} G(\theta_0, 1, a_0; \Psi)(\theta - \theta_0, \omega - 1, a - a_0)^T + \mathcal{O}(|\theta - \theta_0|^2 + |\omega - 1|^2 + |a - a_0|^2),$$

together with the bounds (5.3.5) and (5.3.6) implies the bound (5.3.4) for $|\omega - 1|$ and $|a - a_0|$. From the decomposition (5.3.2), and with use of the triangle inequality for (θ, ω, a) near $(\theta_0, 1, a_0)$, it follows that (U, W) are uniquely defined in H^1_Γ and satisfy the bound in (5.3.4). In addition, $(U, W) \in H^1_\Gamma$ are C^ω with respect to $\Psi \in H^1_\Gamma$. □

Let the initial datum $\Psi_0 \in H^1_\Gamma$ to the Cauchy problem associated with the NLS equation (5.1.1) be defined in the form:

$$\Psi_0(x) = \Phi(x; a_0) + U_0(x) + iW_0(x), \quad \|U_0 + iW_0\|_{H^1(\Gamma)} \leq \delta, \tag{5.3.7}$$

subject to the orthogonality conditions

$$\begin{cases} \langle W_0, \partial_\omega \Phi_\omega|_{\omega=1}(\cdot; a_0)\rangle_{L^2(\Gamma)} = 0, \\ \langle U_0, \Phi(\cdot; a_0)\rangle_{L^2(\Gamma)} = 0, \\ \langle U_0, (\cdot + a_0)\Phi(\cdot; a_0)\rangle_{L^2(\Gamma)} = 0. \end{cases} \tag{5.3.8}$$

Remark 5.13. *By Lemma 5.12, the orthogonal decomposition (5.3.7) with (5.3.8) implies that $\theta(0) = 0$, $\omega(0) = 1$, and $a(0) = a_0$ initially. Although this is not the most general case for the initial datum satisfying (5.1.8), this simplification is used to illustrate the proof of Theorem 5.3. A generalization for initial datum $\Psi_0 \in H^1_\Gamma$ with $\theta(0) \neq 0$, $\omega(0) \neq 1$, and $a(0) \neq a_0$ is straightforward.*

By the well-posedness theory [4, 42], the NLS equation (5.1.1) with the initial datum $\Psi_0 \in H^1_\Gamma$ generates the unique global solution $\Psi \in C(\mathbb{R}, H^1_\Gamma) \cap C^1(\mathbb{R}, H^{-1}_\Gamma)$. By continuous dependence of the solution on the initial datum and by Lemma 5.12, for every $\epsilon \in (0, \delta_0)$ with δ_0 in the bound (5.3.1) there exists $t_0 > 0$ such that the unique solution Ψ satisfies

$$\inf_{\theta\in\mathbb{R}} \|e^{-i\theta}\Psi(t,\cdot) - \Phi\|_{H^1(\Gamma)} \leq \epsilon, \quad t \in [0, t_0] \tag{5.3.9}$$

and can be uniquely decomposed in the form:

$$\Psi(t,x) = e^{i\theta(t)} \left[\Phi_{\omega(t)}(x; a(t)) + U(t,x) + iW(t,x)\right], \tag{5.3.10}$$

subject to the orthogonality conditions

$$\begin{cases} \langle W(t,\cdot), \partial_\omega \Phi_\omega|_{\omega=\omega(t)}(\cdot; a(t))\rangle_{L^2(\Gamma)} = 0, \\ \langle U(t,\cdot), \Phi_{\omega(t)}(\cdot; a(t))\rangle_{L^2(\Gamma)} = 0, \\ \langle U(t,\cdot), (\cdot + a(t))\Phi_\omega(\cdot; a(t))\rangle_{L^2(\Gamma)} = 0. \end{cases} \tag{5.3.11}$$

By the smoothness of the map in Lemma 5.12 and by the well-posedness of the time flow of the NLS equation (5.1.1), we have $U, W \in C([0,t_0], H^1_\Gamma) \cap C^1([0,t_0], H^{-1}_\Gamma)$ and $(\theta, \omega, a) \in C^1([0,t_0], \mathbb{R}\times\mathbb{R}^+\times\mathbb{R})$.

In order to prove Theorem 5.3, we control $\omega(t)$, $U(t,\cdot)$, and $W(t,\cdot)$ from energy estimates and $a(t)$ from modulation equations, whereas $\theta(t)$ plays no role in the bound (5.1.9). Note that the modulation of $a(t)$ captures the irreversible drift of the shifted states along the incoming edge towards the vertex of the balanced star graph. We would not see this drift without using the parameter $a(t)$ and we would not be able to control $\omega(t)$, $U(t,\cdot)$, and $W(t,\cdot)$ from energy estimates without the third constraint in (5.3.11) because of the zero eigenvalue of $L_+(\omega, a)$, see Lemma 5.6.

5.3.2 Monotonicity of $a(t)$

We use the orthogonal decomposition (5.3.10) with (5.3.11) in order to obtain the evolution system for the remainder terms (U, W) and for the modulation parameters (θ, ω, a). By analyzing the modulation equation for $a(t)$, we relate the rate of change of $a(t)$ and the value of the momentum functional $P(\Psi)$ given by (5.1.6).

Lemma 5.14. *Assume that the unique solution $\Psi \in C([0,t_0], H^1_\Gamma) \cap C^1([0,t_0], H^{-1}_\Gamma)$ represented by (5.3.10) and (5.3.11) satisfies*

$$|\omega(t) - 1| + \|U(t,\cdot) + iW(t,\cdot)\|_{H^1(\Gamma)} \leq \epsilon, \quad t \in [0, t_0] \tag{5.3.12}$$

with $\epsilon \in (0, \delta_0)$ and δ_0 defined in (5.3.1). The time evolution of the translation parameter $a(t)$ is given by

$$\dot{a}(t) = -\alpha_1^2 \omega^{-\frac{1}{2}} P(\Psi) \left[1 + \mathcal{O}(\|U + iW\|_{H^1(\Gamma)})\right] + \mathcal{O}(\|U + iW\|^2_{H^1(\Gamma)}), \tag{5.3.13}$$

where $P(\Psi)$ is given by (5.1.6).

Proof. By substituting (5.3.10) into the NLS equation (5.1.1) and by using the rotational and translation symmetries, we obtain the time evolution system for the remainder terms:

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} U \\ W \end{pmatrix} &= \begin{pmatrix} 0 & L_-(\omega,a) \\ -L_+(\omega,a) & 0 \end{pmatrix}\begin{pmatrix} U \\ W \end{pmatrix} + (\dot\theta-\omega)\begin{pmatrix} W \\ -(\Phi_\omega+U) \end{pmatrix} \\ & \quad -\dot\omega\begin{pmatrix} \partial_\omega\Phi_\omega \\ 0 \end{pmatrix} - \dot a\begin{pmatrix} \Phi_\omega' \\ 0 \end{pmatrix} + \begin{pmatrix} -R_U \\ R_W \end{pmatrix}, \end{aligned} \tag{5.3.14}$$

where $\Phi_\omega \equiv \Phi_\omega(x;a)$, the prime denotes derivative in x, the dot denotes derivative in t, and the residual terms are given by

$$\left\{ \begin{array}{l} R_U = 2\alpha^2\left(2\Phi_\omega U + U^2 + W^2\right)W, \\ R_W = 2\alpha^2\left[\Phi_\omega(3U^2+W^2) + (U^2+W^2)U\right]. \end{array} \right. \tag{5.3.15}$$

By using the orthogonality conditions (5.3.11), we obtain the modulation equations for parameters (θ,ω,a) from the system (5.3.14):

$$A\begin{bmatrix} \dot\theta-\omega \\ \dot\omega \\ \dot a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2\langle\Phi_\omega'(\cdot;a),W\rangle_{L^2(\Gamma)} \end{bmatrix} + \begin{bmatrix} \langle\Phi_\omega(\cdot;a),R_U\rangle_{L^2(\Gamma)} \\ \langle\partial_\omega\Phi_\omega,R_W\rangle_{L^2(\Gamma)} \\ -\langle(\cdot+a)\Phi_\omega(\cdot;a),R_W\rangle_{L^2(\Gamma)} \end{bmatrix}, \tag{5.3.16}$$

where the matrix A is given by

$$\begin{aligned} A &= \begin{bmatrix} 0 & D_1(\omega) & 0 \\ -D_1(\omega) & 0 & 0 \\ 0 & 0 & -D_2(\omega) \end{bmatrix} \\ & - \begin{bmatrix} -\langle\Phi_\omega(\cdot;a),W\rangle_{L^2(\Gamma)} & -\langle\partial_\omega\Phi_\omega(\cdot;a),U\rangle_{L^2(\Gamma)} & -\langle\Phi_\omega'(\cdot;a),U\rangle_{L^2(\Gamma)} \\ -\langle\partial_\omega\Phi_\omega(\cdot;a),U\rangle_{L^2(\Gamma)} & \langle\partial_\omega^2\Phi_\omega(\cdot;a),W\rangle_{L^2(\Gamma)} & \langle\partial_\omega\Phi_\omega'(\cdot;a),W\rangle_{L^2(\Gamma)} \\ \langle(\cdot+a)\Phi_\omega(\cdot;a),W\rangle_{L^2(\Gamma)} & \langle(\cdot+a)\partial_\omega\Phi_\omega(\cdot;a),U\rangle_{L^2(\Gamma)} & \langle(\cdot+a)\Phi_\omega(\cdot;a)',U\rangle_{L^2(\Gamma)} \end{bmatrix}, \end{aligned}$$

where $D_1(\omega)$ and $D_2(\omega)$ are given by (5.2.1) and (5.2.2). If $(U,W)=(0,0)$, the matrix A is invertible since

$$A_0 = \begin{bmatrix} 0 & D_1(\omega) & 0 \\ -D_1(\omega) & 0 & 0 \\ 0 & 0 & -D_2(\omega) \end{bmatrix}.$$

Therefore, under the assumption (5.3.12) with small $\epsilon>0$, we have

$$\|A^{-1}\|_{\mathbb{M}_{3\times 3}} \le C \tag{5.3.17}$$

for an ϵ-independent constant $C>0$. This bound implies that the time-evolution of the translation parameter $a(t)$ is given by

$$\dot a = \frac{2\langle\Phi_\omega'(\cdot;a),W\rangle_{L^2(\Gamma)}}{D_1(\omega)}\left[1+\mathcal{O}(\|U+iW\|_{H^1(\Gamma)})\right] + \mathcal{O}(\|U+iW\|^2_{H^1(\Gamma)}). \tag{5.3.18}$$

On the other hand, the momentum functional $P(\Psi)$ in (5.1.6) can be computed at the solution Ψ in the orthogonal decomposition (5.3.10) as follows

$$\begin{aligned} P(\Psi) &= \langle \Phi_\omega(\cdot;a), W' \rangle_{L^2(\Gamma)} - \langle \Phi'_\omega(\cdot;a), W \rangle_{L^2(\Gamma)} + \mathcal{O}(\|U+iW\|^2_{H^1(\Gamma)}) \\ &= -2\langle \Phi'_\omega(\cdot;a), W \rangle_{L^2(\Gamma)} + \mathcal{O}(\|U+iW\|^2_{H^1(\Gamma)}), \end{aligned} \tag{5.3.19}$$

where the integration by parts does not result in any contribution from the vertex at $x = 0$ thanks to the boundary conditions in (5.1.4) and the constraint (2.3.5). Combining (5.3.18) and (5.3.19) with the exact computation (5.2.2) yields expansion (5.3.13). □

Corollary 5.15. *In addition to (5.3.12), assume that Ψ_0 in (5.3.7) is chosen such that $P(\Psi_0) \geq C_0\delta$ with $C_0 > 0$. There exists ϵ_0 sufficiently small such that for every $\epsilon \in (0, \epsilon_0)$ there exists $\delta > 0$ such that the map $t \mapsto a(t)$ is strictly decreasing for $t \in [0, t_0]$.*

Proof. The map $t \mapsto P(\Psi)$ is monotonically increasing, as can be seen from the expression (5.1.7). Therefore, if the initial datum Ψ_0 in (5.3.7) satisfies $P(\Psi_0) \geq C_0\delta$, then

$$P(\Psi) \geq P(\Psi_0) \geq C_0\delta \quad \text{for all } \ t \in [0, t_0]. \tag{5.3.20}$$

It follows from (5.3.12), (5.3.13), (5.3.19), and (5.3.20) that there exist δ- and ϵ-independent constants $C_1, C_2 > 0$ such that

$$-\dot{a} \geq C_1\delta - C_2\epsilon^2.$$

If δ satisfies $\delta \geq C\epsilon^2$ for a given small $\epsilon > 0$ with an ϵ-independent constant $C > C_1^{-1}C_2$ then $-\dot{a} \geq (C_1C - C_2)\epsilon^2 > 0$ so that the map $t \mapsto a(t)$ is strictly decreasing for $t \in [0, t_0]$. □

5.3.3 Energy estimates

The coercivity bound (5.2.6) in Lemma 5.6 allows us to control the time evolution of $\omega(t)$, $U(t, \cdot)$, and $W(t, \cdot)$, as long as $a(t)$ is bounded away from zero. The following result provides this control from energy estimates.

Lemma 5.16. *Let Ψ be the unique solution to the NLS equation (5.1.1) given by (5.3.10)–(5.3.11) for $t \in [0, t_0]$ with some $t_0 > 0$ such that the initial data $\Psi(0, \cdot) = \Psi_0$ satisfies (5.3.7)–(5.3.8). Assume that $a(t) \geq \bar{a}$ for $t \in [0, t_0]$. For every $\bar{a} > 0$, there exists a δ-independent positive constant $K(\bar{a})$ such that*

$$|\omega(t) - 1|^2 + \|U(t, \cdot) + iW(t, \cdot)\|^2_{H^1(\Gamma)} \leq K(\bar{a})\delta^2, \quad t \in [0, t_0]. \tag{5.3.21}$$

Proof. Recall that the shifted state $\Phi_\omega(\cdot;a)$ is a critical point of the action functional $\Lambda_\omega(\Psi) = E(\Psi) + \omega Q(\Psi)$ in (2.4.1). By using the decomposition (5.3.10) and the rotational invariance of the NLS equation (5.1.1), we define the following energy function:

$$\Delta(t) := E(\Phi_{\omega(t)} + U(t,\cdot) + iW(t,\cdot)) - E(\Phi) + \omega(t)\left[Q(\Phi_{\omega(t)} + U(t,\cdot) + iW(t,\cdot)) - Q(\Phi)\right].$$

Expanding Δ into Taylor series, we obtain

$$\Delta = D(\omega) + \langle L_+(\omega,a)U,U\rangle_{L^2(\Gamma)} + \langle L_-(\omega,a)W,W\rangle_{L^2(\Gamma)} + N_\omega(U,W), \tag{5.3.22}$$

where $N_\omega(U,W) = \mathrm{O}(\|U+iW\|^3_{H^1(\Gamma)})$ and $D(\omega)$ is defined by

$$D(\omega) := E(\Phi_\omega) - E(\Phi) + \omega\left[Q(\Phi_\omega) - Q(\Phi)\right].$$

Since $D'(\omega) = Q(\Phi_\omega) - Q(\Phi)$ thanks to the variational problem for the standing wave Φ_ω, we have $D(1) = D'(1) = 0$, and

$$D(\omega) = (\omega-1)^2\langle \Phi, \partial_\omega \Phi_\omega|_{\omega=1}\rangle_{L^2(\Gamma)} + \mathcal{O}(|\omega-1|^3).$$

It follows from the initial decomposition (5.3.7)–(5.3.8) that

$$\Delta(0) = E(\Phi + U_0 + iW_0) - E(\Phi) + Q(\Phi + U_0 + iW_0) - Q(\Phi)$$

satisfies the bound

$$|\Delta(0)| \leq C_0\delta^2 \tag{5.3.23}$$

for a δ-independent constant $C_0 > 0$. On the other hand, the energy and mass conservation in (2.2.4) imply that

$$\Delta(t) = \Delta(0) + (\omega(t)-1)\left[Q(\Phi + U_0 + iW_0) - Q(\Phi)\right], \tag{5.3.24}$$

where the remainder term also satisfies

$$|Q(\Phi + U_0 + iW_0) - Q(\Phi)| \leq C_0\delta^2 \tag{5.3.25}$$

for a δ-independent constant $C_0 > 0$. The representation (5.3.24) together with the expression (5.3.22) allows us to control $\omega(t)$ near $\omega(0) = 1$ and the remainder terms (U,W) in H^1_Γ as follows:

$$\begin{aligned}\Delta(0) \;=\; & (\omega-1)^2\langle \Phi, \partial_\omega\Phi_\omega|_{\omega=1}\rangle_{L^2(\Omega)} - (\omega-1)\left[Q(\Phi+U_0+iW_0) - Q(\Phi)\right] \\ & + \langle L_+(\omega,a)U,U\rangle_{L^2(\Gamma)} + \langle L_-(\omega,a)W,W\rangle_{L^2(\Gamma)} + \mathcal{O}(|\omega-1|^3 + \|U+iW\|^3_{H^1(\Gamma)}).\end{aligned}$$

By using this expansion, the coercivity bound (5.2.6), and the bounds (5.3.23) and (5.3.25), we obtain

$$C_0\delta^2 \geq \frac{1}{2\alpha_1^2|\omega|^{\frac{1}{2}}}(\omega-1)^2 - C_0\delta^2|\omega-1| + C(\omega,a)\|U+iW\|^2_{H^1(\Gamma)} + \mathcal{O}(|\omega-1|^3 + \|U+iW\|^3_{H^1(\Gamma)}),$$

from which the bound (5.3.21) follows. □

Remark 5.17. *By Remark 5.7, for every $\omega > 0$, we have $C(\omega, a) \to 0$ as $a \to 0$. Therefore, we have $K(\bar{a}) \to \infty$ as $\bar{a} \to 0$.*

5.3.4 Monotonic drift towards the vertex

Here we give the proof of the irreversible drift of the shifted states towards the vertex of the star graph.

Proof of Theorem 5.3.

The initial datum satisfies the initial decomposition (5.3.7)–(5.3.8) with small δ and initial conditions $\theta(0) = 0$, $\omega(0) = 1$, and $a(0) = a_0$ with $a_0 > 0$. Thanks to the continuous dependence of the solution of the NLS equation (5.1.1) on initial datum, the solution is represented by the orthogonal decomposition (5.3.10)–(5.3.11) on a short time interval $[0, t_0]$ for some $t_0 > 0$. Hence, it satisfies the apriori bound (5.3.9). The modulation parameters $\theta(t)$, $\omega(t)$, and $a(t)$ are defined for $t \in [0, t_0]$ and $a(t) \geq \bar{a}$ for some $\bar{a} > 0$ for $t \in [0, t_0]$. By energy estimates in Lemma 5.16, the parameter $\omega(t)$ and the remainder terms $(U, W) \in H^1_\Gamma$ satisfy the bound (5.3.21) with a δ-independent positive constant $K(\bar{a})$. For given small $\epsilon > 0$ and $\mathfrak{a} > 0$ in Theorem 5.3, let us define

$$K_{\mathfrak{a}} := \max_{\bar{a} \in [\mathfrak{a}, a_0]} K(\bar{a}), \quad \delta := K_{\mathfrak{a}}^{-\frac{1}{2}} \epsilon. \tag{5.3.26}$$

Then, the bound (5.3.21) provides the bound (5.3.12) of Lemma 5.14 for all $t \in [0, t_0]$. Assume that the initial datum also satisfies $P(\Psi_0) \geq C_0 \delta$. By Corollary 5.15, the map $t \mapsto a$ is strictly decreasing for $t \in [0, t_0]$ if δ satisfies $\delta \geq C\epsilon^2$ for a δ and ϵ-independent constant $C > 0$. The definition of δ in (5.3.26) is compatible with the latter bound if $\epsilon \in (0, \epsilon_0)$ with

$$\epsilon_0 := \frac{1}{C\sqrt{K_{\mathfrak{a}}}}.$$

If in addition $\epsilon_0 \leq \delta_0$, where δ_0 is defined in Lemma 5.12, then the solution $\Psi(t, \cdot) \in H^1_\Gamma$ for $t \in [0, t_0]$ satisfies the conditions of Lemma 5.12 so that the orthogonal decomposition (5.3.10) with (5.3.11) is continued beyond the short time interval $[0, t_0]$ to the maximal time interval $[0, T]$ as long as $a(t) \geq \mathfrak{a}$ for $t \in [0, T]$. Thanks to the monotonicity argument in Lemma 5.14 and Corollary 5.15, for every $\epsilon \in (0, \epsilon_0)$, there exists a finite $T > 0$ such that $\lim_{t \to T} a(t) = \mathfrak{a}$. Note that $T = \mathcal{O}(\epsilon^{-2})$ as $\epsilon \to 0$. Theorem 5.3 is proved. ■

Remark 5.18. *It follows that $K_{\mathfrak{a}} \to \infty$ as $\mathfrak{a} \to 0$ by Remark 5.17 so that $\epsilon_0 \to 0$ as $\mathfrak{a} \to 0$. As a result, it does not follow from Theorem 5.3 that the half-soliton $\Phi(\cdot; a = 0)$ is attained from the drifted shifted state $\Phi(\cdot; a(t))$ in a finite time.*

Remark 5.19. *If the initial datum $\Psi_0 \in H^1_\Gamma$ in (5.3.7) preserves the symmetry constraints (4.4.6), then the map $t \mapsto P(\Psi)$ is constant so that $P(\Psi) = P(\Psi_0)$. The condition $P(\Psi_0) \geq C_0\delta$ with $C_0 > 0$ in Theorem 5.3 ensures that the shifted state $\Phi(\cdot; a(t))$ drifts towards the vertex at $x = 0$ even under the symmetry-preserving perturbations. This already implies the instability according to Definition 5.2.*

5.4 Instability of the half-soliton state

The proof of Theorem 5.4 follows the same steps as the proof of Theorem 3.6. However, since the orthogonal basis of the kernel of $L_+(\omega = 1)$ given by elements in (5.2.9)–(5.2.10) is not compatible with the orthogonal basis defined in Remark 3.10, the truncation of the system (3.4.25) admits a different unstable direction for the zero equilibrium point. That is, to establish the result of Theorem 5.4 it suffices to appropriately extend the result of Section 3.4.3.

5.4.1 Truncated Hamiltonian system, revised

Estimates (3.4.14) in Lemma 3.16 and (3.4.44) in Lemma 3.25, as well as the representation of (R_U, R_W) in (3.4.10)–(3.4.11) applied to the vectors (5.2.9)–(5.2.10) imply that the time-evolution system (3.4.25) is a perturbation of the following Hamiltonian system of degree $N - 1$:

$$\begin{cases} \langle W^{(j)}, U^{(j)} \rangle_{L^2(\Gamma)} \frac{d\gamma_j}{dt} = \frac{\partial H_0}{\partial \beta_j}, \\ \langle W^{(j)}, U^{(j)} \rangle_{L^2(\Gamma)} \frac{d\beta_j}{dt} = -\frac{\partial H_0}{\partial \gamma_j}, \end{cases} \tag{5.4.1}$$

where $H_0(\gamma, \beta)$ is the Hamiltonian given by

$$H_0(\gamma, \beta) = \frac{1}{2} \sum_{j=1}^{N-1} \langle W^{(j)}, U^{(j)} \rangle_{L^2(\Gamma)} \beta_j^2 - 2 \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \sum_{n=1}^{N-1} \langle \alpha^2 \Phi U^{(j)}, U^{(k)} U^{(n)} \rangle_{L^2(\Gamma)} \gamma_j \gamma_k \gamma_n.$$

Direct computation with the help of the representations (5.2.9) and (5.2.10) for $U^{(j)}$ imply that if $j \geq k > n$, then

$$\langle \alpha^2 \Phi U^{(j)}, U^{(k)} U^{(n)} \rangle_{L^2(\Gamma)} = \left(\frac{r_k}{\alpha_k} + \sum_{i=k+1}^{N} \frac{1}{\alpha_i^2} \right) \int_0^\infty \phi(\phi')^3 dx = 0$$

due to the explicit formula for r_k in (5.2.10). Therefore, one can rewrite the Hamiltonian $H_0(\gamma, \beta)$ in the explicit form:

$$H_0(\gamma, \beta) = \frac{1}{2} \sum_{j=1}^{N-1} M_j b_j^2 - 2 \sum_{j=2}^{N-1} R_j \gamma_j^3 - 6 \sum_{j=1}^{N-1} \sum_{k>j}^{N-1} P_k \gamma_j \gamma_k^2, \tag{5.4.2}$$

where

$$\begin{aligned} M_j &:= \langle W^{(j)}, U^{(j)} \rangle_{L^2(\Gamma)}, \\ R_j &:= \langle \alpha^2 \Phi U^{(j)}, U^{(j)} U^{(j)} \rangle_{L^2(\Gamma)}, \\ P_k &:= \langle \alpha^2 \Phi U^{(j)}, U^{(k)} U^{(k)} \rangle_{L^2(\Gamma)}. \end{aligned}$$

Note that the coefficient P_k is independent of j if $k > j$. Thanks to the construction of the eigenvectors in (5.2.9) and (5.2.10), the explicit expressions for coefficients R_j and P_k are given by

$$R_j = \alpha_j^4 \left(\sum_{i=j}^{N} \frac{1}{\alpha_i^2} \right) \left(\sum_{i=j+1}^{N} \frac{1}{\alpha_i^2} \right) \left(\frac{1}{\alpha_j^2} - \sum_{i=j+1}^{N} \frac{1}{\alpha_i^2} \right) \int_0^\infty \phi(\phi')^3 dx, \tag{5.4.3}$$

and

$$P_k = \alpha_k^2 \left(\sum_{i=k}^{N} \frac{1}{\alpha_i^2} \right) \left(\sum_{i=k+1}^{N} \frac{1}{\alpha_i^2} \right) \int_0^\infty \phi(\phi')^3 dx. \tag{5.4.4}$$

It follows from (5.2.21) and (5.4.4) that $M_j > 0$ and $P_k < 0$ since $\phi(\phi')^3 < 0$ on $\mathbb{R}^+$. Also it follows from (2.3.5) and (5.4.3) that $R_1 = 0$.

The following lemma states that the zero equilibrium point is nonlinearly unstable in the reduced system (5.4.1) with the Hamiltonian (5.4.2).

Lemma 5.20. *There exists $\epsilon > 0$ such that for every sufficiently small $\delta > 0$, there is an initial point $(\gamma(0), \beta(0)) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$ with $\|\gamma(0)\| + \|\beta(0)\| \leq \delta$ such that the unique solution of the reduced Hamiltonian system (5.4.1) with (5.4.2) satisfies for some $t_0 > 0$: $\|\gamma(t_0)\| = \epsilon$ and $\|\gamma(t)\| > \epsilon$ for $t > t_0$. Moreover, if $\epsilon > 0$ is small then $t_0 = \mathcal{O}(\epsilon^{-1/2})$, $\gamma(t) = \mathcal{O}(\epsilon)$, and $\beta(t) = \mathcal{O}(\epsilon^{3/2})$ for all $t \in [0, t_0]$.*

Proof. First, we claim that there exists an invariant subspace of solutions of the reduced Hamiltonian system (5.4.1) with (5.4.2) given by

$$S := \{\gamma_1 = C\gamma_2, \gamma_3 = \gamma_4 = \cdots = \gamma_{N-1} = 0\} \tag{5.4.5}$$

for some constant $C \neq 0$. Indeed, eliminating β_j, we close the reduced system (5.4.1) on γ_j for every $j = 1, \dots, N-1$:

$$M_j \frac{d^2 \gamma_j}{dt^2} = 6 R_j \gamma_j^2 + 12 \sum_{i=1}^{j-1} P_j \gamma_i \gamma_j + 6 \sum_{k=j+1}^{N-1} P_k \gamma_k^2. \tag{5.4.6}$$

It follows directly that $\gamma_3 = \gamma_4 = \cdots = \gamma_{N-1} = 0$ is an invariant solution of the last $(N-3)$ equations of system (5.4.6). Since $R_1 = 0$ from (2.3.5) and (5.4.3), the first two (remaining) equations of system (5.4.6) are given by

$$\begin{cases} M_1 \frac{d^2\gamma_1}{dt^2} = 6P_2\gamma_2^2, \\ M_2 \frac{d^2\gamma_2}{dt^2} = 6R_2\gamma_2^2 + 12P_2\gamma_1\gamma_2, \end{cases} \tag{5.4.7}$$

The system is invariant on the subspace S in (5.4.5) if the constant C is a solution of the following quadratic equation:

$$2M_1P_2C^2 + M_1R_2C - M_2P_2 = 0.$$

The quadratic equation admits two nonzero real solutions C if the discriminant is positive:

$$\mathcal{D} := M_1^2R_2^2 + 8M_1M_2P_2^2 > 0,$$

which is true thanks to the positivity of M_1 and M_2 in (5.2.21). The reduced system (5.4.6) on the invariant subspace (5.4.5) yields the following scalar second-order equation:

$$C^2M_1\frac{d^2\gamma_1}{dt^2} - 6P_2\gamma_1^2 = 0, \tag{5.4.8}$$

where $C \neq 0$, $M_1 > 0$ and $P_2 < 0$. The zero equilibrium $(\gamma_1, \dot{\gamma}_1) = (0, 0)$ is a cusp point so that it is unstable in the nonlinear equation (5.4.8).

Next, we prove the assertion of the lemma. For every sufficiently small $\delta > 0$, we choose the initial point $(\gamma(0), \beta(0)) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$ in the invariant subspace S in (5.4.5) satisfying $\|\gamma(0)\| + \|\beta(0)\| \leq \delta$. Since $(0, 0)$ is a cusp point in the reduced equation (5.4.8) there exists a $t_0 > 0$ such that $\|\gamma(t_0)\| = \epsilon$ and $\|\gamma(t)\| > \epsilon$ for $t > t_0$ for any fixed $\epsilon > 0$.

Let us consider a fixed sufficiently small value of $\epsilon > 0$. We have $\gamma(t) = \mathcal{O}(\epsilon)$ for $t \in [0, t_0]$ by the construction Setting $\dot{\gamma}_1 = \beta_1$, we assume that $\beta_1(t) = \mathcal{O}(\epsilon^{3/2})$ for $t \in [0, t_0]$ by the choice of initial condition. The evolution equation (5.4.8) implies that for every $t \in [0, t_0]$ there is an (ϵ, δ)-independent constant $A > 0$ such that

$$\begin{cases} |\gamma_1(t)| \leq \left|\int_0^t \beta_1(s)ds\right| + |\gamma_1(0)| \leq A\epsilon^{3/2}t_0 + \delta \\ |\beta_1(t)| \leq A\left|\int_0^t \gamma_1^2(s)ds\right| + |\beta_1(0)| \leq A\epsilon^2t_0 + \delta. \end{cases}$$

If $\delta \in (0, A\epsilon^{3/2})$, then $\gamma_1(t) = \mathcal{O}(\epsilon)$ and $\beta_1(t) = \mathcal{O}(\epsilon^{3/2})$ holds for $t \in [0, t_0]$ with $t_0 = \mathcal{O}(\epsilon^{-1/2})$. The assertion of the lemma is proven. □

As explained above, all other steps in the proof of Theorem 3.6 apply verbatim to the proof of Theorem 5.4.

Chapter 6

Orbital instability of stationary states in presence of δ interaction

Unlike the previous chapters, here we study the NLS equation on a star graph with a δ interaction at the vertex. The δ interaction appears as the strength parameter γ in the boundary conditions (1.1.7).

For every $\gamma \in \mathbb{R}$, the existence and explicit construction of stationary states to the NLS equation are known [3]. In particular, for $\gamma \neq 0$, the NLS equation admits the unique symmetric stationary state and all other states are asymmetric. By using the classical results in [21] and [37], it has been proven for $\gamma < 0$, that the unique symmetric stationary state is a constrained energy minimizer and is orbitally stable [4, 5]. The energy of each asymmetric state in case of $\gamma < 0$ is higher than the energy of the symmetric state inside the manifold with fixed mass constraint. As a result, the asymmetric states are saddle points of the constrained energy, and one can expect the instability for all asymmetric states. In [54] orbital instability of all asymmetric states has been conjectured, mentioning the difficulty in the explicit computations of the Morse indices of associated operators $L_\pm$.

In this chapter, we overcome such difficulty by using the extension of the Sturm theory to Schrödinger operators on star graphs as in Chapter 4, and give the exact count of the negative and zero eigenvalues of the operators $L_\pm$. We also consider the case $\gamma > 0$ and show the orbital instability of *all* stationary states (including a symmetric one).

6.1 Stationary states

In this chapter we return to the standard parametrization of a star graph Γ by the union of N half-lines connected at the origin. The presence of the δ-interaction at the vertex incorporates an additional term γ into the boundary conditions for the domain of the Laplacian Δ,

$$\mathcal{D}(\Delta) := \left\{ \Psi \in H^2(\Gamma) : \ \psi_1(0) = \cdots = \psi_N(0), \quad \sum_{j=1}^{N} \psi_j'(0) = \gamma \psi_1(0) \right\}, \qquad (6.1.1)$$

where we set $\alpha = 1$ compared to the boundary conditions in (2.1.2).

Remark 6.1. *In case of $\gamma = 0$, $\mathcal{D}(\Delta)$ in (6.1.1) is equivalent to (2.1.2) with $\alpha = 1$.*

The NLS equation (2.2.1) with $\alpha = 1$ is given by

$$i\frac{\partial \Psi}{\partial t} = -\Delta\Psi - (p+1)|\Psi|^{2p}\Psi, \quad t \in \mathbb{R}, \quad x \in \Gamma. \tag{6.1.2}$$

The stationary NLS equation (2.3.1) becomes equivalent to

$$-\Delta\Phi_\omega - (p+1)|\Phi_\omega|^{2p}\Phi_\omega = -\omega\Phi_\omega, \tag{6.1.3}$$

with $(\omega, \Phi_\omega) \in \mathbb{R}^+ \times \mathcal{D}(\Delta)$.

In case of $\gamma \neq 0$, there exists a symmetric stationary state $\Phi_{\omega,0}$ if the parameter ω exceeds $\frac{\gamma^2}{N^2}$ [4]. Additional asymmetric stationary states $\Phi_{\omega,K}$ appear when ω exceeds bifurcation values $\frac{\gamma^2}{(N-2K)^2}$, where $1 \leq K \leq \left[\frac{N-1}{2}\right]$. For sufficiently large positive values of ω, all $\left[\frac{N-1}{2}\right] + 1$ stationary states $\{\Phi_{\omega,0}, \Phi_{\omega,1}, \Phi_{\omega,[(N-1)/2]}\}$ are present. The explicit representation of the stationary states is given in the following lemma which initially was proven in [4].

Lemma 6.2. *Let $p > 0$, $\gamma \in \mathbb{R}\backslash\{0\}$ and $K = 0, ..., \left[\frac{N-1}{2}\right]$. Then, if the condition $\omega > \frac{\gamma^2}{(N-2K)^2}$ is satisfied, there exists a solution $\Phi_{\omega,K}$ to the stationary NLS equation (6.1.3) given, up to permutations of edges, by*

$$(\Phi_{\omega,K})_j(x) = \begin{cases} \phi_\omega(x + a_K), & j = 1, ..., K \\ \phi_\omega(x - a_K), & j = K+1, ..., N \end{cases} \tag{6.1.4}$$

where $\phi_\omega(x) = \omega^{1/2p}\,\mathrm{sech}^{1/p}(p\sqrt{\omega}x)$ and $a_K = \frac{1}{p\sqrt{\omega}}\,\mathrm{arctanh}\left(\frac{\gamma}{(N-2K)\sqrt{\omega}}\right)$.

Proof. The proof is similar in spirit to the proof of Lemma 2.9. A general solution to the stationary NLS equation (6.1.3) is given by $\Phi_\omega = (\phi_1, \phi_2, \ldots, \phi_N)^T$ with components

$$\phi_j(x) = \phi_\omega(x + a_j),$$

where $(a_1, a_2, \ldots, a_N) \in \mathbb{R}^N$ are arbitrary parameters. The continuity conditions in $\mathcal{D}(\Delta)$ yields $|a_1| = |a_2| = \cdots = |a_N|$, so that for every j, there exists $m_j \in \{0, 1\}$ such that $a_j = (-1)^{m_j}a$ for some $a \in \mathbb{R}$. The second boundary condition in $\mathcal{D}(\Delta)$ implies

$$\phi_\omega'(a)\sum_{j=1}^{N}(-1)^{m_j} = \gamma\phi_\omega(a). \tag{6.1.5}$$

Since $\phi'_\omega(a) = -\sqrt{\omega}\tanh(p\sqrt{\omega})\phi_\omega(a)$, the equation (6.1.5) is equivalent to

$$\tanh(p\sqrt{\omega}a)\sum_{j=1}^{N}(-1)^{m_j} = -\frac{\gamma}{\sqrt{\omega}}. \tag{6.1.6}$$

Setting

$$m_j = \begin{cases} 0 & \text{for} \quad 1 \le j \le K \\ 1 & \text{for} \quad K+1 \le j \le N \end{cases}$$

we obtain

$$\tanh(p\sqrt{\omega}a) = \frac{\gamma}{(N-2K)\sqrt{\omega}}.$$

Since the range of the tanh function is $(-1,1)$, we obtain that $a = \frac{1}{p\sqrt{\omega}}\operatorname{arctanh}\left(\frac{\gamma}{(N-2K)\sqrt{\omega}}\right)$ if $\omega > \frac{\gamma^2}{(N-2K)^2}$, and the lemma is proven. $\square$

According to the represention (6.1.4), the profile of $\Phi_{\omega,K}$ on each edge of the graph Γ is either *a bump* (nonmonotonic profile) or *a tail* (monotonic profile). The presence of such bumps or tails on the edges depends on the shift a_K. If $\gamma < 0$, the shift value a_K is negative. Therefore, the solution $\phi_\omega(x + a_K)$ on each edge $1, ..., K$ in (6.1.4) is nonmonotonic and represents a bump, whereas the solution $\phi_\omega(x - a_K)$ on each edge $K+1, ..., N$ is monotonic and represents a tail. Notice that, in this case, the number of bumps in the profile of $\Phi_{\omega,K}$ is equal to K, and since $K < \frac{N}{2}$, there are strictly more tails than bumps. As an example, if $N = 3$ then $K \in \{0, 1\}$ in Lemma 6.2, and there are only two possible stationary states, namely, $\Phi_{\omega,0}$ and $\Phi_{\omega,1}$, see Figure 6.1.

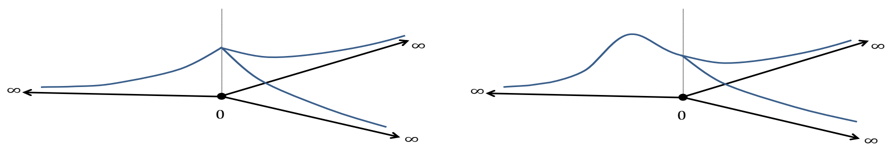

FIGURE 6.1: Case $\gamma < 0$ and $N = 3$: $\Phi_{\omega,0}$ has three tails and no bumps (left) and $\Phi_{\omega,1}$ has two tails and one bump (right).

If $\gamma > 0$, the shift value a_K is positive, and K represents the number of tails in the profile of $\Phi_{\omega,K}$. Here, in contrast to the case with negative γ, the number of tails is strictly less than the number of bumps. Figure 6.2 illustrates the only possible stationary states $\Phi_{\omega,0}$ and $\Phi_{\omega,1}$ when $N = 3$.

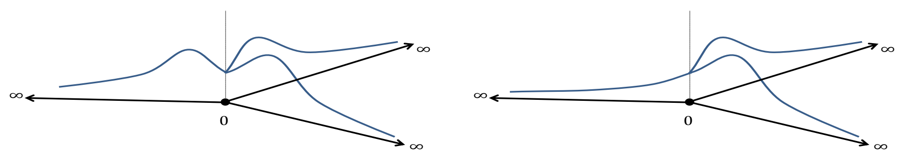

FIGURE 6.2: Case $\gamma > 0$ and $N = 3$: $\Phi_{\omega,0}$ has no tails and three bumps (left) and $\Phi_{\omega,1}$ has one tails and two bumps (right)

6.2 Main results

Let $\Phi_{\omega,K}$ be the stationary state in Lemma 6.2. Recall that the time evolution of the perturbation terms of the stationary state $\Phi_{\omega,K}$ is given by (4.1.1), which is

$$\frac{d}{dt}\begin{bmatrix} U \\ W \end{bmatrix} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}\begin{bmatrix} U \\ W \end{bmatrix},$$

where $L_\pm$ are Hessian operators with domain $\mathcal{D}(\Delta)$ and the differential expression given by

$$L_+ \;:=\; -\Delta + \omega - (2p+1)(p+1)\Phi_{\omega,K}^{2p} \tag{6.2.1}$$
$$L_- \;:=\; -\Delta + \omega - (p+1)\Phi_{\omega,K}^{2p}, \tag{6.2.2}$$

Note that the operators L_+ and L_- are self-adjoint in $L^2(\Gamma)$ [15], and the continuous spectrum is given by $\sigma_c(L_\pm) = [\omega, \infty)$. In what follows, we are interested in the discrete spectrum of these operators in $(-\infty, \omega)$.

The main result is the count of the Morse and degeneracy indices of $L_\pm$ associated with the stationary state $\Phi_{\omega,K}$:

Theorem 6.3. *Let $p > 0$, $\gamma \in \mathbb{R}\backslash\{0\}$, $K = 0, ..., \left[\frac{N-1}{2}\right]$ and $\omega > \frac{\gamma^2}{(N-2K)^2}$. Let L_+ and L_- be the Hessian operators associated with $\Phi_{\omega,K}$ and defined by (6.2.1)-(6.2.2). Then, $\sigma_p(L_-) \geq 0$ and 0 is a simple eigenvalue, whereas the nonpositive part of $\sigma_p(L_+)$ consists of $K+1$ negative eigenvalues (counting multiplicities) for $\gamma < 0$ and $N-K$ negative eigenvalues (counting multiplicities) for $\gamma > 0$. More precisely,*
- *if $\gamma < 0$ and $K = 0$, $n(L_+)$ consists of a simple eigenvalues λ_1;*
- *if $\gamma < 0$ and $K \geq 1$, $n(L_+)$ consists of two simple eigenvalues $\lambda_1 < \lambda_2$ and another eigenvalue $\lambda_* \in (\lambda_1, \lambda_2)$ of multiplicity $K-1$;*
- *if $\gamma > 0$ and $K \geq 0$, $n(L_+)$ consists a simple eigenvalue λ_1 and another eigenvalue $\lambda_* \in (\lambda_1, 0)$ of multiplicity $N-K-1$.*

By using the well-known instability results for the NLS equation [36, 70], Theorem 6.3 implies the following:

Corollary 6.4. *If $\gamma < 0$, then every stationary state $\Phi_{\omega,K}$ with $K \geq 1$ is spectrally and orbitally unstable, in sense of Definitions 3.1 and 4.1. If $\gamma > 0$, then every stationary state $\Phi_{\omega,K}$ with $K \geq 0$ is spectrally and orbitally unstable.*

Remark 6.5. *For $\gamma < 0$ and $K = 0$, our count of Morse and degeneracy indices coincides with the count in Proposition 6.1 in [4]. In this case, the stationary state $\Phi_{\omega,0}$ is orbitally stable for $p \in (0, 2)$ [5].*

Remark 6.6. *Partial results of Theorem 6.4 have been obtained in the recent work [59] by using the extension theory of symmetric operators. In particular, for the case $\gamma < 0$ with $p > 2$, authors showed the existence of some ω_K^* such that the instability result holds* *for all* $\omega \in \left(\frac{\gamma^2}{(N-2K)^2}, \omega_K^*\right)$. *It has been noted by authors that no results were obtained for* $\omega > \omega_K^*$. *Corollary 6.4 extends these results to all* $\omega \in \left(\frac{\gamma^2}{(N-2K)^2}, \infty\right)$ *and for all* $p > 0$.

6.3 The count of the Morse and degeneracy indices

In this section we prove Theorem 6.3 by using the extension of the Sturm theory to star graphs developed in Chapter 4.

Recall that using the scaling (2.3.2), we can transform the unique solutions v given in Lemma 2.19 to the ω-dependent family of solutions v_ω which satisfy

$$\lim_{x \to +\infty} v_\omega(x) e^{\sqrt{\omega - \lambda} x} = 1,$$

and solve the ω-dependent form of the second-order differential equation (2.4.13) given by

$$-v_\omega''(x) + \omega v_\omega(x) - (2p+1)(p+1)\omega \operatorname{sech}^2(p\sqrt{\omega} x) v_\omega(x) = \lambda v_\omega(x), \tag{6.3.1}$$

with $\lambda < \omega$. For simplicity, we denote the ω-dependent solution v_ω as v.

Next two lemmas provide us with the useful tools to compute the Morse and degeneracy indices of the operator L_+ given in (6.2.1).

Lemma 6.7. *Let* $\gamma \neq 0$, $K \geq 0$ *and* v *be the* ω*-dependent solution to (6.3.1) given by Lemma 2.19, and* L_+ *be the Hessian operator (2.4.3) associated with* $\Phi_{\omega,K}$. *Then,* $\lambda \in (-\infty, \omega)$ *is an eigenvalue of* $\sigma_p(L_+)$ *if and only if at least one of the following conditions holds:*

(a) $v(a_K) = 0$ *with* $K \geq 1$,

(b) $v(-a_K) = 0$,

(c) $Kv'(a_K)v(-a_K) + (N-K)v(a_K)v'(-a_K) - \gamma v(a_K)v(-a_K) = 0$.

Moreover, $\lambda \in \sigma_p(L_+)$ *has mutliplicity* $K-1$ *in the case (a),* $N-K-1$ *in the case (b), and is simple in the case (c). If* λ *satisfies several cases, then its multiplicity is the sum of the multiplicities of each case.*

Proof. First, assume $K \geq 1$. Denote $a := a_K$. Let λ be the eigenvalue of L_+ with the eigenvector $U = (u_1, ..., u_N)^T \in \mathcal{D}(\Delta)$. Then, in the eigenvalue problem $L_+U = \lambda U$, each component can be written as the second order differential equation

$$-u_j''(x) + \omega u_j(x) - (2p+1)(p+1)\omega \operatorname{sech}^2\left(p\sqrt{\omega}(x + (-1)^{m_j} a)\right) u_j(x) = \lambda u_j(x), \tag{6.3.2}$$

where $x \in (0, \infty)$ and

$$m_j = \begin{cases} 0 & \text{for} \quad j = 1, ..., K, \\ 1 & \text{for} \quad j = K+1, ..., N. \end{cases}$$

The substitution $u_j(x) = c_j v(x + (-1)^{m_j} a)$ with coefficient c_j transforms (6.3.2) into (2.4.13). By Sobolev embedding of $H^2(\mathbb{R}^+)$ into $C^1(\mathbb{R}^+)$, $u_j(x) \to 0$ and $u_j'(x) \to 0$ as $x \to +\infty$ for each $j = 1, ..., N$. To satisfy the latter condition, we need

$$u_j(x) = \begin{cases} c_j v(x+a), & j = 1, \dots, K, \\ c_j v(x-a), & j = K+1, \dots, N. \end{cases} \tag{6.3.3}$$

The boundary conditions for $U \in \mathcal{D}(\Delta)$ in (6.1.1) imply the homogeneous linear system on the coefficients

$$c_1 v(a) = \cdots = c_K v(a) = c_{K+1} v(-a) = \cdots = c_N v(-a), \tag{6.3.4}$$

and

$$\sum_{j=1}^{K} c_j v'(a) + \sum_{j=K+1}^{N} c_j v'(-a) = \gamma c_N v(-a). \tag{6.3.5}$$

The associated matrix is

$$M = \begin{pmatrix} v(a) & -v(a) & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ v(a) & 0 & -v(a) & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ v(a) & 0 & 0 & \dots & -v(a) & 0 & \dots & 0 & 0 \\ v(a) & 0 & 0 & \dots & 0 & -v(-a) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ v(a) & 0 & 0 & \dots & 0 & 0 & \dots & 0 & v(-a) \\ v'(a) & v'(a) & v'(a) & \dots & v'(a) & v'(-a) & \dots & v'(-a) & v'(-a) - \gamma v(-a) \end{pmatrix}$$

Doing elementary column operations, we can obtain a lower triangular matrix with the determinant

$$\det M = v(a)^{K-1} v(-a)^{N-K-1} \left[K v'(a) v(-a) + (N-K) v'(-a) v(a) - \gamma v(a) v(-a) \right].$$

Therefore, $U \neq 0$ is the eigenvector of L_+ for the eigenvalue $\lambda \in (-\infty, \omega)$ if and only if $\det M = 0$, or equivalently, one of the conditions (a), (b), (c) is true. The multiplicity of λ in cases (a)-(c) comes directly from the linear system (6.3.4)–(6.3.5).

For $K = 0$, the boundary conditions (6.3.4)-(6.3.5) do not contain terms $v(a)$ and $v'(a)$, and the determinant of the associated matrix M becomes

$$\det M = v(-a)^{N-1} \left[N v'(-a) - \gamma v(-a) \right].$$

Then, det$M = 0$ if and only if one of the conditions (b), (c) is true. □

Lemma 6.8. *Let $\gamma \neq 0$, $K \geq 0$ and v be the ω-dependent solution to (6.3.1) given by Lemma 2.19. Consider the function of λ as*

$$F(\lambda) := K\frac{v'(a_K;\lambda)}{v(a_K;\lambda)} + (N-K)\frac{v'(-a_K;\lambda)}{v(-a_K;\lambda)} : (-\infty, 0] \to \mathbb{R}. \tag{6.3.6}$$

Then, the following hold:
- $v(|a_K|;\lambda) > 0$ *for all* $\lambda \in (-\infty, 0]$;
- *there exists unique* $\lambda_* \in (-\infty, 0)$ *such that* $v(-|a_K|;\lambda) = 0$;
- *if* $\gamma < 0$ *with* $K = 0$, *then* $F(\lambda) = \gamma$ *has the unique root* $\lambda_1 \in (-\infty, 0]$;
- *if* $\gamma < 0$ *with* $K \geq 1$, *then* $F(\lambda) = \gamma$ *has exactly two solutions* $\lambda_1 < \lambda_2$ *on* $(-\infty, 0]$. *Moreover,* $\lambda_1 < \lambda_* < \lambda_2 < 0$;
- *if* $\gamma > 0$ *with* $K \geq 0$ *then* $F(\lambda) = \gamma$ *has the unique root* $\lambda_1 \in (-\infty, 0]$, *and* $\lambda_1 < \lambda_*$.

Proof. Since v is the nonzero solution of the second order differential equation (6.3.2), it has only simple zeros which, according to Lemma 2.22, are monotonically increasing functions of λ. At $\lambda = \lambda_0$, by Lemma 2.20, we have positive even $v(x;\lambda_0)$ exponentially decaying as $|x| \to \infty$. Therefore, $v(x;\lambda)$ has the only zero $x_0(\lambda)$ which bifurcates from $x = -\infty$ at $\lambda = \lambda_0$ and moves strictly monotonically towards $x = 0$ as $\lambda \to 0$ with $x_0(0) = 0$, see Figure 2.4. As a result, for $\lambda \leq \lambda_0$, $v(x;\lambda)$ is positive on the entire real line, whereas, for $\lambda \in (\lambda_0, 0]$, $v(x;\lambda)$ is positive for every $x \in (x_0(\lambda), \infty)$ and $v(x_0(\lambda);\lambda) = 0$. We denote λ satisfying $x_0(\lambda) = -|a_K|$ as λ_*. Since $a_K \neq 0$, then $\lambda_* < 0$. The uniqueness of λ_* is guaranteed by the monotonicity of $x_0(\lambda)$. This proves the first two assertions of this Lemma.

By Lemma 2.19, $v(\pm a_K;\lambda)$ is a C^1 function of λ for $\lambda \leq 0$. Therefore, using the first two assertions of this lemma, for $\gamma < 0$ with $K \geq 1$ and for $\gamma > 0$ with $K \geq 0$, $F(\lambda)$ is $C^1((-\infty, 0])\backslash\{\lambda_*\})$ and has a simple pole at $\lambda = \lambda_*$.

To investigate the behaviour of the function F, we first show that F is a monotonically increasing function. Differentiating the equation (6.3.1) in λ, multiplying it by v and integrating by parts on $[c, \infty]$ for some $c \in \mathbb{R}$, we get

$$P(c) := \frac{\partial_\lambda v'(c)v(c) - v'(c)\partial_\lambda v(c)}{v^2(c)} = \frac{1}{v^2(c)}\int_c^\infty v^2(x)dx > 0 \quad \text{if} \quad v(c) \neq 0.$$

Therefore, $F'(\lambda) = KP(a_K) + (N-K)P(-a_K) > 0$ for all $\lambda \in (-\infty, 0]\backslash\{\lambda_*\}$.

By Lemma 2.19, for every $c \in \mathbb{R}$, $\lim_{\lambda\to-\infty}\frac{v'(c;\lambda)}{v(c;\lambda)} = -\infty$. Then, taking $c = \pm a_K$, we have that $\lim_{\lambda\to-\infty} F(\lambda) = -\infty$. By Lemma 2.22 on the monotonicity of a simple zero of v, in case of $\gamma < 0$ with $K \geq 1$ and $\gamma > 0$ with $K \geq 0$, the behaviour of $F(\lambda)$ around the simple pole λ_* is given by

$$\lim_{\lambda\to\lambda_*^-} F(\lambda) = +\infty \quad \text{and} \quad \lim_{\lambda\to\lambda_*^+} F(\lambda) = -\infty.$$

At $\lambda = 0$, the unique solution $v = v(x; 0)$ of (6.3.1) in Lemma 2.19 is known to be $v(x) = -C\phi'_\omega(x)$, where $\phi_\omega =$ is given by Lemma 6.2 and $C = 2^{-1/p}\omega^{-(1+p)/2p}$. Then, using the explicit formulations of v and a_K, direct computations give

$$F(0) = \frac{p(\gamma^2 - (N-2K)^2\omega)}{\gamma} + \gamma.$$

Since $\omega > \frac{\gamma^2}{(N-2K)^2}$, then $p(\gamma^2 - (N-2K)^2\omega) < 0$. Hence, for $\gamma < 0$ we have $F(0) > \gamma$, whereas for $\gamma > 0$ we have $F(0) < \gamma$. As a result, in case of $\gamma < 0$ with $K \geq 1$ and $\gamma > 0$ with $K \geq 0$, the equation $F(\lambda) = \gamma$ has a unique root $\lambda_1 \in (-\infty, \lambda_*)$. Moreover, in case of $\gamma < 0$ with $K \geq 1$, there is an additional root λ_2 which is unique in $(\lambda_*, 0)$, see Figure 6.3.

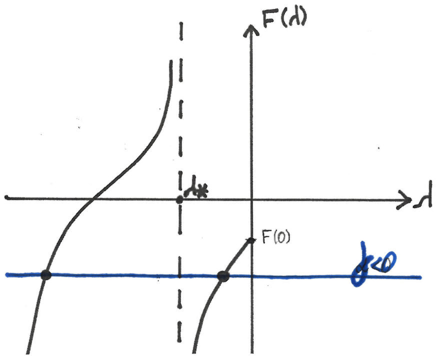

Figure 6.3: The graph of the function F in (6.3.6) for $\lambda < 0$ and $K \geq 1$. The eigenvalue λ_* is the singularity of F. The blue line is the $\gamma < 0$ level. We can see that $F(\lambda) = \gamma$ has exacty two roots on $(-\infty, 0]$ in agreement with the statement of Lemma 6.8.

In case of $\gamma < 0$ with $K = 0$, the first term in (6.3.6) vanishes, and $F(\lambda)$ is $C^1(-\infty, 0])$ since $v(-a_K; \lambda) > 0$ for all $\lambda \in (-\infty, 0]$. In what follows, $F(\lambda)$ is $C^1(-\infty, 0])$ and monotonically increasing with $\lim_{\lambda \to -\infty} F(\lambda) = -\infty$ and $F(0) > \gamma$. Therefore, $F(\lambda) = \gamma$ has the unique root $\lambda_1 \in (-\infty, 0]$. $\square$

Proof of Theorem 6.3. The count of the Morse and degeneracy indices of the operator L_- in (6.2.2) is based on Lemma 2.18. Hence, below we only provide the count for the operator L_+ given in (6.2.1).

Let $\hat{\lambda} \in \sigma_p(L_+) \cap (-\infty, 0]$ be an eigenvalue of $\sigma_p(L_+)$ with the eigenvector $U \in \mathcal{D}(\Delta)$. Then, by Lemma 6.7, one of the conditions *(a), (b), (c)* must be satisfied by $v(x; \hat{\lambda})$.

For the case $\gamma < 0$ and $K = 0$, both parts (a) and (b) of Lemma 6.7 are never true, and the part (c) has a unique root $\lambda_1 \in (-\infty, 0]$ by Lemma 6.8. Thus, $z(L_+) = 0$ and $n(L_+) = 1$.

Next, we consider $\gamma < 0$ and $K \geq 1$ or $\gamma > 0$ and $K \geq 0$. Recall that $a_K < 0$ for negative γ, and $a_K > 0$ for positive γ. Then, for $\gamma < 0$ and $K \geq 1$, by Lemma 6.8, the part (a) of Lemma 6.7 is satisfied for unique $\lambda_* \in (-\infty, 0]$ and the part (b) is never true. For $\gamma > 0$ and $K \geq 0$, the part (a) is never true and the part (b) is satisfied for unique $\lambda_* \in (-\infty, 0]$.

It remains to consider the part (c) of Lemma 6.7, namely, to find all values $\hat{\lambda} \in (-\infty, 0]$ such that $v(x) = v(x; \hat{\lambda})$ will satisfy

$$Kv'(a_K)v(-a_K) + (N - K)v'(-a_K)v(a_K) - \gamma v(a_K)v(-a_K) = 0 \qquad (6.3.7)$$

Since $v(|a_K|) \neq 0$, and $v'(-|a_K|) \neq 0$ if $v(-|a_K|) = 0$, the eigenvalue $\lambda = \lambda_*$ is not a solution of (6.3.7). Therefore, all solutions $\hat{\lambda}$ of (6.3.7) coincide with all solutions of $F(\lambda) = \gamma$, where F is given by (6.3.6). The last two assertions of Lemma 6.8 complete the proof of Theorem 6.3. ∎

Chapter 7

Open problems and future directions

In this thesis we considered the NLS equation with a power-type nonlinearity on the star graphs. In particular, we verified the existence of stationary states for different boundary conditions at the vertex, and investigated their stability. The tools used in the stability analysis depended on the structure of the stationary states, in particular, on the spectrum of the linearized operators associated with the states and the presence of symmetries in the NLS equation. The nature of instabilities in Chapters 3, 4, 5 and 6 is different. In Chapter 3, the constrained linear operator associated with the half-soliton state is nonnegative, however it has a high degeneracy index which produces certain perturbations slowly growing in time. The instability in Chapters 4 and 6 was due to the presence of the negative spectrum whereas in Chapter 5, the constrained linear operator was strictly positive, and the instability appeared due to the lack of translational symmetry.

Below we will list the set of open problems which naturally arise from the results of the thesis.

- The stationary states for the NLS equation on the star graphs has been discussed in Chapter 2, and it was shown that they have a soliton structure on half-lines with the appropriate translational parameter. The interesting question which arise is the existence and structure of stationary states of the NLS equation with a power-type nonlinearity on more complicated graph models with both bounded and unbounded edges. In such case, if half-lines are attached to different vertices, one can expect the soliton profiles on half-lines to depend on different translational parameters which complicates the structure of the stationary states. Moreover, due to the Kirchhoff conditions at the vertex, we also expect that the stationary states admit bifurcations. To obtain more information on possible bifurcations, one need to analyze the spectrum of a linearized Schrödinger operator as L_+.
- The count of the Morse index of the operators $L_\pm$ in Chapters 4 and 6 has been done via the extension of the Sturm theory to the star graphs. However, similar extension might fail for more complicated graphs. One of the ways to overcome such issue is to develop relatively new method for the count of discrete eigenvalues.

The "surgery principles", described initially in [13], allow to modify graphs with finite edges by cutting edges, gluing vertices etc. and observe the change of spectral properties of Laplacian operator under these modifications. Similar results can be obtained for graphs with both finite and infinite edges, and one can further extend these result to linear Schrödinger operators with a potential. The latter result proven for potentials exponentially decaying at infinity will have a big impact to the study of spectral stability of solutions to nonlinear PDEs on graphs. One of the tools to count the Morse index is to use the Courant nodal theorem on graphs, [10, 12, 14] and references therein, which states that the Morse index of the operator depends on the number of internal zeros on the graph of an eigenfunction corresponding to the first nonnegative eigenvalue.

In many applications, the stationary solution Φ to nonlinear PDEs on graphs is known explicitly, and the derivative of the solution, Φ', satisfies the spectral equation for the operator L_+ with 0 (zero) eigenvalue, but often *fails* to satisfy boundary conditions at the vertices. Therefore, Φ' is not an eigenfunction for the spectral problem, and the Courant nodal theorem is not applicable. However, one can extend "surgery principles" to reduce the L_+-associated spectral problem to be defined on simpler graph models. The latter step makes the spectrum of the operators flexible, and playing with the boundary conditions which is allowed by the "surgery principles", we manually create the zero eigenvalue with the explicit eigenfunction. Based on such explicit formulation, Courant nodal theorem gives the approximate count of the Morse index, which might be enough to state spectral stability or instability of the stationary solution.

- The operator L_- in (2.4.4) was nonnegative by Lemma 2.18, and so the spectral stability of stationary states in Chapters 4 and 6 depended on the Morse index of the operator L_+ given by (2.4.3). In particular, the stationary state was spectrally unstable if the number of negative eigenvalues of L_+ exceeded one.

  In general, such approach might be inconclusive for stationary states of the NLS equation on more complicated graphs. As an example, in [55] for certain stationary states of the NLS equation on the tadpole graph, the Morse indices of the operator L_- and the constrained operator L_+ are equal, and so the classical results in [36, 37] are not applicable. In what follows, one can try to extend the spectral stability analysis to such examples, and adopt it to graph models.

- The asymptotic stability is a stronger type of stability compared to spectral or orbital stabilities. The difficulties in asymptotic analysis of standing waves were discussed in [54], and it has been mentioned that the standard analytical tools, e.g. Strichartz estimates, work only for certain nonlinearities. As a result, the asymptotic stability of stationary states for the NLS equation on many simple metric graphs is still an open problem.

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