ON THE KP-II LIMIT OF TWO-DIMENSIONAL FPU LATTICES

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A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS AND STATISTICS
AND THE SCHOOL OF GRADUATE STUDIES
OF MCMASTER UNIVERSITY
IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Doctor of Philosophy (2021)
(Mathematics and Statistics)

McMaster University
Hamilton, Ontario, Canada

TITLE:	On the KP-II limit of two-dimensional FPU lattices
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NUMBER OF PAGES:	viii, 113

Abstract

We study a two-dimensional Fermi-Pasta-Ulam lattice in the long-amplitude, small-wavelength limit. The one-dimensional lattice has been thoroughly studied in this limit, where it has been established that the dynamics of the lattice is well-approximated by the Korteweg–De Vries (KdV) equation for timescales of the order ε^{-3}. Further it has been shown that solitary wave solutions of the FPU lattice in the one dimensional case are well approximated by solitary wave solutions of the KdV equation. A two-dimensional analogue of the KdV equation, the Kadomtsev–Petviashvili (KP-II) equation, is known to be a good approximation of certain two-dimensional FPU lattices for similar timescales, although no proof exists. In this thesis we present a rigorous justification that the KP-II equation is the long-amplitude, small-wavelength limit of a two-dimensional FPU model we introduce, analogous to the one-dimensional FPU system with quadratic nonlinearity. We also prove that the cubic KP-II equation is the limit of a model analogous to a one-dimensional FPU system with cubic nonlinearity. Further we study whether stability of line solitons in the KP-II equation extends to stability of one-dimensional FPU solitary waves in the two-dimensional lattices.

To my parents, Vania and Hristov, and my aunt, Kana.

Acknowledgements

I would like to thank to my supervisors Dr. Dmitry Pelinovsky and the late Dr. Walter Craig. Dr. Pelinovsky advise and expertise was instrumental in getting this thesis to its present form. While this thesis is not based on work done with Dr. Craig, I likely would not have sought a PhD were it not for his mentorship.

I would like to express my deepest gratitude to my committee, Dr. Stan Alama and Dr. Nicholas Kevlahan.

I would like to thank the graduate students and post doctoral fellows of the Mathematics and Statistics department at McMaster University. Particular gratitude goes to my friends and office mates over the years, Tyler Meadows, Ramsha Khan, Pritpal Matharu, Alexandr Chernyavsky, Adilbek Kairzhan, Szymon Sobieszek, and Lee van Brussel. I also wish to express my gratitude to Dr. Livia Corsi, who helped me through the initial difficulties of graduate work.

Finally I'd like to thank the medical staff at St. Joseph's Hospital, and Dr. Andrea Copeland. After getting attacked by two dogs they cleaned me up, and Dr. Copeland repaired any internal damage that the dogs had done. Without them I may not have had full function in my left hand, and so they're credited with the final edits to this thesis.

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Chapter 1

Introduction

1.1 Fermi-Pasta-Ulam system

A Fermi-Pasta-Ulam (FPU) system is comprised of a number of particles connected to their nearest neighbours by identical nonlinear springs. The system was used by the scientists Enrico Fermi, John Pasta, and Stanislaw Ulam in a series of numerical experiments to understand thermalization of gasses. Their numerical experiments and results will be briefly discussed in the next section.

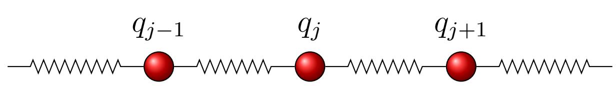


Figure 1.1: A one-dimensional mass-spring system

For a one-dimensional FPU system, we can label the position of the j^{th} particle at time t by $q_j(t)$, where we may take j either in $\mathbb{Z}/(N\mathbb{Z})$ or $\mathbb{Z}$, depending on whether we are looking at a system with finitely many particles in a circle or infinitely many particles on a line. We let $p_j(t) = \dot{q}_j$ be the corresponding momentum of the j^{th} particle. Suppose that the potential energy of each spring is given by some function $V(u)$, where u is the relative displacement between two adjacent particles. The total energy of the FPU system is given by

$$H = \sum_j \frac{1}{2}p_j^2 + V(q_{j+1} - q_j). \tag{1.1}$$

Equations of motion are generated by Hamilton's principle with (p_j, q_j) being the canonical coordinates:

$$\begin{cases} \dot{q}_j &= p_j, \\ \dot{p}_j &= V'(q_{j+1} - q_j) - V'(q_j - q_{j-1}). \end{cases} \tag{1.2}$$

We can take a look first at the linear case where $V(u) = \frac{c_s^2}{2}u^2$, where c_s has the meaning of the speed of sound. Equations of motion reduce in this case to the second-order equation:

$$\ddot{q}_j = c_s^2 \left(q_{j+1} - 2q_j + q_{j-1}\right). \tag{1.3}$$

The second-order equation (1.3) is a discrete wave equation. The continuum limit of this equation is the wave equation. The discrete wave equation (1.3) is solvable using the discrete Fourier transform. Using Dirichlet or periodic boundary conditions, the one-dimensional linear FPU system will experience periodic or quasi-periodic behaviour for any initial condition, and any number of particles.

1.2 Numerical experiments

The Fermi-Pasta-Ulam problem began with a numerical experiment performed by the three scientists, Enrico Fermi, John Pasta, Stanislaw Ulam, at Los Alamos on the MANIAC computer in 1953, using computer code written by Mary Tsingou [13]. The idea was to study thermalization in simple models of gas dynamics. The particular model used in these experiments is a chain of particles interacting with nearest neighbour interactions, as set up in the previous section, using Dirichlet boundary conditions. As discussed before if the system is linear, like the system (1.3), the behaviour of the system is periodic or quasiperiodic for all time, and thermalization does not occur.

The authors instead looked at the behaviour of the system (1.1) once a small nonlinear interaction was added to $V(u)$. They believed that the long time behaviour of such a system should be ergodic and mixing. In particular three cases were studied, all using Dirichlet boundary conditions; the quadratic nonlinearity given by the system

$$\ddot{q}_j = c_s^2 \left(q_{j+1} - 2q_j + q_{j-1}\right) + \alpha \left[(q_{j+1} - q_j)^2 - (q_j - q_{j-1})^2\right], \tag{1.4}$$

the cubic nonlinearity,

$$\ddot{q}_j = c_s^2 \left(q_{j+1} - 2q_j + q_{j-1}\right) + \beta \left[(q_{j+1} - q_j)^3 - (q_j - q_{j-1})^3\right], \tag{1.5}$$

and a piecewise linear system.

The goal of the numerical experiments was to find evidence of ergodic behaviour in a simple nonlinear model of a one-dimensional gas. Informally ergodic behaviour means that the system experiences the same behaviour when averaging over space as when averaging over time.

In the original numerical experiment the scientists looked for a property stronger than ergodicity, called the equipartition of energy, in the long-term behaviour of the system. The property is explained as follows: if all of the energy of the system is placed in the first few Fourier modes, the long time evolution of the modes redistributes the energy more or less evenly between the modes.

The numerical experiments were performed on systems with 16, 32, or 64 particles, using the aforementioned nonlinearities. The models were ran for several characteristic periods of the system. It was observed that most of the energy initially stored in the first mode was first transferred to the other modes of the system; however after some time, the energy of the other modes returned almost fully to the first mode. Thus, the system exhibited a nearly periodic behavior. This phenomenon was referred to as *FPU recurrence* or *the FPU paradox*.

An important question arises from this experiment: does the nearly periodic behaviour continue indefinitely or is there some time after which equipartition of energy occurs? Another natural question is, if certain solutions to this system are in fact nearly periodic, are these solutions exceptional and rare or is there a large probability of initial data yielding a nearly periodic solution?

Figure 1.2 is a reproduction of one of the plots in the original FPU paper. It shows the evolution of the first 5 Fourier modes, the energy of which is defined by

$$E_k = \frac{1}{2}\left(P_k^2 + \omega_k^2 Q_k^2\right), \tag{1.6}$$

where

$$\begin{cases} Q_k = \sqrt{\frac{2}{n}} \sum_{j=1}^{n-1} \sin\left(\frac{jk\pi}{n}\right) q_j \\ P_k = \sqrt{\frac{2}{n}} \sum_{j=1}^{n-1} \sin\left(\frac{jk\pi}{n}\right) p_j \\ \omega_k = 2c_s \sin\left(\frac{k\pi}{n}\right). \end{cases} \tag{1.7}$$

The variables (Q_k, P_k) are the discrete Fourier transforms of (q_j, p_j) subject to the Dirichlet boundary conditions, ω_k are the eigenvalues of the linear equation (1.3). The quantities E_k are sometimes called the harmonic energies, and correspond to the action variables of the simple harmonic oscillator. Since the contributions due to the nonlinear piece is small, as it contributes a few percent of the total energy, these can be taken to be a good approximation for the distribution of energy within the system when trying to track the equipartition hypothesis.

The FPU recurrence can be seen from 1.2: the first few modes return to nearly their initial values after a certain period of time. It is worth noting

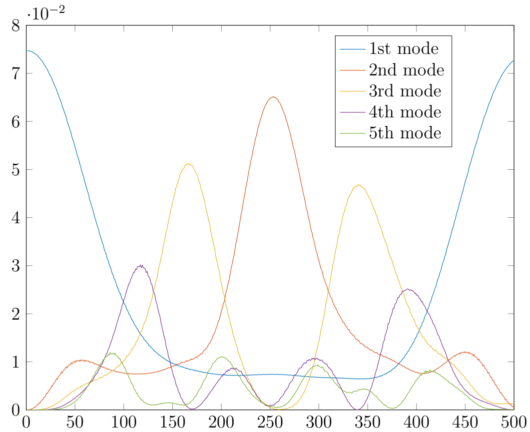


Figure 1.2: Energy in first few Fourier modes for FPU, $\alpha = 0.25$, with initial conditions $x_n = \sin\left(\frac{\pi n}{N+1}\right), \dot{x}_n = 0$

that none of the higher modes exceed the peak of the 5th mode at any point during the simulation with this initial data.

1.3 FPU and Hamiltonian Normal Forms

A potential approach to the FPU recurrence problem is through KAM or Nekhoroshev theory, as discussed in [3]. An important step in this direction is showing that the system has an integrable and nondegenerate Birkhoff normal form. This problem was originally tackled by Nishida [38] under a rather strong nonresonance assumption. Later Bob Rink showed in [41] that this nonresonance condition did not hold for FPU, and the system actually experienced nontrival resonances, however in the case of Dirichlet boundary conditions, nontrivial resonances in the normal form vanish due to symmetries of the system. In 2009 a similar result, but with periodic boundary conditions, was obtained by Kappeler and Henrici in [24]. The authors showed that the resonant normal form for a system with an even number of particles and periodic boundary conditions is integrable and that the resulting normal form is non-degenerate. A consequence is that the KAM theorem applies for some positive measure set of initial conditions.

Most results regarding the Birkhoff normal form, KAM theory, and Nekhoroshev theory for the FPU system have been established in the case of a finite number of particles. Very few papers discuss the thermodynamic limit arising when the number of particles goes to infinity.

One example is [1], where Bambusi and Maspero explored normal forms of the Toda lattice with periodic boundary conditions in the thermodynamic limit, and as a consequence proved a normal form theorem for FPU in a special case where FPU is essentially a truncated Toda lattice.

Nekhoroshev theory, like KAM theory, has been explored in the infinite dimensional context for Hamiltonian PDE, though there are far fewer results in this direction. For example in [2], Bambusi showed the persistence of quasiperiodic solutions of the nonlinear Schrödinger equation for exponential time scales in $\frac{1}{\varepsilon}$, where ε is the size of the initial condition. As a follow-up in [40] Pöschel provided a proof which was technically simpler. Other results have followed, for example for the nonlinear Klein-Gordon equation, however further work has been in polynomial time scales, unlike the exponential time scales for finite dimensional Nekhoroshev theory.

1.4 KdV Approximation to FPU

FPU models for infinitely many particles have been much studied in the limit of small-amplitude and long-wavelength perturbations. The famous Korteweg–de Vries (KdV) equation was derived for the α-model given by (1.4) [5][21]. To derive this limit we introduce the strain variables,

$$r_j(t) = q_{j+1}(t) - q_j(t), \quad j \in \mathbb{Z},$$

then we rewrite the equations of motion (1.4) in strain variables,

$$\ddot{r}_j = c_s^2 \left(r_{j+1} - 2r_j + r_{j-1}\right) + \alpha \left(r_{j+1}^2 - 2r_j^2 + r_{j-1}^2\right). \tag{1.8}$$

We look to approximate a solution of the form

$$r_j(t) = \varepsilon^2 R\left(\varepsilon\left(j - c_s t\right), \varepsilon^3 t\right) + \text{error}, \tag{1.9}$$

where $R(\xi, \tau) : \mathbb{R}^2 \to \mathbb{R}$ is a function on the plane, with $\xi = \varepsilon\left(j - c_s t\right)$, $\tau = \varepsilon^3 t$. Plugging the ansatz (1.9) into equation (1.8) yields to $O(\varepsilon^6)$ the equality

$$-2c_s \partial_\xi \partial_\tau R = \frac{c_s^2}{12}\partial_\xi^4 R + \alpha \partial_\xi^2 \left(R^2\right). \tag{1.10}$$

Integrating with respect to ξ gives the KdV equation,

$$\partial_\tau R + \frac{\alpha}{c_s} R \partial_\xi R + \frac{c_s}{24}\partial_\xi^3 R = 0, \tag{1.11}$$

The Korteweg-de Vries equation was originally introduced as an asymptotic model for water waves in shallow water [32]. The model is widely applicable in physics as a model for nonlinear waves in the long-wavelength limit, for example as a model for collisionless-plasma magnetohydrodynamic waves.

In 1965 Zabusky and Kruskal [48] made the observation that KdV could, at least formally, be viewed as a small-amplitude, long-wavelength limit of FPU. The authors also observed the existence of solitary waves inside numerical solutions of KdV, which could interact nonlinearly, but the individual solitary waves remain unaffected by the interaction. The authors proposed that FPU recurrence and soliton interaction in KdV could be related. As such it's possible to use the KdV approximation to FPU as a means to get long time information about the dynamics of the system for large number of particles. In 1968 Peter Lax proved that the KdV equation is a completely integrable Hamiltonian system [33]. In 1972 Zakharov and Faddeev constructed a canonical transformation of the KdV into action-angle coordinates [50]. See also [34] for a review of the literature.

The first rigorous justification of the KdV approximation was provided by Schneider and Wayne [43] in 1999 as an exercise related to a technique the authors had developed for providing a rigorous justification for the KdV limit of the water wave problem [44]. The authors showed that for a solution to the FPU system (1.4) in strain variables there exists a pair of solutions to the KdV equation (1.11), propagating left and right, the sum of which stays close to $r(j,t)$ in ℓ^∞ norm for long times. The solutions of KdV are assumed to have initial data in $H^{12,0} \cap H^{7,2}$, where $H^{s,n}$ denotes the weighted Sobolev space with functions satisfying $\sqrt{1+x^{2n}} f(x) \in H^s(\mathbb{R})$. The authors controlled the linear piece of the approximation using the long-wavelength assumption on the approximating function, while the nonlinearity was controlled using standard energy methods. An application of Gronwall's inequality gave them an approximation result on the time scales for which the KdV dynamics is observed.

This technique has been successfully extended to a number of different cases, e.g. in [9] Dumas and Pelinovsky gave a rigorous justification that using a Hertzian potential function for the FPU system yields a log-KdV equation in the small-amplitude, long-wavelength limit. In [31] Khan and Pelinovsky proved that for a potential function of the form

$$V(u) = \frac{c_s^2}{2} u^2 + \frac{\varepsilon}{p+1} u^{p+1}$$

the short-amplitude, long-wavelength limit is the generalized KdV equation. They also extended the time scale for this approximation to logarithmic timescale in ε, if a global solution to the generalized KdV equation exists, by improving on the Gronwall inequality argument.

1.5 FPU Solitary Waves

In a series of papers, [14]-[17], Friesecke and Pego extended results for the KdV approximation to FPU and showed that KdV solitary waves are a good approximation for FPU solitary waves, provided they are in the long-wavelength, small-amplitude regime, and the speed of the wave is close to the speed of sound $c_s = \sqrt{V''(0)}$. By solitary waves we mean solutions of the FPU system (4.1) of the form $r_j(t) = r_c(j - ct)$ such chat $r_c(j) \to 0$ as $|j| \to \infty$, where $r_j(t)$ is in strain variables, and c is the speed of the solitary wave. In this section we will discuss these results as applied to the α-model (1.4) which is given by the Hamiltonian (1.1) with potential function $V(u) = \frac{c_s^2}{2}u^2 + \frac{\alpha}{3}u^3$.

We will first discuss solitary waves of the KdV equation (1.11). We will look at a family of solutions of the form

$$R(\xi, \tau) = \phi_\gamma(\xi - \gamma\tau), \tag{1.12}$$

parametrized by $\gamma > 0$. Plugging the solution form (1.12) into the KdV equation (1.11) we get

$$\frac{c_s}{24}\phi_\gamma'''(x) - \gamma\phi_\gamma'(x) + \frac{\alpha}{c_s}\phi_\gamma(x)\phi_\gamma'(x) = 0. \tag{1.13}$$

Integrating with respect to x, with zero boundary conditions, yields,

$$\frac{c_s}{24}\phi_\gamma''(x) - \gamma\phi_\gamma(x) + \frac{\alpha}{2c_s}\phi_\gamma(x)^2 = 0. \tag{1.14}$$

Equation (1.14) admits the following family of solutions

$$\phi_\gamma(\xi - \gamma\tau) = \frac{3c_s\gamma}{\alpha}\operatorname{sech}^2\left(\sqrt{\frac{6\gamma}{c_s}}(\xi - \gamma\tau)\right). \tag{1.15}$$

In light of the small-amplitude, long-wavelength limit discussed in section 1.4 we may look for an approximate solitary wave solution of the FPU system of the form,

$$r_c(x) = \frac{3c_s\gamma\varepsilon^2}{\alpha}\operatorname{sech}^2\left(\varepsilon\sqrt{\frac{6\gamma}{c_s}}\left(j - c_st - \gamma\varepsilon^2 t\right)\right) + \text{error}, \tag{1.16}$$

with wave speed $c = c_s + \gamma\varepsilon^2$.

The continuum limit was initially developed in the first paper [14]. Here the authors, using a displacement profile of a single-pulse wave with speed c,

$$r_j(t) = r_c(j - ct),$$

show that if the wave speed is within $\frac{\varepsilon^2}{24}c_s$ of the speed of sound, that is,

$$0 \le c - c_s \le \frac{\varepsilon^2}{24}c_s$$

then, as $\varepsilon \to 0$, $\frac{1}{\varepsilon^2}r_c(\frac{\xi}{\varepsilon})$ converges uniformly to $\phi_1(\xi)$, with $\phi_1(x)$ given by (1.15) for $0 < \gamma \le \frac{c_s}{24}$.

In the second paper [15], the authors establish nonlinear stability for perturbations of solitary waves. In particular the authors show that if solitary waves of the lattice satisfy a "exponential linear stability" condition, then solutions of nearby initial data stay close to the limiting solitary wave of the KdV equation for all time. The exponential linear stability requires that a solution to the evolution equation, linearized at a solitary wave, decay exponentially in a weighted ℓ^2_a-space defined as

$$\ell^2_a = \left\{u : \mathbb{Z} \to \mathbb{R}^2 | e^{aj}u(j) \in \ell^2\right\}.$$

Specifically the exponential stability condition is given for a particular solitary wave as;

(L) There exist positive constants K, b such that for any solution of the linearized evolution equation (1.17) in ℓ^2_a, we have the estimate

$$\left\|e^{a(j-ct)}w(t)\right\| \le Ke^{-b(t-s)}\left\|e^{a(j-cs)}w(s)\right\|,$$

where $\|\cdot\|$ is the usual ℓ^2-norm, provided that $t \ge s$ and perturbation $w(t)$ is symplectically orthogonal to the modes

$$w_1(t) = \partial_t u_c(j - ct),$$

$$w_2(t) = \partial_c u_c(j - ct),$$

in the sense that $\omega(w_k, w(s)) = 0$ for $k = 1, 2$. Where $\omega(u, v)$ is the Symplectic form given by $\omega(u, v) = \langle J^{-1}u, v\rangle$ with

$$J^{-1} = \begin{bmatrix} 0 & \sum_{n=-\infty}^{0} e^{n\partial_j} \\ \sum_{n=-\infty}^{-1} e^{n\partial_j} & 0 \end{bmatrix}.$$

Here $w(t)$ is a solution to the FPU equations of motion, linearized at a solitary wave. That is to say that if $w(t)$ is a perturbation to the solitary wave

$$u_c(j - ct) = (r_c(j - ct), p_c(j - ct)),$$

then $w(t)$ is a solution to the linearized equation,

$$\partial_t w = B(j-ct)w, \tag{1.17}$$

with

$$B(j-ct) = JH''(u_c(j-ct))$$

where $H(u)$ is the α-model Hamiltonian, and the Symplectic matrix J is given by

$$J = \begin{bmatrix} 0 & e^{\partial_j}-1 \\ 1-e^{-\partial_j} & 0 \end{bmatrix}.$$

In the third paper [16], the authors develop the Floquet theory on the lattice which compares time evolution on the lattice with a continuous group of operators on the real line. This helps the authors reduce the exponential linear stability condition to an eigenvalue condition of a differential-difference operator on the real line. The authors determine the operator's essential spectrum, which corresponds to a continuous Floquet spectrum on the lattice. Through this the authors are able to characterize the condition (L) in terms of an eigenvalue condition on the differential-difference operator, and a requirement that the travelling wave be supersonic. The authors study solutions of the linearized system (1.17). Introducing $w(j,t) = e^{\lambda t}W(j-ct)$ yields the following eigenvalue problem

$$\left(c\partial_x + B(x)\right)W(x) = \lambda W(x), \tag{1.18}$$

where $x - ct$. The authors additionally assume that, for some $a > 0$, the solitary wave, $u_c(\cdot - \tau)$, satisfies the following property from [15],

(P) $u_c(\cdot - ct)$ belongs to a family of solitary waves $u_{\hat{c}}(\cdot - ct)$ such that the map $(\tau, \hat{c}) \to u_{\hat{c}}(\cdot - ct)$ is C^2 from $\mathbb{R} \times (c_-, c_+)$ into the exponentially weighted spaces ℓ^2_a and ℓ^2_{-a} , for some interval (c_-, c_+) containing c.

The authors then prove the following statement

Theorem 1.1 (Friesecke-Pego)**.** *Assume that $u_c(\cdot - ct)$ is a solitary wave speed $c > 0$ which satisfies (P) for some $a > 0$. Then the stability condition (L) is equivalent to the following two conditions:*

1. *$c > c_s$ and $a < a_c$, where $a_c > 0$ is the solution of* $\sinh(\frac{1}{2}a_c)\left(\frac{1}{2}a_c\right)^{-1} = \frac{c}{c_s}$.
2. *Whenever* $\operatorname{Re}\lambda \geq 0$ *and* $|\operatorname{Im}\lambda| \leq \pi$, *equation* (1.18) *has no nonzero solution in H^1_a that is symplectically orthogonal to all neutral modes $e^{2\pi inx}\partial_x u_c(x)$ and $e^{2\pi inx}\partial_c u_c(x)$.*

An important step of the proof of this theorem is an analysis of the spectrum of the operator $c\partial_x + B$ in an exponentially weighted space L^2_a. The authors do this by breaking down the operator into pieces $A + \tilde{A}$, where A is a closed operator with constant coefficients, whose spectrum can be computed explicitly, and $\tilde{A}$ is a relatively compact perturbation.

The series of papers concludes with the final paper [17], where the authors show that low-energy solitary waves are linearly and hence nonlinearly stable. This latter result is sufficient for a recurrence theorem for the FPU system, relating the results of this series to the original numerical experiments of Fermi, Pasta, and Ulam [13]. Specifically the paper proves the following result

Theorem 1.2 (Friesecke-Pego)**.** *Suppose the interaction potential is given by* $V(u) = \frac{c_s^2}{2}u^2 + \frac{\alpha}{3}u^3$. *On any energy surface* $H = E$ *with* $E > 0$ *sufficiently small the unique supersonic single-pulse solitary wave* u_c *has the following property. There exists a parameter* $a > 0$*, a decay exponent* $\beta > 0$ *and a constant* $C > 0$ *such that if the initial data* u_0 *satisfy*

$$\begin{aligned} \|u_0 - u_c(\cdot)\| &\leq \sqrt{\delta}, \\ \|u_0 - u_c(\cdot)\|_{\ell^2_a} &\leq \delta \end{aligned} \tag{1.19}$$

and $\delta > 0$ *is sufficiently small, then the solution* $u(\cdot, t)$ *to the FPU equations, given by Hamiltonian* (4.1)*, satisfies*

$$\begin{aligned} \|u(\cdot, t) - u_{c_*}(\cdot - c_* t - \tau_*)\| &\leq C\sqrt{\delta} \\ \left\| e^{a(\cdot - c_* t - \tau_*)}\left(u(\cdot, t) - u_{c_*}(\cdot - c_* t - \tau_*)\right)\right\| &\leq C\delta e^{-\beta(t - t_0)} \end{aligned} \tag{1.20}$$

for all $t \geq 0$*, where* u_{c_*} *is a solitary wave of speed* c_* *and* τ_* *a phase shift which satisfy*

$$|c_* - c| + |\tau_*| + \|u_{c_*} - u_c\| \leq C\delta.$$

The strategy for the proof of this theorem is to verify that the conditions specified in theorem 1.1 hold for low energy waves. The authors had already verified that (P) holds for all $a \in (0, a_c)$ and that $c > c_s$, what remained was to check that property 2 as stated in theorem 1.1 held.

Related to the work of Friesecke and Pego [14]-[17] is the work of Herrmann and Matthies in [25]-[27] on high-energy solitary travelling waves. The authors studied potentials of the form

$$V(u) = \frac{1}{m(m+1)}\left(\frac{1}{(1-u)^m} - mu - 1\right),$$

where $m > 1$ is a constant.The authors initially looked at asymptotic expressions for the wave profiles in [25]. In [26], the authors showed that solitary waves in the high energy case are unique, at least locally. Finally in [27], the authors study the stability of solitary waves in the high energy case, in particular extending the results of Friesecke and Pego [17].

1.6 Diatomic FPU

Diatomic FPU models have also been recently studied in many publications. The diatomic FPU system has two particles of different masses, $m_1 > m_2$, appear at alternating lattice sites. In addition to varying masses, the springs, meaning the interaction term between particles, may vary between lattice sites as well. The diatomic case is illustrated in figure 1.3.

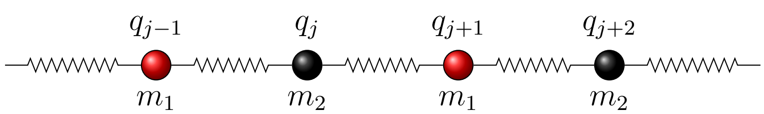


Figure 1.3: A one-dimensional diatomic mass-spring system

In [20] J. Gaison, S. Moskow, J. D. Wright, and Q. Zhang extended the KdV approximation of the FPU system to poly-atomic lattices. Masses of particles in the poly-atomic model may differ from site to site as can the interaction between nearest neighbours. This result was not restricted to the diatomic model, so there may be more than two distinct masses. Newton's equations for the poly-atomic model used are given as

$$m_j \ddot{q}_j = V_j' \left(q_{j+1} - q_j - l\right) - V_{j-1}' \left(q_j - q_{j-1} - l\right),$$

where l is the relaxation length of the spring, and $m(j)$ is the mass of the particle at lattice site $j \in \mathbb{Z}$. There are N distinct masses and potential functions, which repeat periodically, that is there is an $N \in \mathbb{N}$ so that

$$m_{j+N} = m_j, \quad V_{j+N}(u) = V_j(u).$$

The potential function is also assumed to take the form

$$V_j(u) = \frac{1}{2} c_s(j)^2 u^2 + \frac{1}{3}\alpha(j) u^3 + O(u^3).$$

Define $\mathbf{r}(j,t) = (r(j,t), p(j,t))$ to be a solution of the FPU system with the given potential and initial conditions $r(j,0) = \frac{\varepsilon^2}{c_s(j)^2}\phi\left(\varepsilon j\right), p(j,0) = \varepsilon^2 \psi\left(\varepsilon j\right)$. Here ϕ, ψ are functions which are in H^5 and whose anti-derivative is in L^∞. Let A, B solve the KdV equations

$$\frac{1}{c} A_t + a A_{xxx} + b A A_x = 0,$$

$$-\frac{1}{c} B_t + a B_{xxx} + b B B_x = 0,$$

with initial conditions

$$A(x,0) = \frac{1}{2}\left(\phi(x) - \sqrt{\overline{m}\tilde{c}_s}\psi(x)\right), B(x,0) = \frac{1}{2}\left(\phi(x) + \sqrt{\overline{m}\tilde{c}_s}\psi(x)\right),$$

where the quantities above are given by: $\overline{m} = \frac{1}{N}\sum_{j=1}^{N} m(j)$, $\tilde{c}_s = \frac{N}{\sum_{j=1}^{N} c_s(j)^{-2}}$, $b = \frac{\tilde{c}_s}{N}\sum_{j=1}^{N}\frac{\alpha(j)}{c_s(j)^6}$, and $c = \sqrt{\frac{\tilde{c}_s}{\bar{m}}}$, $a = \frac{1}{24}\left(1 - 12\gamma_0 - 12\gamma_0^2 - 24\gamma_1 + 12\gamma_2 + 12\gamma_3\right)$. For the coefficient a define

$$\chi_1(1) = -\frac{1}{N}\sum_{k=2}^{N}\sum_{j=1}^{k-1}\left(\frac{\tilde{c}_s}{c_s^2(j)} - 1\right), \chi_2(1) = -\frac{1}{N}\sum_{k=2}^{N}\sum_{j=1}^{k-1}\left(\frac{\tilde{m}(j+1)}{\bar{m}} - 1\right)$$

and for $2 \leq k \leq N$

$$\chi_1(k) = \sum_{j=1}^{k-1}\left(\frac{\tilde{c}_s}{c_s^2(j)} - 1\right), \chi_2(k) = \sum_{j=1}^{k-1}\left(\frac{\tilde{m}(j+1)}{\bar{m}} - 1\right).$$

Set $\gamma_0 = \langle\frac{\tilde{c}_s}{c_s(j)^2}, \chi_2\rangle, \gamma_1 = \langle\chi_1, \chi_2\rangle, \gamma_2 = \langle\chi_1^2, \frac{m}{\bar{m}}\rangle, \gamma_3 = \langle\chi_2^2, \frac{\tilde{c}_s}{c_s(j)^2}\rangle$, where the inner product is given by

$$\langle f(j), g(j)\rangle = \frac{1}{N}\sum_{j=1}^{N} f(j)g(j)$$

. Define

$$A_\varepsilon(j,t) = \left(\frac{1}{c_s^2(j)}\left[A\left(\varepsilon(j-ct), \varepsilon^3 t\right) + B\left(\varepsilon(j+ct), \varepsilon^3 t\right)\right], \sqrt{\frac{1}{\overline{m}\tilde{c}_s}}\left[-A\left(\varepsilon(j-ct), \varepsilon^3 t\right) + B\left(\varepsilon(j+ct), \varepsilon^3 t\right)\right]\right).$$

It was shown in [20] that

$$\sup_{|t|\leq T_0\varepsilon^{-3}}\left\|\mathbf{r}(t) - \varepsilon^2 A_\varepsilon(t)\right\|_{\ell^2} \leq C\varepsilon^{\frac{5}{2}}.$$

In [39] Pelinovsky and Schneider show that a diatomic FPU system can be approximated by a monatomic FPU system if the ratio of the small mass to the large mass is small enough. In [29], Hoffman and Wright studied the existence of travelling waves in a diatomic FPU system under the same assumption. In this limit the authors of [29] construct "nanopteron" solutions, which are a superposition of a solitary wave and a periodic wave. More specifically if we label the heavy particles by $Q_j(t)$ and the light particles by $q_j(t)$ the authors find an expression for this solution of the form

$$Q_j(t) = \sigma_c\,(j - ct) + \Upsilon_1\,(j - ct) + \Phi_1\,(j - ct)\,,$$

$$q_j(t) = \Upsilon_2\,(j - ct) + \Phi_2\,(j - ct)\,.$$

Here $\sigma_c\,(\cdot)$ is a solitary wave solution of the system when the lighter particle is of mass zero, the pair of Υ functions decay exponentially in their variable, and the functions Φ are periodic and their magnitude is smaller than any power of the small parameter $\mu = \frac{m_2}{m_1}$, the ratio of the masses.

Analogous results to those of Friesecke and Pego were extended to diatomic lattices by Faver and Wright in [12], this is a continuation of the work by Hoffman and Wright in [29]. This is not a straightforward generalization since the dispersion relationship for the linearized equations of motions have two parts in the diatomic model. The first part is called the "acoustic" band, which is similar to the dispersion relation of the monatomic model, and a similar justification analysis works. The second piece, called the "optical" band, is not present in the dispersion relation of the monatomic model, and the equation is classically singularly perturbed. Since the problem is nonlocal the authors weren't able to use traditional methods for dealing with singular perturbations, and adapted a functional analytic approach which was developed to prove existence of solitary capillary-gravity waves in [4]. They prove that for wavespeeds, c, close to but larger than the speed of sound of the lattice c_s, which depends on the two masses, there is a traveling wave which is a superposition of two pieces. They show that the first piece is localized and solves a KdV travelling wave equation, it has an amplitude proportional to $(c - c_s)$ and wavelength proportional to $(c - c_s)^{-\frac{1}{2}}$. The additional piece is a periodic function, whose amplitude can be made smaller than any power of $(c - c_s)$.

In [11] Faver and Hupkes study "micropteron" travelling waves for a diatomic FPU system in the equal mass limit. The distinction between a micropteron and the nanopteron solutions is that the oscillations in a micropteron solution are not necessarily smaller than all orders of the mass ratio.

1.7 The Kadomtsev-Petviashvili Equation

The KdV equation models a wide-range of one-dimensional, nonlinear waves, in the small-amplitude, long-wavelength limit. The KdV equation also possesses a class of special solutions, known as solitary waves, which are stable in the one-dimensional dynamics. In [30] Kadomtsev and Petviashvili studied whether this stability is preserved for solitary waves whose amplitude and phase are allowed to vary slowly in the transverse direction, a property called transverse stability. To do this the authors added a small transverse perturbation to the KdV equation by introducing a function $\phi(x, y, t)$ on the right hand side of (1.11),

$$c_1\partial_t u + \alpha u \partial_x u + \frac{c_1^2}{24}\partial_x^3 u = \partial_y \phi. \tag{1.21}$$

A linear dispersion analysis showed that the function $\phi(x, y, t)$ must satisfy $\partial_x \phi = \mp\frac{c_2^2}{2}\partial_y u$. The signs correspond to negative and positive dispersion respectively. In the literature the positive dispersion case is called the KP-I equation, while the negative dispersion case is called the KP-II equation. Solitary waves of the KdV equations were studied formally in both versions, and the authors found that solitary waves were unstable with respect to transverse perturbations in KP-I, while "bending" of solitary waves leads to harmonic oscillations with a weak damping in the case of KP-II.

Analogous to the KdV limit of the two-dimensional water-wave problem, the KP-II equation can be observed in the long-wave limit of the three-dimensional water wave problem. A rigorous justification in this direction was presented in [22] by Gallay and Schneider, where the authors showed that a KP-II equation can be found in the two-dimensional Boussinesq equation, which is a realistic model for the three-dimensional water wave problem.

A rigorous result about the stability of KdV solitons as solutions to the KP-II equation was obtained by Mizumachi and Tzvetkov [37]. Specifically the authors studed the KP-II equation

$$\partial_\xi\left(\partial_\tau u + \partial_\xi^3 u + 3\partial_\xi\left(u^2\right)\right) + 3\partial_\eta^2 u = 0, \tag{1.22}$$

with initial data $u_0(\xi, \eta) \in H^s\left(\mathbb{R}_\xi \times \mathbb{T}_\eta\right)$ for $s \geq 0$. Such an equation is known to be globally well posed, with a unique solution in the space $C^0\left(\mathbb{R}; \left(\mathbb{R}_\xi \times \mathbb{T}_\eta\right)\right)$. Let

$$\phi_c(\xi) = c\,\mathrm{sech}^2\left(\sqrt{\frac{c}{2}}\xi\right), c > 0,$$

then $\phi_c(\xi - 2c\tau)$ is a solitary wave solution of the KdV equation and a solution to (1.22). Mizumachi and Tzvetkov show that $\phi_c(\xi - 2c\tau)$ is stable as a solution to the KP-II equation subject to perturbations which are periodic in the transverse direction. In particular they prove the following result:

Theorem 1.3 (Mizumachi-Tzvetkov). *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if the initial data of* (1.22) *satisfies $\|u_0 - \phi_c\|_{L^2\left(\mathbb{R}_\xi \times \mathbb{T}_\eta\right)} < \delta$, the corresponding solution of* (1.22) *satisfies*

$$\inf_{\gamma \in \mathbb{R}} \|u(\tau, \xi, \eta) - \phi_c(\xi + \gamma)\|_{L^2\left(\mathbb{R}_\xi \times \mathbb{T}_\eta\right)} < \varepsilon, \qquad \forall t \in \mathbb{R}.$$

Moreover, there exists a constant $\tilde{c}$ *satisfying* $\tilde{c} - c = O(\delta)$ *and a modulation parameter* $\xi(\tau)$ *satisfying* $\lim_{\tau\to\infty} \dot{\xi}(\tau) = 2\tilde{c}$ *and such that*

$$\lim_{t\to\infty} \|u(\tau,\xi,\eta) - \phi_{\tilde{c}}(\xi - \xi(\tau))\|_{L^2((\xi\geq c\tau)\times\mathbb{T}_\eta)} = 0.$$

The stability of such solitary waves was extended by Mizumachi to solutions in the space where transverse perturbations are exponentially localized as $x \to \infty$ [35], and with perturbations in either $(1+\xi)^{-\frac{1}{2}}H^1(\mathbb{R}^2)$ or $H^1(\mathbb{R}^2)\cap\partial_\xi L^2(\mathbb{R}^2)$ [36]. In [23] Haragus, Li, and Pelinovsky prove that periodic travelling waves of the KdV equation are tranvsersely linearly stable with respect to periodic perturbations.

1.8 Two-dimensional FPU

It is natural to ask to what extent do the one dimensional results for FPU extend to a two-dimensional case, where the mass-spring system is arranged along some lattice in $\mathbb{R}^2$ instead of a line. Figure 1.4 illustrates the system where the particles in the lattice are connected by horizontal and vertical springs, there are no diagonal springs.

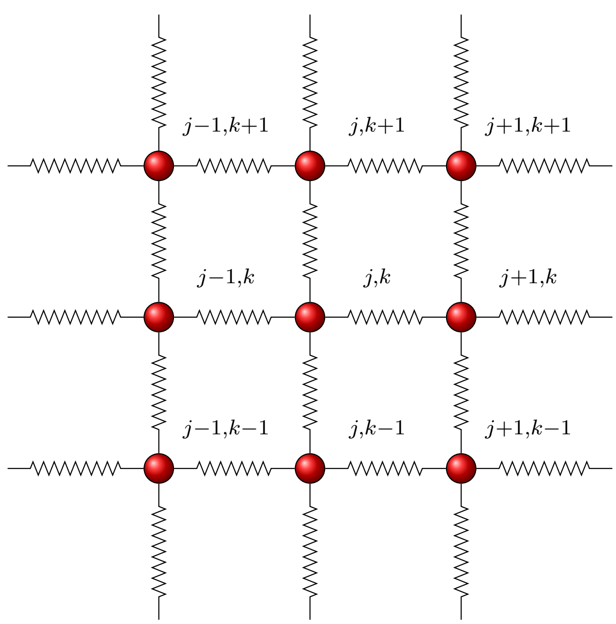


Figure 1.4: A mass spring system arranged in a square lattice

There are few results addressing whether the two-dimensional FPU problem can be approximated by an integrable Hamiltonian PDE, similar to the KdV limit in the one-dimensional case. Friesecke and Matthies [19] have proven the existence of longitudinal solitary waves in the KdV limit of a two-dimensional FPU like lattice. Interestingly the lattice only included linear terms, but the waves observed were nonlinear. The nonlinearity presenting itself by virtue of the two-dimensional Euclidean norm, as a consequence the authors referred to these as "geometric waves". More recently, Chen and Herrmann [8] generalized this result by allowing for nonlinear coupling and arbitrary propagation direction of such one-dimensional waves.

Analogously to the KdV limit of the one-dimensional FPU system, we expect of the two-dimensional FPU system that an appropriate continuum approximation is given by the KP equation. One such limit was formally studied by Duncan, Eilbeck, and Zakharov in [10], where the authors used a strongly anisotropic model whose dynamics are given by the Hamiltonian

$$\begin{aligned} H = &\sum_{j,k} \frac{1}{2}\dot{u}_{j,k} + \frac{1}{2}\left(u_{j+1,k} - u_{j,k}\right)^2 + \frac{1}{2}\varepsilon^2\left(u_{j,k+1} - u_{j,k}\right)^2 \\ &+ \sum_{j,k} \frac{1}{3} a\varepsilon \left(u_{j+1,k} - u_{j,k}\right)^3 . \end{aligned}$$

Seeking a continuous function $u_{j,k}(t) = U\left(\varepsilon\left(x - t\right), \varepsilon y, \varepsilon^3 t\right)$ which satisfies the equations of motion given by the above Hamiltonian, the team found that the function then satisfies a KP-II equation to $O(\varepsilon^6)$.

It is the purpose of this thesis to study rigorously transverse variations of FPU solitary waves propagating along a certain direction in the lattice. We will derive the KP-II equation in the case of horizontal and diagonal propagation in the α-model. We will also derive the cubic KP-II equation for horizontal propagation in the β-model. The two-dimensional FPU models studied in this thesis will be introduced in the subsequent section

1.9 Two-Dimensional FPU Models

This subsection will introduce one of the two-dimensional models studied in this thesis, and introduce their Hamiltonians and the equations of motion. The particles in the lattice will be indexed by $(j, k) \in \mathbb{Z}^2$. We will introduce the vector quantities $q_{j,k} = (x_{j,k}, y_{j,k})$ and $p_{j,k} = (w_{j,k}, z_{j,k})$, as well as the vector quantities

$$r^x_{j,k} = (x_{j+1,k} - x_{j,k}, y_{j+1,k} - y_{j,k})$$

and

$$r^y_{j,k} = (x_{j,k+1} - x_{j,k}, y_{j,k+1} - y_{j,k}).$$

1.9.1 Two-Dimensional α-model

The first model we will look at is what we will call the two-dimensional α-model. This is analogous to a highly anisotropic model studied in [10], with additional nonlinear terms, and the anisotropy introduced through scaling, rather than differences in the springs. We write the Hamiltonian,

$$H = \frac{1}{2}\sum_{j,k}\left(w_{j,k}^2 + z_{j,k}^2\right) + \sum_{j,k} V_h(r_{j,k}^x) + \sum_{j,k} V_v(r_{j,k}^y). \tag{1.23}$$

The potential functions will be defined as,

$$\begin{aligned} V_h(u,v) &= \frac{c_1^2}{2}u^2 + \frac{c_2^2}{2}v^2 + \frac{\alpha_1}{3}u^3 + \frac{\alpha_2}{2}uv^2, \\ V_v(u,v) &= V_h(v,u). \end{aligned} \tag{1.24}$$

The potential function is chosen so that it resembles the dynamics of models built on the displacement, measured in the sense of Euclidean norm, while also maintaining algebraic simplicity of the model in [10]. The uv^2 term is added due to mechanical considerations, it ensures that horizontal displacements are symmetric with respect to the sign of displacements in the vertical direction, while vertical displacements are symmetric with respect to the sign of displacements in the horizontal direction. This is a nearest neighbour model, so it will fail to capture some properties of materials, such as elasticity. As discussed in Friesecke and Theil, [18] and Friesecke and Matthies [19], a next nearest neighbour model, which in the case of a mass-spring lattice system involves including diagonal springs, is required to describe elasticity. The lattice studied by these authors is similar to the lattice studied by Chen and Herrmann [8] for their KdV limit for a two-dimensional FPU system with diagonal springs. A KP-II limit analogous to our result should be possible in the lattice, however additional terms due to diagonal interactions in the lattice will make calculations less clear.

We compute the equations of motion,

$$\begin{aligned} \dot{x}_{j,k} &= w_{j,k}, \\ \dot{w}_{j,k} &= c_1^2\left(x_{j+1,k} - 2x_{j,k} + x_{j-1,k}\right) + c_2^2\left(x_{j,k+1} - 2x_{j,k} + x_{j,k-1}\right) \\ &\quad + \alpha_1\left[\left(x_{j+1,k} - x_{j,k}\right)^2 - \left(x_{j,k} - x_{j-1,k}\right)^2\right] \\ &\quad + \frac{\alpha_2}{2}\left[\left(y_{j+1,k} - y_{j,k}\right)^2 - \left(y_{j,k} - y_{j-1,k}\right)^2\right] \\ &\quad + \alpha_2\left[\left(x_{j,k+1} - x_{j,k}\right)\left(y_{j,k+1} - y_{j,k}\right) - \left(x_{j,k} - x_{j,k-1}\right)\left(y_{j,k} - y_{j,k-1}\right)\right], \end{aligned} \tag{1.25}$$

and

$$\begin{aligned}\dot{y}_{j,k} &= z_{j,k},\\ \dot{z}_{j,k} &= c_2^2\left(y_{j+1,k} - 2y_{j,k} + y_{j-1,k}\right) + c_1^2\left(y_{j,k+1} - 2y_{j,k} + y_{j,k-1}\right)\\ &\quad + \frac{\alpha_2}{2}\left[\left(x_{j,k+1} - x_{j,k}\right)^2 - \left(x_{j,k} - x_{j,k-1}\right)^2\right]\\ &\quad + \alpha_1\left[\left(y_{j,k+1} - y_{j,k}\right)^2 - \left(y_{j,k} - y_{j,k-1}\right)^2\right]\\ &\quad + \alpha_2\left[\left(x_{j+1,k} - x_{j,k}\right)\left(y_{j+1,k} - y_{j,k}\right) - \left(x_{j,k} - x_{j-1,k}\right)\left(y_{j,k} - y_{j-1,k}\right)\right].\end{aligned} \tag{1.26}$$

In order to study propagation along the horizontal direction we will seek a continuous approximating function of the form

$$x_{j,k} = \varepsilon X\left(\xi, \eta, \tau\right) + \text{error},$$

with $\xi = \varepsilon(j - c_1 t), \eta = \varepsilon^2 k, \tau = \varepsilon^3 t$. We will show that $x_{j,k}$ will satisfy the equations of motion if $X\left(\xi, \eta, \tau\right)$ solves a KP-II equation

$$2c_1\partial_\xi\partial_t X + \frac{c_1^2}{12}\partial_\xi^4 X + c_2^2\partial_\eta^2 X + \frac{\alpha}{2}\partial_\xi\left(\left(\partial_\xi X\right)^2\right) = 0. \tag{1.27}$$

This limit is first studied through a linear dispersion analysis in section 2.1 and formally derived in section 2.2. A rigorous justification is provided in chapter 4. In the justification we look at the system in "strain variables", which are defined by the relative displacements between adjacent particles, as in the following ansatz:

$$x_{j+1,k} - x_{j,k} = \varepsilon^2 A\left(\varepsilon(j - c_1 t), \varepsilon^2 k, \varepsilon^3 t\right) + \text{error}.$$

The reason for the different scaling here is that we can formally consider the relationship between the function A and X through a Taylor expansion

$$A\left(\varepsilon(j - c_1 t), \varepsilon^2 k, \varepsilon^3 t\right) = \partial_\xi X\left(\varepsilon(j - c_1 t), \varepsilon^2 k, \varepsilon^3 t\right) + O(\varepsilon),$$

so that the KP-II (1.27) can be rewritten with $\partial_\xi X$ replaced by A.

1.9.2 Diagonal Propagation in an α-model

In chapter 5 we will study the propagation of waves along the diagonal of the 2D-FPU lattice in the small-amplitude long-wavelength limit. In order to study propagation along a diagonal we will take the existing α-model with equations of motion (1.25)-(1.26) and introduce a new coordinate system on the lattice by $m = \frac{j+k}{2}, n = \frac{j-k}{2}$. Under the new coordinate system the particle experiences nearest-neighbour interactions with neighbours located a half lattice site away. Due to this we redefine $x_{m,n}$ and introduce $\chi_{m,n} = x_{m+\frac{1}{2},n+\frac{1}{2}}$.

The system becomes a diatomic system where $x_{m,n}$ particles communicate with four $\chi_{m,n}$ nearest-neighbour particles and vice versa, see figure 1.5 for an illustration. In order to study propagation along the diagonals we will seek a continuous approximating function of the form

$$x_{m,n} = \varepsilon X\left(\varepsilon(m - c_1^* t), \varepsilon^2(n - c_2^* t), \varepsilon^3 t\right) + \text{error},$$

where $c_1^* = \frac{\sqrt{c_1^2+c_2^2}}{2}, c_2^* = \frac{\sqrt{c_1^2-c_2^2}}{2}$.

A computation of the linear dispersion relationship in section 2.3 shows that some propagation in the transverse direction is unavoidable in this case, unless some careful choices are made in the parameters of the model. We will study the case where $\alpha_1 = \frac{\alpha_2}{2} = \alpha$, and $c_1^2 = c_2^2 = c^2$, so that $c_2^* = 0$. Further these parameters will ensure that we have the reduction $x_{j,k} = y_{j,k}$, which allows us to describe symmetric diagonal motion. In section 2.4 we formally derive the nonlinear KP-II equation for $X(\xi, \eta, \tau)$:

$$c\sqrt{2}\partial_\xi\partial_\tau X + \frac{c^2}{96}\partial_\xi^4 X + \frac{c^2}{2}\partial_\eta^2 X + \frac{\alpha}{2}\partial_\xi\left((\partial_\xi X)^2\right) = 0$$

from the equations of motion. A rigorous justification is performed in chapter 5.

Next we introduce $\eta_{m,n} = y_{m+\frac{1}{2},n+\frac{1}{2}}$, as well as the velocities

$$\begin{aligned} u_{m,n} &= \dot{x}_{m,n} \\ v_{m,n} &= \dot{\chi}_{m,n} \\ w_{m,n} &= \dot{y}_{m,n} \\ z_{m,n} &= \dot{\eta}_{m,n} \end{aligned} \tag{1.28}$$

and the strain variables

$$\begin{aligned} a^l_{m,n} &= \chi_{m,n} - x_{m,n}, \\ a^d_{m,n} &= x_{m+1,n+1} - \chi_{m,n}, \\ a^x_{m,n} &= x_{m+1,n} - \chi_{m,n}, \\ a^y_{m,n} &= x_{m,n+1} - \chi_{m,n}, \end{aligned} \tag{1.29}$$

as well as,

$$\begin{aligned} b^l_{m,n} &= \eta_{m,n} - y_{m,n}, \\ b^d_{m,n} &= y_{m+1,n+1} - \eta_{m,n}, \\ b^x_{m,n} &= y_{m+1,n} - \eta_{m,n}, \\ b^y_{m,n} &= y_{m,n+1} - \eta_{m,n}. \end{aligned} \tag{1.30}$$

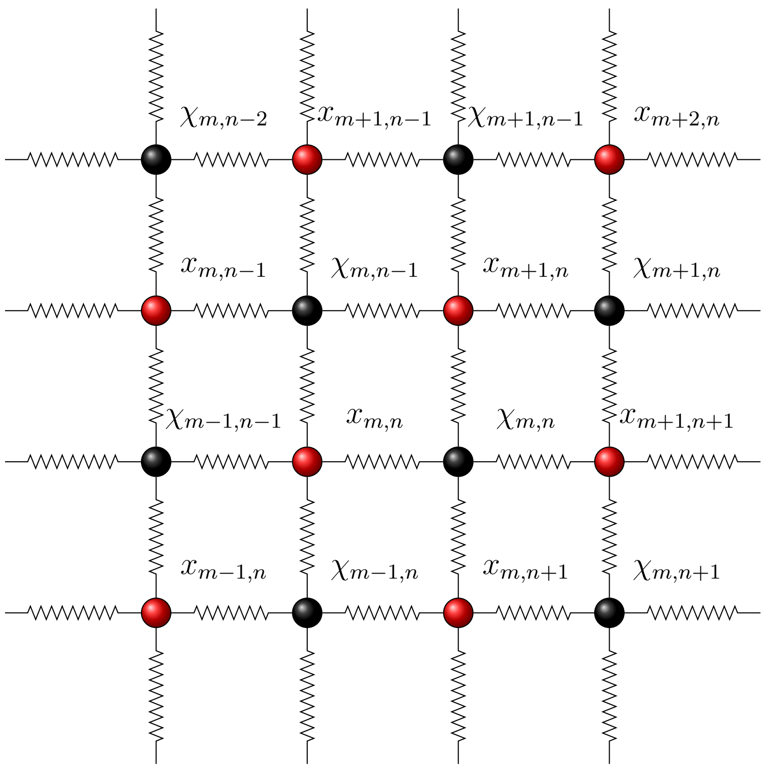


Figure 1.5: A diatomic mass spring system arranged in a square lattice

From (1.23) rewrite the Hamiltonian in the relabeled coordinates as,

$$\begin{aligned} H =&\frac{1}{2}\sum_{m,n}\left(u_{m,n}^2+v_{m,n}^2+w_{m,n}^2+z_{m,n}^2\right)+\sum_{m,n}V(a_{m,n}^l,b_{m,n}^l) \\ &+\sum_{m,n}V(a_{m,n}^d,b_{m,n}^d)+\sum_{m,n}V(a_{m,n}^x,b_{m,n}^x)+\sum_{m,n}V(a_{m,n}^y,a_{m,n}^y). \end{aligned} \tag{1.31}$$

From this we have the equations of motion

$$\begin{aligned} \dot{u}_{m,n} =&c^2\left(\chi_{m,n}-2x_{m,n}+\chi_{m-1,n-1}\right)+c^2\left(\chi_{m-1,n}-2x_{m,n}+\chi_{m,n-1}\right) \\ &+\alpha\left[\left(\chi_{m,n}-x_{m,n}\right)^2-\left(x_{m,n}-\chi_{m-1,n-1}\right)^2\right] \\ &+\alpha\left[\left(\eta_{m,n}-y_{m,n}\right)^2-\left(y_{m,n}-\eta_{m-1,n-1}\right)^2\right] \\ &+2\alpha\left(x_{m,n}-\chi_{m,n-1}\right)\left(y_{m,n}-\eta_{m,n-1}\right) \\ &-2\alpha\left(\chi_{m-1,n}-x_{m,n}\right)\left(\eta_{m-1,n}-y_{m,n}\right), \end{aligned} \tag{1.32}$$

$$\begin{aligned} \dot{v}_{m,n} =&c^2\left(x_{m+1,n+1}-2\chi_{m,n}+x_{m,n}\right)+c^2\left(x_{m+1,n}-2\chi_{m,n}+x_{m,n+1}\right) \\ &+\alpha\left[\left(x_{m+1,n+1}-\chi_{m,n}\right)^2-\alpha\left(\chi_{m,n}-x_{m,n}\right)^2\right] \\ &+\alpha\left[\left(y_{m+1,n+1}-\eta_{m,n}\right)^2-\alpha\left(\eta_{m,n}-y_{m,n}\right)^2\right] \\ &+2\alpha\left(x_{m+1,n}-\chi_{m,n}\right)\left(y_{m+1,n}-\eta_{m,n}\right) \\ &-2\alpha\left(\chi_{m,n}-x_{m,n+1}\right)\left(\eta_{m,n}-y_{m,n+1}\right) \end{aligned} \tag{1.33}$$

$$\begin{aligned} \dot{w}_{m,n} =&c^2\left(\eta_{m,n}-2y_{m,n}+\eta_{m-1,n-1}\right)+c^2\left(\eta_{m-1,n}-2y_{m,n}+\eta_{m,n-1}\right) \\ &+\alpha\left[\left(\eta_{m,n-1}-y_{m,n}\right)^2-\left(y_{m,n}-\eta_{m-1,n}\right)^2\right] \\ &+\alpha\left[\left(\chi_{m,n-1}-x_{m,n}\right)^2-\left(x_{m,n}-\chi_{m-1,n}\right)^2\right] \\ &+2\alpha\left(\eta_{m,n}-y_{m,n}\right)\left(\chi_{m,n}-x_{m,n}\right) \\ &-\alpha\left(y_{m,n}-\eta_{m-1,n-1}\right)\left(x_{m,n}-\chi_{m-1,n-1}\right) \end{aligned} \tag{1.34}$$

$$\begin{aligned} \dot{z}_{m,n} =&c^2\left(y_{m+1,n+1}-2\eta_{m,n}+y_{m,n}\right)+c^2\left(y_{m+1,n}-2\eta_{m,n}+y_{m,n+1}\right) \\ &+\alpha\left[\left(y_{m+1,n}-\eta_{m,n}\right)^2-\left(\eta_{m,n}-y_{m,n+1}\right)^2\right] \\ &+\alpha\left[\left(x_{m+1,n}-\chi_{m,n}\right)^2-\left(\chi_{m,n}-x_{m,n+1}\right)^2\right] \\ &+2\alpha\left(y_{m+1,n+1}-\eta_{m,n}\right)\left(x_{m+1,n+1}-\chi_{m,n}\right) \\ &-2\alpha\left(\eta_{m,n}-y_{m,n}\right)\left(\chi_{m,n}-x_{m,n}\right) \end{aligned} \tag{1.35}$$

Note that under the assumption that $x_{j,k} = y_{j,k}$ we have equations of motion (1.32) and (1.34) coincide, as well as equations of motion (1.33) and (1.35). As a result for this choice of parameters if we set our initial conditions so that $x_{j,k} = y_{j,k}$ and $u_{j,k} = w_{j,k}$, these equalities will hold for all time. With this reduction we may write the equations of motion, in our displacement variables, as

$$\begin{aligned}
\dot{u}_{m,n} =& c^2 \left(a^l_{m,n} - a^d_{m-1,n-1} \right) + c^2 \left(-a^x_{m-1,n} - a^y_{m,n-1} \right) \\
&+ 2\alpha \left[\left(a^l_{m,n} \right)^2 - \left(a^d_{m-1,n-1} \right)^2 + \left(a^y_{m,n-1} \right)^2 - \left(a^x_{m-1,n} \right)^2 \right] \\
\dot{v}_{m,n} =& c^2 \left(a^d_{m,n} - a^l_{m,n} \right) + c^2 \left(a^x_{m,n} + a^y_{m,n} \right) \\
&+ 2\alpha \left[\left(a^d_{m,n} \right)^2 - \left(a^l_{m,n} \right)^2 + \left(a^x_{m,n} \right)^2 - \left(a^y_{m,n} \right)^2 \right]
\end{aligned} \tag{1.36}$$

1.9.3 Two-dimensional β-Model

For the two dimensional analogue of the β-model we introduce the Hamiltonian

$$H = \frac{1}{2} \sum_{j,k} \left(w^2_{j,k} + z^2_{j,k} \right) + \sum_{j,k} V_\beta(||r^x_{j,k}||) + \sum_{j,k} V_\beta(||r^y_{j,k}||), \tag{1.37}$$

where $\|\cdot\|$ is the standard Euclidean norm of vectors in the plane. The potential function is given by

$$V_\beta(r) = \frac{1}{2} r^2 + \frac{\beta}{4} r^4, \tag{1.38}$$

Since the nonlinear term is the product of a polynomial in $x_{j,k}$ with the square of the Euclidean norm, the final result is a polynomial. As a result for the β model we have,

$$\begin{aligned}
\dot{w}_{j,k} =& (x_{j+1,k} - 2x_{j,k} + x_{j-1,k}) + (x_{j,k+1} - 2x_{j,k} + x_{j,k-1}) \\
&+ \beta \left((x_{j+1,k} - x_{j,k})^3 - (x_{j,k} - x_{j-1,k})^3 \right) \\
&+ \beta \left(x_{j+1,k} - x_{j,k} \right) \left(y_{j+1,k} - y_{j,k} \right)^2 \\
&- \beta \left(x_{j,k} - x_{j-1,k} \right) \left(y_{j,k} - y_{j-1,k} \right)^2 \\
&+ \beta \left(x_{j,k+1} - x_{j,k} \right) \left(y_{j,k+1} - y_{j,k} \right)^2 \\
&- \beta \left(x_{j,k} - x_{j,k-1} \right) \left(y_{j,k} - y_{j,k-1} \right)^2 \\
&+ \beta \left((x_{j,k+1} - x_{j,k})^3 - (x_{j,k} - x_{j,k-1})^3 \right).
\end{aligned} \tag{1.39}$$

The second equation of motion is given by,

$$\begin{aligned}\dot{z}_{j,k} =& \left(x_{j+1,k} - 2x_{j,k} + x_{j-1,k}\right) + \left(x_{j,k+1} - 2x_{j,k} + x_{j,k-1}\right) \\ &+ \beta \left(x_{j+1,k} - x_{j,k}\right)^2 \left(y_{j+1,k} - y_{j,k}\right) \\ &- \beta \left(x_{j,k} - x_{j-1,k}\right)^2 \left(y_{j,k} - y_{j-1,k}\right) \\ &+ \beta \left(\left(y_{j+1,k} - y_{j,k}\right)^3 - \left(y_{j,k} - y_{j-1,k}\right)^3\right) \\ &+ \beta \left(\left(y_{j,k+1} - y_{j,k}\right)^3 - \left(y_{j,k} - y_{j,k-1}\right)^3\right) \\ &+ \beta \left(x_{j,k+1} - x_{j,k}\right)^2 \left(y_{j,k+1} - y_{j,k}\right) \\ &- \beta \left(x_{j,k} - x_{j,k-1}\right)^2 \left(y_{j,k} - y_{j,k-1}\right).\end{aligned} \tag{1.40}$$

The small-amplitude, long-wavelength limit in this case is not given by a usual KP-II equation, but by a cubic KP-II equation, which is the case when the nonlinearity is cubic. Formally if we seek a continuous approximation of the form

$$x_{j+1,k} - x_{j,k} = \varepsilon A\left(\varepsilon\left(j-t\right), \varepsilon^2 k, \varepsilon^3 t\right) + \text{error},$$

and the function satisfies

$$2\partial_\xi \partial_\tau A + \frac{1}{12}\partial_\xi^4 A + \partial_\eta^2 A + \beta\ \partial_\xi^2\left(A^3\right) = 0,$$

then the function satisfies the equations of motion with an error of $O(\varepsilon^2)$. A rigorous justification of this limit is provided in chapter 6. There is a different scaling for the continuous function in the β-model than for the α-model due to the different nonlinearity. It should be possible to extend this to a more general nonlinearity, with a small modification. For example one would expect that the small-amplitude, long-wavelength limit of a FPU system with potential function like in [31]:

$$V_\gamma(r) = \frac{1}{2}r^2 + \frac{\gamma\varepsilon^2}{p+1}r^{p+1},$$

be given by a generalized KP-II equation,

$$2\partial_\xi \partial_\tau A + \frac{1}{12}\partial_\xi^4 A + \partial_\eta^2 A + \gamma\ \partial_\xi^2\left(A^p\right) = 0.$$

1.10 Outline of Results

The goal of this thesis is to study two-dimensional variants of the FPU problem in the small-amplitude, long-wavelength limit. In particular, our goal is to give a rigorous justification for the KP-II equation as an approximation of the FPU system in this limit.

In Chapter 2 we present the linear dispersion relationships of the various models discussed in section 1.9, and show the presence of the linear KP-II equation. In addition, we provide formal derivations of the KP-II equation and cubic KP-II equation in the small-amplitude, long-wavelength limits for these models.

In Chapter 3 we review local well-posedness results for the KP-II equations required in the study of the long-wavelength, small-amplitude results. Namely this chapter extends the regularity in time for solutions the KP-II equation obtained by Gallay and Schneider in [22]. This chapter also extends the regularity in time for solutions of a cubic KP-II equation, this is an extension on the local-wellposedness result of Saut from [42]. The regularity in time of solutions to the KP-II and cubic KP-II equations need to be improved in order to control the second time derivatives of

$$\partial_\xi^{-1} A(\xi, \eta, \tau) = \int_{-\infty}^{\xi} A(\xi', \eta, \tau) d\xi',$$

which shows up in the residual terms of the asymptotic expansions. This chapter also includes a review of some results by Mizumachi [35] on transverse stability of a KP-II equation linearized around a KdV soliton.

In Chapter 4 we provide a rigorous justification of the KP-II equation as the long-wavelength, small-amplitude limit of the two-dimensional FPU system on a square lattice. A precise statement of our result is given in theorem 4.1. Roughly speaking our result states that if the initial conditions for a two-dimensional FPU α-model in strain coordinates is initially $\varepsilon^{\frac{5}{2}}$-close to a sufficiently smooth, and appropriately scaled, solution of the KP-II equation, it remains $\varepsilon^{\frac{5}{2}}$-close for timescales of $O(\varepsilon^{-3})$. The justification analysis will be largely the same as the one-dimensional case considered in [43]. There are additional difficulties in introducing the strain variables, as we will need a new variable for both displacement in j and k. Additional book keeping will be required since $q_{j,k}$ is a vector quantity here. Also, some lemmas need to be extended to two dimensions, for example in comparing the ℓ^2 norm on the lattice with the H^s norm in $\mathbb{R}^2$. The asymptotic expansions also include non-local terms, in the form of anti-derivatives of $A(\xi, \eta, \tau)$, as mentioned previously. These non-local terms complicate getting error bounds on the residual terms of the expansion, but are handled in part with regularity assumptions of solutions to KP-II.

In Chapter 5 we provide a rigorous justification of the presence of a KP-II equation for the diagonal propagation of the two-dimensional FPU system. Here we need to make a careful choice of parameters for the system, discussed in section 1.9.2. A precise statement of our result is given in theorem 5.1. Roughly what we prove is that, in the appropriate variables, if the system is initially $\varepsilon^{\frac{5}{2}}$-close to a collection of functions, which depend only on a solution of the KP-II equation and its derivatives, then it remains $\varepsilon^{\frac{5}{2}}$-close for time scales of $O(\varepsilon^{-3})$. One fundamental problem that comes up with other choices of parameters is that other parameters introduce motion in the transverse direction, which can be shown using a linear dispersion analysis. A formal expansion also shows that instead of the KP-II equation satisfying the equations of motion to $O(\varepsilon^2)$, it satisfies them to $O(\varepsilon)$, which means our justification analysis would require the introduction of a perturbed KP-II equation. Additional difficulties include the presence of terms of the form $\partial_\xi^{-2} A$ in the formal expansion, for which we'd need two derivatives in time, and it isn't obvious that we could prove this for solutions to the KP-II equation, or the perturbation we have in this case. Another choice of parameters also prevents us from using the reduction $x_{j,k} = y_{j,k}$, which is used to handle some of the nonlinear terms in the justification analysis.

In Chapter 6 we provide a rigorous justification of the cubic KP-II equation in the small-amplitude short-wavelength limit of a two-dimensional cubic FPU system. The result is similar to that of chapter 4, though adapted for a cubic nonlinearity, a precise statement of the result is given in Theorem 6.1. This chapter is very similar to Chapter 4, as the expansion did not give rise to additional mathematical difficulties. The main differences are that the scaling on the amplitude and the energy in energy estimates had to be modified to accommodate the nonlinearity. The new energy estimates also grew quadratically, instead of linearly like for the α-model, so a modification had to be made to the Gronwall lemma argument. However note that the cubic KP-II equation is not integrable, and may display different stability properties of line solitary waves.

In Chapter 7 we outline a proof that the KdV solitary wave converges to a line solitary wave of the two-dimensional FPU $\alpha-$model, and that solitary waves of the FPU system are linearly stable with respect to transverse perturbations, at least under flows of the FPU system linearized at a solitary wave. A precise statement of the expected result is summarized in conjecture 7.1. In this chapter we linearize the two-dimensional FPU system in the neighbourhood of a soliton of the one dimensional FPU system. In strain variables this results in a six-by-six system, which can be block diagonalized into two three-by-three systems. We show that the real part of the spectrum of this system is bounded above by the real part of the spectrum of a one-dimensional FPU system, which is known to be asymptotically stable due to work by Friesecke and Pego [16].

In Chapter 8 we give a summary of the results obtained in the thesis. We briefly discuss how the work in this thesis could be extended.

Chapter 2

Formal Expansions in the KP limit

2.1 Dispersion relation for the α-model

Computing the equations of motion from Hamiltonian (1.23), and linearizing we get the linearized equation of motion,

$$\ddot{x}_{j,k} = c_1^2\left(x_{j+1,k} - 2x_{j,k} + x_{j-1,k}\right) + c_2^2\left(x_{j,k+1} - 2x_{j,k} + x_{j,k-1}\right). \tag{2.1}$$

Next we can compute the linear dispersion relation for the system (2.1) by introducing

$$x_{j,k}(t) = \hat{x}(\theta, \phi)\exp\left(i(\theta j + \phi k - \omega t)\right). \tag{2.2}$$

We get the linear dispersion relationship

$$\omega^2 = 4c_1^2 \sin^2\left(\frac{\theta}{2}\right) + 4c_2^2 \sin^2\left(\frac{\phi}{2}\right) \tag{2.3}$$

Expanding (2.3) in power series in θ and ϕ gives formally

$$\begin{aligned} \omega^2 = c_1^2\theta^2 + c_2^2\phi^2 - \frac{1}{12}c_1^2\theta^4 \\ - \frac{1}{12}c_2^2\phi^4 + O(\theta^6 + \phi^6). \end{aligned} \tag{2.4}$$

We suppose that that θ is of order O(ε), and ϕ is smaller and of order $O(\varepsilon^2)$. Set $\omega = c_1\theta + \Omega$. Plugging this into equation (2.4) and collecting the leading-order terms yields

$$2c_1\theta\Omega = c_2^2\phi^2 - \frac{c_1^2}{12}\theta^4 + h.o.t. \tag{2.5}$$

Neglecting the higher order terms and taking the inverse Fourier transform of (2.5) we get the linearized KP-II equation,

$$2c_1\partial_\xi\partial_\tau X = -c_2^2\partial_\eta^2 X - \frac{c_1^2}{12}\partial_\xi^4 X \tag{2.6}$$

2.2 Formal Derivation of KP-II in the $\alpha-$model

We start with the system (1.25)

$$\begin{aligned}
\ddot{x}_{j,k} &= c_1^2(\Delta_j x)_{j,k} + c_2^2(\Delta_k x)_{j,k} + \alpha_1\left[(x_{j+1,k} - x_{j,k})^2 - (x_{j,k} - x_{j-1,k})^2\right] \\
&\quad + \alpha_2\left[\frac{1}{2}\left(y_{j+1,k} - y_{j,k}\right)^2 - \frac{1}{2}\left(y_{j,k} - y_{j-1,k}\right)^2\right] \\
&\quad + \alpha_2\left[\left(x_{j,k+1} - x_{j,k}\right)\left(y_{j,k+1} - y_{j,k}\right) - \left(x_{j,k} - x_{j,k-1}\right)\left(y_{j,k} - y_{j,k-1}\right)\right] + O(3) \\
&= c_1^2(\Delta_j x)_{j,k} + c_2^2(\Delta_k x)_{j,k} + \textcircled{1} + \textcircled{2}.
\end{aligned}$$

We can approximate the lattice equations by functions $(X(\xi,\eta,\tau), Y(\xi,\eta,\tau)) : \mathbb{R}^3 \to \mathbb{R}^2$ using the following substitutions:

$$\begin{aligned}
x_{jk}(t) &= \varepsilon X(\xi,\eta,\tau) + O(\varepsilon^3), \\
y_{jk}(t) &= O(\varepsilon^3),
\end{aligned}$$

where $\xi = \varepsilon(j - c_1 t), \eta = \varepsilon^2 k, \tau = \varepsilon^3 t$. Plugging this ansatz into the equations of motion of the lattice we end up with the following for the time derivatives,

$$\ddot{x}_{j,k} = \varepsilon^3 c_1^2\partial_\xi^2 X - 2c_1\varepsilon^5\partial_\xi\partial_t X + \varepsilon^7\partial_t^2 X.$$

And expanding the finite differences in Taylor series gives

$$\begin{aligned}
c_1^2(\Delta_j x)_{j,k} &= \varepsilon c_1^2(X(\xi+\varepsilon,\eta) + X(\xi-\varepsilon,\eta) - 2X(\xi,\eta)) \\
&= \varepsilon c_1^2(\varepsilon^2\partial_\xi^2 X(\xi,\eta) + \frac{1}{12}\varepsilon^4\partial_\xi^4 X(\xi,\eta)) + O(\varepsilon^7),
\end{aligned}$$

$$\begin{aligned}
c_2^2(\Delta_k x)_{j,k} &= \varepsilon(X(\xi,\eta+\varepsilon^2) + X(\xi,\eta-\varepsilon^2) - 2X(\xi,\eta)) \\
&= \varepsilon^5 c_2^2\partial_\eta^2 X(\xi,\eta) + O(\varepsilon^7),
\end{aligned}$$

$$
\begin{aligned}
\text{①} =&\alpha_1\varepsilon^2\left[(X(\xi+\varepsilon,\eta)-X(\xi,\eta))^2-(X(\xi,\eta)-X(\xi-\varepsilon,\eta)x)^2\right) \\
&\alpha_2\varepsilon^2\left(\frac{1}{2}\varepsilon^2\left(Y(\xi+\varepsilon,\eta)-Y(\xi,\eta)\right)^2-\frac{1}{2}\varepsilon^2\left(Y(\xi,\eta)-Y(\xi-\varepsilon,\eta)\right)^2\right] \\
=&\alpha_1\varepsilon^5\partial_\xi X(\xi,\eta)\partial_\xi^2X(\xi,\eta)\ +O(\varepsilon^7),
\end{aligned}
$$

$$
\begin{aligned}
\text{②} = \alpha_2\varepsilon^4&\left[\left(X(\xi,\eta+\varepsilon^2)-X(\xi,\eta)\right)\left(Y(\xi,\eta)-Y(\xi,\eta+\varepsilon^2)\right)\right. \\
&-\left.\left(X(\xi,\eta)-X(\xi,\eta-\varepsilon^2)\right)\left(Y(\xi,\eta)-Y(\xi,\eta-\varepsilon^2)\right)\right] = O(\varepsilon^9).
\end{aligned}
$$

Bringing these together we get,

$$
\begin{aligned}
\varepsilon^3c_1^2\partial_\xi^2X-2c_1\varepsilon^5\partial_\xi\partial_tX =&\varepsilon^3c_1^2\partial_\xi^2X(\xi,\eta)+\frac{c_1^2}{12}\varepsilon^5\partial_\xi^4X(\xi,\eta)+c_2^2\varepsilon^5\partial_\eta^2X(\xi,\eta) \\
&+\alpha\varepsilon^5\partial_\xi X(\xi,\eta)\partial_\xi^2X(\xi,\eta)+O(\varepsilon^7).
\end{aligned}
$$

Dividing by ε^5, we see that the function $X(\xi,\eta,\tau)$ must solve the following KP-II equation,

$$
2c_1\partial_\xi\partial_tX+\frac{c_1^2}{12}\partial_\xi^4X+c_2^2\partial_\eta^2X+\frac{\alpha}{2}\partial_\xi\left((\partial_\xi X)^2\right)=O(\varepsilon^2).
$$

2.3 Dispersion Relation for Diagonal Propagation

The linearized equations of motion (1.32) and (1.33) are given by

$$
\begin{aligned}
\ddot{x}_{m,n} =&c^2\left(\chi_{m,n}+\chi_{m-1,n-1}-2x_{m,n}\right)+c^2\left(\chi_{m,n-1}+\chi_{m-1,n}-2x_{m,n}\right) \\
\ddot{\chi}_{m,n} =&c^2\left(x_{m+1,n+1}+x_{m,n}-2\chi_{m,n}\right)+c^2\left(x_{m+1,n}+x_{m,n+1}-2\chi_{m,n}\right)
\end{aligned}
\tag{2.7}
$$

To compute the dispersion relationship we introduce

$$
\begin{aligned}
x_{m,n}(t) &= \hat{x}\left(\theta,\phi\right)\exp\left(i\left(\theta m+\phi n-\omega t\right)\right), \\
\chi_{m,n}(t) &= \hat{\chi}\left(\theta,\phi\right)\exp\left(i\left(\theta m+\phi n-\omega t\right)\right).
\end{aligned}
\tag{2.8}
$$

Plugging the expression (2.8) into the linearized equations of motion (2.7) and simplifying we get

$$
\begin{bmatrix}
\omega^2+4c^2 & c^2\left(1+e^{-i\theta-i\phi}+e^{-i\theta}+e^{-i\phi}\right) \\
c^2\left(1+e^{i\theta+i\phi}+e^{i\theta}+e^{i\phi}\right) & \omega^2+4c^2
\end{bmatrix}
\begin{bmatrix}\hat{x}\\ \hat{\chi}\end{bmatrix}=0
$$

The determinant of this linear homogeneous equation must be zero. Computing the determinant yields

$$\left(\omega^2 + 4c^2\right)^2 - 4c^4\left(1 + \cos\theta\right)\left(1 + \cos\phi\right) = 0 \tag{2.9}$$

The dispersion relationship simplifies to,

$$\omega^2 = -\,4c^2 \pm 2c^2\left(\left(1 + \cos\theta\right)\left(1 + \cos\phi\right)\right)^{\frac{1}{2}}. \tag{2.10}$$

We will assume that θ is small, of order $O(\varepsilon)$, and ϕ is smaller, and of order $O(\varepsilon^2)$. Expanding (2.10) in Taylor series of (θ, ϕ), then truncating at order $O(\varepsilon^4)$ we have

$$\omega^2 = -4c^2 \pm c^2\left(4 - \frac{1}{2}\theta^2 - \frac{1}{2}\phi^2 + \frac{1}{96}\theta^4\right) + O(\theta^6 + \theta^2\phi^2 + \phi^4), \tag{2.11}$$

where the negative root gives the optical branch and the positive root gives the acoustic branch of the dispersion relationship. Looking at the acoustic branch of (2.11) we have

$$\omega^2 = \frac{c^2}{2}\theta^2 + \frac{c^2}{2}\phi^2 - \frac{c^2}{96}\theta^4 + O(\theta^6 + \theta^2\phi^2 + \phi^4). \tag{2.12}$$

Setting $\omega = \frac{c}{\sqrt{2}}\theta + \Omega$, and plugging this into (2.12), we have

$$\sqrt{2}c\theta\Omega = \frac{c^2}{2}\phi^2 - \frac{c^2}{96}\theta^4 + h.o.t., \tag{2.13}$$

which in light of the inverse Fourier transform gives us

$$\delta\frac{\partial^2 u}{\partial\xi\partial\tau} = -\frac{\sqrt{2}c}{4}\frac{\partial^2 u}{\partial\eta^2} - \frac{\sqrt{2}c}{192}\frac{\partial^4 u}{\partial\xi^4} \tag{2.14}$$

Remark. Recall we made a choice of parameters for the diagonal propagation case in section 1.9.2. An issue which arises for other parameters can be seen in the linear dispersion analysis. Other choices of parameters introduce a $\theta\phi$ term which appears an order lower than the terms in equation (2.13), for which we need to introduce a small propagation in the transverse direction.

2.4 Formal Derivation of KP Equation for Diagonal Propagation

In this section we will formally derive the KP-II equation in the small-amplitude, short-wavelength limit of equations (1.32) and (1.33). For the formal derivation we will let

$$\begin{aligned} x_{m,n} &= \varepsilon X\left(\varepsilon\left(m - \frac{c}{\sqrt{2}}t\right), \varepsilon^2 n, \varepsilon^3 t\right) \\ &= \varepsilon X(\xi, \eta, \tau) + O(\varepsilon^2), \end{aligned} \tag{2.15}$$

where $X : \mathbb{R}^3 \to \mathbb{R}$. The wave speed $\frac{c}{\sqrt{2}}$ is chosen to correspond to the wave speed in the linear dispersion relationship from the section 2.3. Recall that $\chi_{m,n} = x_{m+\frac{1}{2},n+\frac{1}{2}}$, so that we can use the continuous approximation,

$$\begin{aligned} \chi_{m,n} &= \varepsilon X\left(\varepsilon\left(m - \frac{c}{\sqrt{2}}t\right) + \frac{\varepsilon}{2}, \varepsilon^2 n + \frac{\varepsilon^2}{2}, \varepsilon^3 t\right) + O(\varepsilon^2) \\ &= \varepsilon X\left(\xi + \frac{\varepsilon}{2}, \eta + \frac{\varepsilon^2}{2}, \tau\right) + O(\varepsilon^2). \end{aligned} \tag{2.16}$$

We will show that plugging this ansatz into equation (1.32), and assuming that X satisfies an appropriate chosen KP-II equation, the εX is a good approximation of $x_{m,n}$ up to $O(\varepsilon^6)$. We will verify that the assumption in χ are consistent with the system of equations (1.33), at least on the linear level, before proceeding with a rigorous justification analysis in chapter 5.

$$\ddot{x}_{m,n} = \frac{c^2}{2}\varepsilon^3 \partial_\xi^2 X - \varepsilon^5 c\sqrt{2}\partial_\xi \partial_\tau X + \varepsilon^7 \partial_\tau^2 X$$

On the right hand side of (1.32) we expand each term in Taylor series,

$$\begin{aligned} c^2\left(\chi_{m,n-1} + \chi_{m-1,n} - 2x_{m,n}\right) &= c^2\left(x_{m+\frac{1}{2},n-\frac{1}{2}} + x_{m-\frac{1}{2},n+\frac{1}{2}} - 2x_{m,n}\right) \\ &= c^2\left(\varepsilon^3 \frac{1}{4}\partial_\xi^2 X - \frac{\varepsilon^4}{2}\partial_\xi \partial_\eta X + \frac{\varepsilon^5}{4}\partial_\eta^2 X + \varepsilon^5 \frac{1}{4!}\frac{1}{2^3}\partial_\xi^4 X - \varepsilon^6 \frac{1}{2^3}\frac{1}{3!}\partial_\xi^3 \partial_\eta X\right) + O\left(\varepsilon^7\right) \end{aligned}$$

$$\begin{aligned} c^2\left(\chi_{m,n} + \chi_{m-1,n-1} - 2x_{m,n}\right) &= c_1^2\left(x_{m+\frac{1}{2},n+\frac{1}{2}} + x_{m-\frac{1}{2},n-\frac{1}{2}} - 2x_{m,n}\right) \\ &= c^2\left(\varepsilon^3 \frac{1}{4}\partial_\xi^2 X + \frac{\varepsilon^4}{2}\partial_\xi \partial_\eta X + \frac{\varepsilon^5}{4}\partial_\eta^2 X + \varepsilon^5 \frac{1}{4!}\frac{1}{2^3}\partial_\xi^4 X + \varepsilon^6 \frac{1}{2^3}\frac{1}{3!}\partial_\xi^3 \partial_\eta X\right) + O\left(\varepsilon^7\right) \end{aligned}$$

$$
\begin{aligned}
(\chi_{m,n}- x_{m,n})^2 - (x_{m,n} - \chi_{m-1,n-1})^2 &= \left(x_{m+\frac{1}{2},n+\frac{1}{2}} - x_{m,n}\right)^2 - \left(x_{m,n} - x_{m-\frac{1}{2},n-\frac{1}{2}}\right)^2 \\
&= \left(\frac{1}{2}\varepsilon^2\partial_\xi X + \frac{1}{2}\varepsilon^3\partial_\eta X + \frac{1}{8}\varepsilon^3\partial_\xi^2 X + \frac{\varepsilon^4}{48}\partial_\xi^3 X + \frac{\varepsilon^4}{8}\partial_\xi\partial_\eta X + O(\varepsilon^5)\right)^2 \\
&\quad - \left(\frac{1}{2}\varepsilon^2\partial_\xi X + \frac{1}{2}\varepsilon^3\partial_\eta X - \frac{1}{8}\varepsilon^3\partial_\xi^2 X + \frac{\varepsilon^4}{48}\partial_\xi^3 X - \frac{\varepsilon^4}{8}\partial_\xi\partial_\eta X + O(\varepsilon^5)\right)^2 \\
&=\varepsilon^5\left(\frac{1}{4}\partial_\xi X\partial_\xi^2 X\right) + \varepsilon^6\left(\frac{1}{4}\partial_\eta X\partial_\xi^2 X + \frac{1}{4}\partial_\xi X\partial_\xi\partial_\eta X\right) + O(\varepsilon^7)
\end{aligned}
$$

$$
\begin{aligned}
(\chi_{m,n-1}- x_{m,n})^2 - (x_{m,n} - \chi_{m-1,n})^2 &= \left(x_{m+\frac{1}{2},n-\frac{1}{2}} - x_{m,n}\right)^2 - \left(x_{m,n} - x_{m-\frac{1}{2},n+\frac{1}{2}}\right)^2 \\
&= \left(\frac{1}{2}\varepsilon^2\partial_\xi X - \frac{1}{2}\varepsilon^3\partial_\eta X + \frac{1}{8}\varepsilon^3\partial_\xi^2 X + \frac{\varepsilon^4}{48}\partial_\xi^3 X - \frac{\varepsilon^4}{8}\partial_\xi\partial_\eta X + O(\varepsilon^5)\right)^2 \\
&\quad - \left(\frac{1}{2}\varepsilon^2\partial_\xi X - \frac{1}{2}\varepsilon^3\partial_\eta X - \frac{1}{8}\varepsilon^3\partial_\xi^2 X + \frac{\varepsilon^4}{48}\partial_\xi^3 X + \frac{\varepsilon^4}{8}\partial_\xi\partial_\eta X + O(\varepsilon^5)\right)^2 \\
&=\varepsilon^5\left(\frac{1}{4}\partial_\xi X\partial_\xi^2 X\right) - \varepsilon^6\left(\frac{1}{4}\partial_\eta X\partial_\xi^2 X + \frac{1}{4}\partial_\xi X\partial_\xi\partial_\eta X\right) + O(\varepsilon^7)
\end{aligned}
$$

Bringing all expansions to equation (1.32) and dividing by ε^5 yields the following equation

$$
c\sqrt{2}\partial_\xi\partial_\tau X + \frac{c^2}{96}\partial_\xi^4 X + \frac{c^2}{2}\partial_\eta^2 X + \frac{\alpha}{2}\partial_\xi\left((\partial_\xi X)^2\right) = O(\varepsilon^2) \tag{2.17}
$$

2.5 Derivation of the Cubic KP Equation for β-model

For the formal expansion in the cubic case we will first introduce the following strain variables:

$$
\begin{aligned}
u_{j,k}^{(1)} &= x_{j+1,k} - x_{j,k}, \\
u_{j,k}^{(2)} &= x_{j,k+1} - x_{j,k}, \\
v_{j,k}^{(1)} &= y_{j+1,k} - y_{j,k}, \\
v_{j,k}^{(2)} &= y_{j,k+1} - y_{j,k},
\end{aligned} \tag{2.18}
$$

and the second order finite difference operators $\Delta_j x_{j,k} = x_{j+1,k} - 2x_{j,k} + x_{j-1,k}$ and $\Delta_k x_{j,k} = x_{j,k+1} - 2x_{j,k} + x_{j,k-1}$.

From (1.39) we can write the equation of motion

$$\begin{aligned}
\ddot{u}^{(1)}_{j,k} =& \ddot{x}_{j+1,k} - \ddot{x}_{j,k} \\
=& \Delta_j \left(u^{(1)}_{j,k}\right) + \Delta_k \left(u^{(1)}_{j,k}\right) + \beta\Delta_j \left(\left(u^{(1)}_{j,k}\right)^3\right) \\
&+ \beta \left(\left(u^{(2)}_{j+1,k}\right)^3 - \left(u^{(2)}_{j+1,k-1}\right)^3 - \left(u^{(2)}_{j,k}\right)^3 + \left(u^{(2)}_{j,k-1}\right)^3\right) \\
&+ \beta\Delta_j \left(\left(u^{(1)}_{j,k}\right)\left(v^{(1)}_{j,k}\right)^2\right) \\
&+ \beta \left(\left(u^{(2)}_{j+1,k}\right)\left(v^{(2)}_{j+1,k}\right)^2 - \left(u^{(2)}_{j+1,k-1}\right)\left(v^{(2)}_{j+1,k-1}\right)^2\right. \\
&\left. - \left(u^{(2)}_{j,k}\right)\left(v^{(2)}_{j,k}\right)^2 + \left(u^{(2)}_{j,k-1}\right)\left(v^{(2)}_{j,k-1}\right)^2\right)
\end{aligned} \tag{2.19}$$

We set

$$\begin{aligned}
u^{(1)}_{j,k} &= \varepsilon U + O(\varepsilon^2), \\
u^{(2)}_{j,k} &= \varepsilon^2 \tilde{U} + O(\varepsilon^3), \\
v^{(1)}_{j,k} &= \varepsilon^2 V + O(\varepsilon^3), \\
v^{(2)}_{j,k} &= \varepsilon^2 \tilde{V} + O(\varepsilon^3),
\end{aligned} \tag{2.20}$$

and, $\xi = \varepsilon\,(j - t)\,, \eta = \varepsilon^2 k, \tau = \varepsilon^3 t$. Expanding the left hand side we have

$$\ddot{u}^{(1)}_{j,k} = \varepsilon^3 \partial^2_\xi U - 2\varepsilon^5 \partial_\xi \partial_\tau U + \varepsilon^7 \partial^2_\tau U.$$

Expanding the right hand side we have,

$$\Delta_j \left(u^{(1)}_{j,k}\right) = \varepsilon^3 \partial^2_\xi U + \varepsilon^5 \frac{1}{12} \partial^4_\xi U + O(\varepsilon^7),$$

$$\Delta_k \left(u^{(1)}_{j,k}\right) = \varepsilon^5 \partial^2_\xi U + O(\varepsilon^7),$$

$$\beta\Delta_j \left(\left(u^{(1)}_{j,k}\right)^3\right) = \varepsilon^5 \partial^2_\xi \left(U^3\right) + O(\varepsilon^7),$$

$$\beta\Delta_j \left(\left(u^{(1)}_{j,k}\right)\left(v^{(1)}_{j,k}\right)^2\right) = O(\varepsilon^7),$$

$$\beta\left(\left(u^{(2)}_{j+1,k}\right)^3-\left(u^{(2)}_{j+1,k-1}\right)^3\right)=O(\varepsilon^8),$$

$$\beta\left(-\left(u^{(2)}_{j,k}\right)^3+\left(u^{(2)}_{j,k-1}\right)^3\right)=O(\varepsilon^8),$$

$$\beta\left(\left(u^{(2)}_{j+1,k}\right)\left(v^{(2)}_{j+1,k}\right)^2-\left(u^{(2)}_{j,k}\right)\left(v^{(2)}_{j,k}\right)^2\right)=O(\varepsilon^7),$$

$$\beta\left(\left(u^{(2)}_{j,k-1}\right)\left(v^{(2)}_{j,k-1}\right)^2\right)-\left(u^{(2)}_{j+1,k-1}\right)\left(v^{(2)}_{j+1,k-1}\right)^2=O(\varepsilon^7).$$

Dividing through by ε^5 we see that $U(\xi,\eta,\tau)$ must satisfy the following cubic KP-II equation,

$$2\partial_\xi\partial_\tau U+\frac{1}{12}\partial_\xi^4 U+\partial_\eta^2 U+\beta\ \partial_\xi^2\left(U^3\right)=O(\varepsilon^2). \qquad (2.21)$$

Chapter 3

Properties of the KP-II Equation

3.1 Well-Posedness for the KP-II Equation

Well-posedness of the KP-II equation

$$2\partial_\tau A + \partial_\xi \left(A^2\right) + \partial_\xi^3 A + \partial_\xi^{-1}\partial_\eta^2 A = 0. \tag{3.1}$$

has been thoroughly studied. Bourgain has established that the equation is globally well posed for initial data in $H^s\left(\mathbb{T}^2\right)$ or $H^s\left(\mathbb{R}^2\right)$ for $s \geq 0$ in [6], provided that the initial data satisfies $\int_{\mathbb{R}} u\left(\xi, \eta, 0\right) d\xi = 0$ for every η. The result was proven by combining local well-posedness and conservation laws, namely conservation of the L^2 norm.

In [46] Tzvetkov shows that the local well-posedness result can be extended to Sobolev spaces of the type $H^{s_1,s_2}\left(\mathbb{R}^2\right)$, with $s_1 > -\frac{1}{4}, s_2 \geq 0$, the global result can be obtain by conservation laws again provided that $s_1 \geq 0$. In [45] Takaoka shows that the zero mean constraint can be dropped in the local-wellposedness result.

Since we require some regularity of the local solutions in time, we will be building on the local well-posedness result by S. Ukai [47], which was extended by Gallay and Schneider[22].

Theorem 3.1 ([22][47])**.** *Let $s \geq 0$. For any $A_0 \in H^{s+6}\left(\mathbb{R}^2\right)$ such that $\partial_\xi^{-2}\partial_\eta^2 A_0 \in H^{s+6}\left(\mathbb{R}^2\right)$, there exists $\tau_0 > 0$ such that the KP-II equation* (3.1) *has a unique solution*

$$A \in C^0\left(\left[-\tau_0, \tau_0\right], H^{s+6}\right) \cap C^1\left(\left[-\tau_0, \tau_0\right], H^{s+3}\right) \cap C^2\left(\left[-\tau_0, \tau_0\right], H^s\right),$$
$$\text{and } \partial_\xi^{-1}\partial_\eta A \in C^1\left(\left[-\tau_0, \tau_0\right], H^{s+2}\right),$$

with initial data $A\left(\cdot, \cdot, 0\right) = A_0$.

In the next section we will extend this result so that we have a solution in $C^3([-\tau_0,\tau_0],H^s)$.

In chapter 6 we study the β-model for FPU in two dimensions, the limit for which is a cubic KP-II equation. The cubic KP-II equation in chapter 6 takes the form

$$\partial_\tau A + \frac{1}{3}\partial_\xi\left(A^3\right) + \partial_\xi^3 A + \partial_\xi^{-1}\partial_\eta^2 A = 0. \tag{3.2}$$

We will require the same regularity for equation (3.2) as we do for (3.1). The well-posedness of Cauchy problem for the generalized KP equations is established by J.C. Saut in [42], which for the cubic KP-II equation (3.2) is given in the following theorem.

Theorem 3.2 ([42])**.** *Let $s \geq 0$, and A_0 be a function such that $A_0 \in H^{s+3}(\mathbb{R}^2), \partial_\xi^{-1}A_0 \in H^{s+3}(\mathbb{R}^2)$ and $\partial_\xi^{-2}\partial_\eta^2 A_0 \in L^2(\mathbb{R}^2),.$ Then there exists a $\tau_0 > 0$ such that the cubic $KP-II$ equation (3.2) has a unique solution*

$$\begin{aligned} &A \in C^0\left([-\tau_0,\tau_0],H^{s+3}\right) \cap C^1\left([-\tau_0,\tau_0],H^s\right) \\ &\text{and } \partial_\xi^{-1}\partial_\eta A \in C^1\left([-\tau_0,\tau_0],H^{s+2}\right), \end{aligned}$$

with initial data $A(\cdot,\cdot,0) = A_0$.

The goal of section 3.3 is to extend this result to the appropriate regularity.

3.2 Extended Results on the Well-Posedness of the KP-II Equation

The next lemma will allow for control of the L^2 norm of terms of the form $\partial_\xi^{-1}\partial_\tau^2 A$, where A is a solution to the KP-II equation (3.1). These terms arise in the Taylor remainder of Res^W, which will be dealt with rigorously in section 4.3.

Beyond just existence and uniqueness of solutions to the associated KP-II equation we will require that the antiderivative term $\partial_\xi^{-1}A$ as a function of τ is not only continuous, but is twice continuously differentiable, in some Sobolev space. Theorem 3.1, of Gallay-Schneider [22], gives us that as a function of τ the antiderivative term is once continuously differentiable. Our goal is, given some additional constraints on the initial data, to extend this result to C^3, which by taking two time derivatives of (3.1) will give us that the antiderivative term is in the appropriate space.

Lemma 3.1. *Let $s \geq 0$. For any $A_0 \in H^{s+9}(\mathbb{R}^2)$ such that $\partial_\xi^{-2}\partial_\eta^2 A_0 \in H^{s+9}(\mathbb{R}^2)$ and $\partial_\xi^{-1}\partial_\eta^2\left[\partial_\xi^{-2}\partial_\eta^2 A(0) + A(0)^2\right] \in H^{s+2}(\mathbb{R}^2)$ then there exists some*

$\tau_0 > 0$ *such that the KP-II equation given by*

$$2\partial_\tau A + \partial_\xi \left(A^2\right) + \partial_\xi^3 A + \partial_\xi^{-1}\partial_\eta^2 A = 0 \tag{3.3}$$

has a unique solution with

$$A \in C^0\left(\left[-\tau_0, \tau_0\right], H^{s+9}\right) \cap C^1\left(\left[-\tau_0, \tau_0\right], H^{s+6}\right) \cap C^2\left(\left[-\tau_0, \tau_0\right], H^{s+3}\right) \cap C^3\left(\left[-\tau_0, \tau_0\right], H^{s}\right),$$

moreover

$$\partial_\xi^{-1}\partial_\eta^2\partial_\tau^2 A \in C^0\left(\left[-\tau_0, \tau_0\right], H^s\right).$$

Proof. Setting $D = \partial_\xi^{-2}\partial_\eta^2 A$, where A solves the KP-II equation (3.1). By theorem 3.1, since the initial data satisfies any $A_0 \in H^{s+9}\left(\mathbb{R}^2\right)$ and $\partial_\xi^{-2}\partial_\eta^2 A_0 \in H^{s+9}\left(\mathbb{R}^2\right)$, the KP-II equation has a solution with $A \in C^0\left(\left[-\tau_0, \tau_0\right], H^{s+9}\right) \cap C^1\left(\left[-\tau_0, \tau_0\right], H^{s+6}\right) \cap C^2\left(\left[-\tau_0, \tau_0\right], H^{s+3}\right)$ and $\partial_\xi^{-1}\partial_\eta A \in C^1\left(\left[-\tau_0, \tau_0\right], H^{s+5}\right)$. Taking $\partial_\eta^2\left(\cdot\right)$ of the KP-II equation (3.1) we get

$$2\partial_\xi^2\partial_\tau D + \partial_\xi\partial_\eta^2\left(A^2\right) + \partial_\xi^5 D + \partial_\xi\partial_\eta^2 D = 0$$

If we introduce $\tilde{D} = D + A^2$ we can rewrite the above as

$$2\partial_\xi^2\partial_\tau \tilde{D} + \partial_\xi^5 \tilde{D} + \partial_\xi\partial_\eta^2 \tilde{D} - \left(2\partial_\xi^2\partial_\tau\left(A^2\right) + \partial_\xi^5\left(A^2\right)\right) = 0$$

Taking $\partial_\xi^{-2}\left(\cdot\right)$ of the above equation yields the evolution equation

$$\partial_\tau \tilde{D} + \frac{1}{2}\partial_\xi^3 \tilde{D} + \frac{1}{2}\partial_\xi^{-1}\partial_\eta^2 \tilde{D} - \left(\partial_\tau\left(A^2\right) + \frac{1}{2}\partial_\xi^3\left(A^2\right)\right) = 0$$

Defining $S(\tau) = e^{\tau\Omega}$ where $\Omega = \frac{1}{2}\left(\partial_\xi^{-1}\partial_\eta^2 + \partial_\xi^3\right)$, note that $S\left(\tau\right)$ is unitary in L^2. Using Duhamel's principle we can write the above PDE in integral form as

$$\tilde{D}\left(\tau\right) = S(\tau)\tilde{D}\left(0\right) + \int_0^\tau S(\tau - s)\left(\partial_s\left(A^2\right) + \frac{1}{2}\partial_\xi^3\left(A^2\right)\right) ds.$$

Expressing back $D = \tilde{D} - A$ gives us an integral equation for $D(\tau)$ of the form

$$D\left(\tau\right) = S(\tau)\tilde{D}\left(0\right) - A(\tau)^2 + \int_0^\tau S(\tau - s)\left(\partial_s\left(A^2\right) + \frac{1}{2}\partial_\xi^3\left(A^2\right)\right) ds.$$

Taking the time derivative, rewriting $\frac{d}{d\tau}S(\tau - s) = -\frac{d}{ds}S(\tau - s)$, we get

$$\begin{aligned}\partial_\tau D(\tau) = &- S(\tau)\Omega\tilde{D}(0) + S(\tau)\left(2A(0)\partial_\tau A(0) + \frac{1}{2}\partial_\xi^3\left(A^2(0)\right)\right)\\ &- 2A\partial_\tau A(\tau) + \int_0^\tau S(\tau - s)\,\partial_\xi^3\left(A(s)\frac{d}{ds}(A(s))\right)ds\\ &+ 2\int_0^\tau S(\tau - s)\frac{d}{ds}\left(A(s)\frac{d}{ds}(A(s))\right)ds.\end{aligned}$$

Since by assumption $\Omega\tilde{D}(0) \in H^{s+2}$, then we have that $D'(\tau) \in C^0\left([-\tau_0, \tau_0], H^{s+2}\right)$. We compute $\partial_\tau^3 A$ below:

$$\partial_\tau A = -\frac{1}{2}\left(\partial_\xi\left(A^2\right) + \partial_\xi^3 A + \partial_\xi^{-1}\partial_\eta^2 A\right)$$

$$\partial_\tau^2 A = -\frac{1}{2}\left(2\partial_\xi\left(AA_\tau\right) + \partial_\xi^3 A_\tau + \partial_\xi^{-1}\partial_\eta^2 A_\tau\right)$$

$$\partial_\tau^3 A = -\frac{1}{2}\left(2\partial_\xi\left((A_\tau)^2 + AA_{\tau\tau}\right) + \partial_\xi^3 A_{\tau\tau} + \partial_\xi^{-1}\partial_\eta^2 A_{\tau\tau}\right)$$

Since $A \in C^0\left([-\tau_0, \tau_0], H^{s+9}\right) \cap C^1\left([-\tau_0, \tau_0], H^{s+6}\right) \cap C^2\left([-\tau_0, \tau_0], H^{s+3}\right)$, by theorem 3.1, then all but the last term in $\partial_\tau^3 A$ are in $C^0\left([-\tau_0, \tau_0], H^s\right)$. We check the final term

$$\begin{aligned}\partial_\xi^{-1}\partial_\eta^2\partial_\tau^2 A &= -\frac{1}{2}\left(2\partial_\eta^2\left(A\partial_\tau A\right) + \partial_\xi^2\partial_\eta^2\partial_\tau A + \partial_\xi^{-2}\partial_\eta^4\partial_\tau A\right)\\ &= -\frac{1}{2}\left(2\partial_\eta^2\left(A\partial_\tau A\right) + \partial_\xi^2\partial_\eta^2\partial_\tau A + \partial_\eta^2\partial_\tau D(\tau)\right),\end{aligned}$$

which is in $C^0\left([-\tau_0, \tau_0], H^s\right)$, and the result follows. □

Note, equation (3.3) differs slightly from (4.3), in the choice of constants. This is because the constants in the KP-II equation can be changed, as long as each constant is positive, through a scaling of the variables, and the argument involves less bookkeeping without having to carry through the constants in equation (4.3).

3.3 Extended results on well-posedness of the cubic KP-II equation

The following lemma is a direct generalization of a lemma in [22], and builds on the well-posedness result from [42].

Lemma 3.2. *Let $s \geq 0$. For any $A_0 \in H^{s+6}(\mathbb{R}^2)$ such that $\partial_\xi^{-2}\partial_\eta^2 A_0 \in H^{s+6}(\mathbb{R}^2)$, there exists a $\tau_0 > 0$ such that the cubic cKP-II equation* (6.3) *has a unique solution*

$$\begin{aligned} &A \in C^0\left([-\tau_0,\tau_0], H^{s+6}\right) \cap C^1\left([-\tau_0,\tau_0], H^{s+3}\right) \cap C^2\left([-\tau_0,\tau_0], H^{s}\right) \\ &\partial_\xi^{-1}\partial_\eta A \in C^0\left([-\tau_0,\tau_0], H^{s+2}\right) \end{aligned} \tag{3.4}$$

Proof. Setting $B = \partial_\xi^{-1}\partial_\eta A$ we write the cubic KP equation as the equivalent system

$$\partial_\tau A + A^2\partial_\xi A + \partial_\xi^3 A + \partial_\eta B = 0, \partial_\eta A = \partial_\xi B. \tag{3.5}$$

By a result by J.C. Saut [42] the system (3.5) has a solution with $A \in C^0([-\tau_0,\tau_0], H^{s+6}) \cap C^1([-\tau_0,\tau_0], H^{s+3})$, $B \in C^0([-\tau_0,\tau_0], H^{s+5})$. Introducing the operator $\Omega = \partial_\xi^3 + \partial_\xi^{-1}\partial_\eta^2$ we can write $B(\xi,\eta,\tau)$, using Duhamel's principle,

$$B(\xi,\eta,\tau) = \exp(\Omega\tau) B_0 - \frac{1}{3}\int_0^\tau \exp(\Omega(\tau - s))\,\partial_\eta\left(A^3\right) ds.$$

Taking a derivative in τ, since $\Omega B_0 \in H^3$ by hypothesis, we see that $B \in C^1([-\tau_0,\tau_0], H^{s+2})$. Taking a derivative in τ of (3.4) we see that $A \in C^2([-\tau_0,\tau_0], H^s)$ as well. □

The following lemma ensures that for a solution to the cubic KP-II equation (3.2) we have three time derivatives of A and two time derivatives of the antiderivative term $\partial_\xi^{-1}A$ in an appropriate Sobolev space. In chapter 6 these terms come appear in the asymptotic expansions necessary to justify the cubic KP-II equation in the small-amplitude long-wavelength limit of the β-model.

Lemma 3.3. *Let $s \geq 0$. For any $A_0 \in H^{s+9}(\mathbb{R}^2)$ such that $\partial_\xi^{-2}\partial_\eta^2 A_0 \in H^{s+9}(\mathbb{R}^2)$ and $\partial_\xi^{-1}\partial_\eta^2\left[\partial_\xi^{-2}\partial_\eta^2 A(0) + A(0)^2\right] \in H^{s+2}(\mathbb{R}^2)$ then there exists some $\tau_0 > 0$ such that the cKP-II equation* (6.3) *has a unique solution with*

$$A \in C^0\left([-\tau_0,\tau_0], H^{s+9}\right) \cap C^1\left([-\tau_0,\tau_0], H^{s+6}\right) \cap C^2\left([-\tau_0,\tau_0], H^{s+3}\right) \cap C^3\left([-\tau_0,\tau_0], H^{s}\right),$$

moreover

$$\partial_\xi^{-1}\partial_\eta^2\partial_\tau^2 A \in C^0\left([-\tau_0,\tau_0], H^{s}\right).$$

Proof. Setting $D = \partial_\xi^{-2}\partial_\eta^2 A$, where A solves the KP-II equation (6.3). By theorem 3.1, since the initial data satisfies any $A_0 \in H^{s+9}(\mathbb{R}^2)$ and $\partial_\xi^{-2}\partial_\eta^2 A_0 \in H^{s+9}(\mathbb{R}^2)$, the KP-II equation has a solution with $A \in C^0([-\tau_0, \tau_0], H^{s+9}) \cap C^1([-\tau_0, \tau_0], H^{s+6}) \cap C^2([-\tau_0, \tau_0], H^{s+3})$ and $\partial_\xi^{-1}\partial_\eta A \in C^1([-\tau_0, \tau_0], H^{s+5})$. Taking $\partial_\eta^2(\cdot)$ of the KP-II equation (3.1) we get

$$\partial_\xi^2\partial_\tau D + \partial_\xi\partial_\eta^2\left(A^3\right) + \partial_\xi^5 D + \partial_\xi\partial_\eta^2 D = 0$$

If we introduce $\tilde{D} = D + A^3$ we can rewrite the above as

$$\partial_\xi^2\partial_\tau\tilde{D} + \partial_\xi^5\tilde{D} + \partial_\xi\partial_\eta^2\tilde{D} - \left(2\partial_\xi^2\partial_\tau\left(A^3\right) + \partial_\xi^5\left(A^3\right)\right) = 0$$

Taking $\partial_\xi^{-2}(\cdot)$ of the above equation yields the evolution equation

$$\partial_\tau\tilde{D} + \frac{1}{2}\partial_\xi^3\tilde{D} + \frac{1}{2}\partial_\xi^{-1}\partial_\eta^2\tilde{D} - \left(\partial_\tau\left(A^3\right) + \frac{1}{2}\partial_\xi^3\left(A^3\right)\right) = 0$$

The remainder of the proof is similar to lemma 3.1. □

3.4 Stability of line solitons in the KP-II equation

As a part of a larger result on the stability of line solitons for the KP-II equation in $\mathbb{R}^2$ in [35], Mizumachi studied the stability of the KP-II equation linearized near a solitary wave of the KdV equation. Specifically they looked at the KP-II equation

$$\partial_\xi\left(\partial_\tau u + \partial_\xi^3 u + 3\partial_\xi(u)^2\right) + 3\partial_\eta^2 u = 0, \tag{3.6}$$

for $\tau > 0$, and $(x, y) \in \mathbb{R}^2$. Looking for a solution of the form $u(\xi, \eta, \tau) = \phi(\xi - 4\tau) + U(\xi - 4\tau, \eta, \tau)$, where $\phi(x)$ is the solitary wave solution of the associated KdV equation, and linearizing the KP-II equation around $U = 0$ yields the linearized KP-II equation

$$\partial_\tau U = \mathcal{L}_M U, \qquad \mathcal{L}_M U = -\partial_\xi^3 U + 4\partial_\xi U - 3\partial_\xi^{-1}\partial_\eta^2 U - 6\partial_\xi(\phi U) \tag{3.7}$$

Taking a Fourier transform in η gives the operator

$$\mathcal{L}_M(\hat{\eta})U = -\partial_\xi^3 U + 4\partial_\xi U - 3\hat{\eta}^2\partial_\xi^{-1} U - 6\partial_\xi(\phi U). \tag{3.8}$$

It was known since the work of V. Zakharov [49] and S. Burtsev [7] that the spectral stability problem $\mathcal{L}_M u = \lambda u$ has a pair of resonant modes, which are exponentially decaying as $\xi \to +\infty$ but exponentially growing as $\xi \to -\infty$.

The rest of the continuous spectrum is located for $\lambda \in i\mathbb{R}$. Mizumachi [35] analyzed the resonant modes and the spectral projections in the exponentially weighted spaces in order to obtain linearized stability of the line solitions in the KP-II equation.

The space studied in the stability result is $L_a^2(\mathbb{R}^2) = \{g \mid e^{ax}g \in L^2(\mathbb{R}^2)\}$, endowed with the norm $\|e^{ax}g\|_{L^2}$. The resonant modes are functions $g(\cdot,\hat{\eta}) \in L_a^2(\mathbb{R})$ of the form $\left\{g(\xi,\hat{\eta})e^{i\eta\hat{\eta}}\right\}$ such that g is a solution to $\mathcal{L}_M(\hat{\eta})u = \lambda u$. The expectation is that the continuous spectrum is contained in the left complex half plane, with a pair of isolated eigenvalues for resonant modes.

Lemma 3.4. *Let* $\hat{\eta} \in \mathbb{R} \setminus \{0\}, \lambda(\hat{\eta}) = 4i\hat{\eta}\sqrt{1+i\hat{\eta}}$ *and*

$$g(\xi,\hat{\eta}) = -\frac{i}{2\hat{\eta}\sqrt{1+i\hat{\eta}}}\partial_\xi^2\left(e^{-\sqrt{1+i\hat{\eta}}\xi}\operatorname{sech}\xi\right),$$

$$g^*(\xi,\hat{\eta}) = \partial_\xi\left(e^{\sqrt{1-i\hat{\eta}}\xi}\operatorname{sech}\xi\right).$$

Then

$$\mathcal{L}_M(\hat{\eta})g(\xi,\pm\hat{\eta}) = \lambda(\pm\hat{\eta})g(\xi,\pm\eta), \tag{3.9}$$

$$\mathcal{L}_M^*(\hat{\eta})g^*(\xi,\pm\hat{\eta}) = \lambda(\pm\hat{\eta})g^*(\xi,\pm\eta), \tag{3.10}$$

$$\int_{\mathbb{R}} g(\xi,\hat{\eta})\overline{g^*(\xi,\hat{\eta})}d\xi = 1, \qquad \int_{\mathbb{R}} g(\xi,\hat{\eta})\overline{g^*(\xi,-\hat{\eta})}d\xi = 0. \tag{3.11}$$

Here $\mathcal{L}_M^*$ *is the adjoint operator of* $\mathcal{L}_M$*, i.e.* $\langle\mathcal{L}_M f, g\rangle = \langle f, \mathcal{L}_M^* g\rangle$*, for* $f \in H_a^3$*,* $g \in L_a^2$*, and* $\langle\cdot,\cdot\rangle$ *denotes the* L_a^2 *inner product.* g^* *denotes eigenfunctions of the adjoint operator* $\mathcal{L}_M^*$*.*

To resolve the singularity of $g(\xi,\hat{\eta})$ and the degeneracy of $g^*(\xi,\hat{\eta})$ the resonant modes were decomposed by defining

$$g_1(\xi,\hat{\eta}) = g(\xi,\hat{\eta}) + g(\xi,-\hat{\eta}), g_2(\xi,\hat{\eta}) = i\hat{\eta}\left[g(\xi,\hat{\eta}) - g(\xi,-\hat{\eta})\right],$$

$$g_1^*(\xi,\hat{\eta}) = \frac{1}{2}\left[g(\xi,\hat{\eta}) + g(\xi,-\hat{\eta})\right], g_2^*(\xi,\hat{\eta}) = \frac{i}{2\hat{\eta}}\left[g^*(\xi,\hat{\eta}) - g^*(\xi,-\hat{\eta})\right].$$

The essential spectrum of the linearized KP-II operator in L_a^2 is given in the following lemma

Lemma 3.5. *Define the Fourier symbol* $p(\hat{\xi},\hat{\eta}) = \hat{\xi}^3 + 4\hat{\xi} - 3\frac{\hat{\eta}^2}{\hat{\xi}}$*, and* $\nu(\hat{\eta}) = \operatorname{Re}\beta(\hat{\eta}) - 1$*. Let* $a \in (0,2)$ *and* η_* *be a positive number satisfying* $\nu(\eta_*) = a$*.*

1. *If* $\hat{\eta} \in (-\eta_*,\eta_*)$*, then* $\mathcal{L}(\hat{\eta})$ *has no eigenvalue other than* $\lambda(\pm\hat{\eta})$ *and*

$$\sigma(\mathcal{L}(\hat{\eta})) = \{\lambda(\pm\hat{\eta})\} \cup \{ip(\hat{\xi}+ia,\hat{\eta}) | \hat{\xi} \in \mathbb{R}\}.$$

2. If $\hat{\eta} \in \mathbb{R} \setminus [-\eta_*, \eta_*]$*, then*

$$\sigma(\mathcal{L}(\hat{\eta})) = \{ip(\hat{\xi} + ia, \hat{\eta}) | \hat{\xi} \in \mathbb{R}\}.$$

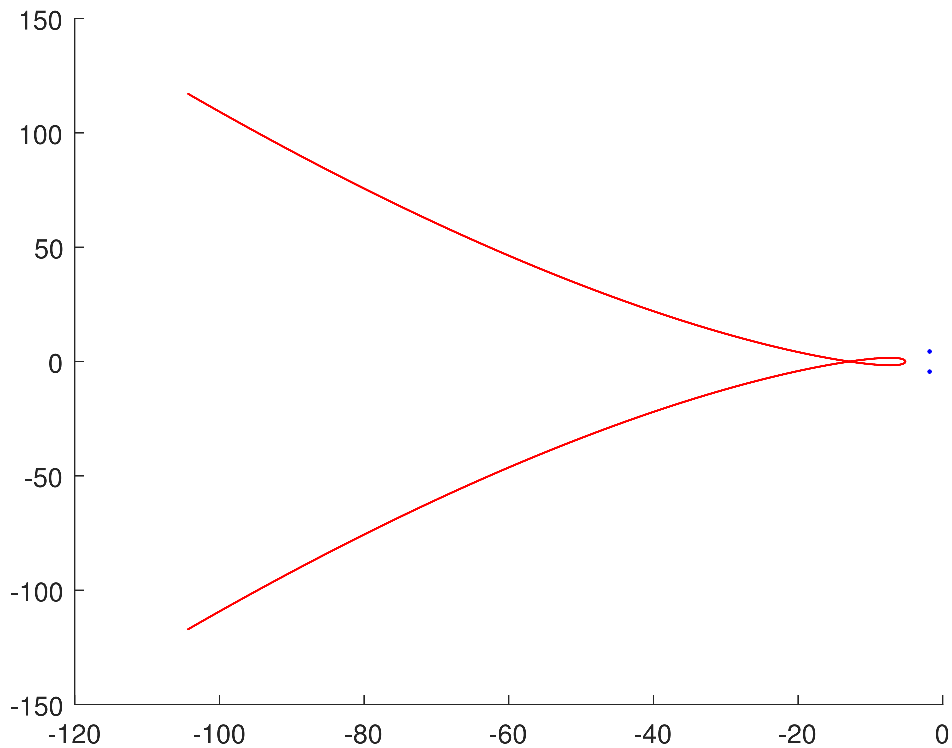


Figure 3.6: Plot of of the essential spectrum of $\mathcal{L}(\hat{\eta})$ when $a = 1.35, \hat{\eta} = 1$ for which $\eta_* \approx 10$ satisfies $\nu(\eta_*) = a$.

The projection onto the low frequency resonant modes is given by

$$P_0(\hat{\eta}_0) f(\xi, \eta) = \frac{1}{2\pi} \sum_{k=1,2} \int_{-\hat{\eta}_0}^{\hat{\eta}_0} a_k(\hat{\eta}) g_k(\xi, \hat{\eta}) e^{i\hat{\eta}\eta} d\hat{\eta},$$

where,

$$a_k(\hat{\eta}) = \sqrt{2\pi} \int_{\mathbb{R}} \left(\mathcal{F}_\eta f\right)(\xi, \hat{\eta}) g_k^*(\bar{\xi}, \hat{\eta}) d\xi.$$

Let $0 < \hat{\eta}_1 \leq \hat{\eta}_2 \leq \infty$, $P_1(\hat{\eta}_1, \hat{\eta}_2)$ and $P_2(\hat{\eta}_1, \hat{\eta}_2)$ are the projections defined by

$$P_1(\hat{\eta}_1, \hat{\eta}_2) u(\xi, \eta) = \frac{1}{2\pi} \int_{\hat{\eta}_1 \leq |\hat{\eta}| \leq \hat{\eta}_2} \int_{\mathbb{R}} u(\xi, \eta_1) e^{i\hat{\eta}(\eta - \eta_1)} d\eta_1 d\hat{\eta},$$

$$P_2(\hat{\eta}_1, \hat{\eta}_2) = P_1(0, \hat{\eta}_2) - P_0(\hat{\eta}_1).$$

Here $P_2(\hat{\eta}_1, \hat{\eta}_2)$ is the projection onto the low frequency non-resonant modes. Mizumachi then proves that solutions projected on the non-resonant modes decay exponentially as $t \to \infty$, which is summarized in the following lemma,

Lemma 3.6. *Let $a \in (0, 2)$ and $\hat{\eta}_1$ be a positive number satisfying $\nu(\hat{\eta}_1) < a$. Then there exists positive constants K and b such that for any $\hat{\eta}_0 \in (0, \hat{\eta}_1]$, $f \in X = L^2(\mathbb{R}^2; e^{2a\xi} d\xi d\eta)$ and $t \geq 0$,*

$$\left\| e^{t\mathcal{L}_M} P_2(\hat{\eta}_0, \infty) f \right\|_x \leq K \left(\hat{\eta}_0^{-1} e^{\mathrm{Re}\,\lambda(\hat{\eta}_0)t} + e^{-bt} \right) \|f\|_X .$$

Chapter 4

KP-II limit of 2D-FPU

4.1 Formulation and Main Result

The goal of this chapter is to study the behaviour of the two-dimensional Fermi-Pasta-Ulam (FPU) lattices in the small-amplitude, long-wave limit. We will give rigorous justification of the Kadomtsev-Petviashvili (KP-II) equation arising in this limit, in analogy with the Korteweg-de Vries (KdV) equation in the one-dimensional case. We will work with the α model of the two-dimensional FPU system on a square lattice, as described in Section 1.9.1. The position of a particle is given by $(x_{j,k}, y_{j,k}) \in \mathbb{R}^2$ with velocities $(w_{j,k}, z_{j,k}) = (\dot{x}_{j,k}, \dot{y}_{j,k})$, where $(j,k) \in \mathbb{Z}^2$. The energy is given by:

$$H = \frac{1}{2}\sum_{j,k}\left(w_{j,k}^2 + z_{j,k}^2\right) + \sum_{j,k} V_h\left(x_{j+1,k} - x_{j,k}, y_{j+1,k} - y_{j,k}\right) + \sum_{j,k} V_v\left(x_{j,k+1} - x_{j,k}, y_{j,k+1} - y_{j,k}\right),$$

where, the two potential functions V_h and V_v are given by (1.24).

Introducing the strain variables,

$$\begin{aligned} u_{j,k}^{(1)} &= x_{j+1,k} - x_{j,k}, \\ u_{j,k}^{(2)} &= x_{j,k+1} - x_{j,k}, \\ v_{j,k}^{(1)} &= y_{j+1,k} - y_{j,k}, \\ v_{j,k}^{(2)} &= y_{j,k+1} - y_{j,k}, \end{aligned}$$

we can rewrite the Hamiltonian as

$$H = \frac{1}{2}\sum_{j,k}\left(w_{j,k}^2 + z_{j,k}^2\right) + \sum_{j,k} V_h(u_{j,k}^{(1)}, v_{j,k}^{(1)}) + V_v(u_{j,k}^{(2)}, v_{j,k}^{(2)}). \tag{4.1}$$

As shown in section 2.2, using an asymptotic multi-scale expansion of the form

$$u_{j,k}^{(1)} = \varepsilon^2 A\left(\varepsilon\left(j - c_1 t\right), \varepsilon^2 k, \varepsilon^3 t\right) + error, \tag{4.2}$$

yields a KP-II equation

$$2c_1\partial_\xi\partial_\tau A + \frac{c_1^2}{12}\partial_\xi^4 A + 2\alpha\partial_\xi\left(A\partial_\xi A\right) + c_2^2\partial_\eta^2 A = 0, \tag{4.3}$$

where $\xi = \varepsilon\left(j - c_1 t\right), \eta = \varepsilon^2 k, \tau = \varepsilon^3 t, c_1^2, c_2^2 \geq 1$.

We define the anti-derivative of a function $u \in H^s\left(\mathbb{R}^2\right)$ by

$$\partial_\xi^{-1} u = \int_{-\infty}^{\xi} u(\xi', \eta) d\xi'.$$

We are looking for solutions to the KP-II equation, for which both $u \in H^s\left(\mathbb{R}^2\right)$ and $\partial_\xi^{-1} u \in H^s\left(\mathbb{R}^2\right)$. We will require that the solution has enough regularity so that $\partial_\xi^{-1}\partial_\tau^2 A \in C\left(\left[-\tau_0, \tau_0\right], H^s\left(\mathbb{R}^2\right)\right)$. This modification of arguments from [22] is presented in lemma 3.1 which is proven in Section 3.2, and built on results discussed in Section 3.1. Due to lemma 3.1 we will require more stringent constraints on the initial data, which are summarized in the statement of theorem 4.1.

Given a solution to the KP-II equation (4.3), we define

$$W_\varepsilon = -c_1 A + \varepsilon\frac{c_1}{2}\partial_\xi A + \varepsilon^2\left(\partial_\xi^{-1}\partial_\tau A - \frac{c_1}{12}\partial_\xi^2 A\right) - \varepsilon^3\frac{1}{2}\partial_\tau A \tag{4.4}$$

and

$$U_\varepsilon = \varepsilon\partial_\xi^{-1}\partial_\eta A - \varepsilon^2\frac{1}{2}\partial_\eta A + \varepsilon^3\left(\frac{1}{2}\partial_\xi^{-1}\partial_\eta^2 A + \frac{1}{12}\partial_\eta\partial_\xi A\right). \tag{4.5}$$

The following theorem presents the main result of this chapter.

Theorem 4.1. *Let* $A \in C^0\left(\left[-\tau_0, \tau_0\right], H^9\left(\mathbb{R}^2\right)\right)$ *be a solution to the KP-II equation* (4.3)*, whose initial data satisfies*

$$A(\xi, \eta, 0) = A_0 \in H^9\left(\mathbb{R}^2\right)$$

such that

$$\partial_\xi^{-2}\partial_\eta^2 A_0 \in H^9\left(\mathbb{R}^2\right)$$

and

$$\partial_\xi^{-1}\partial_\eta^2\left[\partial_\xi^{-2}\partial_\eta^2 A_0 + A_0^2\right] \in H^2\left(\mathbb{R}^2\right).$$

Then there are constants $C_0, C_1, \varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$ if the initial conditions of the two-dimensional FPU system satisfies

$$\begin{aligned}&\left\|u_{in}^{(1)} - \varepsilon^2 A(\xi,\eta,0)\right\|_{\ell^2} + \left\|u_{in}^{(2)} - \varepsilon^2 U_\varepsilon(\xi,\eta,0)\right\|_{\ell^2}\\ &+\left\|w_{in} - \varepsilon^2 W_\varepsilon(\xi,\eta,0)\right\|_{\ell^2} + \left\|v_{in}^{(1)}\right\|_{\ell^2} + \left\|v_{in}^{(2)}\right\|_{\ell^2} + \|z_{in}\|_{\ell^2} \leq C_0\varepsilon^{2+\frac{1}{2}}\end{aligned} \tag{4.6}$$

then the solution to the two-dimensional FPU system satisfies

$$\begin{aligned}&\left\|u^{(1)} - \varepsilon^2 A(\xi,\eta,t)\right\|_{\ell^2} + \left\|u^{(2)} - \varepsilon^2 U_\varepsilon(\xi,\eta,t)\right\|_{\ell^2}\\ &+\left\|w - \varepsilon^2 W_\varepsilon(\xi,\eta,t)\right\|_{\ell^2} + \left\|v^{(1)}\right\|_{\ell^2} + \left\|v^{(2)}\right\|_{\ell^2} + \|z\|_{\ell^2} \leq C_1\varepsilon^{2+\frac{1}{2}},\end{aligned} \tag{4.7}$$

for $t \in [-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}]$.

Remark. Extending this result to $\varepsilon^{\frac{7}{2}}$ is difficult as the next order of the asymptotic expansion has terms which are not removed by seeking solutions to the KP-II alone. They could be removed by seeking a function of the form $A(\xi,\eta,\tau) = A_0(\xi,\eta,\tau) + \varepsilon^2 A_1(\xi,\eta,\tau)$, where A_0 solves the KP-II equation (4.3) and A_1 solves an appropriately chosen linearized version of KP-II. However the linearized version is a nonhomogeneous linear PDE, where the nonhomogeneous piece contains higher order antiderivative terms of A_0 which is not controlled by lemma 3.1.

Remark. As discussed in chapter 1, a strongly anisotropic version of FPU was studied. [10] One distinction is that the KP limit in this model uses the scaling $\eta = \varepsilon k$. With this scaling the comparison between the ℓ^2 norm and the Sobolev norm would lose us only ε^{-1}, whereas the scaling needed for our model loses $\varepsilon^{-\frac{3}{2}}$. Performing the justification analysis on a version of [10] should also yield KP-II solutions which are ε^3-close rather than $\varepsilon^{\frac{5}{2}}$-close to the FPU system.

4.2 Preliminary Results

The dynamics of the 2D FPU system in the strain variables is described by equations of motion:

$$
\begin{aligned}
\dot{u}_{j,k}^{(1)} =& w_{j+1,k} - w_{j,k}, \\
\dot{u}_{j,k}^{(2)} =& w_{j,k+1} - w_{j,k}, \\
\dot{v}_{j,k}^{(1)} =& z_{j+1,k} - z_{j,k}, \\
\dot{v}_{j,k}^{(2)} =& z_{j,k+1} - z_{j,k}, \\
\dot{w}_{j,k} =& c_1^2 \left(u_{j,k}^{(1)} - u_{j-1,k}^{(1)} \right) + c_2^2 \left(u_{j,k}^{(2)} - u_{j,k-1}^{(2)} \right) \\
&+ \alpha_1 \left[\left(u_{j,k}^{(1)} \right)^2 - \left(u_{j-1,k}^{(1)} \right)^2 \right] \\
&+ \alpha_2 \left[u_{j,k}^{(2)} v_{j,k}^{(2)} - u_{j,k-1}^{(2)} v_{j,k-1}^{(2)} + \frac{1}{2} \left(v_{j,k}^{(1)} \right)^2 - \frac{1}{2} \left(v_{j-1,k}^{(1)} \right)^2 \right] \\
\dot{z}_{j,k} =& c_1^2 \left(v_{j,k}^{(2)} - v_{j,k-1}^{(2)} \right) + c_2^2 \left(v_{j,k}^{(1)} - v_{j-1,k}^{(1)} \right) \\
&+ \alpha_1 \left[\left(v_{j,k}^{(2)} \right)^2 - \left(v_{j,k-1}^{(2)} \right)^2 \right] \\
&+ \alpha_2 \left[u_{j,k}^{(1)} v_{j,k}^{(1)} - u_{j-1,k}^{(1)} v_{j-1,k}^{(1)} + \frac{1}{2} \left(u_{j,k}^{(2)} \right)^2 - \frac{1}{2} \left(u_{j,k-1}^{(2)} \right)^2 \right]
\end{aligned}
\tag{4.8}
$$

Let us use the following decomposition,

$$
\begin{aligned}
u_{j,k}^{(1)} &= \varepsilon^2 A(\xi, \eta, \tau) + \varepsilon^2 U_{j,k}^{(1)} \\
u_{j,k}^{(2)} &= \varepsilon^2 U_\varepsilon(\xi, \eta, \tau) + \varepsilon^2 U_{j,k}^{(2)} \\
v_{j,k}^{(1)} &= \varepsilon^2 V_{j,k}^{(1)} \\
v_{j,k}^{(2)} &= \varepsilon^2 V_{j,k}^{(2)} \\
w_{j,k} &= \varepsilon^2 W_\varepsilon(\xi, \eta, \tau) + \varepsilon^2 W_{j,k} \\
z_{j,k} &= \varepsilon^2 Z_{j,k}
\end{aligned}
\tag{4.9}
$$

where $\xi = \varepsilon (j - c_1 t), \eta = \varepsilon^2 k, \tau = \varepsilon^3 t, (j,k) \in \mathbb{Z}^2$. Here $A(\xi, \eta, \tau)$ is a suitable solution to the KP-II equation (4.3). U_ε and W_ε are a pair of ε-dependent functions, which will allow us to eliminate lower order terms in ε arising from time derivatives and finite differences of A in the equations of motion. Neglecting the error terms, $W_{j,k}, U_{j,k}^{(1)}$, and rewriting the first equation of (4.8) using the decomposition (4.9) we end up with the ε-dependent function $W_\varepsilon(\xi, \eta, \tau)$ satisfying the equality

$$W_\varepsilon\left(\xi+\varepsilon,\eta\right)-W_\varepsilon\left(\xi,\eta\right)=-\varepsilon c_1\partial_\xi A+\varepsilon^3\partial_\tau A. \tag{4.10}$$

We look for an approximate solution to (4.10) up to and including order $O(\varepsilon^3)$,

$$W_\varepsilon=W^{(0)}+\varepsilon W^{(1)}+\varepsilon^2W^{(2)}+\varepsilon^3W^{(3)}, \tag{4.11}$$

with the functions $W^{(j)}$ decaying to zero as ξ,η go to infinity. Plugging equation (4.11) into (4.10), expanding each $W^{(j)}$ in Taylor series, we get:

$$\begin{aligned}
&\varepsilon\partial_\xi W^{(0)}+\varepsilon^2\frac{1}{2}\partial_\xi^2W^{(0)}+\varepsilon^3\frac{1}{6}\partial_\xi^3W^{(0)}+\varepsilon^4\frac{1}{24}\partial_\xi^4W^{(0)}\\
&+\varepsilon^2\partial_\xi W^{(1)}+\varepsilon^3\frac{1}{2}\partial_\xi^2W^{(1)}+\varepsilon^4\frac{1}{6}\partial_\xi^3W^{(1)}+\varepsilon^3\partial_\xi W^{(2)}\\
&+\varepsilon^4\frac{1}{2}\partial_\xi^2W^{(2)}+\varepsilon^4\partial_\xi W^{(3)}+O(\varepsilon^5)=-\varepsilon c_1\partial_\xi A+\varepsilon^3\partial_\tau A.
\end{aligned} \tag{4.12}$$

Grouping terms by order in ε yields a sequence of equations with their relevant solutions:

$$\begin{aligned}
O(\varepsilon):\ &\partial_\xi W^{(0)}=-c_1\partial_\xi A\\
&\implies W^{(0)}=-c_1A\\
O(\varepsilon^2):\ &\frac{1}{2}\partial_\xi^2W^{(0)}+\partial_\xi W^{(1)}=0\\
&\implies W^{(1)}=\frac{c_1}{2}\partial_\xi A\\
O(\varepsilon^3):\ &\frac{1}{6}\partial_\xi^3W^{(0)}+\frac{1}{2}\partial_\xi^2W^{(1)}+\partial_\xi W^{(2)}=\partial_\tau A\\
&\implies W^{(2)}=\partial_\xi^{-1}\partial_\tau A-\frac{c_1}{12}\partial_\xi^2A\\
O(\varepsilon^4):\ &\frac{1}{24}\partial_\xi^4W^{(0)}+\frac{1}{6}\partial_\xi^3W^{(1)}+\frac{1}{2}\partial_\xi^2W^{(2)}+\partial_\xi W^{(3)}=0\\
&\implies W^{(3)}=-\frac{1}{2}\partial_\tau A.
\end{aligned} \tag{4.13}$$

With the choice in (4.13), this construction ensures that terms in (4.10) will vanish up to and including order $O(\varepsilon^4)$, we will later control the error on this approximation by an application of Taylor's theorem. Note that (4.10) with the choice in (4.13) yields (4.4).

We can similarly rewrite the second equation of (4.8) and neglect the error terms to end up with the following equation for a second $\varepsilon-$dependant function $U_\varepsilon(\xi,\eta,\tau)$,

$$W_\varepsilon\left(\xi,\eta+\varepsilon^2\right)-W_\varepsilon\left(\xi,\eta\right)=-\varepsilon c_1\partial_\xi U_\varepsilon+\varepsilon^3\partial_\tau U_\varepsilon. \tag{4.14}$$

Here W_ε is defined by the expansion (4.11) with the choice in (4.13). We look for an approximate solution to (4.14) up to and including order $O(\varepsilon^3)$, which takes the form,

$$U_\varepsilon=\varepsilon U^{(1)}+\varepsilon^2U^{(2)}+\varepsilon^3U^{(3)}. \tag{4.15}$$

Plugging the approximate solution (4.15) into the right hand side of (4.14) we get:

$$\begin{aligned}-\varepsilon c_1\partial_\xi U_\varepsilon+\varepsilon^3\partial_\tau U_\varepsilon=&\varepsilon^2\left(-c_1\partial_\xi U^{(1)}\right)+\varepsilon^3\left(-c_1\partial_\xi U^{(2)}\right)\\&+\varepsilon^4\left(-c_1\partial_\xi U^{(3)}+\partial_\tau U^{(1)}\right)+O(\varepsilon^5)\end{aligned} \tag{4.16}$$

Using the approximate solution (4.11) and expanding the left hand side of the equation in Taylor series,

$$\begin{aligned}W_\varepsilon\left(\xi,\eta+\varepsilon^2\right)-W_\varepsilon\left(\xi,\eta\right)=&\ \varepsilon^2\partial_\eta W^{(0)}+\varepsilon^3\partial_\eta W^{(1)}\\&+\varepsilon^4\left(\partial_\eta W^{(2)}+\frac{1}{2}\partial_\eta^2W^{(0)}\right)+O(\varepsilon^5)\end{aligned} \tag{4.17}$$

Now we match orders of ε, and using the values for W_ε found in (4.13), we get,

$$\begin{aligned}O(\varepsilon^2):-c_1\partial_\eta U=\partial_\eta W^{(0)}&=-c_1\partial_\xi U^{(1)}\\&\Longrightarrow\ U^{(1)}=\partial_\xi^{-1}\partial_\eta A\\O(\varepsilon^3):\frac{c_1}{2}\partial_\xi\partial_\eta U=\partial_\eta W^{(1)}&=-c_1\partial_\xi U^{(2)}\\&\Longrightarrow\ U^{(2)}=-\frac{1}{2}\partial_\eta A\\O(\varepsilon^4):\partial_\eta W^{(2)}+\frac{1}{2}\partial_\eta^2W^{(0)}&=-c_1\partial_\xi U^{(3)}+\partial_\tau U^{(1)}\\&\Longrightarrow\ U^{(3)}=\frac{1}{2}\partial_\xi^{-1}\partial_\eta^2A+\frac{1}{12}\partial_\eta\partial_\xi A\end{aligned} \tag{4.18}$$

Substituting (4.18) into (4.15) yields (4.5).

Substituting decomposition (4.9) into the equations of motion (4.8) we get evolution equations for the error term.

$$\begin{aligned}\dot U_{j,k}^{(1)}&=W_{j+1,k}-W_{j,k}+Res_{j,k}^{U^{(1)}},\\\dot U_{j,k}^{(2)}&=W_{j,k+1}-W_{j,k}+Res_{j,k}^{U^{(2)}},\end{aligned} \tag{4.19}$$

where

$$\begin{aligned} Res_{j,k}^{U^{(1)}}(t) &= c_1 \varepsilon \partial_\xi A - \varepsilon^3 \partial_\tau A + W_\varepsilon(\xi+\varepsilon,\eta) - W_\varepsilon(\xi,\eta), \\ Res_{j,k}^{U^{(2)}}(t) &= c_1 \varepsilon \partial_\xi U_\varepsilon - \varepsilon^3 \partial_\tau U_\varepsilon + W_\varepsilon\left(\xi,\eta+\varepsilon^2\right) - W_\varepsilon(\xi,\eta). \end{aligned} \tag{4.20}$$

We also have the following unchanged equations

$$\begin{aligned} \dot{V}_{j,k}^{(1)} &= Z_{j+1,k} - Z_{j,k}, \\ \dot{V}_{j,k}^{(2)} &= Z_{j,k+1} - Z_{j,k}. \end{aligned} \tag{4.21}$$

Finally, we also have two more equations

$$\begin{aligned} \dot{W}_{j,k} &= c_1^2\left[U_{j,k}^{(1)} - U_{j-1,k}^{(1)}\right] + c_2^2\left[U_{j,k}^{(2)} - U_{j-1,k}^{(2)}\right] \\ &+ \alpha_1\varepsilon^2\left[2AU_{j,k}^{(1)} - 2A(\xi-\varepsilon,\eta)U_{j-1,k}^{(1)} + \left(U_{j,k}^{(1)}\right)^2 - \left(U_{j-1,k}^{(1)}\right)^2\right] \\ &+ \alpha_2\varepsilon^2\left[U_\varepsilon(\xi,\eta)V_{j,k}^{(2)} - U_\varepsilon\left(\xi,\eta-\varepsilon^2\right)V_{j,k-1}^{(2)}\right] \\ &+ \alpha_2\varepsilon^2\left[\frac{1}{2}\left(V_{j,k}^{(1)}\right)^2 - \frac{1}{2}\left(V_{j-1,k}^{(1)}\right)^2\right] \\ &+ \alpha_2\varepsilon^2\left[U_{j,k}^{(2)}V_{j,k}^{(2)} - U_{j,k-1}^{(2)}V_{j,k-1}^{(2)}\right] + Res_{j,k}^W \\ \dot{Z}_{j,k} &= c_2^2\left[V_{j,k}^{(1)} - V_{j-1,k}^{(1)}\right] + c_1^2\left[V_{j,k}^{(2)} - V_{j,k-1}^{(2)}\right] \\ &+ \alpha_2\varepsilon^2\left[U_\varepsilon(\xi,\eta)U_{j,k}^{(2)} - U_\varepsilon\left(\xi,\eta-\varepsilon^2\right)U_{j,k-1}^{(2)}\right] \\ &+ \alpha_2\varepsilon^2\left[A(\xi,\eta)V_{j,k}^{(1)} - A(\xi-\varepsilon,\eta)V_{j-1,k}^{(1)}\right] \\ &+ \alpha_2\varepsilon^2\left[\frac{1}{2}\left(U_{j,k}^{(2)}\right)^2 - \frac{1}{2}\left(U_{j,k-1}^{(2)}\right)^2\right] \\ &+ \alpha_1\varepsilon^2\left[\left(V_{j,k}^{(2)}\right)^2 - \left(V_{j,k-1}^{(2)}\right)^2\right] \\ &+ \alpha_2\varepsilon^2\left[V_{j,k}^{(1)}U_{j,k}^{(1)} - V_{j-1,k}^{(1)}U_{j-1,k}^{(1)}\right] + Res_{j,k}^Z \end{aligned} \tag{4.22}$$

where,

$$\begin{aligned} Res_{j,k}^W &= c_1\varepsilon\partial_\xi W_\varepsilon - \varepsilon^3\partial_\tau W_\varepsilon + c_1^2\left[A(\xi,\eta) - A(\xi-\varepsilon,\eta)\right] \\ &+ c_2^2\left[U_\varepsilon(\xi,\eta) - U_\varepsilon\left(\xi,\eta-\varepsilon^2\right)\right] \\ &+ \alpha_1\varepsilon^2\left[\left(A(\xi,\eta)\right)^2 - \left(A(\xi-\varepsilon,\eta)\right)^2\right] \\ Res_{j,k}^Z &= \frac{\alpha_2\varepsilon^2}{2}\left[U_\varepsilon(\xi,\eta)^2 - U_\varepsilon\left(\xi,\eta-\varepsilon^2\right)^2\right] \end{aligned} \tag{4.23}$$

The residual terms, Res^W and Res^Z, contains functions defined on all of $\mathbb{R}^2$, and shown to be small in ℓ^2-norm through Taylor's theorem in lemma 4.2.

4.3 Control of Residual Terms

In this section we will get bounds on the residual terms in equations (4.20) and (4.23). We will measure these in ℓ^2 norm, however, since they are given by smooth functions on $\mathbb{R}^2$ we will need to compare the ℓ^2 norm with Sobolev norms on $\mathbb{R}^2$. The following lemma, which is a generalization of a similar result from [9] to 2 dimensions, will allow us to do this.

Lemma 4.1. *Let $u_{j,k} = U(\varepsilon j, \varepsilon^2 k)$, with $U \in H^s(\mathbb{R}^2)$, $s > 1$. Then, there is a constant $C_s > 0$, such that for every $\varepsilon \in (0,1)$ we have*

$$\|u\|_{\ell^2(\mathbb{Z}^2)} \leq C_s \varepsilon^{-\frac{3}{2}} \|U\|_{H^s(\mathbb{R}^2)}, \qquad \forall U \in H^s\left(\mathbb{R}^2\right). \tag{4.24}$$

Proof. Let $\hat{u}(\theta,\phi) = \sum_{(j,k)\in\mathbb{Z}^2} u_{j,k} e^{-i(j\theta+k\phi)}$ so that

$$u_{j,k} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \hat{u}(\theta,\phi)\, e^{i(j\theta+k\phi)} d\theta d\phi \tag{4.25}$$

Since $U \in H^s(\mathbb{R}^2)$ for $s > 1$ then $\hat{U} \in L^1(\mathbb{R}^2)$, so that the inverse Fourier transform can be defined through the traditional formula. Then we have

$$\begin{aligned} u_{j,k} = U(\varepsilon j, \varepsilon^2 k) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{U}(\tilde{p},\tilde{q}) e^{i(\varepsilon j\tilde{p}+\varepsilon^2 k\tilde{q})} d\tilde{p}d\tilde{q} \\ &= \frac{1}{(2\pi)^2\varepsilon^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{U}\left(\frac{p}{\varepsilon},\frac{q}{\varepsilon^2}\right) e^{i(jp+kq)} dpdq \\ &= \frac{1}{(2\pi)^2\varepsilon^3}\sum_{(n,m)\in\mathbb{Z}^2}\int_{(2n-1)\pi}^{(2n+1)\pi}\int_{(2m-1)\pi}^{(2m+1)\pi} \hat{U}\left(\frac{p}{\varepsilon},\frac{q}{\varepsilon^2}\right) e^{i(jp+kq)} dpdq \\ &= \frac{1}{(2\pi)^2\varepsilon^3}\sum_{(n,m)\in\mathbb{Z}^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \hat{U}\left(\frac{\theta+2\pi m}{\varepsilon},\frac{\phi+2\pi n}{\varepsilon^2}\right) e^{i(j\theta+k\phi)} d\theta d\phi. \end{aligned} \tag{4.26}$$

Notice that we have for any finite subset $\Lambda \subset \mathbb{Z}^2$

$$\begin{aligned} \sum_{(n,m)\in\Lambda}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\hat{U}\left(\frac{\theta+2\pi m}{\varepsilon},\frac{\phi+2\pi n}{\varepsilon^2}\right)\right| d\theta d\phi &\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\hat{U}\left(\frac{p}{\varepsilon},\frac{q}{\varepsilon^2}\right)\right| dpdq \\ &= ||\hat{U}||_{L^1(\mathbb{R}^2)}, \end{aligned}$$

hence

$$\sum_{(n,m)\in\mathbb{Z}^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\hat{U}\left(\frac{\theta+2\pi m}{\varepsilon},\frac{\phi+2\pi n}{\varepsilon^2}\right)\right|d\theta d\phi\leq||\hat{U}||_{L^1(\mathbb{R}^2)}<\infty.$$

Then we can interchange summation and integration in (4.26) by the Fubini-Tonelli theorem. Comparing (4.25) and (4.26) yields

$$\hat{u}\left(\theta,\phi\right)=\frac{1}{\varepsilon^3}\sum_{(n,m)\in\mathbb{Z}^2}\hat{U}\left(\frac{\theta+2\pi m}{\varepsilon},\frac{\phi+2\pi n}{\varepsilon^2}\right).$$

Using Parseval's identity we get,

$$||u||^2_{\ell^2(\mathbb{Z}^2)}=\frac{1}{(2\pi)^2\varepsilon^6}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\sum_{(n,m)\in\mathbb{Z}^2}\hat{U}\left(\frac{\theta+2\pi m}{\varepsilon},\frac{\phi+2\pi n}{\varepsilon^2}\right)\right|^2d\theta d\phi$$

$$\leq\frac{1}{(2\pi)^2\varepsilon^6}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum_{\substack{(n_1,m_1)\in\mathbb{Z}^2\\(n_2,m_2)\in\mathbb{Z}^2}}\left|\hat{U}\left(\frac{\theta+2\pi m_1}{\varepsilon},\frac{\phi+2\pi n_1}{\varepsilon^2}\right)\right|\times\left|\hat{U}\left(\frac{\theta+2\pi m_2}{\varepsilon},\frac{\phi+2\pi n_2}{\varepsilon^2}\right)\right|d\theta d\phi$$

$$=\frac{1}{(2\pi)^2\varepsilon^6}\sum_{\substack{(n_1,m_1)\in\mathbb{Z}^2\\(n_2,m_2)\in\mathbb{Z}^2}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\hat{U}\left(\frac{\theta+2\pi m_1}{\varepsilon},\frac{\phi+2\pi n_1}{\varepsilon^2}\right)\right|\times\left|\hat{U}\left(\frac{\theta+2\pi m_2}{\varepsilon},\frac{\phi+2\pi n_2}{\varepsilon^2}\right)\right|d\theta d\phi.$$

Denote $\langle x,y\rangle_2=\sqrt{1+x^2+y^2}$. Inserting the weights $\left\langle\frac{\pi m_1}{\varepsilon},\frac{\pi n_1}{\varepsilon^2}\right\rangle_2^{-2s}$ and $\left\langle\frac{\pi m_2}{\varepsilon},\frac{\pi n_2}{\varepsilon^2}\right\rangle_2^{-2s}$, then applying Young's inequality, $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, yields

$$\begin{aligned}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} &\left|\hat{U}\left(\frac{\theta+2\pi m_1}{\varepsilon}, \frac{\phi+2\pi n_1}{\varepsilon^2}\right)\right|\left|\hat{U}\left(\frac{\theta+2\pi m_2}{\varepsilon}, \frac{\phi+2\pi n_2}{\varepsilon^2}\right)\right| d\theta d\phi \\ &\leq \left\langle \frac{\pi m_1}{\varepsilon}, \frac{\pi n_1}{\varepsilon^2}\right\rangle_2^{-2s}\left\langle \frac{\pi m_2}{\varepsilon}, \frac{\pi n_2}{\varepsilon^2}\right\rangle_2^{-2s} \\ &\times \left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left\langle \frac{\pi m_1}{\varepsilon}, \frac{\pi n_1}{\varepsilon^2}\right\rangle_2^{4s}\left|\hat{U}\left(\frac{\theta+2\pi m_1}{\varepsilon}, \frac{\phi+2\pi n_1}{\varepsilon^2}\right)\right|^2\right. \\ &\quad \left. + \frac{1}{2}\left\langle \frac{\pi m_2}{\varepsilon}, \frac{\pi n_2}{\varepsilon^2}\right\rangle_2^{4s}\left|\hat{U}\left(\frac{\theta+2\pi m_2}{\varepsilon}, \frac{\phi+2\pi n_2}{\varepsilon^2}\right)\right|^2 d\theta d\phi\right).\end{aligned}$$

Hence, by symmetry of coefficients, we obtain:

$$\begin{aligned}||u||_{\ell^2(\mathbb{Z}^2)}^2 \leq \frac{1}{(2\pi)^2\varepsilon^6}&\left(\sum_{(n_1,m_1)\in\mathbb{Z}^2}\left\langle \frac{\pi m_1}{\varepsilon}, \frac{\pi n_1}{\varepsilon^2}\right\rangle_2^{-2s}\right) \\ &\times\left(\sum_{(n_2,m_2)\in\mathbb{Z}^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left\langle \frac{\pi m_2}{\varepsilon}, \frac{\pi n_2}{\varepsilon^2}\right\rangle_2^{2s}\left|\hat{U}\left(\frac{\theta+2\pi m_2}{\varepsilon}, \frac{\phi+2\pi n_2}{\varepsilon^2}\right)\right|^2 d\theta d\phi\right).\end{aligned}$$

When $\varepsilon \in (0,1]$ the double series converges for $s > 1$ by the integral test, hence $\exists C_s > 0$ so that

$$\sum_{(n_1,m_1)\in\mathbb{Z}^2}\left\langle \frac{\pi m_1}{\varepsilon}, \frac{\pi n_1}{\varepsilon^2}\right\rangle_2^{-2s} < C_s^2.$$

which converges whenever $s > 1$. The second term in $\|u\|_{\ell^2(\mathbb{Z})}$ is related to the H^s norm of U given by,

$$\begin{aligned}||U||_{H^s}^2 &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \tilde{p}, \tilde{q}\rangle_2^{2s}\left|\hat{U}(\tilde{p},\tilde{q})\right|^2 d\tilde{p}d\tilde{q} \\ &= \frac{1}{(2\pi)^2\varepsilon^3}\sum_{(n_2,m_2)\in\mathbb{Z}^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left\langle \frac{\theta+2\pi m_2}{\varepsilon}, \frac{\phi+2\pi n_2}{\varepsilon^2}\right\rangle_2^{2s} \\ &\qquad\qquad \times\left|\hat{U}\left(\frac{\theta+2\pi m_2}{\varepsilon}, \frac{\phi+2\pi n_2}{\varepsilon^2}\right)\right|^2 d\theta d\phi.\end{aligned}$$

Since $\left\langle \frac{\pi m_2}{\varepsilon}, \frac{\pi n_2}{\varepsilon^2}\right\rangle_2^{2s} \leq \left\langle \frac{\theta+2\pi m_2}{\varepsilon}, \frac{\phi+2\pi n_2}{\varepsilon^2}\right\rangle_2^{2s}$ for every $\theta \in [-\pi,\pi]$, $\phi \in [-\pi,\pi]$, the second term in $\|u\|_{\ell^2(\mathbb{Z})}$ is bounded above by $(2\pi)^2\varepsilon^3||U||_{H^s}^2$. Hence, we obtain (4.24). □

The next lemma gives us estimates of the ℓ^2-norm for the residual terms.

The residual terms are handled by truncating using Taylor's theorem, and estimating the ℓ^2 norm of the remainder term in Sobolev space by an application of lemma 4.1.

Lemma 4.2. *(Estimates for Residual terms) Let A be a solution to the KP-II equation* (4.3) *in the class of functions of lemma 3.1 for $s = 0$. Then there are constants $C(A) > 0$ such that for all $\varepsilon \in (0, 1]$, we have*

$$\left\|Res_{j,k}^{W}\right\|_{\ell^2} + \left\|Res_{j,k}^{Z}\right\|_{\ell^2} + \left\|Res_{j,k}^{U^{(1)}}\right\|_{\ell^2} + \left\|Res_{j,k}^{U^{(2)}}\right\|_{\ell^2} \leq C\varepsilon^{\frac{7}{2}}$$

Proof. By construction all terms in

$$Res_{j,k}^{U^{(1)}}(t) = c_1\varepsilon\partial_\xi A - \varepsilon^3\partial_\tau A + W_\varepsilon\left(\xi+\varepsilon,\eta\right) - W_\varepsilon\left(\xi,\eta\right)$$

below $O(\varepsilon^5)$ vanish. From the Taylor remainder theorem the nonzero terms are given by the integrals:

$$\varepsilon^5\int_0^1 \partial_\xi^{5-l}W^{(l)}\left(\varepsilon\left(j+r\right),\varepsilon^2 k,\tau\right)\frac{\left(1-r\right)^{5-l-1}}{\left(5-l-1\right)!}dr,$$

for $0 \leq l \leq 3$. In light of our calculation in (4.13), the error is then given by a linear combination of the following integrals,

$$\varepsilon^5\int_0^1 \partial_\xi^5 A\left(\varepsilon\left(j+r\right),\varepsilon^2 k,\tau\right)\left(1-r\right)^{5-l-1}dr,$$
$$\varepsilon^5\int_0^1 \partial_\xi^2\partial_\tau A\left(\varepsilon\left(j+r\right),\varepsilon^2 k,\tau\right)\left(1-r\right)^{5-l-1}dr,$$

where $1 \leq l \leq 3$. Using lemma 4.1 we can estimate the ℓ^2 norm by $\varepsilon^{\frac{7}{2}}\left(\|A\|_{H^5} + \|\partial_\tau A\|_{H^2}\right)$.

Similarly all terms in

$$Res_{j,k}^{U^{(2)}}(t) = \varepsilon c_1\partial_\xi U_\varepsilon - \varepsilon^3\partial_\tau U_\varepsilon + W_\varepsilon\left(\xi,\eta+\varepsilon^2\right) - W_\varepsilon\left(\xi,\eta\right)$$

below $O(\varepsilon^5)$ vanish. From the Taylor remainder theorem and calculations in (4.13) the nonzero terms are a linear combination of the following integrals:

$$\varepsilon^5\int_0^1 \partial_\eta^2\partial_\xi A\left(\varepsilon j,\varepsilon^2(k+r),\tau\right)(1-r)\,dr,$$
$$\varepsilon^5\int_0^1 \partial_\eta\partial_\xi^2 A\left(\varepsilon j,\varepsilon^2(k+r),\tau\right)dr,$$
$$\varepsilon^5\int_0^1 \partial_\eta\partial_\tau A\left(\varepsilon j,\varepsilon^2(k+r),\tau\right)dr.$$

Again using lemma 4.1 we can estimate the ℓ^2 norm by $\varepsilon^{\frac{7}{2}}\left(\|A\|_{H^3}+\|\partial_\tau A\|_{H^1}\right)$.

The next residual term is given by,

$$\begin{aligned} Res^W_{j,k} = &\quad c_1\varepsilon\partial_\xi W_\varepsilon - \varepsilon^3\partial_\tau W_\varepsilon + c_1^2\left[A\left(\xi,\eta\right)-A\left(\xi-\varepsilon,\eta\right)\right] \\ &+c_2^2\left[U_\varepsilon\left(\xi,\eta\right)-U_\varepsilon\left(\xi,\eta-\varepsilon^2\right)\right]+\alpha_1\varepsilon^2\left[A\left(\xi,\eta\right)^2-A\left(\xi-\varepsilon,\eta\right)^2\right]. \end{aligned}$$

We can handle the above term by term:

$$\begin{aligned} c_1\varepsilon\partial_\xi W_\varepsilon &= c_1\varepsilon\partial_\xi\left(W^{(0)}+\varepsilon W^{(1)}+\varepsilon^2 W^{(2)}+\varepsilon^3 W^{(3)}\right) \\ &= -c_1^2\varepsilon\partial_\xi A+\frac{c_1^2}{2}\varepsilon^2\partial_\xi^2 A+c_1\varepsilon^3\left[\partial_\tau A-\frac{c_1}{12}\partial_\xi^3 A\right] \\ &\quad+\varepsilon^4 c_1\left[-\frac{1}{2}\partial_\xi\partial_\tau A\right], \end{aligned}$$

$$\begin{aligned} -\varepsilon^3\partial_\tau W_\varepsilon &= -\varepsilon^3\partial_\tau\left(W^{(0)}+\varepsilon W^{(1)}+\varepsilon^2 W^{(2)}+\varepsilon^3 W^{(3)}\right) \\ &= c_1\varepsilon^3\partial_\tau A-\frac{c_1}{2}\varepsilon^4\partial_\tau\partial_\xi A+\varepsilon^5\left(-\partial_\xi^{-1}\partial_\tau^2 A+\frac{c_1}{12}\partial_\tau\partial_\xi^2 A\right) \\ &\quad+\varepsilon^6\left(\frac{1}{2}\partial_\tau^2 A\right), \end{aligned}$$

$$\begin{aligned} c_1^2\left(A\left(\xi,\eta\right)-A\left(\xi-\varepsilon,\eta\right)\right) =& c_1^2\varepsilon\partial_\xi A-\varepsilon^2\frac{c_1^2}{2}\partial_\xi^2 A+\varepsilon^3\frac{c_1^2}{6}\partial_\xi^3 A \\ &-\varepsilon^4\frac{c_1^2}{24}\partial_\xi^4 A+\varepsilon^5\mathcal{A}_{\varepsilon,5}, \end{aligned}$$

$$c_2^2\left(U_\varepsilon\left(\xi,\eta\right)-U_\varepsilon\left(\xi,\eta-\varepsilon^2\right)\right)=c_2^2\left(\varepsilon^3\partial_\eta U^{(1)}+\varepsilon^4\partial_\eta U^{(2)}+\varepsilon^5\mathcal{U}_\varepsilon\right),$$

and

$$\begin{aligned} \alpha_1\varepsilon^2\left[A\left(\xi,\eta\right)^2-A\left(\xi-\varepsilon,\eta\right)^2\right] =& \alpha_1\varepsilon^2\Bigg[A^2-\Bigg(A-\varepsilon\partial_\xi A+\varepsilon^2\frac{1}{2}\partial_\xi^2 A \\ &-\varepsilon^3\mathcal{A}_{\varepsilon,3}\Bigg)^2\Bigg] \\ =& \alpha_1\left[2\varepsilon^3 A\partial_\xi A-\varepsilon^4\left(A\partial_\xi^2 A+\left(\partial_\xi A\right)^2\right)\right. \\ &\left.+\varepsilon^5\left(\partial_\xi A\partial_\xi^2 A+2A\mathcal{A}_{\varepsilon,3}\right)\right]+O(\varepsilon^6), \end{aligned}$$

where

$$\mathcal{A}_{\varepsilon,5}=\frac{1}{4!}\int_0^1\partial_\xi^5 A\left(\xi+\varepsilon(r-1),\eta\right)\left(1-r\right)^4 dr,$$

$$\mathcal{U}_\varepsilon = \int_0^1 \partial_\eta^2 U^{(1)}\left(\xi, \eta + \varepsilon^2(r-1)\right)(1-r)\,dr + \int_0^1 \partial_\eta U^{(3)}\left(\xi, \eta + \varepsilon^2(r-1)\right)dr$$
$$+\varepsilon \int_0^1 \partial_\eta^2 U^{(2)}\left(\xi, \eta + \varepsilon^2(r-1)\right)(1-r)\,dr,$$

and

$$\mathcal{A}_{\varepsilon,3} = \frac{1}{2}\int_0^1 \partial_\xi^3 A\left(\xi + \varepsilon(r-1), \eta\right)(1-r)^2\,dr,$$

are the Lagrange form of the Taylor remainders. Putting the above together we get

$$\begin{aligned} Res_{j,k}^W = \; & \varepsilon^3\left[2c_1\partial_\tau A + \frac{c_1^2}{12}\partial_\xi^3 A + c_2^2\partial_\xi^{-1}\partial_\eta^2 A + \alpha_1\partial_\xi\left(A^2\right)\right] \\ & -\varepsilon^4\left[c_1\partial_\xi\partial_\tau A + \frac{c_1^2}{24}\partial_\xi^4 A + \frac{c_2^2}{2}\partial_\eta^2 A + \alpha_1\partial_\xi\left(A\partial_\xi A\right)\right] \\ & +\varepsilon^5\Big[\alpha_1\partial_\xi A\partial_\xi^2 A + 2\alpha A\mathcal{A}_{\varepsilon,3} + \mathcal{U}_\varepsilon - \partial_\xi^{-1}\partial_\tau^2 A \\ & \quad + \frac{c_1}{12}\partial_\tau\partial_\xi^2 A + \mathcal{A}_{\varepsilon,5}\Big] + O(\varepsilon^6). \end{aligned}$$

We seek a function A requiring that it is a solution to the KP-II equation (4.3), we can see that then $O(\varepsilon^3)$ and $O(\varepsilon^4)$ terms of $Res_{j,k}^W$ will vanish. Using lemma 4.1 the remaining terms can be estimated by

$$\begin{aligned} \varepsilon^{\frac{7}{2}}\Big(\|\partial_\tau A\|_{H^2} + \|A\|_{H^5} + \left\|\partial_\xi^{-1} A\right\|_{H^3} + \|A\|_{H^3}^2 \\ + \|A\|_{H^3}^3 + \left\|\partial_\xi^{-1} A\right\|_{H^3} + \left\|\partial_\xi^{-1}\partial_\tau^2 A\right\|_{L^2}\Big). \end{aligned}$$

For the $Res_{j,k}^Z$ term with the Taylor remainder term we write

$$\begin{aligned} Res_{j,k}^Z &= \frac{\alpha_2\varepsilon^2}{2}\left[U_\varepsilon(\xi,\eta)^2 - U_\varepsilon\left(\xi, \eta - \varepsilon^2\right)^2\right] \\ &= \frac{\alpha_2}{2}\varepsilon^4\left[\left(U^{(1)} + \varepsilon U^{(2)} + \varepsilon^2 U^{(3)} + \varepsilon^3 U^{(4)}\right)^2 - \left(U^{(1)} + \varepsilon U^{(2)} + \varepsilon^2 U^{(3)} + \varepsilon^3 U^{(4)} + \varepsilon^2\tilde{\mathcal{U}}\right)^2\right] \\ &= \alpha_2\varepsilon^6 U^{(1)}\tilde{\mathcal{U}} + O(\varepsilon^7) \end{aligned}$$

where $\tilde{\mathcal{U}} = \int_0^1 \partial_\eta U^{(1)}\left(\xi, \eta + \varepsilon^2(r-1)\right)dr$. The ℓ^2 norm of Res^Z is then estimated by $\varepsilon^{\frac{9}{2}}\left\|\partial_\xi^{-1} A\right\|_{H^2}^2$. □

4.4 Energy Estimates

To control the growth of the approximation error we will introduce the following energy type quantity:

$$\begin{aligned} E(t) =& \frac{1}{2}\sum_{j,k\in\mathbb{Z}^2} W_{j,k}^2 + Z_{j,k}^2 + c_1^2\left(U_{j,k}^{(1)}\right)^2 + c_2^2\left(U_{j,k}^{(2)}\right)^2 \\ &+ c_1^2\left(V_{j,k}^{(1)}\right)^2 + c_2^2\left(V_{j,k}^{(2)}\right)^2 + 2\alpha_2\varepsilon^2 U_\varepsilon U_{j,k}^{(2)} V_{j,k}^{(2)} \\ &+ \alpha_1\varepsilon^2\left[2A\left(U_{j,k}^{(1)}\right)^2 + \frac{2}{3}\left(U_{j,k}^{(1)}\right)^3 + \frac{2}{3}\left(V_{j,k}^{(2)}\right)^3\right] \\ &+ \alpha_2\varepsilon^2\left[A\left(V_{j,k}^{(1)}\right)^2 + U_{j,k}^{(1)}\left(V_{j,k}^{(1)}\right)^2 + \left(U_{j,k}^{(2)}\right)^2 V_{j,k}^{(2)}\right]. \end{aligned} \tag{4.27}$$

The quantity is chosen such that the the growth $\dot{E}(t)$ does not contain terms of lower order than $O(\varepsilon^3)$, which we will require for the Gronwall lemma argument.

The following lemma establishes coercivity of the energy type quantity with respect to the ℓ^2 norm of the lattice equations.

Lemma 4.3. *Let $A \in C^0\left([-\tau_0, \tau_0], H^4\left(\mathbb{R}^2\right)\right)$ and $\partial_\xi^{-1} A \in C^0\left([-\tau_0, \tau_0], H^3\left(\mathbb{R}^2\right)\right)$. Define*

$$\mathcal{A} = \sup_{\tau\in[-\tau_0,\tau_0]} \|A(\cdot,\cdot,\tau)\|_{L^\infty}$$

and

$$\mathcal{U} = \sup_{\tau\in[-\tau_0,\tau_0]} \left(\left\|\partial_\xi^{-1} A\right\|_{H^3} + 2\|A\|_{H^4}\right).$$

Then, there exists some $\varepsilon_1(\mathcal{A},\mathcal{U},H) > 0$ and $K > 0$, such that

$$\|W\|_{\ell^2}^2 + \|Z\|_{\ell^2}^2 + \left\|U^{(1)}\right\|_{\ell^2}^2 + \left\|U^{(2)}\right\|_{\ell^2}^2 + \left\|V^{(1)}\right\|_{\ell^2}^2 + \left\|V^{(2)}\right\|_{\ell^2}^2 \leq 2KE(t),$$

for each $\varepsilon \in (0, \varepsilon_1]$ and $t \in \left[\frac{-\tau_0}{\varepsilon^3}, \frac{\tau_0}{\varepsilon^3}\right]$.

Proof. By decomposition (4.9) we have

$$\begin{aligned} \varepsilon^2\left\|U^{(1)}\right\|_{\ell^\infty} &\leq \left\|u^{(1)}\right\|_{\ell^\infty} + \varepsilon^2\|A\|_{L^\infty} \\ &\leq \left\|u^{(1)}\right\|_{\ell^2} + \varepsilon^2\|A\|_{L^\infty} \\ &\leq H^{\frac{1}{2}} + \varepsilon^2\|A\|_{L^\infty}, \end{aligned}$$

where H is the Hamiltonian, (4.1), of the two-dimensional FPU system. From this we have the estimate,

$$\varepsilon^2 \sup_{\tau\in[-\tau_0,\tau_0]} \left\|U^{(1)}\right\|_{\ell^\infty} \leq H^{\frac{1}{2}} + \varepsilon^2\mathcal{A}.$$

A similar argument shows that

$$\begin{aligned}\varepsilon^2 \sup_{\tau\in[-\tau_0,\tau_0]} \left\|V^{(2)}\right\|_{\ell^\infty} &\leq H^{\frac{1}{2}},\\ \varepsilon^2 \sup_{\tau\in[-\tau_0,\tau_0]} \left\|U^{(2)}\right\|_{\ell^\infty} &\leq H^{\frac{1}{2}} + \varepsilon^2\mathcal{U}.\end{aligned}$$

We can also estimate U_ε by,

$$\begin{aligned}\left\|U_\varepsilon\right\|_{L^\infty} &\leq \varepsilon\left\|\partial_\xi^{-1}\partial_\eta A\right\|_{L^\infty} + \frac{\varepsilon^2}{2}\left\|\partial_\eta A\right\|_{L^\infty} + \frac{\varepsilon^3}{2}\left\|\partial_\eta^2 A\right\|_{L^\infty} + \frac{\varepsilon^3}{12}\left\|\partial_\xi\partial_\eta A\right\|_{L^\infty}\\ &\leq \left\|\partial_\xi^{-1}A\right\|_{H^3} + 2\left\|A\right\|_{H^4},\end{aligned}$$

so that

$$\sup_{\tau\in[-\tau_0,\tau_0]} \left\|U_\varepsilon\right\|_{L^\infty} \leq \mathcal{U}.$$

$$\begin{aligned}2E(t) \geq \|W\|_{\ell^2}^2 + \|Z\|_{\ell^2}^2 &+ \left(c_1^2 - \frac{8}{3}\alpha_1\varepsilon^2\mathcal{A} - \frac{2}{3}\alpha_1 H^{\frac{1}{2}}\right)\left\|U^{(1)}\right\|_{\ell^2}^2\\ &+ \left(c_2^2 - \alpha_2\varepsilon^2\mathcal{U} - \alpha_2 H^{\frac{1}{2}}\right)\left\|U^{(2)}\right\|_{\ell^2}^2\\ &+ \left(c_1^2 - \alpha_2\varepsilon^2\mathcal{A} - \alpha_2 H^{\frac{1}{2}}\right)\left\|V^{(1)}\right\|_{\ell^2}^2\\ &+ \left(c_2^2 - \alpha_2\varepsilon^2\mathcal{U} - \frac{2}{3}\alpha_1 H^{\frac{1}{2}}\right)\left\|V^{(2)}\right\|_{\ell^2}^2\end{aligned}$$

For a fixed $c_1, c_2 > 0$ and a sufficiently small H there is a $K > 0$ such that the following quantity is positive

$$\begin{aligned}\varepsilon_1 = \min\Bigg(1, &\mathcal{A}^{-\frac{1}{2}}\left(\frac{3}{8\alpha_1}\left(c_1^2 - \frac{1}{K} - \frac{2\alpha_1}{3}H^{\frac{1}{2}}\right)\right)^{\frac{1}{2}},\\ &\alpha_2^{-\frac{1}{2}}\mathcal{U}^{-\frac{1}{2}}\left(c_2^2 - \frac{1}{K} - \alpha_2 H^{\frac{1}{2}}\right)^{\frac{1}{2}},\\ &\alpha_2^{-\frac{1}{2}}\mathcal{A}^{-\frac{1}{2}}\left(c_1^2 - \frac{1}{K} - \alpha_2 H^{\frac{1}{2}}\right)^{\frac{1}{2}},\\ &\alpha_2^{-\frac{1}{2}}\mathcal{U}^{-\frac{1}{2}}\left(c_2^2 - \frac{1}{K} - \frac{2}{3}\alpha_1 H^{\frac{1}{2}}\right)^{\frac{1}{2}}\Bigg).\end{aligned}$$

Then we have

$$2E(t) \geq \|W\|_{\ell^2}^2 + \|Z\|_{\ell^2}^2 + \frac{1}{K}\left\|U^{(1)}\right\|_{\ell^2}^2 + \frac{1}{K}\left\|U^{(2)}\right\|_{\ell^2}^2 + \frac{1}{K}\left\|V^{(1)}\right\|_{\ell^2}^2 + \frac{1}{K}\left\|V^{(2)}\right\|_{\ell^2}^2$$

for each $\varepsilon \in (0, \varepsilon_1]$, and the result follows. ☐

The following lemma uses the coercivity of the energy type quantity to establish the rate at which it grows in time. We will be able to use this, along with a Gronwall lemma argument, in order to get a bound on the size of the energy quantity on times scales of $\frac{1}{\varepsilon^3}$. This will in turn gives a bound on how far solutions of the KP-II equation (4.3) drift away from solutions of the FPU system (4.8).

Lemma 4.4 (Growth of E(t))**.** *Let A be a solution to the KP-II equation* (4.3) *in the class of functions of lemma 3.1 for $s = 0$. Then there is a $K_{\alpha A} > 0$ such that,*

$$\left|\frac{dE}{dt}\right| \leq K_{\alpha A}\left(E^{\frac{1}{2}}\varepsilon^{\frac{7}{2}} + E\varepsilon^3\right).$$

Proof. By differentiating $E(t)$, defined by (4.27), in time, we obtain

$$\begin{aligned}\dot{E}(t) = \sum_{j,k\in\mathbb{Z}^2} &\dot{W}_{j,k}W_{j,k} + \dot{Z}_{j,k}Z_{j,k} + c_1^2\dot{U}_{j,k}^{(1)}U_{j,k}^{(1)} + c_2^2\dot{U}_{j,k}^{(2)}U_{j,k}^{(2)} + c_1^2\dot{V}_{j,k}^{(1)}V_{j,k}^{(1)} + c_2^2\dot{V}_{j,k}^{(2)}V_{j,k}^{(2)} \\ &+ \alpha_1\varepsilon^2\left[\dot{A}\left(U_{j,k}^{(1)}\right)^2 + 2A\dot{U}_{j,k}^{(1)}U_{j,k}^{(1)} + \dot{U}_{j,k}^{(1)}\left(U_{j,k}^{(1)}\right)^2 + \dot{V}_{j,k}^{(2)}\left(V_{j,k}^{(2)}\right)^2\right] \\ &+ \alpha_2\varepsilon^2\left[\dot{U}_\varepsilon U_{j,k}^{(2)}V_{j,k}^{(2)} + \frac{1}{2}\dot{U}_{j,k}^{(1)}\left(V_{j,k}^{(1)}\right)^2 + U_{j,k}^{(1)}\dot{V}_{j,k}^{(1)}V_{j,k}^{(1)} + \frac{1}{2}\left(U_{j,k}^{(2)}\right)^2\dot{V}_{j,k}^{(2)}\right. \\ &\left.+\dot{U}_{j,k}^{(2)}U_{j,k}^{(2)}V_{j,k}^{(2)} + U_\varepsilon\dot{U}_{j,k}^{(2)}V_{j,k}^{(2)} + U_\varepsilon U_{j,k}^{(2)}\dot{V}_{j,k}^{(2)} + \frac{1}{2}\dot{A}\left(V_{j,k}^{(1)}\right)^2 + A\dot{V}_{j,k}^{(1)}V_{j,k}^{(1)}\right]\end{aligned}$$

We expand $\dot{E}(t)$ term by term using equations (4.19) to (4.23), then by summing across $(j,k) \in \mathbb{Z}^2$, $\dot{E}(t)$ simplifies to

$$\begin{aligned}\dot{E}(t) = \sum_{j,k\in\mathbb{Z}^2} & W_{j,k} Res^{W}_{j,k}(t) + Z_{j,k} Res^{Z}_{j,k}(t) + c_1^2 U^{(1)}_{j,k} Res^{U^{(1)}}_{j,k}(t) \\ & + c_2^2 U^{(2)}_{j,k} Res^{U^{(2)}}_{j,k}(t) + \alpha_1 \varepsilon^2 \left(-c_1 \varepsilon \partial_\xi A + \varepsilon^3 \partial_\tau A\right) \left(U^{(1)}_{j,k}\right)^2 \\ & + 2\alpha_1 \varepsilon^2 Res^{U^{(1)}}_{j,k}(t)\, A U^{(1)}_{j,k} + \alpha_2 \varepsilon^2 \left(-c_1 \varepsilon \partial_\xi U_\varepsilon + \varepsilon^3 \partial_\tau U_\varepsilon\right) U^{(2)}_{j,k} V^{(2)}_{j,k} \\ & + \alpha_2 \varepsilon^2 Res^{U^{(2)}}_{j,k}(t)\, U_\varepsilon V^{(2)}_{j,k} + \frac{\alpha_2}{2} \varepsilon^2 \left(c_1 \varepsilon \partial_\xi A - \varepsilon^3 \partial_\tau A\right) \left(V^{(1)}_{j,k}\right)^2 \\ & + \alpha_1 \varepsilon^2 Res^{U^{(1)}}_{j,k}(t) \left(U^{(1)}_{j,k}\right)^2 + \alpha_2 \varepsilon^2 Res^{U^{(1)}}_{j,k}(t) \left(V^{(1)}_{j,k}\right)^2 + \alpha_2 \varepsilon^2 Res^{U^{(2)}}_{j,k}(t) U^{(2)}_{j,k} V^{(2)}_{j,k}\end{aligned}$$

Setting $\alpha = \max(\alpha_1, \alpha_2)$, and applying the Cauchy-Schwartz inequality yields,

$$\begin{aligned}\left|\dot{E}(t)\right| \le & \|W\|_{\ell^2} \left\|Res^W\right\|_{\ell^2} + \|Z\|_{\ell^2} \left\|Res^Z\right\|_{\ell^2} \\ & + c_1^2 \left\|U^{(1)}\right\|_{\ell^2} \left\|Res^{U^{(1)}}\right\|_{\ell^2} + c_2^2 \left\|U^{(2)}\right\|_{\ell^2} \left\|Res^{U^{(2)}}\right\|_{\ell^2} \\ & + \alpha \varepsilon^2 \Big[\left\|c_1 \varepsilon \partial_\xi A - \varepsilon^3 \partial_\tau A\right\|_{L^\infty} \left(\left\|U^{(1)}\right\|^2_{\ell^2} + \left\|V^{(1)}\right\|^2_{\ell^2} \right) \\ & + \left\|\varepsilon c_1 \partial_\xi U_\varepsilon - \varepsilon^3 \partial_\tau U_\varepsilon\right\|_{L^\infty} \left\|U^{(2)}\right\|_{\ell^2} \left\|V^{(2)}\right\|_{\ell^2} \\ & + \|A\|_{L^\infty} \left\|Res^{U^{(1)}}\right\|_{\ell^2} \left\|U^{(1)}\right\|_{\ell^2} + \|U_\varepsilon\|_{L^\infty} \left\|Res^{U^{(2)}}\right\|_{\ell^2} \left\|V^{(2)}\right\|_{\ell^2} \\ & + \left\|Res^{U^{(1)}}\right\|_{\ell^2} \left(\left\|U^{(1)}\right\|^2_{\ell^2} + \left\|V^{(1)}\right\|^2_{\ell^2} \right) \\ & + \left\|Res^{U^{(2)}}\right\|_{\ell^2} \left\|U^{(2)}\right\|_{\ell^2} \left\|V^{(2)}\right\|_{\ell^2} \Big]\end{aligned}$$

Applying lemmas 4.2 and 4.3 we get,

$$\left|\dot{E}(t)\right| \le K_{\alpha A} \left(E^{\frac{1}{2}} \varepsilon^{\frac{7}{2}} + E \varepsilon^3 \right),$$

which completes the proof. □

By making the substitution $E(t)^{\frac{1}{2}} = Q(t)$, then

$$\left|\dot{Q}(t)\right| \le K_{\alpha A} \left(\varepsilon^{\frac{7}{2}} + \varepsilon^3 Q \right).$$

We can estimate the above using the Gronwall lemma.

Lemma 4.5. *Suppose that*

$$\left|\dot{Q}(t)\right| \leq K_{\alpha A}\left(\varepsilon^{\frac{7}{2}} + \varepsilon^3 Q\right),$$

and $Q(0) \leq C\varepsilon^{\frac{1}{2}}$ *then there is an* $\varepsilon_1 > 0$ *such that*

$$Q(t) \leq \varepsilon^{\frac{1}{2}}\left(C+1\right)\exp\left(K_{\alpha A}\tau_0\right)$$

for $\varepsilon \in (0, \varepsilon_1]$.

Proof. For $\varepsilon \in [0, \varepsilon_1]$, where ε_1 is defined in the proof of lemma 4.3, we have

$$\left|\dot{Q}\right| \leq \left(\varepsilon^3 Q + \varepsilon^{\frac{7}{2}}\right), \tag{4.28}$$

where the constant $K_{\alpha A}$ depends on α, A but not ε. We can now perform a Gronwall type argument to (4.28). First we rewrite (4.28) as

$$\frac{d}{dt}\left[\exp\left(-\varepsilon^3 K_{\alpha A} t\right) Q\right] \leq K_{\alpha A}\varepsilon^{\frac{7}{2}}\exp\left(-\varepsilon^3 K_{\alpha A} t\right). \tag{4.29}$$

Integrating (4.29) we have the inequality

$$Q(t) \leq \left(Q(0) + \varepsilon^{\frac{1}{2}}\right)\exp\left(\varepsilon^3 K_{\alpha A} t\right). \tag{4.30}$$

Since $Q(0) \leq C\varepsilon^{\frac{1}{2}}$, we have

$$Q(t) \leq \varepsilon^{\frac{1}{2}}\left(C+1\right)\exp\left(\varepsilon^3 K_{\alpha A} t\right) \tag{4.31}$$

For $t \in [-\varepsilon^{-3}\tau_0, \varepsilon^{-3}\tau_0]$ we have

$$Q(t) \leq \varepsilon^{\frac{1}{2}}\left(C+1\right)\exp\left(K_{\alpha A}\tau_0\right), \tag{4.32}$$

which is the desired result. □

We finish the chapter with a proof of the main result.

Proof of Theorem 4.1. Note that

$$\begin{aligned}\varepsilon^2 Q(0) \leq C_A\varepsilon^2 \big(&\left\|U^{(1)}(0)\right\|_{\ell^2} + \left\|U^{(2)}(0)\right\|_{\ell^2} + \left\|V^{(1)}(0)\right\|_{\ell^2} \\ &+ \left\|V^{(2)}(0)\right\|_{\ell^2} + \|W(0)\|_{\ell^2} + \|Z(0)\|_{\ell^2}\big).\end{aligned}$$

In light of decomposition (4.9) and the hypothesis (4.6) we have that $\varepsilon^2 Q(0) \leq C\varepsilon^{2+\frac{1}{2}}$, so that lemma 4.5 applies. With decomposition (4.9) and lemma 4.3

we have that

$$\begin{aligned}&\left\|u^{(1)}-\varepsilon^2 A\left(\xi,\eta,t\right)\right\|_{\ell^2}+\left\|u^{(2)}-\varepsilon^2 U_\varepsilon\left(\xi,\eta,t\right)\right\|_{\ell^2}\\+&\left\|w-\varepsilon^2 W_\varepsilon\left(\xi,\eta,t\right)\right\|_{\ell^2}+\left\|v^{(1)}\right\|_{\ell^2}+\left\|v^{(2)}\right\|_{\ell^2}+\|z\|_{\ell^2}\leq 6K\varepsilon^2 Q(t).\end{aligned}$$

By lemma 4.5 we have that $Q(t) \leq \varepsilon^{\frac{1}{2}}\left(C+1\right)\exp\left(K_{\alpha A}\tau_0\right)$, and the result follows. $\square$

Chapter 5

Propagation Along a Diagonal

5.1 Formulation and Main Result

In this chapter we will study the propagation of waves along the diagonal of a two-dimensional FPU lattice, as discussed in section 1.9.2, in the small-amplitude long-wavelength limit. In order to study propagation along a diagonal we will introduce a new coordinate system on the lattice by $m = \frac{j+k}{2}, n = \frac{j-k}{2}$. Under the new coordinate system the particle experiences nearest-neighbour interactions with neighbours located a half lattice site away. Due to this we introduce $\chi_{m,n} = x_{m+\frac{1}{2},n+\frac{1}{2}}$. The system becomes a diatomic system where x particles communicate with 4 χ nearest-neighbour particles and vice versa, see figure 1.5 for an illustration. As noted in section 1.9.2, the choice of parameters allows us to perform the reduction $x_{j,k} = y_{j,k}$, this will be reflected in our strain variables in this context,

$$\begin{aligned}
a^l_{m,n} &= \chi_{m,n} - x_{m,n}, \\
a^d_{m,n} &= x_{m+1,n+1} - \chi_{m,n}, \\
a^x_{m,n} &= x_{m+1,n} - \chi_{m,n}, \\
a^y_{m,n} &= x_{m,n+1} - \chi_{m,n}, \\
u_{m,n} &= \dot{x}_{m,n}, \\
v_{m,n} &= \dot{\chi}_{m,n}.
\end{aligned} \tag{5.1}$$

From (1.31) rewrite the Hamiltonian in the relabeled coordinates as,

$$\begin{aligned}
H =& \sum_{m,n} \left(u^2_{m,n} + v^2_{m,n}\right) + \sum_{m,n} V(a^l_{m,n}, a^l_{m,n}) \\
&+ \sum_{m,n} V(a^d_{m,n}, a^d_{m,n}) + \sum_{m,n} V(a^x_{m,n}, a^x_{m,n}) + \sum_{m,n} V(a^y_{m,n}, a^y_{m,n}).
\end{aligned} \tag{5.2}$$

As we showed formally in section 2.4, using an asymptotic multi-scale expansion of the form

$$a^x_{m,n} + a^l_{m,n} = \varepsilon^2 A\left(\varepsilon\left(m - \frac{c}{\sqrt{2}}t\right), \varepsilon^2 n, \varepsilon^3 t\right) + error, \tag{5.3}$$

yields a KP-II equation

$$c\sqrt{2}\partial_\xi\partial_\tau A + \frac{c^2}{96}\partial_\xi^4 A + \frac{c^2}{2}\partial_\eta^2 A + \frac{\alpha}{2}\partial_\xi^2\left(A^2\right) = 0, \tag{5.4}$$

where $\xi = \varepsilon\left(m - ct\right), \eta = \varepsilon^2 n, \tau = \varepsilon^3 t, c^2 \geq 1$. The sum

$$a^x_{m,n} + a^l_{m,n} = x_{m+1,n} - x_{m,n}$$

corresponds to a displacement along the main diagonal, and is a natural choice for the amplitude of a wave here. Other choices give rise to technical difficulties, including terms in the residuals of the expansion for which we don't have control in Sobolev space.

Remark. The continuous function in our expansion (5.3) does not correspond to any of the coordinates, but a linear combination. If one of the coordinates, say

$$a^l_{m,n} = \varepsilon^2 A\left(\varepsilon\left(m - \frac{c}{\sqrt{2}}t\right), \varepsilon^2 n, \varepsilon^3 t\right) + error, \tag{5.5}$$

is chosen as the approximation, the asymptotic expansions are much more complicated, and contain non-local terms, which are difficult to control in Sobolev norm. These terms can be transformed away by near identity transformations, however using the expansion (5.3) is simpler, and more natural as $a^x_{m,n} + a^l_{m,n}$ corresponds to a displacement along the main diagonal.

Remark. There is another alternative way of performing the expansion, which is to introduce a different set of coordinates, where a pair of coordinates connect the adjacent lattice sites, and a pair of coordinates correspond to the diagonal and its transverse direction. This however greatly complicates the equations of motion.

Given a solution $A\left(\xi, \eta, \tau\right)$ of the KP-II equation (5.4) we define the following functions

$$\begin{aligned} L_\varepsilon =& \frac{1}{2}A + \varepsilon\left(\frac{1}{2}\partial_\xi^{-1}\partial_\eta A - \frac{1}{8}\partial_\xi A\right) \\ &+ \varepsilon^3\left(-\frac{1}{48}\partial_\xi\partial_\eta A + \frac{1}{384}\partial_\xi^3 A + \frac{1}{8}\partial_\xi^{-1}\partial_\eta^2 A\right), \end{aligned} \tag{5.6}$$

$$\begin{aligned} D_\varepsilon =&\frac{1}{2}A + \varepsilon\left(\frac{1}{2}\partial_\xi^{-1}\partial_\eta A + \frac{1}{8}\partial_\xi A\right) \\ &+ \frac{\varepsilon^2}{2}\partial_\eta A + \varepsilon^3\left(\frac{5}{48}\partial_\xi\partial_\eta A - \frac{1}{384}\partial_\xi^3 A + \frac{3}{8}\partial_\xi^{-1}\partial_\eta^2 A\right), \end{aligned} \tag{5.7}$$

$$\begin{aligned} X_\varepsilon =&\frac{1}{2}A + \varepsilon\left(-\frac{1}{2}\partial_\xi^{-1}\partial_\eta A + \frac{1}{8}\partial_\xi A\right) \\ &+ \varepsilon^3\left(\frac{1}{48}\partial_\xi\partial_\eta A - \frac{1}{384}\partial_\xi^3 A - \frac{1}{8}\partial_\xi^{-1}\partial_\eta^2 A\right), \end{aligned} \tag{5.8}$$

$$\begin{aligned} Y_\varepsilon =&-\frac{1}{2}A + \varepsilon\left(\frac{1}{2}\partial_\xi^{-1}\partial_\eta A + \frac{1}{8}\partial_\xi A\right) \\ &- \frac{\varepsilon^2}{2}\partial_\eta A + \varepsilon^3\left(\frac{5}{48}\partial_\xi\partial_\eta A - \frac{1}{384}\partial_\xi^3 A + \frac{3}{8}\partial_\xi^{-1}\partial_\eta^2 A\right), \end{aligned} \tag{5.9}$$

$$U_\varepsilon = -\frac{c}{\sqrt{2}}A + \frac{\varepsilon}{2}\frac{c}{\sqrt{2}}\partial_\xi A + \varepsilon^2\left(\partial_\xi^{-1}\partial_\tau A - \frac{1}{12}\frac{c}{\sqrt{2}}\partial_\xi^2 A\right) - \frac{\varepsilon^3}{2}\partial_\tau A, \tag{5.10}$$

$$V_\varepsilon = U_\varepsilon + \frac{dL_\varepsilon}{dt}. \tag{5.11}$$

The functions in equations (5.6)-(5.11) correspond to approximations of the system (5.1) with domain in $\mathbb{R}^3$. The equations are arrived by taking asymptotic expansions, under the constraint given by (5.3), the specific decomposition is defined by equation (5.16) in section 5.2.

Theorem 5.1. *Let $A \in C^0\left([-\tau_0, \tau_0], H^9\left(\mathbb{R}^2\right)\right)$ be a solution to the cubic KP-II equation* (5.4)*, whose initial data satisfies*

$$A(\xi, \eta, 0) = A_0 \in H^9\left(\mathbb{R}^2\right)$$

such that

$$\partial_\xi^{-2}\partial_\eta^2 A_0 \in H^9\left(\mathbb{R}^2\right)$$

and

$$\partial_\xi^{-1}\partial_\eta^2\left[\partial_\xi^{-2}\partial_\eta^2 A_0 + A_0^2\right] \in H^2\left(\mathbb{R}^2\right).$$

Then there are constants $C_0, C_1, \varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$ if the initial *conditions of the FPU system* (5.2) *satisfies*

$$\begin{aligned}&\left\|a^l_{in}-\varepsilon^2 L_\varepsilon\left(\xi,\eta,0\right)\right\|_{\ell^2}+\left\|a^d_{in}-\varepsilon^2 D_\varepsilon\left(\xi,\eta,0\right)\right\|_{\ell^2}\\&+\left\|a^x_{in}-\varepsilon^2 X_\varepsilon\left(\xi,\eta,0\right)\right\|_{\ell^2}+\left\|a^y_{in}-\varepsilon^2 Y_\varepsilon\left(\xi,\eta,0\right)\right\|_{\ell^2}\\&+\left\|u_{in}-\varepsilon^2 U_\varepsilon\left(\xi,\eta,0\right)\right\|_{\ell^2}+\left\|v_{in}-\varepsilon^2 V_\varepsilon\left(\xi,\eta,0\right)\right\|_{\ell^2}\leq C_0\varepsilon^{\frac{5}{2}}\end{aligned}\tag{5.12}$$

then the solution to the FPU system (5.2) *satisfies*

$$\begin{aligned}&\left\|a^l-\varepsilon^2 L_\varepsilon\left(\xi,\eta,\tau\right)\right\|_{\ell^2}+\left\|a^d-\varepsilon^2 D_\varepsilon\left(\xi,\eta,\tau\right)\right\|_{\ell^2}\\&+\left\|a^x-\varepsilon^2 X_\varepsilon\left(\xi,\eta,\tau\right)\right\|_{\ell^2}+\left\|a^y-\varepsilon^2 Y_\varepsilon\left(\xi,\eta,\tau\right)\right\|_{\ell^2}\\&+\left\|u-\varepsilon^2 U_\varepsilon\left(\xi,\eta,\tau\right)\right\|_{\ell^2}+\left\|v-\varepsilon^2 V_\varepsilon\left(\xi,\eta,\tau\right)\right\|_{\ell^2}\leq C_1\varepsilon^{\frac{5}{2}}\end{aligned}\tag{5.13}$$

for $t\in[-\tau_0\varepsilon^{-3},\tau_0\varepsilon^{-3}]$.

5.2 Preliminary Results

The next several sections will be dedicated to the justification analysis for the KP-II limit in this case. The velocities in these variables are given by,

$$\begin{aligned}\dot{a}^l_{m,n}&=v_{m,n}-u_{m,n},\\\dot{a}^d_{m,n}&=u_{m+1,n+1}-v_{m,n},\\\dot{a}^x_{m,n}&=u_{m+1,n}-v_{m,n},\\\dot{a}^y_{m,n}&=u_{m,n+1}-v_{m,n}.\end{aligned}\tag{5.14}$$

The remaining equations of motion in these variables are given by,

$$\begin{aligned}\dot{u}_{m,n}=&c^2\left(a^l_{m,n}-a^d_{m-1,n-1}\right)+c^2\left(-a^x_{m-1,n}-a^y_{m,n-1}\right)\\&+2\alpha\left[\left(a^l_{m,n}\right)^2-\left(a^d_{m-1,n-1}\right)^2+\left(a^y_{m,n-1}\right)^2-\left(a^x_{m-1,n}\right)^2\right]\\\dot{v}_{m,n}=&c^2\left(a^d_{m,n}-a^l_{m,n}\right)+c^2\left(a^x_{m,n}+a^y_{m,n}\right)\\&+2\alpha\left[\left(a^d_{m,n}\right)^2-\left(a^l_{m,n}\right)^2+\left(a^x_{m,n}\right)^2-\left(a^y_{m,n}\right)^2\right]\end{aligned}\tag{5.15}$$

We will seek a continuous approximation of the lattice equations by first setting $\xi=\varepsilon\left(m-\frac{c}{\sqrt{2}}t\right),\eta=\varepsilon^2 n,\tau=\varepsilon^3 t$. We will study displacements of the form $x_{m+1,n}-x_{m,n}$, to do so we will take a look at the continuous function

$$a^x_{m,n}+a^l_{m,n}=\varepsilon^2 A(\xi,\eta,\tau)+error.$$

Notice that this comes with a corresponding pair at the "half" lattice site $\chi_{m,n}$, here we have the displacement term

$$a^{l}_{m+1,n} + a^{x}_{m,n} = \varepsilon^2 A_\varepsilon\left(\xi,\eta,\tau\right) + error.$$

Since we'd like to think of the $\chi_{m,n} = x_{m+\frac{1}{2},n+\frac{1}{2}}$, a natural expectation is that $A_\varepsilon = A(\xi + \frac{\varepsilon}{2}, \eta + \frac{\varepsilon^2}{2})$. Another relationship we could exploit in our formal derivation will be that

$$a^{l}_{m+1,n} - a^{l}_{m,n} = A_\varepsilon\left(\xi,\eta,\tau\right) - A\left(\xi,\eta,\tau\right) + error.$$

However it is more convenient to use the first equation in (5.14), and write

$$V_\varepsilon\left(\xi,\eta,\tau\right) = U_\varepsilon\left(\xi,\eta,\tau\right) + L_\varepsilon\left(\xi,\eta,\tau\right) + error,$$

which will be sufficient for our justification here.

We will use the following decomposition,

$$\begin{aligned} a^{l}_{m,n} &= \varepsilon^2 L_\varepsilon\left(\xi,\eta,\tau\right) + \varepsilon^2 L_{m,n}, \\ a^{d}_{m,n} &= \varepsilon^2 D_\varepsilon\left(\xi,\eta,\tau\right) + \varepsilon^2 D_{m,n}, \\ a^{x}_{m,n} &= \varepsilon^2 X_\varepsilon\left(\xi,\eta,\tau\right) + \varepsilon^2 X_{m,n}, \\ a^{y}_{m,n} &= \varepsilon^2 Y_\varepsilon\left(\xi,\eta,\tau\right) + \varepsilon^2 Y_{m,n}, \\ u_{m,n} &= \varepsilon^2 U_\varepsilon\left(\xi,\eta,\tau\right) + \varepsilon^2 U_{m,n}, \\ v_{m,n} &= \varepsilon^2 V_\varepsilon\left(\xi,\eta,\tau\right) + \varepsilon^2 V_{m,n}. \end{aligned} \tag{5.16}$$

Here A will be a solution of a KP-II equation, and the remaining terms $L_\varepsilon, ..., V_\varepsilon$ are ε-dependent functions. We will, for the time being, ignore the errors and use the equations of motion (5.14) and (5.15) to find approximations for the ε-dependent functions in (5.16).

First note that using this decomposition the first equation in (5.14) gives us the relationship $F_\varepsilon = E_\varepsilon + \frac{dL_\varepsilon}{dt}$.

Next we use the relationship

$$\frac{dA}{dt} = U_\varepsilon\left(\xi+\varepsilon,\eta,\tau\right) - U_\varepsilon\left(\xi,\eta,\tau\right), \tag{5.17}$$

we seek an approximate solution of the form

$$U_\varepsilon = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3. \tag{5.18}$$

We get the solution

$O(1) : U_0 = -\frac{c}{\sqrt{2}}A,$

$O(\varepsilon) : U_1 = \frac{c}{2\sqrt{2}}\partial_\xi A,$

$O(\varepsilon^2) : U_2 = \partial_\xi^{-1}\partial_\tau A - \frac{c}{12\sqrt{2}}\partial_\xi^2 A,$

$O(\varepsilon^3) : U_3 = -\frac{1}{2}\partial_\tau A.$

From the third equation in (5.14) we can write,

$$\begin{aligned} \frac{dX_\varepsilon}{dt} &= U_\varepsilon(\xi+\varepsilon,\eta) - V_\varepsilon(\xi,\eta) \\ &= U_\varepsilon(\xi+\varepsilon,\eta) - U_\varepsilon(\xi,\eta) - \frac{dL_\varepsilon}{dt}. \end{aligned} \tag{5.19}$$

We will look for approximate solutions to (5.19) by expanding both $X_\varepsilon, U_\varepsilon$ in terms of epsilon,

$$X_\varepsilon = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \varepsilon^3 X_3, \tag{5.20}$$

$$U_\varepsilon = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3. \tag{5.21}$$

First expanding the left hand side of (5.19) we have

$$\begin{aligned} \frac{dX_\varepsilon}{dt} =& -\frac{c}{\sqrt{2}}\varepsilon\partial_\xi X_\varepsilon + \varepsilon^3\partial_\tau X_\varepsilon \\ =&\varepsilon\left(-\frac{c}{\sqrt{2}}\partial_\xi X_0\right) + \varepsilon^2\left(-\frac{c}{\sqrt{2}}\partial_\xi X_1\right) \\ &+ \varepsilon^3\left(\partial_\tau X_0 - \frac{c}{\sqrt{2}}\partial_\xi X_2\right) \\ &+ \varepsilon^4\left(\partial_\tau X_1 - \frac{c}{\sqrt{2}}\partial_\xi X_3\right) + O(\varepsilon^5). \end{aligned} \tag{5.22}$$

We expand the right hand side of (5.19) in terms of ε and in Taylor series,

and simplifying using the known expression for U_ε we have

$$\begin{aligned} U_\varepsilon(\xi+\varepsilon,\eta) - U_\varepsilon(\xi,\eta) - \frac{dL_\varepsilon}{dt} = \varepsilon &\left(-\frac{c}{\sqrt{2}}\partial_\xi A + \frac{c}{\sqrt{2}}\partial_\xi L_0\right) \\ &+ \varepsilon^2 \left(\frac{c}{\sqrt{2}}\partial_\xi L_1\right) \\ &+ \varepsilon^3 \left(\partial_\tau A + \frac{c}{\sqrt{2}}\partial_\xi L_2 - \partial_\tau L_0\right) \\ &+ \varepsilon^4 \left(\frac{c}{\sqrt{2}}\partial_\xi L_3 - \partial_\tau L_1\right) + O(\varepsilon^5). \end{aligned} \tag{5.23}$$

Comparing equations (5.22) and (5.23) in powers of ε we get solutions for $X_0, ..., X_3$ in terms of $L_0, ..., L_3$ and A.

$O(\varepsilon) : X_0 = -L_0 + A,$
$O(\varepsilon^2) : X_1 = -L_1,$
$O(\varepsilon^3) : X_2 = -L_2,$
$O(\varepsilon^4) : X_3 = -L_3.$

From the fourth equation in (5.14) we have,

$$\begin{aligned} \frac{dY_\varepsilon}{dt} &= U_\varepsilon\left(\xi, \eta+\varepsilon^2\right) - V_\varepsilon(\xi,\eta) \\ &= U_\varepsilon\left(\xi, \eta+\varepsilon^2\right) - U_\varepsilon(\xi,\eta) - \frac{dL_\varepsilon}{dt}. \end{aligned} \tag{5.24}$$

First expanding the left hand side of (5.24) we have

$$\begin{aligned} \frac{dY_\varepsilon}{dt} &= -\frac{c}{\sqrt{2}}\varepsilon\partial_\xi Y_\varepsilon + \varepsilon^3\partial_\tau Y_\varepsilon \\ &= \varepsilon\left(-\frac{c}{\sqrt{2}}\partial_\xi Y_0\right) + \varepsilon^2\left(-\frac{c}{\sqrt{2}}\partial_\xi Y_1\right) \\ &\quad + \varepsilon^3\left(\partial_\tau Y_0 - \frac{c}{\sqrt{2}}\partial_\xi Y_2\right) \\ &\quad + \varepsilon^4\left(\partial_\tau Y_1 - \frac{c}{\sqrt{2}}\partial_\xi Y_3\right) + O(\varepsilon^5). \end{aligned} \tag{5.25}$$

We expand the right hand side of (5.24) in terms of ε and in Taylor series,

$$\begin{aligned}U_\varepsilon\left(\xi,\eta+\varepsilon^2\right)-U_\varepsilon\left(\xi,\eta\right)-\frac{dL_\varepsilon}{dt}&=\varepsilon\left(\frac{c}{\sqrt{2}}\partial_\xi L_0\right)\\&+\varepsilon^2\left(-\frac{c}{\sqrt{2}}\partial_\eta A+\frac{c}{\sqrt{2}}\partial_\xi L_1\right)\\&+\varepsilon^3\left(\frac{c}{\sqrt{2}}\frac{1}{2}\partial_\xi\partial_\eta A+\frac{c}{\sqrt{2}}\partial_\xi L_2-\partial_\tau L_0\right)\\&+\varepsilon^4\left(\partial_\xi^{-1}\partial_\eta\partial_\tau A-\frac{1}{12}\frac{c}{\sqrt{2}}\partial_\xi^2\partial_\eta A-\frac{1}{2}\frac{c}{\sqrt{2}}\partial_\eta^2 A\right.\\&\left.+\frac{c}{\sqrt{2}}\partial_\xi L_3-\partial_\tau L_1\right)+O(\varepsilon^5).\end{aligned}\tag{5.26}$$

Comparing equations (5.25) and (5.26) in powers of ε we get solutions for $Y_0,...,Y_3$ in terms of $L_0,...,L_2$ and A.

$O(\varepsilon):Y_0=-L_0,$
$O(\varepsilon^2):Y_1=\partial_\xi^{-1}\partial_\eta A-L_1,$
$O(\varepsilon^3):Y_2=\frac{1}{2}\partial_\eta A-L_2,$
$O(\varepsilon^4):Y_3=\frac{1}{12}\partial_\xi\partial_\eta A+\frac{1}{2}\partial_\xi^{-1}\partial_\eta^2 A-L_3.$

From the second equation in (5.14) we have,

$$\begin{aligned}\frac{dD_\varepsilon}{dt}&=U_\varepsilon\left(\xi+\varepsilon,\eta+\varepsilon^2\right)-V_\varepsilon\left(\xi,\eta\right)\\&=U_\varepsilon\left(\xi+\varepsilon,\eta+\varepsilon^2\right)-U_\varepsilon\left(\xi,\eta\right)-\frac{dL_\varepsilon}{dt}.\end{aligned}\tag{5.27}$$

First expanding the left hand side of (5.27) we have

$$\begin{aligned}\frac{dD_\varepsilon}{dt}=&-\frac{c}{\sqrt{2}}\varepsilon\partial_\xi D_\varepsilon+\varepsilon^3\partial_\tau D_\varepsilon\\=&\varepsilon\left(-\frac{c}{\sqrt{2}}\partial_\xi D_0\right)+\varepsilon^2\left(-\frac{c}{\sqrt{2}}\partial_\xi D_1\right)\\&+\varepsilon^3\left(\partial_\tau D_0-\frac{c}{\sqrt{2}}\partial_\xi D_2\right)\\&+\varepsilon^4\left(\partial_\tau D_1-\frac{c}{\sqrt{2}}\partial_\xi D_3\right)+O(\varepsilon^5).\end{aligned}\tag{5.28}$$

We expand the right hand side of (5.27) in terms of ε and in Taylor series,

$$\begin{aligned} U_\varepsilon\left(\xi+\varepsilon,\eta+\varepsilon^2\right)-U_\varepsilon\left(\xi,\eta\right)-\frac{dL_\varepsilon}{dt} &= \varepsilon\left(-\frac{c}{\sqrt{2}}\partial_\xi A+\frac{c}{\sqrt{2}}\partial_\xi L_0\right)\\ &+\varepsilon^2\left(-\frac{c}{\sqrt{2}}\partial_\eta A+\frac{c}{\sqrt{2}}\partial_\xi L_1\right)\\ &+\varepsilon^3\left(\partial_\tau A-\frac{1}{2}\frac{c}{\sqrt{2}}\partial_\xi\partial_\eta A+\frac{c}{\sqrt{2}}\partial_\xi L_2-\partial_\tau L_0\right)\\ &+\varepsilon^4\left(\partial_\xi^{-1}\partial_\eta\partial_\tau A-\frac{1}{12}\frac{c}{\sqrt{2}}\partial_\xi^2\partial_\eta A-\frac{1}{2}\frac{c}{\sqrt{2}}\partial_\eta^2 A\right.\\ &\left.+\frac{c}{\sqrt{2}}\partial_\xi L_3-\partial_\tau L_1\right)+O(\varepsilon^5). \end{aligned} \tag{5.29}$$

Comparing equations (5.28) and (5.29) in powers of ε we get solutions for $D_0,...,D_3$ in terms of $L_0,...,L_3$ and A.

$O(\varepsilon): D_0 = A - L_0,$
$O(\varepsilon^2): D_1 = \partial_\xi^{-1}\partial_\eta A - L_1,$
$O(\varepsilon^3): D_2 = -\frac{1}{2}\partial_\eta A - L_2,$
$O(\varepsilon^4): D_3 = \frac{1}{12}\partial_\xi\partial_\eta A + \frac{1}{2}\partial_\xi^{-1}\partial_\eta^2 A - L_3.$

So far we have found the the terms up to unknown functions $L_1,...,L_3$. We can use either of the remaining equations of motion to find these values. From the first equation in (5.15) we have

$$\begin{aligned} -\frac{c}{\sqrt{2}}\varepsilon\partial_\xi U_\varepsilon+\varepsilon^3\partial_\tau U_\varepsilon &= c^2\left(L_\varepsilon(\xi,\eta)-D_\varepsilon\left(\xi-\varepsilon,\eta-\varepsilon^2\right)\right)\\ &+c^2\left(-X_\varepsilon(\xi-\varepsilon,\eta)-Y_\varepsilon\left(\xi,\eta-\varepsilon^2\right)\right)\\ &+2\alpha\varepsilon^2\left[\left(L_\varepsilon(\xi,\eta)\right)^2-\left(D_\varepsilon\left(\xi-\varepsilon,\eta-\varepsilon^2\right)\right)^2\right]\\ &+2\alpha\varepsilon^2\left[-\left(X_\varepsilon(\xi-\varepsilon,\eta)\right)^2+\left(Y_\varepsilon\left(\xi,\eta-\varepsilon^2\right)\right)^2\right] \end{aligned} \tag{5.30}$$

From the second equation in (5.15) we have

$$\begin{aligned} \frac{dV_\varepsilon}{dt}=\frac{dU_\varepsilon}{dt}+\frac{d^2L_\varepsilon}{dt^2} =& c^2\left(D_\varepsilon(\xi,\eta)-L_\varepsilon(\xi,\eta)\right)\\ &+c^2\left(X_\varepsilon(\xi,\eta)+Y_\varepsilon(\xi,\eta)\right)\\ &+2\alpha\varepsilon^2\left[D_\varepsilon(\xi,\eta)^2-L_\varepsilon(\xi,\eta)^2\right]\\ &+2\alpha\varepsilon^2\left[X_\varepsilon(\xi,\eta)^2-Y_\varepsilon(\xi,\eta)^2\right] \end{aligned} \tag{5.31}$$

By expanding equation (5.31) and comparing in terms of ϵ we have the following solutions, which were checked for consistency with equation (5.30):

$$
\begin{aligned}
O(1): L_0 &= \frac{1}{2}A, \\
D_0 &= \frac{1}{2}A, \\
X_0 &= \frac{1}{2}A, \\
Y_0 &= -\frac{1}{2}A \\
O(\varepsilon): L_1 &= \frac{1}{2}\partial_\xi^{-1}\partial_\eta A - \frac{1}{8}\partial_\xi A \\
D_1 &= \frac{1}{2}\partial_\xi^{-1}\partial_\eta A + \frac{1}{8}\partial_\xi A \\
X_1 &= -\frac{1}{2}\partial_\xi^{-1}\partial_\eta A + \frac{1}{8}\partial_\xi A \\
Y_1 &= \frac{1}{2}\partial_\xi^{-1}\partial_\eta A + \frac{1}{8}\partial_\xi A \\
O(\varepsilon^2): L_2 &= 0 \\
D_2 &= \frac{1}{2}\partial_\eta A \\
X_2 &= 0 \\
Y_2 &= -\frac{1}{2}\partial_\eta A \\
O(\varepsilon^3): L_3 &= -\frac{1}{48}\partial_\xi\partial_\eta A + \frac{1}{384}\partial_\xi^3 A + \frac{1}{8}\partial_\xi^{-1}\partial_\eta^2 A \\
D_3 &= \frac{5}{48}\partial_\xi\partial_\eta A - \frac{1}{384}\partial_\xi^3 A + \frac{3}{8}\partial_\xi^{-1}\partial_\eta^2 A \\
X_3 &= \frac{1}{48}\partial_\xi\partial_\eta A - \frac{1}{384}\partial_\xi^3 A - \frac{1}{8}\partial_\xi^{-1}\partial_\eta^2 A \\
Y_3 &= \frac{5}{48}\partial_\xi\partial_\eta A - \frac{1}{384}\partial_\xi^3 A + \frac{3}{8}\partial_\xi^{-1}\partial_\eta^2 A
\end{aligned}
$$

5.3 Justification Analysis

Plugging our decomposition (5.16) into the equations of motion (5.14) gives us, up to some residual terms, the following equations of motion

$$\begin{aligned}\dot{L}_{m,n} &= V_{m,n} - U_{m,n}\\ \dot{D}_{m,n} &= U_{m+1,n+1} - V_{m,n} + Res^D_{m,n}\\ \dot{X}_{m,n} &= U_{m+1,n} - V_{m,n} + Res^X_{m,n}\\ \dot{Y}_{m,n} &= U_{m,n+1} - V_{m,n} + Res^Y_{m,n}\end{aligned} \tag{5.32}$$

where the residuals are

$$\begin{aligned}Res^D_{m,n} =& \varepsilon\frac{c}{\sqrt{2}}\partial_\xi D_\varepsilon - \varepsilon^3\partial_\tau D_\varepsilon + \varepsilon\frac{c}{\sqrt{2}}\partial_\xi L_\varepsilon\\ &- \varepsilon^3\partial_\tau L_\varepsilon + U_\varepsilon\left(\xi+\varepsilon, \eta+\varepsilon^2\right) - U_\varepsilon\\ Res^X_{m,n} =& \varepsilon\frac{c}{\sqrt{2}}\partial_\xi X_\varepsilon - \varepsilon^3\partial_\tau X_\varepsilon + \varepsilon\frac{c}{\sqrt{2}}\partial_\xi L_\varepsilon\\ &- \varepsilon^3\partial_\tau L_\varepsilon + U_\varepsilon\left(\xi+\varepsilon, \eta\right) - U_\varepsilon\\ Res^Y_{m,n} =& \varepsilon\frac{c}{\sqrt{2}}\partial_\xi Y_\varepsilon - \varepsilon^3\partial_\tau Y_\varepsilon + \varepsilon\frac{c}{\sqrt{2}}\partial_\xi L_\varepsilon\\ &- \varepsilon^3\partial_\tau L_\varepsilon + U_\varepsilon\left(\xi, \eta+\varepsilon^2\right) - U_\varepsilon.\end{aligned} \tag{5.33}$$

Similarly plugging (5.16) into (5.15) gives

$$\begin{aligned}\dot{U}_{m,n} =& c^2\left(L_{m,n} - D_{m-1,n-1} - X_{m-1,n} - Y_{m,n-1}\right)\\ &+ 2\alpha\varepsilon^2\left[(L_{m,n})^2 - (D_{m-1,n-1})^2 + (Y_{m,n-1})^2 - (X_{m-1,n})^2\right]\\ &+ 4\alpha\varepsilon^2\left[L_{m,n}L_\varepsilon(\xi,\eta) - D_{m-1,n-1}D_\varepsilon\left(\xi-\varepsilon,\eta-\varepsilon^2\right)\right]\\ &+ 4\alpha\varepsilon^2\left[Y_{m,n-1}Y_\varepsilon\left(\xi,\eta-\varepsilon^2\right) - X_{m-1,n}X_\varepsilon(\xi-\varepsilon,\eta)\right]\\ &+ Res^U_{m,n}\\ \dot{V}_{m,n} =& c^2\left(D_{m,n} - L_{m,n} + X_{m,n} + Y_{m,n}\right)\\ &+ 2\alpha\varepsilon^2\left[(D_{m,n})^2 - (L_{m,n})^2 - (Y_{m,n})^2 + (X_{m,n})^2\right]\\ &+ 4\alpha\varepsilon^2\left[D_{m,n}D_\varepsilon(\xi,\eta) - L_{m,n}L_\varepsilon(\xi,\eta)\right]\\ &+ 4\alpha\varepsilon^2\left[X_{m,n}X_\varepsilon(\xi,\eta) - Y_{m,n}Y_\varepsilon(\xi,\eta)\right] + Res^V_{m,n}\end{aligned} \tag{5.34}$$

where the residuals are

$$
\begin{aligned}
Res^U_{m,n} =& \frac{c}{\sqrt{2}}\varepsilon\partial_\xi U_\varepsilon - \varepsilon^3\partial_\tau U_\varepsilon \\
&+ c^2\left(L_\varepsilon(\xi,\eta) - D_\varepsilon\left(\xi-\varepsilon,\eta-\varepsilon^2\right)\right) \\
&+ c^2\left(-X_\varepsilon(\xi-\varepsilon,\eta) - Y_\varepsilon\left(\xi,\eta-\varepsilon^2\right)\right) \\
&+ 2\alpha\varepsilon^2\left[\left(L_\varepsilon(\xi,\eta)\right)^2 - \left(D_\varepsilon\left(\xi-\varepsilon,\eta-\varepsilon^2\right)\right)^2\right] \\
&+ 2\alpha\varepsilon^2\left[-\left(X_\varepsilon(\xi-\varepsilon,\eta)\right)^2 + \left(Y_\varepsilon\left(\xi,\eta-\varepsilon^2\right)\right)^2\right] \\
Res^V_{m,n} =& \varepsilon\frac{c}{\sqrt{2}}\partial_\xi U_\varepsilon - \varepsilon^3\partial_\tau U_\varepsilon - \varepsilon^2\frac{c^2}{2}\partial_\xi^2 L_\varepsilon \\
&+ 2\frac{c}{\sqrt{2}}\varepsilon^4\partial_\xi\partial_\tau L_\varepsilon - \varepsilon^6\partial_\tau^2 L_\varepsilon \\
&+ c^2\left(D_\varepsilon(\xi,\eta) - L_\varepsilon(\xi,\eta)\right) \\
&+ c^2\left(X_\varepsilon(\xi,\eta) + Y_\varepsilon(\xi,\eta)\right) \\
&+ 2\alpha\varepsilon^2\left[D_\varepsilon(\xi,\eta)^2 - L_\varepsilon(\xi,\eta)^2\right] \\
&+ 2\alpha\varepsilon^2\left[X_\varepsilon(\xi,\eta)^2 - Y_\varepsilon(\xi,\eta)^2\right]
\end{aligned} \tag{5.35}
$$

As in chapter 4 we would like to estimate the ℓ^2 norm of the residuals using the H^s norm of the approximating function $A(\xi,\eta,\tau)$, and their first few anti-derivatives. This is summarized in the following lemma,

Lemma 5.1. *Let $A(\xi,\eta,\tau)$ be in the same class of functions as in theorem 5.1. Then there are constants $C(A) > 0$ such that for all $\varepsilon \in (0,1]$, we have*

$$\left\|Res^D\right\|_{\ell^2} + \left\|Res^X\right\|_{\ell^2} + \left\|Res^Y\right\|_{\ell^2} + \left\|Res^U\right\|_{\ell^2} + \left\|Res^V\right\|_{\ell^2} \le C\varepsilon^{\frac{7}{2}}. \tag{5.36}$$

Proof. Using the calculations of section 5.2 this proof is similar to the proof of lemma 4.2. □

As in chapter 4, in order to control how far away the approximating functions drift from solutions to the system (5.2) we introduce the following energy function,

$$
\begin{aligned}
E(t) =& \sum_{m,n}\frac{1}{2}\left(U^2_{m,n} + V^2_{m,n}\right) + \frac{c^2}{2}\left(L^2_{m,n} + D^2_{m,n} + X^2_{m,n} + Y^2_{m,n}\right) \\
&+ \sum_{m,n} 2\alpha\varepsilon^2\left(L^2_{m,n}L_\varepsilon + D^2_{m,n}D_\varepsilon + X^2_{m,n}X_\varepsilon + Y^2_{m,n}Y_\varepsilon\right) \\
&+ \sum_{m,n}\frac{2}{3}\alpha\varepsilon^2\left(L^3_{m,n} + D^3_{m,n} + X^3_{m,n} + Y^3_{m,n}\right).
\end{aligned} \tag{5.37}
$$

For this energy function we have the following lemma, which shows that the energy is coercive with respect to the ℓ^2-norm of the lattice equations.

Lemma 5.2. *Let* $A \in C^0\left([-\tau_0, \tau_0], H^4\left(\mathbb{R}^2\right)\right)$ *and* $\partial_\xi^{-1} A \in C^0\left([-\tau_0, \tau_0], H^3\left(\mathbb{R}^2\right)\right)$. *Define*

$$\mathcal{A} = \sup_{\tau \in [-\tau_0, \tau_0]} \|A(\cdot, \cdot, \tau)\|_{L^\infty}$$

and

$$\mathcal{L} = \sup_{\tau \in [-\tau_0, \tau_0]} \left(\left\|\partial_\xi^{-1} A\right\|_{H^4} + 2\|A\|_{H^5} \right).$$

Then, there exists some

$$\varepsilon_1(\mathcal{A}, \mathcal{L}) > 0$$

such that

$$\|U\|_{\ell^2}^2 + \|V\|_{\ell^2}^2 + \|L\|_{\ell^2}^2 + \|D\|_{\ell^2}^2 + \|X\|_{\ell^2}^2 + \|Y\|_{\ell^2}^2 \leq 4E(t),$$

for each $\varepsilon \in (0, \varepsilon_1]$ *and* $t \in \left[\frac{-\tau_0}{\varepsilon^3}, \frac{\tau_0}{\varepsilon^3}\right]$.

Proof. The proof is similar to lemma 4.3. □

For the growth of the energy (5.37) we have

$$\begin{aligned} \dot{E}(t) = & \sum_{m,n} U_{m,n} Res_{m,n}^U + V_{m,n} Res_{m,n}^V \\ & + c^2 \sum_{m,n} \left(D_{m,n} Res_{m,n}^D + Y_{m,n} Res_{m,n}^Y + X_{m,n} Res_{m,n}^X \right) \\ & + 4\alpha\varepsilon^2 \sum_{m,n} \left(D_{m,n}^2 Res_{m,n}^D + Y_{m,n}^2 Res_{m,n}^Y + X_{m,n}^2 Res_{m,n}^X \right) \\ & + 4\alpha\varepsilon^2 \sum_{m,n} \left(D_{m,n} Res_{m,n}^D D_\varepsilon(\xi, \eta) + Y_{m,n} Res_{m,n}^Y Y_\varepsilon(\xi, \eta) \right. \\ & \qquad \left. + X_{m,n} Res_{m,n}^X X_\varepsilon(\xi, \eta) \right) \\ & + 2\alpha\varepsilon^2 \sum_{m,n} \left(L_{m,n}^2 \dot{L}_\varepsilon + D_{m,n}^2 \dot{D}_\varepsilon + X_{m,n}^2 \dot{X}_\varepsilon + Y_{m,n}^2 \dot{Y}_\varepsilon \right). \end{aligned} \tag{5.38}$$

In light of this and lemmas 5.1 and 5.2, a simple calculation shows that we can bound the growth of the energy (5.37) by

$$\left|\dot{E}(t)\right| \leq K_{\alpha A} \left(E^{\frac{1}{2}} \varepsilon^{\frac{7}{2}} + E\varepsilon^3 \right). \tag{5.39}$$

As in chapter 4 we use the substitution $Q(t) = E^{\frac{1}{2}}$, and an application lemma 4.5 gives the result of Theorem 5.1.

Chapter 6

KP-II limit of 2D-β Model

6.1 Formulation and Main Result

The goal of this chapter is to study the behaviour of two-dimensional Fermi-Pasta-Ulam (FPU) β-model lattices in the small-amplitude long-wave limit. We will give rigorous justification of the cubic Kadomtsev-Petviashvili (cKP-II) equation. We will study a 2D FPU system on a square lattice with a Hamiltonian given by equation (1.23) using the potential function (1.37)-(1.38). This chapter largely follows the structure of chapter 4, as such many of the computations will not be redone, the results will be stated with the relevant computations referenced.

Introducing the strain variables,

$$u^{(1)}_{j,k} = x_{j+1,k} - x_{j,k},$$
$$u^{(2)}_{j,k} = x_{j,k+1} - x_{j,k},$$
$$v^{(1)}_{j,k} = y_{j+1,k} - y_{j,k},$$
$$v^{(2)}_{j,k} = y_{j,k+1} - y_{j,k},$$

we can rewrite the Hamiltonian as

$$H = \frac{1}{2}\sum_{j,k}\left(w^2_{j,k} + z^2_{j,k}\right) + \sum_{j,k} V_\beta(||(u^{(1)}_{j,k}, v^{(1)}_{j,k})||) + V_\beta(||(u^{(2)}_{j,k}, v^{(2)}_{j,k})||). \tag{6.1}$$

Using an asymptotic multi-scale expansion, analogous to the expansion used for the one-dimensional FPU system in [43], of the form

$$u^{(1)}_{j,k} = \varepsilon A\left(\varepsilon\left(j - c_1 t\right), \varepsilon^2 k, \varepsilon^3 t\right) + error, \tag{6.2}$$

yields a cKP-II equation

$$2\partial_\xi\partial_\tau A + \frac{1}{12}\partial_\xi^4 A + \beta\partial_\xi^2\left(A^3\right) + \partial_\eta^2 A = 0, \tag{6.3}$$

where $\xi = \varepsilon\,(j-t)\,, \eta = \varepsilon^2 k, \tau = \varepsilon^3 t$.

We define the anti-derivative of a function $u \in H^s\left(\mathbb{R}^2\right)$ by

$$\partial_\xi^{-1} u = \int_{-\infty}^{\xi} u(\xi', \eta, \tau)d\xi'.$$

We are looking for solutions to the cKP-II equation, for which both $u \in H^s\left(\mathbb{R}^2\right)$ and $\partial_\xi^{-1} u \in H^s\left(\mathbb{R}^2\right)$. We will require that the solution has enough regularity so that $\partial_\xi^{-1}\partial_\tau^2 A \in C\left(\left[-\tau_0, \tau_0\right], H^s\left(\mathbb{R}^2\right)\right)$. This modification of arguments from [22] is presented in lemma 3.1 which is proven in Section 3.3, and built on results from [42]. For the extension we will require more stringent constraints on the initial data, which are summarized in the statement of theorem 4.1, the main result of the chapter.

Given a solution to the cKP-II equation (6.3), we define

$$W_\varepsilon\left(\xi, \eta, \tau\right) = -A + \frac{\varepsilon}{2}\partial_\xi A + \varepsilon^2\left(\partial_\xi^{-1}\partial_\tau A - \frac{1}{12}\partial_\xi^2 A\right) - \frac{\varepsilon^3}{2}\partial_\tau A \tag{6.4}$$

and

$$U_\varepsilon\left(\xi, \eta, \tau\right) = -\varepsilon\partial_\xi^{-1}\partial_\eta A - \frac{\varepsilon^2}{2}\partial_\eta A + \varepsilon^3\left(\frac{1}{2}\partial_\xi^{-1}\partial_\eta^2 A + \frac{1}{12}\partial_\xi\partial_\eta A\right). \tag{6.5}$$

The following theorem presents the main result of this chapter.

Theorem 6.1. *Let* $A \in C^0\left(\left[-\tau_0, \tau_0\right], H^9\left(\mathbb{R}^2\right)\right)$ *be a solution to the cubic KP-II equation* (6.3)*, whose initial data satisfies*

$$A(\xi, \eta, 0) = A_0 \in H^9\left(\mathbb{R}^2\right)$$

such that

$$\partial_\xi^{-2}\partial_\eta^2 A_0 \in H^9\left(\mathbb{R}^2\right)$$

and

$$\partial_\xi^{-1}\partial_\eta^2\left[\partial_\xi^{-2}\partial_\eta^2 A_0 + A_0^2\right] \in H^2\left(\mathbb{R}^2\right).$$

Define

$$\mathcal{A} = \sup_{\tau\in[-\tau_0,\tau_0]} \left\|A(\cdot,\cdot,\tau)\right\|_{L^\infty}$$

and

$$\mathcal{U} = \sup_{\tau\in[-\tau_0,\tau_0]} \left(\left\|\partial_\xi^{-1} A\right\|_{H^3} + 2\left\|A\right\|_{H^4}\right).$$

Then there are constants $C_0, C_1, \varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$ if the initial conditions of the 2D FPU system satisfies

$$\begin{aligned} &\left\| u_{in}^{(1)} - \varepsilon A(\xi, \eta, 0) \right\|_{\ell^2} + \left\| u_{in}^{(2)} - \varepsilon U_\varepsilon(\xi, \eta, 0) \right\|_{\ell^2} \\ &+ \left\| w_{in} - \varepsilon W_\varepsilon(\xi, \eta, 0) \right\|_{\ell^2} + \left\| v_{in}^{(1)} \right\|_{\ell^2} + \left\| v_{in}^{(2)} \right\|_{\ell^2} + \| z_{in} \|_{\ell^2} \leq C_0 \varepsilon^{\frac{3}{2}} \end{aligned} \tag{6.6}$$

then the solution to the 2D FPU system satisfies

$$\begin{aligned} &\left\| u^{(1)} - \varepsilon A(\xi, \eta, t) \right\|_{\ell^2} + \left\| u^{(2)} - \varepsilon U_\varepsilon(\xi, \eta, t) \right\|_{\ell^2} \\ &+ \left\| w - \varepsilon W_\varepsilon(\xi, \eta, t) \right\|_{\ell^2} + \left\| v^{(1)} \right\|_{\ell^2} + \left\| v^{(2)} \right\|_{\ell^2} + \| z \|_{\ell^2} + \leq C_1 \varepsilon^{\frac{3}{2}}, \end{aligned} \tag{6.7}$$

for $t \in [-\tau_0 \varepsilon^{-3}, \tau_0 \varepsilon^{-3}]$.

The proof of theorem 6.1 is similar to that of theorem 4.1, however some modifications need to be made to accommodate the cubic nonlinearity. The scaling of the continuous limit is of order ε here instead of ϵ^2. A more complicated energy function needs to be introduced to perform the energy estimates step. In the energy estimate the derivative of the quantity, $Q(t)$, we need to control using a Gronwall type lemma is quadratic in $Q(t)$ here, rather than linear. Consequently the Gronwall lemma needs to be modified, namely we show that for a sufficiently small ε we have that $Q(t) \leq 1$, so that the derivative of the term is linear, for the time scale of our approximation.

6.2 Preliminary Results

The equations of motion in the strain variables are given by

$$
\begin{aligned}
\dot{u}_{j,k}^{(1)} =& w_{j+1,k} - w_{j,k}, \\
\dot{u}_{j,k}^{(2)} =& w_{j,k+1} - w_{j,k}, \\
\dot{v}_{j,k}^{(1)} =& z_{j+1,k} - z_{j,k}, \\
\dot{v}_{j,k}^{(2)} =& z_{j,k+1} - z_{j,k}, \\
\dot{w}_{j,k} =& \left(u_{j,k}^{(1)} - u_{j-1,k}^{(1)}\right) + \left(u_{j,k}^{(2)} - u_{j,k-1}^{(2)}\right) \\
&+ \beta\left[\left(u_{j,k}^{(1)}\right)^3 - \left(u_{j-1,k}^{(1)}\right)^3 + \left(u_{j,k}^{(2)}\right)^3 - \left(u_{j,k-1}^{(2)}\right)^3\right] \\
&+ \beta\left[\left(u_{j,k}^{(1)}\right)\left(v_{j,k}^{(1)}\right)^2 - \left(u_{j-1,k}^{(1)}\right)\left(v_{j-1,k}^{(1)}\right)^2\right] \\
&+ \beta\left[\left(u_{j,k}^{(2)}\right)\left(v_{j,k}^{(2)}\right)^2 - \left(u_{j,k-1}^{(2)}\right)\left(v_{j,k-1}^{(2)}\right)^2\right], \\
\dot{z}_{j,k} =& \left(v_{j,k}^{(2)} - v_{j,k-1}^{(2)}\right) + \left(v_{j,k}^{(1)} - v_{j-1,k}^{(1)}\right) \\
&+ \beta\left[\left(v_{j,k}^{(2)}\right)^3 - \left(v_{j,k-1}^{(2)}\right)^3 + \left(v_{j,k}^{(1)}\right)^3 - \left(v_{j-1,k}^{(1)}\right)^3\right] \\
&+ \beta\left[\left(v_{j,k}^{(1)}\right)\left(u_{j,k}^{(1)}\right)^2 - \left(v_{j-1,k}^{(1)}\right)\left(u_{j-1,k}^{(1)}\right)^2\right] \\
&+ \beta\left[\left(v_{j,k}^{(2)}\right)\left(u_{j,k}^{(2)}\right)^2 - \left(v_{j,k-1}^{(2)}\right)\left(u_{j,k-1}^{(2)}\right)^2\right].
\end{aligned}
\tag{6.8}
$$

Let us use the following decomposition,

$$
\begin{aligned}
u_{j,k}^{(1)} &= \varepsilon A(\xi,\eta,\tau) + \varepsilon U_{j,k}^{(1)} \\
u_{j,k}^{(2)} &= \varepsilon U_\varepsilon(\xi,\eta,\tau) + \varepsilon U_{j,k}^{(2)} \\
v_{j,k}^{(1)} &= \varepsilon V_{j,k}^{(1)} \\
v_{j,k}^{(2)} &= \varepsilon V_{j,k}^{(2)} \\
w_{j,k} &= \varepsilon W_\varepsilon(\xi,\eta,\tau) + \varepsilon W_{j,k} \\
z_{j,k} &= \varepsilon Z_{j,k}
\end{aligned}
\tag{6.9}
$$

where $\xi = \varepsilon(j-t), \eta = \varepsilon^2 k, \tau = \varepsilon^3 t$, $(j,k) \in \mathbb{Z}^2$. Here $A(\xi,\eta,\tau)$ is a suitable solution to the cKP-II equation (6.3). U_ε and W_ε are a pair of ε-dependent functions, which will allow us to eliminate lower order terms in ε arising from time derivatives and finite differences of A in the equations of motion. Neglecting the error terms, $W_{j,k}, U_{j,k}^{(1)}$, and rewriting the first equation of (6.8) using the decomposition (6.9) we end up with the ε-dependent function $W_\varepsilon(\xi, \eta, \tau)$ satisfying the equality

$$W_\varepsilon(\xi + \varepsilon, \eta) - W_\varepsilon(\xi, \eta) = -\varepsilon \partial_\xi A + \varepsilon^3 \partial_\tau A. \tag{6.10}$$

This calculation was already performed in chapter 4, the result is given by equation (6.4).

We can similarly rewrite the second equation of (6.8) and neglect the error termsto end up with the following equation for a second $\varepsilon-$dependant function $U_\varepsilon(\xi, \eta, \tau)$,

$$W_\varepsilon\left(\xi, \eta + \varepsilon^2\right) - W_\varepsilon(\xi, \eta) = -\varepsilon c_1 \partial_\xi U_\varepsilon + \varepsilon^3 \partial_\tau U_\varepsilon. \tag{6.11}$$

In light of our calculations in chapter 4 we have the solution given by equation (6.5).

6.3 Justification Analysis

Substituting decomposition (6.9) into the equations of motion (6.8) we get evolution equations for the error term.

$$\begin{aligned} \dot{U}^{(1)}_{j,k} &= W_{j+1,k} - W_{j,k} + Res^{U^{(1)}}_{j,k}, \\ \dot{U}^{(2)}_{j,k} &= W_{j,k+1} - W_{j,k} + Res^{U^{(2)}}_{j,k}, \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} Res^{U^{(1)}}_{j,k}(t) &= \varepsilon \partial_\xi A - \varepsilon^3 \partial_\tau A + W_\varepsilon(\xi + \varepsilon, \eta) - W_\varepsilon(\xi, \eta), \\ Res^{U^{(2)}}_{j,k}(t) &= \varepsilon \partial_\xi U_\varepsilon - \varepsilon^3 \partial_\tau U_\varepsilon + W_\varepsilon\left(\xi, \eta + \varepsilon^2\right) - W_\varepsilon(\xi, \eta). \end{aligned} \tag{6.13}$$

We also have the following unchanged equations,

$$\begin{aligned} \dot{V}^{(1)}_{j,k} &= Z_{j+1,k} - Z_{j,k}, \\ \dot{V}^{(2)}_{j,k} &= Z_{j,k+1} - Z_{j,k}. \end{aligned} \tag{6.14}$$

Substituting the decomposition (6.9) into the last two equations (6.8) yields

$$\begin{aligned}
\dot{W}_{j,k} =& U^{(1)}_{j,k} - U^{(1)}_{j-1,k} + U^{(2)}_{j,k} - U^{(2)}_{j,k-1} \\
&+ \beta\varepsilon^2 \left[\left(U^{(1)}_{j,k}\right)^3 - \left(U^{(1)}_{j-1,k}\right)^3 \right] + \beta\varepsilon^2 \left[\left(U^{(2)}_{j,k}\right)^3 - \left(U^{(2)}_{j,k-1}\right)^3 \right] \\
&+ 3\beta\varepsilon^2 \left[U^{(1)}_{j,k} A(\xi,\eta)^2 - U^{(1)}_{j-1,k} A(\xi-\varepsilon,\eta)^2 \right] \\
&+ 3\beta\varepsilon^2 \left[\left(U^{(1)}_{j,k}\right)^2 A(\xi,\eta) - \left(U^{(1)}_{j-1,k}\right)^2 A(\xi-\varepsilon,\eta) \right] \\
&+ 3\beta\varepsilon^2 \left[U^{(2)}_{j,k} U_\varepsilon(\xi,\eta)^2 - U^{(2)}_{j,k-1} U_\varepsilon\left(\xi,\eta-\varepsilon^2\right)^2 \right] \\
&+ 3\beta\varepsilon^2 \left[\left(U^{(2)}_{j,k}\right)^2 U_\varepsilon(\xi,\eta) - \left(U^{(2)}_{j,k-1}\right)^2 U_\varepsilon\left(\xi,\eta-\varepsilon^2\right) \right] \\
&+ \beta\varepsilon^2 \left[A(\xi,\eta) \left(V^{(1)}_{j,k}\right)^2 - A(\xi-\varepsilon,\eta) \left(V^{(1)}_{j-1,k}\right)^2 \right] \\
&+ \beta\varepsilon^2 \left[\left(U^{(1)}_{j,k}\right) \left(V^{(1)}_{j,k}\right)^2 - \left(U^{(1)}_{j-1,k}\right) \left(V^{(1)}_{j-1,k}\right)^2 \right] \\
&+ \beta\varepsilon^2 \left[U_\varepsilon(\xi,\eta) \left(V^{(2)}_{j,k}\right)^2 - U_\varepsilon\left(\xi,\eta-\varepsilon^2\right) \left(V^{(2)}_{j,k-1}\right)^2 \right] \\
&+ \beta\varepsilon^2 \left[\left(U^{(2)}_{j,k}\right) \left(V^{(2)}_{j,k}\right)^2 - \left(U^{(2)}_{j,k-1}\right) \left(V^{(2)}_{j,k-1}\right)^2 \right] + Res^W_{j,k},
\end{aligned} \tag{6.15}$$

$$\begin{aligned}
\dot{Z}_{j,k} =& \left(V^{(2)}_{j,k} - V^{(2)}_{j,k-1}\right) + \left(V^{(1)}_{j,k} - V^{(1)}_{j-1,k}\right) \\
&+ \beta\varepsilon^2 \left[\left(V^{(2)}_{j,k}\right)^3 - \left(V^{(2)}_{j,k-1}\right)^3 + \left(V^{(1)}_{j,k}\right)^3 - \left(V^{(1)}_{j-1,k}\right)^3 \right] \\
&+ \beta\varepsilon^2 \left(V^{(1)}_{j,k}\right) \left(A(\xi,\eta) + U^{(1)}_{j,k}\right)^2 \\
&- \beta\varepsilon^2 \left(V^{(1)}_{j-1,k}\right) \left(A(\xi-\varepsilon,\eta) + U^{(1)}_{j-1,k}\right)^2 \\
&+ \beta\varepsilon^2 \left(V^{(2)}_{j,k}\right) \left(U_\varepsilon(\xi,\eta) + U^{(2)}_{j,k}\right)^2 \\
&- \beta\varepsilon^2 \left(V^{(2)}_{j,k-1}\right) \left(U_\varepsilon\left(\xi,\eta-\varepsilon^2\right) + U^{(2)}_{j,k-1}\right)^2,
\end{aligned} \tag{6.16}$$

where

$$\begin{aligned} Res^W_{j,k} =& \varepsilon\partial_\xi W_\varepsilon(\xi,\eta) - \varepsilon^3\partial_\tau W_\varepsilon(\xi,\eta) + A(\xi,\eta) - A(\xi-\varepsilon,\eta) \\ &+ U_\varepsilon(\xi,\eta) - U_\varepsilon\left(\xi,\eta-\varepsilon^2\right) \\ &+ \beta\varepsilon^2\left(A(\xi,\eta)^3 - A(\xi-\varepsilon,\eta)^3\right) \\ &+ \beta\varepsilon^2\left(U_\varepsilon(\xi,\eta)^3 - U_\varepsilon\left(\xi,\eta-\varepsilon^2\right)^3\right). \end{aligned} \tag{6.17}$$

Lemma 6.1. *(Estimates for Residual terms) Let A be a solution to the KP-II equation* (6.3) *whose initial data satisfies $A_0 \in H^9(\mathbb{R}^2)$ such that $\partial_\xi^{-2}\partial_\eta^2 A_0 \in H^9(\mathbb{R}^2)$ and $\partial_\xi^{-1}\partial_\eta^2\left[\partial_\xi^{-2}\partial_\eta^2 A(0) + A(0)^2\right] \in H^2(\mathbb{R}^2)$. Then there are constants $C(A) > 0$ such that for all $\varepsilon \in (0,1]$, we have*

$$\left\|Res^W_{j,k}\right\|_{\ell^2} + \left\|Res^{U^{(1)}}_{j,k}\right\|_{\ell^2} + \left\|Res^{U^{(2)}}_{j,k}\right\|_{\ell^2} \le C\varepsilon^{\frac{7}{2}}.$$

The proof is similar to lemma 4.2, we estimate the residual terms (6.13), (6.17) by computing the Taylor remainder term and applying lemma 4.1.

As we did in section 4.4 we introduce an energy function analogous to equation (4.27), modified for the cubic case,

$$\begin{aligned} E(t) =& \frac{1}{2}\sum_{j,k\in\mathbb{Z}^2} W^2_{j,k} + Z^2_{j,k} + \left(U^{(1)}_{j,k}\right)^2 + \left(U^{(2)}_{j,k}\right)^2 + \left(V^{(1)}_{j,k}\right)^2 \\ &+ \sum_{j,k\in\mathbb{Z}^2} \frac{1}{2}\left(V^{(2)}_{j,k}\right)^2 + \frac{\beta}{4}\varepsilon^2\left[\left(U^{(1)}_{j,k}\right)^4 + \left(U^{(2)}_{j,k}\right)^4\right] \\ &\quad + \frac{3}{2}\beta\varepsilon^2\left(U^{(1)}_{j,k}\right)^2 A(\xi,\eta)^2 + \beta\varepsilon^2\left(U^{(1)}_{j,k}\right)^3 A(\xi,\eta) \\ &\quad + \frac{3}{2}\beta\varepsilon^2\left(U^{(2)}_{j,k}\right)^2 U_\varepsilon(\xi,\eta)^2 + \beta\varepsilon^2\left(U^{(2)}_{j,k}\right)^3 U_\varepsilon(\xi,\eta) \\ &\quad + \beta\varepsilon^2 A(\xi,\eta)\, U^{(1)}_{j,k}\left(V^{(1)}_{j,k}\right)^2 + \frac{\beta}{2}\varepsilon^2\left(U^{(1)}_{j,k}\right)^2\left(V^{(1)}_{j,k}\right)^2 \\ &\quad + \beta\varepsilon^2 U_\varepsilon(\xi,\eta)\, U^{(2)}_{j,k}\left(V^{(2)}_{j,k}\right)^2 + \frac{\beta}{2}\varepsilon^2\left(U^{(2)}_{j,k}\right)^2\left(V^{(2)}_{j,k}\right)^2 \\ &\quad + \frac{\beta}{4}\varepsilon^2\left[\left(V^{(1)}_{j,k}\right)^4 + \left(V^{(2)}_{j,k}\right)^4\right] \\ &\quad + \frac{\beta}{2}\varepsilon^2\left(A(\xi,\eta)^2\left(V^{(1)}_{j,k}\right)^2 + U_\varepsilon(\xi,\eta)^2\left(V^{(2)}_{j,k}\right)^2\right) \end{aligned} \tag{6.18}$$

We'd like to show a coercivity property for this energy function, which will help us give a bound on its growth. The property is summarized in the following lemma.

Lemma 6.2. *Let $A \in C^0\left([-\tau_0,\tau_0], H^4(\mathbb{R}^2)\right)$ and $\partial_\xi^{-1}A \in C^0\left([-\tau_0,\tau_0], H^3(\mathbb{R}^2)\right)$.*

Define

$$\mathcal{A} = \sup_{\tau \in [-\tau_0, \tau_0]} \|A(\cdot, \cdot, \tau)\|_{L^\infty}$$

and

$$\mathcal{U} = \sup_{\tau \in [-\tau_0, \tau_0]} \left(\left\| \partial_\xi^{-1} A \right\|_{H^3} + 2 \|A\|_{H^4} \right).$$

Then, there exists some

$$\varepsilon_1 (\mathcal{A}, \mathcal{U}) > 0$$

such that

$$\|W\|_{\ell^2}^2 + \varepsilon^2 \|Z\|_{\ell^2}^2 + \left\|U^{(1)}\right\|_{\ell^2}^2 + \left\|U^{(2)}\right\|_{\ell^2}^2 + \varepsilon^2 \left\|V^{(1)}\right\|_{\ell^2}^2 + \varepsilon^2 \left\|V^{(2)}\right\|_{\ell^2}^2 \leq 4E(t),$$

for each $\varepsilon \in (0, \varepsilon_1]$ *and* $t \in \left[\frac{-\tau_0}{\varepsilon^3}, \frac{\tau_0}{\varepsilon^3}\right]$.

Proof. The computation is similar to lemma 4.3. Since many of the terms are positive semi-definite, the energy is bounded below by

$$\begin{aligned} 2E(t) \geq & \|W\|_{\ell^2}^2 + \|Z\|_{\ell^2}^2 \\ & + \left(\left\|U^{(1)}\right\|_{\ell^2}^2 + \left\|V^{(1)}\right\|_{\ell^2}^2 \right) \left(1 - \beta\varepsilon^2 \left\|U^{(1)}\right\|_\infty \|A\|_\infty\right) \\ & + \left(\left\|U^{(2)}\right\|_{\ell^2}^2 + \left\|V^{(2)}\right\|_{\ell^2}^2 \right) \left(1 - \beta\varepsilon^2 \left\|U^{(2)}\right\|_\infty \|U_\varepsilon\|_\infty\right) \end{aligned} \tag{6.19}$$

With decomposition (6.9) we write,

$$\left(1 - \beta\varepsilon^2 \left\|U^{(1)}\right\|_\infty \|A\|_\infty\right) \geq \left(1 - \beta\varepsilon \left(H^{\frac{1}{2}} + \varepsilon\mathcal{A}\right) \mathcal{A}\right). \tag{6.20}$$

For sufficiently small H we have that

$$\left(1 - \beta\varepsilon^2 \left\|U^{(1)}\right\|_\infty \|A\|_\infty\right) \geq \left(1 - \beta\varepsilon \left(1 + \varepsilon\mathcal{A}\right) \mathcal{A}\right), \tag{6.21}$$

similarly

$$\left(1 - \beta\varepsilon^2 \left\|U^{(2)}\right\|_\infty \|U_\varepsilon\|_\infty\right) \geq \left(1 - \beta\varepsilon \left(1 + \varepsilon\mathcal{U}\right) \mathcal{U}\right). \tag{6.22}$$

Choose

$$\varepsilon_1 = \min\left(1, \frac{1}{\beta\left(1 + \mathcal{A}\right)\mathcal{A}}, \frac{1}{\beta\left(1 + \mathcal{U}\right)\mathcal{U}}\right), \tag{6.23}$$

then for $\varepsilon \in (0, \varepsilon_1]$ we have that

$$2E(t) \geq \|W\|_{\ell^2}^2 + \|Z\|_{\ell^2}^2 + \frac{1}{2}\left(\left\|U^{(1)}\right\|_{\ell^2}^2 + \left\|V^{(1)}\right\|_{\ell^2}^2 + \left\|U^{(2)}\right\|_{\ell^2}^2 + \left\|V^{(2)}\right\|_{\ell^2}^2 \right) \tag{6.24}$$

and the result follows. □

By differentiating (6.18) and applying lemmas 6.2 and 6.1 that

$$\left|\dot{E}(t)\right| \leq K_{\beta A}\left(\varepsilon^{\frac{7}{2}}E(t)^{\frac{1}{2}} + \varepsilon^{3}E(t) + \varepsilon^{3}E(t)^{\frac{3}{2}}\right) \tag{6.25}$$

Performing the substitution $Q(t) = E(t)^{\frac{1}{2}}$ we get

$$\left|\dot{Q}(t)\right| \leq K_{\beta A}\left(\varepsilon^{3}Q^{2} + \varepsilon^{\frac{7}{2}} + \varepsilon^{3}Q\right),$$

To get a bound on this quantity we will need the following modification on the Gronwall lemma.

Lemma 6.3. *Suppose that*

$$\left|\dot{Q}(t)\right| \leq K_{\beta A}\left(\varepsilon^{3}Q^{2} + \varepsilon^{\frac{7}{2}} + \varepsilon^{3}Q\right),$$

and $Q(0) \leq C\varepsilon^{\frac{1}{2}}$ *then there is an* $\varepsilon_0 > 0$ *such that*

$$Q(t) \leq \varepsilon^{\frac{1}{2}}\left(C + \frac{1}{2}\right)\exp\left(2K_{\beta A}\tau_0\right)$$

for $\varepsilon \in (0, \varepsilon_0]$.

Proof. Define

$$T = \sup\left\{t \in [-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}] : Q(t) \leq 1\right\}.$$

Assume $Q(0) \leq C\varepsilon^{\frac{1}{2}}$ for some $C > 0$. Since Q is continuous in t, then there is some $1 \geq \varepsilon_2 > 0$ such that for $\varepsilon \in (0, \varepsilon_2]$ the set is nonempty and $T > 0$. Suppose that $t \in [-T, T]$, then we have

$$\left|\dot{Q}\right| \leq K_{\beta A}\left(\varepsilon^{3}\left(Q+1\right)Q + \varepsilon^{\frac{7}{2}}\right) \leq K_{\beta A}\left(2\varepsilon^{3}Q + \varepsilon^{\frac{7}{2}}\right), \tag{6.26}$$

where the constant $K_{\beta A}$ depends on α, A but not ε. We can now perform a Gronwall type argument to (6.26). First we rewrite (6.26) as

$$\frac{d}{dt}\left[\exp\left(-2\varepsilon^{3}K_{\beta A}t\right)Q\right] \leq K_{\beta A}\varepsilon^{\frac{7}{2}}\exp\left(-2\varepsilon^{3}K_{\beta A}t\right). \tag{6.27}$$

Integrating (6.27) we have the inequality

$$Q(t) \leq \left(Q(0) + \frac{1}{2}\varepsilon^{\frac{1}{2}}\right)\exp\left(2\varepsilon^{3}K_{\beta A}t\right). \tag{6.28}$$

Since $Q(0) \leq C\varepsilon^{\frac{1}{2}}$, we have

$$Q(t) \leq \varepsilon^{\frac{1}{2}} \left(C + \frac{1}{2} \right) \exp\left(2\varepsilon^3 K_{\beta A} t\right) \tag{6.29}$$

For $t \in [-T, T]$ we have

$$Q(t) \leq \varepsilon^{\frac{1}{2}} \left(C + \frac{1}{2} \right) \exp\left(2 K_{\beta A} \tau_0\right). \tag{6.30}$$

The above holds as long as

$$\varepsilon_2 = \frac{1}{\left(C + \frac{1}{2}\right)^2 \exp\left(4K_{\beta A}\tau_0\right)}.$$

Choosing $\varepsilon_0 = \min\left(\varepsilon_1, \varepsilon_2\right)$, where ε_1 is defined in the proof of lemma 5.2, then $Q(t) \leq 1$ for $t \in [-T, T]$, and T can be extended to $T = \tau_0 \varepsilon^{-3}$. □

We finish the chapter with a proof of the main result.

Proof of Theorem 6.1. Note that

$$\begin{aligned} \varepsilon Q(0) \leq C_A \varepsilon \Big(& \left\|U^{(1)}(0)\right\|_{\ell^2} + \left\|U^{(2)}(0)\right\|_{\ell^2} + \left\|V^{(1)}(0)\right\|_{\ell^2} \\ & + \left\|V^{(2)}(0)\right\|_{\ell^2} + \left\|W(0)\right\|_{\ell^2} + \left\|Z(0)\right\|_{\ell^2} \Big). \end{aligned}$$

In light of decomposition (6.9) and the hypothesis (4.6) we have that $\varepsilon Q(0) \leq C\varepsilon^{\frac{3}{2}}$, so that lemma 6.3 applies. With decomposition (6.9) and lemma 6.2 we have that

$$\begin{aligned} & \left\|u^{(1)} - \varepsilon A\left(\xi, \eta, t\right)\right\|_{\ell^2} + \left\|u^{(2)} - \varepsilon U_\varepsilon\left(\xi, \eta, t\right)\right\|_{\ell^2} \\ & + \left\|w - \varepsilon W_\varepsilon\left(\xi, \eta, t\right)\right\|_{\ell^2} + \left\|v^{(1)}\right\|_{\ell^2} + \left\|v^{(2)}\right\|_{\ell^2} + \left\|z\right\|_{\ell^2} \leq 6\varepsilon Q(t). \end{aligned}$$

By lemma 6.3 we have that $Q(t) \leq \varepsilon^{\frac{1}{2}} \left(C + \frac{1}{2}\right) \exp\left(2K_{\beta A}\tau_0\right)$, and the result follows. □

Chapter 7

Linear Stability for Solitary Waves in 2D FPU

7.1 Introduction

In this chapter we will study FPU travelling waves in two-dimensions. From Friesecke and Pego's work [14] we know that with appropriate scaling the FPU solitary wave converges to the KdV solitary wave, $\phi_\gamma(x)$, uniformly as $\varepsilon \to 0$. Here we will show that we can use the one-dimensional FPU solitary wave to construct a solitary wave of the two-dimensional FPU system, which is linearly stable under transverse perturbations. The analysis is a generalization of Friesecke and Pego's work in [16] and [17], and depends on the linear stability of a KdV soliton as a solution to the KP-II equation under transverse perturbations developed by Mizumachi in [36].

We are interested in the behaviour of solutions of the system (4.8) in the neighbourhood of a solitary wave solution. In particular we are interested in seeing if a result analogous to theorems 1.1 and 1.2 can be extended to a two-dimensional FPU system. We begin by denoting a solution to the α-model of the FPU 2D lattice given by the dynamical system (4.8) by

$$r\left(j,k,t\right) = \begin{bmatrix} u_{j,k}^{(1)} & u_{j,k}^{(2)} & w_{j,k} & v_{j,k}^{(1)} & v_{j,k}^{(2)} & z_{j,k} \end{bmatrix}^T . \tag{7.1}$$

Introducing the coordinate $x = j - ct$, and the delay and advance operators $\Delta^{\pm} f(x) = \pm\left(f(x \pm 1) - f(x)\right)$, we look for solutions of the form $r(j,k,t) = r_c\left(x\right)$. For each variable we plug this ansatz into the equations of motion (4.8) to get the travelling wave reduction. Note that functions of this form are constant in the transverse direction, so that $f(j,k+1,t) = f(j,k,t)$

$$\begin{aligned}
-c\frac{d}{dx}u_c^{(1)}(x) =& \Delta^+ w_c(x), \\
-c\frac{d}{dx}u_c^{(2)}(x) =& 0, \\
-c\frac{d}{dx}w_c(x) =& c_1^2 \Delta^- u_c^{(1)}(x) + \alpha \Delta^- \left[\left(u_c^{(1)}(x) \right)^2 \right], \\
-c\frac{d}{dx}v_c^{(1)}(x) =& \Delta^+ z_c(x), \\
-c\frac{d}{dx}v_c^{(2)}(x) =& 0, \\
-c\frac{d}{dx}z_c(x) =& c_2^2 \Delta^- v_c^{(1)}(x).
\end{aligned} \tag{7.2}$$

For simplicity, we study decaying solutions of the system (7.2) in space $H^1(\mathbb{R})$ and denote $u_c(x) = u_c^{(1)}(x)$.

Lemma 7.1. *There exists the unique solution of the system* (7.2) *in* $H^1(\mathbb{R})$ *for* $c > c_1$ *in the form*

$$r_c(x) = \begin{bmatrix} u_c(x) & 0 & w_c(x) & 0 & 0 & 0 \end{bmatrix}^T, \tag{7.3}$$

where $u_c, w_c \in H^1(\mathbb{R})$ *satisfy the following system*

$$\begin{aligned}
-c\frac{d}{dx}u_c(x) =& \Delta^+ w_c(x), \\
-c\frac{d}{dx}w_c(x) =& c_1^2 \Delta^- u_c(x) + \alpha \Delta^- \left[(u_c(x))^2 \right].
\end{aligned} \tag{7.4}$$

Proof. From the second and fifth equations, it follows that the solution is constant and the constant is zero in space $H^1(\mathbb{R})$.

From the fourth and six equations, we get the linear wave equation solutions of which are given by exponential functions (either oscillatory or divergent at one infinity). In either case, the only admissible solution in $H^1(\mathbb{R})$ is zero.

From the first and third equation, we get exactly the same system as in Theorem 1.1(a) of [14], which states existence and uniqueness (up to spatial translation) of solutions in $H^1(\mathbb{R})$ for every $c > c_1$. □

We linearize in the neighbourhood of solitary wave $r_c(x)$ with the perturbation $r(x,k,t)$, and taking a discrete Fourier transform in the transverse variable k,

$$\hat{r}(x,\hat{k},t) = \sum_{m \in \mathbb{Z}} e^{im\hat{k}} r(x,m,t)$$

which will take differences in k to $\pm\left(e^{\pm i\hat{k}}-1\right)$, for simplicity we will write $r(x,t)=\hat{r}(x,\hat{k},t)$. The linearized equations of motion are given by,

$$\partial_t r - c\partial_x r = Lr = L_0 r + L_1 r, \tag{7.5}$$

where L_0 corresponds to contributions from the linear terms of the equations of motion, whereas L_1 is contributions from the nonlinear terms, given by

$$L_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \Delta^+ & 0 \\ 0 & 0 & 0 & 0 & e^{i\hat{k}}-1 & 0 \\ c_1^2\Delta^- & c_2^2\left(1-e^{-i\hat{k}}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta^+ \\ 0 & 0 & 0 & 0 & 0 & e^{i\hat{k}}-1 \\ 0 & 0 & c_2^2\Delta^- & c_1^2\left(1-e^{-i\hat{k}}\right) & 0 & 0 \end{bmatrix}, \tag{7.6}$$

and

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2\alpha\Delta^- u_c(x) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7.7}$$

The decomposition $L = L_0 + L_1$ is useful because L_0 is an operator with constant coefficients and L_1 has potentials $u_c(x)$ decaying to zero at infinity. Recall the exponentially weighted space ℓ^2_a:

$$\ell^2_a = \left\{u : \mathbb{Z} \to \mathbb{R}^2 | e^{aj}u(j) \in \ell^2\right\}.$$

We will say the linear system (7.5) is asymptotically stable in ℓ^2_a if there exist positive constants K, β such that for any solution of the linearized evolution equation (7.5) in ℓ^2_a satisfies the estimate:

$$\|w(t)\|_{\ell^2_a} \leq K e^{-\beta t}\|w(s)\|_{\ell^2_a}, \tag{7.8}$$

provided $t \geq 0$.

Conjecture 7.1. *Suppose that $\hat{k} \in [-\pi, \pi]$. On any energy surface $H = E$ with $E > 0$ sufficiently small the unique supersonic solitary wave r_c with $c > c_1$ has the following property. There exists a, β, $K > 0$ in* (7.8) *such that if the* *initial data r_0 satisfy*

$$\begin{aligned} \|r_0 - r_c(\cdot - ct_0)\| &\leq \sqrt{\delta}, \\ \|r_0 - r_c(\cdot - ct_0)\|_{\ell^2_a} &\leq \delta \end{aligned} \tag{7.9}$$

and $\delta > 0$ is sufficiently small, then the solution $r(\cdot, t)$ to the linearized FPU equation (7.5) *is asymptotically stable in ℓ^2_a.*

Note that given the discrete Fourier transform of the transverse forward shift operator on the lattice $\hat{\Delta}^+_k = e^{i\hat{k}} - 1$, we have two cases. The first is when $\hat{k} = 0$, which is the one-dimensional FPU system. This can be handled using theorem 1.1 and 1.2 proven in [14], [16], and [17]. As we will see this case breaks down into two three-by-three cases, one of which is does not involve the solitary wave profile, the other further breaks down into the two-by-two system for one dimensional FPU and a constant function, which is zero in ℓ^2_a. The case when $\hat{k} \in [-\pi, \pi] \setminus \{0\}$ the analysis is similar to the one-dimensional case. However, as noted before the proof is incomplete as analytic estimates in one of the lemmas are not finished.

7.2 Case $\hat{k} = 0$

When $\hat{k} = 0$, the system $\dot{r} = Lr$, with $L = L_0 + L_1$ as given in (7.6) and (7.7), can be expressed as four decoupled systems of equations,

$$\frac{d}{dt}\begin{bmatrix} r_1 \\ r_3 \end{bmatrix} = \begin{bmatrix} 0 & \Delta^+ \\ c_1^2\Delta^- + 2\alpha\Delta^- u_c(x) & 0 \end{bmatrix}\begin{bmatrix} r_1 \\ r_3 \end{bmatrix}, \tag{7.10}$$

$$\frac{d}{dt} r_2 = 0, \tag{7.11}$$

$$\frac{d}{dt}\begin{bmatrix} r_4 \\ r_6 \end{bmatrix} = \begin{bmatrix} 0 & \Delta^+ \\ c_2^2\Delta^- & 0 \end{bmatrix}\begin{bmatrix} r_4 \\ r_6 \end{bmatrix}, \tag{7.12}$$

$$\frac{d}{dt} r_5 = 0. \tag{7.13}$$

Solutions for r_2, r_5 are constant functions, for which only $r_2 = 0 = r_5$ are in the solutions space ℓ^2.

The system (7.10) is the one-dimensional FPU system with a quadratic potential, a special case of the system studied in [14]-[17]. For this system we have the stability result given by theorem 1.2. The third system is constant coefficient equation, and introducing $\begin{bmatrix} r_4 & r_6 \end{bmatrix} = e^{\lambda t}\begin{bmatrix} R_4 & R_6 \end{bmatrix}$, we find that this two-by-two system has the same spectrum as the one-dimensional FPU system (7.10).

7.3 Eigenvalue Problem

Introducing $r(x,t) = e^{\lambda t} R(x,t)$, we get the eigenvalue problem

$$\lambda R(x) = \left(c\frac{d}{dx} + L \right) R(x) = (A + L_1) R(x), \tag{7.14}$$

with $A = c\frac{d}{dx} + L_0$. Here L_1 is a relatively compact perturbation to the linear operator A due to the exponential decay of $u_c(x)$ to zero at infinity.

The system (7.14) is in block diagonal form, for clarity we will define

$$\lambda \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = D_1 \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}, \quad \lambda \begin{bmatrix} R_4 \\ R_5 \\ R_6 \end{bmatrix} = D_2 \begin{bmatrix} R_4 \\ R_5 \\ R_6 \end{bmatrix}. \tag{7.15}$$

where

$$D_1 = \begin{bmatrix} c\frac{d}{dx} & 0 & \Delta^+ \\ 0 & c\frac{d}{dx} & e^{i\hat{k}} - 1 \\ c_1^2\Delta^- + 2\alpha\Delta^- u_c(x) & c_2^2\left(1 - e^{-i\hat{k}}\right) & c\frac{d}{dx} \end{bmatrix},$$

and

$$D_2 = \begin{bmatrix} c\frac{d}{dx} & 0 & \Delta^+ \\ 0 & c\frac{d}{dx} & e^{i\hat{k}} - 1 \\ c_2^2\Delta^- & c_1^2\left(1 - e^{-i\hat{k}}\right) & c\frac{d}{dx} \end{bmatrix}.$$

Note that by symmetry the continuous spectrum for D_2 will be shown to be identical to D_1, with c_1 and c_2 interchanged. The argument for D_2 is simpler since it is a linear operator. For D_1 the nonlinearity is handled by an application of Weyl's theorem.

In the case when $\hat{k} = 0$, $R_2(x)$ decoupled from the remainder of the system, and we had the same system and spectrum as [16]. This will be handled in section 7.2. In the case when $\hat{k} \neq 0$ we can reduce the first system in (7.15) to a closed second order system in terms of $R_2(x)$ in order to study the continuous spectrum.

Lemma 7.2. *Suppose that $\hat{k} \in [-\pi, \pi] \setminus \{0\}$. Then the first system in* (7.15) *reduces to the following second order equation*

$$\begin{aligned}\left(\lambda - c\frac{d}{dx}\right)^2 R_1 = &\left(c_1^2\Delta^+\Delta^- + 2\alpha\Delta^+\Delta^- u_c(x)\right) R_1 \\ &- 4c_2^2 \sin^2\left(\hat{k}\right) R_1 + (1 - e^{-i\hat{k}})r_2,\end{aligned} \tag{7.16}$$

where r_2 *satisfies*

$$\left(\lambda - c\frac{d}{dx}\right) r_2 = 0. \tag{7.17}$$

Proof. Applying the operator $\lambda - c\frac{d}{dx}$ to R_1, R_2, R_3 yields,

$$\begin{aligned}\left(\lambda - c\frac{d}{dx}\right) R_1 &= \Delta^+ R_3(x), \\ \left(\lambda - c\frac{d}{dx}\right) R_2 &= \left(e^{i\hat{k}} - 1\right) R_3(x), \\ \left(\lambda - c\frac{d}{dx}\right) R_3 &= \left(c_1^2\Delta^- + 2\alpha\Delta^- u_c(x)\right) R_1 + c_2^2(1 - e^{-i\hat{k}})R_2.\end{aligned} \tag{7.18}$$

We seek a solution of the form

$$R_2 = (\Delta^+)^{-1}\left[(e^{i\hat{k}} - 1)R_1 + r_2\right], \tag{7.19}$$

where r_2 satisfies

$$\left(\lambda - c\frac{d}{dx}\right) r_2 = 0. \tag{7.20}$$

The first equation of (7.18) gives

$$R_3 = (\Delta^+)^{-1}\left(\lambda - c\frac{d}{dx}\right) R_1. \tag{7.21}$$

From the expressions (7.19)-(7.21) we see that the second equation of (7.18) is satisfied. In light of equations (7.19) and (7.21) the third equation of (7.18) gives,

$$\begin{aligned}\left(\lambda - c\frac{d}{dx}\right)^2 R_1 = &\left(c_1^2\Delta^+\Delta^- + 2\alpha\Delta^+\Delta^- u_c(x)\right) R_1 \\ &- 4c_2^2 \sin^2\left(\hat{k}\right) R_1 + (1 - e^{-i\hat{k}})r_2.\end{aligned} \tag{7.22}$$

□

7.4 Essential Spectrum

In this section we will study the essential spectrum of the first system in (7.15). In particular we show that the real part of the spectrum is negative. To prove this we use a result about the spectrum of one-dimensional FPU from [16], which coincides with the spectrum of the operator we study when $\hat{k} = 0$.

Lemma 7.3. *Suppose that $\hat{k} \in [-\pi, \pi] \setminus \{0\}$. If $c > c_1$ and $0 < a < a_c$ where $a_c > 0$ is the solution of $\sinh(\frac{1}{2}a_c)\left(\frac{1}{2}a_c\right)^{-1} = \frac{c}{c_1}$ then the essential spectrum of the operator $c\partial_x + L$ in L^2_a does not intersect the closed right half plane.*

Proof. Since $u_c(x)$ vanishes at infinity we have that L_1 is relatively compact, so by Weyl's theorem (theorem 14.6 in [28]) the essential spectrum of L_0 and $L = L_0 + L_1$ coincide, and it suffices to compute the spectrum of L_0. To study the eigenvalue problem we look at equation (7.16) with $r_2 = 0$,

$$\begin{aligned}\left(\lambda - c\frac{d}{dx}\right)^2 R_1 = &\left(c_1^2\Delta^+\Delta^- + 2\alpha\Delta^+\Delta^- u_c(x)\right) R_1 \\ &- 4c_2^2 \sin^2\left(\hat{k}\right) R_1.\end{aligned} \tag{7.23}$$

Take a Fourier transform in the x variable of we get

$$(\lambda - ic\hat{x} + ac)^2 = \left(-4c_1^2 \sin^2\left(\frac{\hat{x} + ia}{2}\right) - 4c_2^2 \sin^2(\frac{\hat{k}}{2})\right). \tag{7.24}$$

Setting $\lambda = \lambda_R + i\lambda_I$, then expanding and comparing the real and imaginary parts we have the following pair of equations,

$$\begin{aligned}(\lambda_R + ac)^2 - (\lambda_I - c\hat{x})^2 = &-4c_2^2 \sin^2\left(\frac{\hat{k}}{2}\right) \\ &- 4c_1^2\left[\sin^2\left(\frac{\hat{x}}{2}\right)\cosh^2\left(\frac{a}{2}\right) - \cos^2\left(\frac{\hat{x}}{2}\right)\sinh^2\left(\frac{a}{2}\right)\right]\end{aligned}$$

and

$$(\lambda_R + ac)(\lambda_I - c\hat{x}) = -c_1^2 \sin(\hat{x})\sinh(a).$$

The second equation allows us to eliminate $(\lambda_I - c\hat{x})$, after which the first equation gives a quadratic equation in terms of $(\lambda_R + ac)^2$ with roots,

$$(\lambda_R + ac)^2 = c_1^2 \left(\cos\left(\hat{x}\right)\cosh\left(a\right) - 1\right) - 2c_2^2 \sin^2\left(\frac{\hat{k}}{2}\right)$$

$$\pm \sqrt{c_1^4 \sinh^2\left(a\right)\sin^2\left(\hat{x}\right) + \left(c_1^2\left(1 - \cos\left(\hat{x}\right)\cosh\left(a\right)\right) + 2c_2^2 \sin^2\left(\frac{\hat{k}}{2}\right)\right)^2}.$$

The negative root can be rejected as $\lambda_R + ac$ is a real number.

We'd like to show that $\lambda_R < 0$. The case for $\lambda_R(0)$ is the spectrum studied in [16] where it is shown that $\lambda_R(0) < 0$, we'd like to show that this continues to hold for $\lambda_R(\hat{k})$ when $\hat{k} \in [-\pi, \pi] \setminus \{0\}$. Note that if for the negative root this is trivially satisfied, so we consider the case when $\lambda_R(\hat{k}) + ac > 0$. We will show that

$$\lambda_R(\hat{k}) + ac \leq \lambda_R(0) + ac < ac,$$

since $\lambda_R(\hat{k}) \geq 0$ and $\lambda_R(0) \geq 0$ it is sufficient to show that

$$\left(\lambda_R(\hat{k}) + ac\right)^2 \leq \left(\lambda_R(0) + ac\right)^2.$$

To do this we define the function

$$f\left(\hat{k}\right) = -\gamma_1 - 2c_2^2 \sin^2\left(\frac{\hat{k}}{2}\right) + \sqrt{\gamma_2^2 + \left(\gamma_1 + 2c_2^2 \sin^2\left(\frac{\hat{k}}{2}\right)\right)^2},$$

with $\gamma_1 = c_1^2\left(1 - \cos\left(\hat{x}\right)\cosh\left(a\right)\right), \gamma_2 = c_1^2 \sinh\left(a\right)\sin\left(\hat{x}\right)$. Taking a derivative with respect to $\hat{k}$ we have,

$$f'(\hat{k}) = 2c_2^2 \sin\left(\frac{\hat{k}}{2}\right)\cos\left(\frac{\hat{k}}{2}\right)\left(\frac{\gamma_1 + 2c_2^2\sin^2\left(\frac{\hat{k}}{2}\right)}{\sqrt{\left(\gamma_1 + 2c_2^2\sin^2\left(\frac{\hat{k}}{2}\right)\right)^2 + \gamma_2^2}} - 1\right).$$

For any collection of parameters the following inequality holds,

$$\left|\frac{\gamma_1 + 2c_2^2\sin^2\left(\frac{\hat{k}}{2}\right)}{\sqrt{\left(\gamma_1 + 2c_2^2\sin^2\left(\frac{\hat{k}}{2}\right)\right)^2 + \gamma_2^2}}\right| \leq 1,$$

then for $\hat{k} \in [0, \pi]$ the derivative is non-positive, $f'(\hat{k}) \leq 0$. Integrating this inequality from 0 to $\hat{k}$ yields the result for $\hat{k} \in [0, \pi]$, and since f is an even function in $\hat{k}$ we have the same inequality for $\hat{k} \in [-\pi, 0]$.

To finish studying the continuous spectrum we look for solutions of (7.16) when $r_2 \neq 0$ but satisfies (7.17). In this case the continuous spectrum in L^2_a is given by $\lambda = ic\hat{x} - ac, r_2(x) = e^{(i\hat{x}-a)x}$, equation (7.16) becomes,

$$\left(\lambda - c\frac{d}{dx}\right)^2 R_1 = \left(c_1^2\Delta^- + 2\alpha\Delta^- u_c(x)\right)\left(\Delta^+ R_1 + e^{(i\hat{x}-a)x}\right) - 4c_2^2 \sin^2\left(\hat{k}\right) R_1. \tag{7.25}$$

We are concerned with the growth of R_1 as $|x| \to \infty$, since $u_c(x) \to 0$ in this limit we have

$$\left(\lambda - c\frac{d}{dx}\right)^2 R_1 = \left(c_1^2\Delta^-\right)\left(\Delta^+ R_1 + e^{(i\hat{x}-a)x}\right) - 4c_2^2 \sin^2\left(\hat{k}\right) R_1. \tag{7.26}$$

In this limit the equation is satisfied by $\lambda = ic\hat{x} - ac$, $R_2(x) = re^{(i\hat{x}-a)x}$, where r is constant in x. Plugging this form into the equation yields,

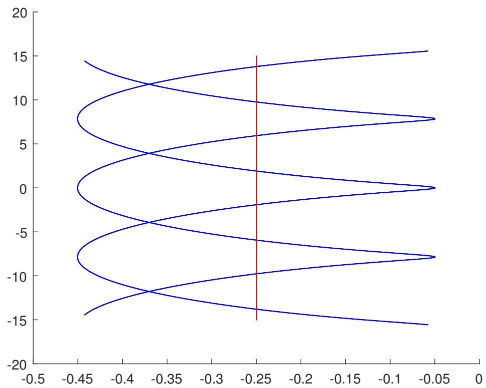
$$0 = (-4(c_1^2 \sin^2\left(\frac{\hat{x}+ia}{2}\right) + c_2^2 \sin^2\left(\frac{\hat{k}}{2}\right)) + c_1^2(1 - e^{-i\hat{x}+a}))re^{i\hat{x}-a}. \tag{7.27}$$

Simplifying we find that r is given by,

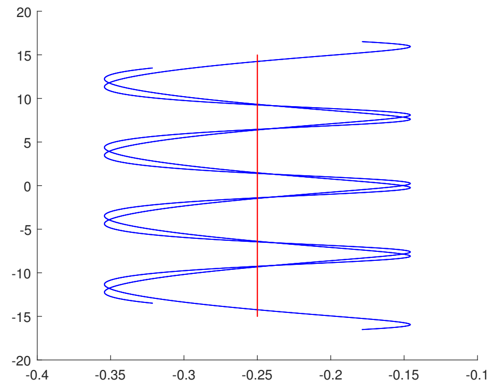
$$r = \frac{c_1^2\left(1 - e^{-i\hat{x}+a}\right)}{4\left(c_1^2 \sin^2\left(\frac{\hat{x}+ia}{2}\right) + c_2^2 \sin^2\left(\frac{\hat{k}}{2}\right)\right)}. \tag{7.28}$$

The spectrum of (7.25) is in the left half-plane. Equations (7.27)-(7.28) are valid when the denominator is non-zero, this condition holds away from the intersections of $ic\hat{\xi} - ac$ and solutions to (7.24). This is illustrated in figures 7.7 and 7.8, the condition holds away from the intersection between the blue curve and vertical red line. At these intersections a resonance occurs, however since at these points we still have that $\operatorname{Re}\lambda < 0$, the growth due to the resonance will be dominated by the exponential decay of solutions in time. □

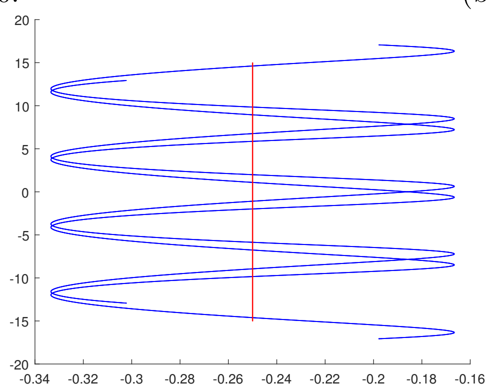
The case when $\hat{k} = 0$ reduces to the spectrum studied in the case of one-dimensional FPU, by Friesecke and Pego in [16]. Plotting the spectrum in a similar scenario to the case of Friesecke and Pego, where $a = 0.2$ or $a = 2$, $c_1 = c_2$, $\frac{c}{c_1} = 1.25$, we find that the effect of the $\sin\left(\hat{k}\right)$ term is to decrease the maximum of $\operatorname{Re}(\lambda)$, this is illustrated in figures 7.7 and 7.8. Note that the blue curves and the red line respresent distinct branches of the continuous spectrum, the blue curve represent solutions of (7.24), while the red vertical line represents the line $\lambda = ic\hat{\xi} - ac$. These intersect infinitely many points due to the periodicity of the spectrum in $\mathrm{Im}(\lambda)$. At points of intersection resonances occur, resonances however will not be an issue for stability in an exponentally weighted space as the linear growth at a resonance will be dominated by the exponential decay rate.



(a) $\hat{k} = 0$.

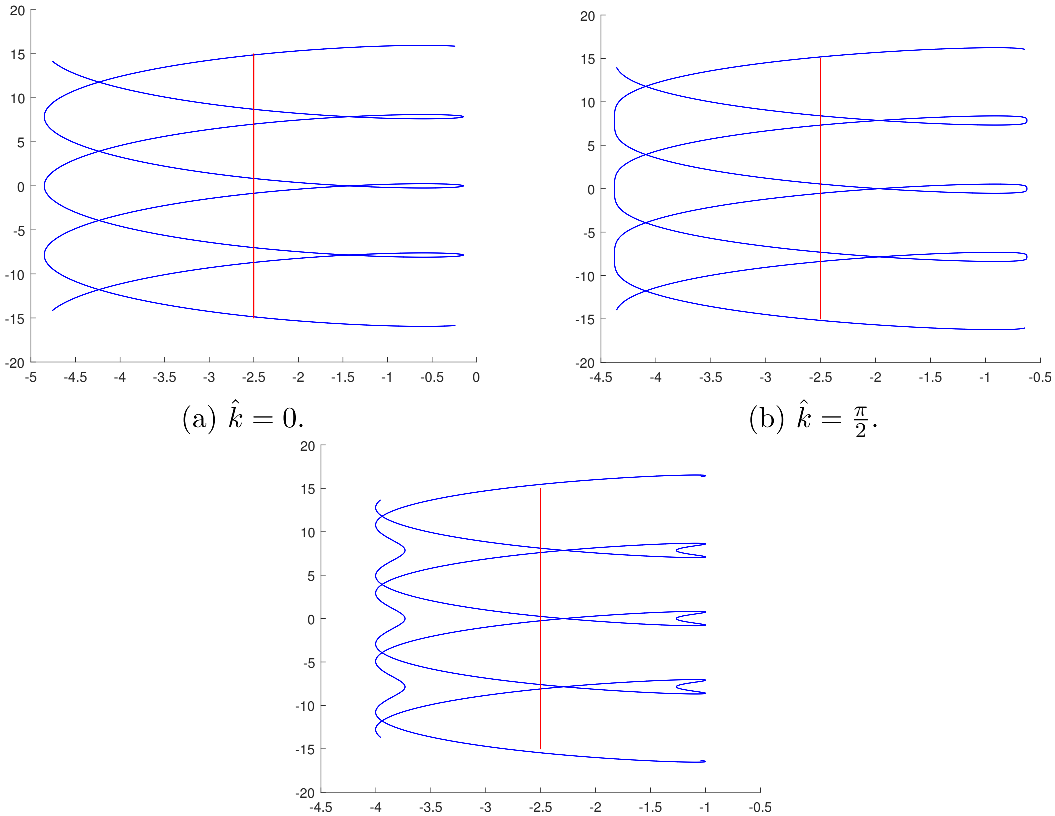


(b) $\hat{k} = \frac{\pi}{2}$.



(c) $\hat{k} = \pi$.

Figure 7.7: Plot of $\lambda_+ \cup \lambda_- \cup \{i\hat{k}c - ac | \hat{k} \in \mathbb{R}\}$ when $a = 0.2$ for varied values of $\hat{k}$



(a) $\hat{k} = 0$.

(b) $\hat{k} = \frac{\pi}{2}$.

(c) $\hat{k} = \pi$.

Figure 7.8: Plot of $\lambda_+ \cup \lambda_- \cup \{i\hat{k}c - ac | \hat{k} \in \mathbb{R}\}$ when $a = 2$ for varied values of $\hat{k}$

7.5 Reduction to the linearized KP-II equation

In this section we will formally derive the linearized KP-II equation from the linearized two-dimensional FPU system. Before proceeding recall that given the KdV equation

$$-2c_s \partial_\xi \partial_\tau R = \frac{c_s^2}{12} \partial_\xi^4 R + \alpha \partial_\xi^2 \left(R^2\right), \tag{7.29}$$

we may seek a solitary wave solution of the form

$$R\left(\xi, \tau\right) = \phi_\gamma \left(\xi - \gamma\tau\right), \tag{7.30}$$

parametrized by $\gamma > 0$. The solitary wave satisfies the ODE,

$$\frac{c_s}{24}\phi_\gamma'''(\xi) - \gamma\phi_\gamma'(\xi) + \frac{\alpha}{c_s}\phi_\gamma(\xi)\phi_\gamma'(\xi) = 0, \tag{7.31}$$

which a solution

$$\phi_\gamma(\xi) = \frac{3c_1\gamma}{\alpha}\operatorname{sech}^2\left(\sqrt{\frac{6\gamma}{c_1}}(\xi - \gamma\tau)\right). \tag{7.32}$$

It was proven by Friesecke and Pego in [14] that a solitary wave of the FPU system $u_c(x)$ converges uniformly to $\varepsilon^2\phi_\gamma(\varepsilon x)$ as $\varepsilon \to 0$, where $c = c_1 + \varepsilon^2\gamma$.

The linearized equations of motion in the neighbourhood of an FPU soliton $u_c(x)$ are given by,

$$\begin{aligned}
\lambda R_1(x) =& cR_1'(x) + R_5(x+1) - R_5(x),\\
\lambda R_2(x) =& \left(e^{i\hat{k}} - 1\right)R_5(x) + cR_2'(x),\\
\lambda R_5(x) =& c_1^2\left(R_1(x) - R_1(x-1)\right) + c_2^2\left(1 - e^{-i\hat{k}}\right)R_2(x)\\
&+ cR_5'(x) + 2\alpha\left(u_c(x)R_1(x) - u_c(x-1)R_1(x-1)\right).
\end{aligned} \tag{7.33}$$

Introducing the scaling $x = \varepsilon^{-1}\xi, \hat{k} = \varepsilon^2\hat{\eta}, \lambda = \varepsilon^3\Lambda, c = c_1 + \varepsilon^2\gamma$ and the rescaled functions $\mathcal{R}_j(\xi) = \mathcal{R}_j(\varepsilon x), .0j = 1, 2, 5$. We write the equations of motion (7.33) with this scaling,

$$\begin{aligned}
\varepsilon^3\Lambda\mathcal{R}_1(\xi) =& \varepsilon\left(c_1 + \varepsilon^2\gamma\right)\mathcal{R}_1'(\xi) + \mathcal{R}_5(\xi+\varepsilon) - \mathcal{R}_5(\xi),\\
\varepsilon^3\Lambda\mathcal{R}_2(\xi) =& \left(e^{i\varepsilon^2\hat{\eta}} - 1\right)\mathcal{R}_5(\xi) + \varepsilon\left(c_1 + \varepsilon^2\gamma\right)\mathcal{R}_2'(\xi),\\
\varepsilon^3\Lambda\mathcal{R}_5(\xi) =& c_1^2\left(\mathcal{R}_1(\xi) - \mathcal{R}_1(\xi-\varepsilon)\right)\\
&+ c_2^2\left(1 - e^{-i\varepsilon^2\hat{\eta}}\right)\mathcal{R}_2(\xi) + \varepsilon\left(c_1 + \varepsilon^2\gamma\right)\mathcal{R}_5'(\xi)\\
&+ 2\alpha\varepsilon^2\left(\phi_\gamma(\xi)\mathcal{R}_1(\xi) - \phi_\gamma(\xi-\varepsilon)\mathcal{R}_1(\xi-\varepsilon)\right),
\end{aligned} \tag{7.34}$$

where $\phi_\gamma(\xi)$ is the KdV solitary wave given by (7.32). Expanding the first equation in (7.34) in Taylor series yields

$$\varepsilon^3\Lambda\mathcal{R}_1(\xi) = \varepsilon\left(c_1 + \varepsilon^2\gamma\right)\mathcal{R}_1'(\xi) + \varepsilon\mathcal{R}_5'(\xi) + \frac{\varepsilon^2}{2}\mathcal{R}_5''(\xi) + \frac{\varepsilon^3}{6}\mathcal{R}_5'''(\xi). \tag{7.35}$$

Seeking an approximate solution for $\mathcal{R}_1$ of the form

$$\mathcal{R}_1^{(\varepsilon)} = \mathcal{R}_1^{(0)} + \varepsilon\mathcal{R}_1^{(1)} + \varepsilon^2\mathcal{R}_1^{(2)}. \tag{7.36}$$

Plugging the approximate solution (7.36) into the first equation of (7.35) and comparing orders of epsilon yields,

$$\begin{aligned}\mathcal{R}_1^{(0)} &= -\frac{1}{c_1}\mathcal{R}_5(\xi),\\ \mathcal{R}_1^{(1)} &= -\frac{1}{2c_1}\mathcal{R}_5'(\xi),\\ \mathcal{R}_1^{(2)} &= \frac{\gamma}{c_1^2}\mathcal{R}_5(\xi) - \frac{1}{6c_1}\mathcal{R}_5''(\xi) - \frac{\Lambda}{c_1^2}\partial_\xi^{-1}R_5(\xi).\end{aligned} \tag{7.37}$$

Expanding the second equation in (7.34) in Taylor series yields

$$\varepsilon^3\Lambda\mathcal{R}_2(\xi) = i\varepsilon^2\hat{\eta}\mathcal{R}_5(\xi) + \varepsilon\left(c_1 + \varepsilon^2\gamma\right)\mathcal{R}_2'(\xi), \tag{7.38}$$

Seeking an approximate solution for $\mathcal{R}_2$ of the form

$$\mathcal{R}_2^{(\varepsilon)} = \mathcal{R}_2^{(0)} + \varepsilon\mathcal{R}_2^{(1)} + \varepsilon^2\mathcal{R}_2^{(2)}. \tag{7.39}$$

Plugging the approximate solution (7.39) into the first equation of (7.38) and comparing orders of epsilon yields,

$$\mathcal{R}_2^{(0)} = 0, \qquad \mathcal{R}_2^{(1)} = -\frac{i\hat{\eta}}{c_1}\partial_\xi^{-1}\mathcal{R}_5, \qquad \mathcal{R}_2^{(2)} = 0. \tag{7.40}$$

Expanding the last equation of (7.34) in Taylor series,

$$\Lambda\varepsilon^3\mathcal{R}_5 = \varepsilon c_1\mathcal{R}_5' + \varepsilon^3\gamma\mathcal{R}_5' + c_1^2\left(\varepsilon\mathcal{R}_1' - \frac{\varepsilon^2}{2}\mathcal{R}_1'' + \frac{\varepsilon^3}{6}\mathcal{R}_1'''\right) + c_2^2\varepsilon^2\left(i\hat{\eta}\right)\mathcal{R}_2 \tag{7.41}$$

We use (7.36) and (7.39), discard higher order terms to find that $\mathcal{R}_5$ satisfies the eigenvalue problem for the KP-II equation linearized in the neighbourhood of a KdV solitary wave given by,

$$\Lambda\mathcal{R}_5 = \mathcal{L}\mathcal{R}_5, \tag{7.42}$$

where

$$\mathcal{L}\mathcal{R}_5 = \gamma\frac{d}{d\xi}\mathcal{R}_5 - \frac{c_1}{24}\partial_\xi^3\mathcal{R}_5 + \frac{c_2^2}{2c_1}\hat{\eta}^2\partial_\xi^{-1}\mathcal{R}_5 - \frac{\alpha}{c_1}\partial_\xi\left(\phi_\gamma(\xi)\mathcal{R}_5(\xi)\right). \tag{7.43}$$

For comparison we can expand the essential spectrum of the two-dimensional FPU system in this limit. In the asymptotic limit we replace $\hat{x} = \varepsilon\hat{\xi}, \hat{k} = \varepsilon^2\hat{\eta}, \lambda = \varepsilon^3\Lambda, a = \varepsilon\tilde{a}$, expanding equation (7.24) in Taylor series and truncating terms of $O(\varepsilon^6)$ and higher, we get that the essential spectrum of FPU is given by

$$\left(\varepsilon^3\Lambda - ic\varepsilon\hat{\xi} + \varepsilon\tilde{a}c\right)^2 = c_1^2\left(-\varepsilon^2\left(\hat{\xi} + i\tilde{a}\right)^2 + \frac{\varepsilon^2}{12}\left(\hat{\xi} + i\tilde{a}\right)^4\right) - c_2^2\hat{\eta}^2\varepsilon^4. \quad (7.44)$$

Expanding the left hand side and truncating terms of $O(\varepsilon^6)$ and higher we have

$$\Lambda = i\frac{c^2 - c_1^2}{2\varepsilon^2 c}\left(\hat{\xi} + i\tilde{a}\right) + \frac{ic_1^2}{24c}\left(\hat{\xi} + i\tilde{a}\right)^3 - i\frac{c_2^2}{2c}\frac{\hat{\eta}^2}{\hat{\xi} + i\tilde{a}} \quad (7.45)$$

Using $c = c_1 + \varepsilon^2\gamma$ we have

$$\Lambda = \frac{1}{c_1 + \varepsilon^2\gamma}\left(i(c_1\gamma + O(\varepsilon^2))\left(\hat{\xi} + i\tilde{a}\right) + \frac{ic_1^2}{24}\left(\hat{\xi} + i\tilde{a}\right)^3 - i\frac{c_2^2}{2}\frac{\hat{\eta}^2}{\hat{\xi} + i\tilde{a}}\right). \quad (7.46)$$

Formally the limit as $\varepsilon \to 0$ is given by

$$\Lambda = i\gamma\left(\hat{\xi} + i\tilde{a}\right) + \frac{ic_1}{24}\left(\hat{\xi} + i\tilde{a}\right)^3 - i\frac{c_2^2}{2c_1}\frac{\hat{\eta}^2}{\hat{\xi} + i\tilde{a}}, \quad (7.47)$$

which coincides with the linearized KP-II equation (7.42) at infinities where the potential terms are small.

7.6 FPU eigenvalue problem in the asymptotic limit

Introducing the operators

$$\begin{aligned} K_1 &= \left(\lambda - c\frac{d}{dx}\right)^2 - c_1^2\Delta^+\Delta^- + 4c_2^2\sin^2(\frac{\hat{k}}{2}) \\ K_2 &= 2\alpha\Delta^+\Delta^- u_c(x) \end{aligned} \quad (7.48)$$

we may rewrite equation (7.16) as the system

$$R_1 = (K_1)^{-1}K_2R_1. \quad (7.49)$$

Applying the scaling $x = \varepsilon^{-1}\xi, a = \varepsilon\tilde{a}, \lambda = \frac{\varepsilon^3}{24}c\Lambda, \hat{k} = \varepsilon^2\hat{\eta}$, and the scaled finite difference operators $\Delta_\varepsilon^\pm f(x) = \pm(f(x \pm \varepsilon) - f(x))$, we may write the system as

$$\mathcal{R}_1 = (K_1^{(\varepsilon)})^{-1} K_2^{(\varepsilon)} \left(\varepsilon^{-2} u_c(\frac{\xi}{\varepsilon}) \right) \mathcal{R}_1 \tag{7.50}$$

where

$$\begin{aligned} K_1^{(\varepsilon)} = c^2 \left(\frac{\varepsilon^3}{24}\Lambda - \varepsilon \partial_\xi \right)^2 - c_1^2 \Delta_\varepsilon^+ \Delta_\varepsilon^- + 4c_1^2 \sin^2(\varepsilon^2 \frac{\hat{\eta}}{2}) \\ K_2^{(\varepsilon)} = 8\alpha \Delta_\varepsilon^+ \Delta_\varepsilon^- \end{aligned} \tag{7.51}$$

Note that in the limit $\varepsilon \to 0$ we have $\varepsilon^{-2} u_c(\frac{\xi}{\varepsilon}) \to \phi_c(\xi)$, the solitary wave of the KdV equation. For this system we have the corresponding Fourier symbol

$$\begin{aligned} m^{(\varepsilon)} = \Bigg[c^2 \left(\frac{\varepsilon^3}{24}\Lambda - \varepsilon \left(i\hat{\xi} - \tilde{a} \right) \right)^2 - 4c_1^2 \sinh^2(\varepsilon \frac{i\hat{\xi} - \tilde{a}}{2}) \\ + 4c_1^2 \sin^2(\varepsilon^2 \frac{\hat{\eta}}{2}) \Bigg]^{-1} (8\alpha\varepsilon^2 \sinh^2(\varepsilon \frac{i\hat{\xi} - \tilde{a}}{2})) \end{aligned} \tag{7.52}$$

Denoting $\nu = i\hat{\xi} - \tilde{a}$, setting $c = c_1 + \varepsilon^2 \frac{c_1}{24}$, and expanding the above in Taylor series yields

$$\begin{aligned} m^{(\varepsilon)} &= \left[\frac{c_1^2}{12}\nu^2 - \frac{c_1^2}{12}\Lambda\nu - \frac{c_1^2}{12}\nu^4 + c_2^2 \hat{\eta}^2 + O(\varepsilon^2(\nu^6 + \hat{\eta}^4 + \Lambda^2 + \nu\Lambda)) \right]^{-1} \\ &\quad \cdot \left(2\tilde{a}\nu^2 + O(\varepsilon^2 \nu^4) \right) \\ &= \left(\nu - \Lambda - \nu^3 + 12\frac{c_2^2}{c_1^2}\nu^{-1}\hat{\eta}^2 + O(\varepsilon^2) \right)^{-1} \left(\frac{24\alpha}{c_1^2}\nu + O(\varepsilon^2) \right) \end{aligned} \tag{7.53}$$

Formally the limit as $\varepsilon \to 0$ of $m^{(\varepsilon)}$ is given by

$$m^{(0)} = \left(\nu - \Lambda - \nu^3 + 12\frac{c_2^2}{c_1^2}\nu^{-1}\hat{\eta}^2 \right)^{-1} \left(\frac{24\alpha}{c_1^2}\nu \right). \tag{7.54}$$

The limit equation

$$m^{(0)}\phi(\xi)\mathcal{R}_1 = \mathcal{R}_1, \tag{7.55}$$

is the KP-II equation linearized in the neighbourhood of a KdV soliton.

For this section we will use the following notations. For the continuous spectrum of the one-dimensional FPU we will write,

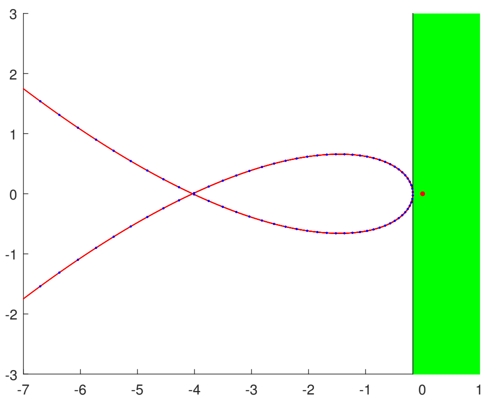
$$\mathcal{P}_\pm(\varepsilon(i\hat{\xi} - \tilde{a})) = \varepsilon(i\hat{\xi} - \tilde{a}) \pm 2\frac{c_1}{c}\sinh(\frac{\varepsilon}{2}(i\hat{\xi} - \tilde{a})) \tag{7.56}$$

For the continuous spectrum of the two-dimensional FPU, we will write

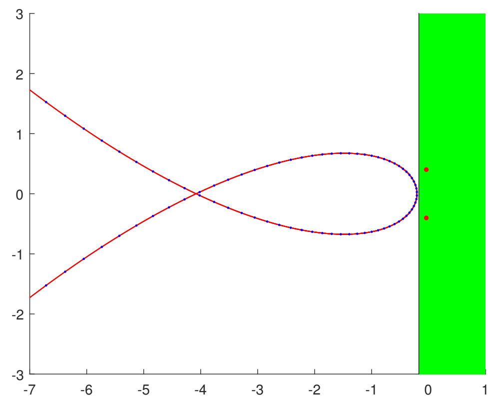
$$\sigma_{\pm}(\varepsilon(i\hat{\xi}-\tilde{a}),\varepsilon^2\eta) = \varepsilon(i\hat{\xi}-\tilde{a}) \pm 2\frac{c_1}{c}\mu(\varepsilon(i\hat{\xi}-\tilde{a}),\varepsilon^2\eta), \tag{7.57}$$

where

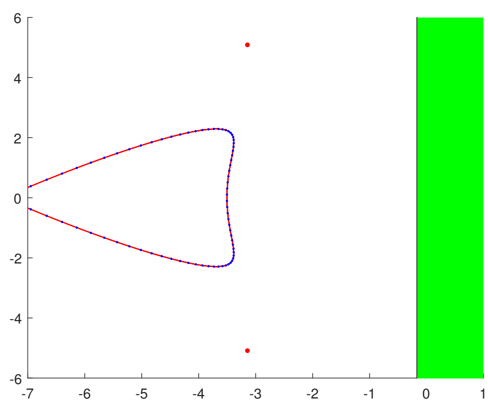
$$\mu(\varepsilon(i\hat{\xi}-\tilde{a}),\varepsilon^2\eta) = \sqrt{\sinh^2(\frac{\varepsilon}{2}(i\hat{\xi}-\tilde{a})) - \frac{c_2^2}{c_1^2}\sin^2(\frac{\varepsilon^2}{2}\hat{\eta})} \tag{7.58}$$



(a) $\hat{\eta} = 0$.



(b) $\hat{\eta} = 0.05$.



(c) $\hat{\eta} = 0.5$.

Figure 7.9: Plot of the spectrum of the linearized KP-II equation (red) with the continuous spectrum of the FPU system (blue dots) when $\tilde{a} = 0.9$ and $\varepsilon = 10^{-2}$ for various values of $\hat{\eta}$. The green shaded region represents $\operatorname{Re}\Lambda > \tilde{a}^3 - \tilde{a}, |\operatorname{Im}\Lambda| \leq 24\pi\varepsilon^{-3}$

Figure 7.9 illustrates that for small epsilon the spectrum of the two-dimensional FPU system converges to the spectrum of the linearized KP-II equation. Both spectra are contained in the left half plane for $\tilde{a} > 0$. The green shaded region represents $\operatorname{Re}\Lambda > \tilde{a}^3 - \tilde{a}, |\operatorname{Im}\Lambda| \leq 24\pi\varepsilon^{-3}$, a region which does not contain the KdV spectrum, and for small *êta* contains two isolated points of the linearized KP-II spectra. This region will be studied in lemma 7.4, the goal is to establish that the spectrum of FPU problem in the small epsilon limit has no points in the green region.

The following lemma is a generalization of proposition 3.2 in [17] for nonzero $\hat{\eta}$. The goal of the lemma is to establish that within the green strip in figure 7.9 either the truncated Taylor series of $m^{(\varepsilon)}$ converges uniformly to $m^{(0)}$, the linearized KP-II equation, or both $m^{(\varepsilon)}$ and $m^{(0)}$ converge to zero as $\varepsilon \to 0$.

Lemma 7.4. *Fix* $\tilde{a} \in (0,1)$, $c_1 = c_2$, *let* β_* *be a number in the interval* $(0, \tilde{a} - \tilde{a}^3)$, *and* $0 < \delta < \frac{1}{6}$

Let

$$\Omega^\varepsilon = \{(\Lambda, \hat{\xi}, \hat{\eta}) | \hat{\xi} \in \mathbb{R}, |\hat{\eta}| \leq \pi\varepsilon^{-2}, \operatorname{Re}\Lambda \geq -\beta_*, |\operatorname{Im}\Lambda| \leq \pi\varepsilon^{-3}\},$$

$$\tilde{\Omega}^\varepsilon = \{(\Lambda, \hat{\xi}, \hat{\eta}) | \hat{\xi} \in \mathbb{R}, \left|\varepsilon\hat{\xi}\right| \geq \varepsilon^{1-\delta}, |\hat{\eta}| \leq \pi\varepsilon^{-2}, \operatorname{Re}\Lambda \geq -\beta_*, |\operatorname{Im}\Lambda| \leq \pi\varepsilon^{-3}\},$$

and

$$\Omega_0^\varepsilon = \{(\Lambda, \hat{\xi}, \hat{\eta}) | \left|\varepsilon^{\frac{1}{3}}\hat{\xi}\right| \leq \varepsilon^\delta, \left|\varepsilon^{\frac{1}{2}}\hat{\eta}\right| \leq \varepsilon^\delta, |\varepsilon\Lambda| \leq \varepsilon^\delta\} \cap \Omega^\varepsilon.$$

Then

$$\sup_{(\Lambda,\hat{\xi},\hat{\eta}) \in \tilde{\Omega}^\varepsilon} \left|m^{(\varepsilon)} - m^{(0)}\right| \to 0 \qquad \text{as } \varepsilon \to 0.$$

Further,

$$\sup_{(\Lambda,\hat{\xi},\hat{\eta}) \in \tilde{\Omega}^\varepsilon \setminus \Omega_0^\varepsilon} \left|m^{(\varepsilon)}\right| \to 0 \qquad \text{as } \varepsilon \to 0,$$

and

$$\sup_{(\Lambda,\hat{\xi},\hat{\eta}) \in \Omega^\varepsilon \setminus \Omega_0^\varepsilon} \left|m^{(0)}\right| \to 0 \qquad \text{as } \varepsilon \to 0.$$

Proof. The proof is broken down into a number of regions. The first of which is the regime where $m^{(\varepsilon)}$ converges to the KP-II equation.

Region 0: We introduce a small parameter $\delta > 0$, and define the subset

$$\Omega_0^\varepsilon = \{(\Lambda, \hat{\xi}, \hat{\eta}) | \left|\varepsilon^{\frac{1}{3}}\hat{\xi}\right| \leq \varepsilon^\delta, \left|\varepsilon^{\frac{1}{2}}\hat{\eta}\right| \leq \varepsilon^\delta, |\varepsilon\Lambda| \leq \varepsilon^\delta\} \cap \Omega^\varepsilon.$$

On this region we have for the Taylor remainder terms that $\varepsilon^2\Lambda^2 = O(\varepsilon^{2\delta}), \varepsilon^2\nu^6 = O(\varepsilon^{6\delta}), \varepsilon^2\hat{\eta}^4 = O(\varepsilon^{4\delta}), \varepsilon^2\Lambda\nu = O(\varepsilon^{\frac{2}{3}+2\delta}) + O(\varepsilon^{1+\delta})$, so that the numerator and denominator of $m^{(\varepsilon)}$ converge uniformly to the numerator and denominator of $m^{(0)}$ as $\varepsilon \to 0$. We'd like to show that $m^{(0)}$ is bounded by showing that the denominator is uniformly bounded away from zero. Note that

$$\begin{aligned}
&\mathrm{Re}\left(\nu - \Lambda - \nu^3 + 12\frac{c_2^2}{c_1^2}|\nu|^{-2}\bar{\nu}\hat{\eta}^2\right) \\
&= -\tilde{a} - \mathrm{Re}\,\Lambda + \left(\tilde{a}^3 - 3\tilde{a}\hat{\xi}^2\right) - \tilde{1}2a\frac{c_2^2}{c_1^2\,|\nu|^2}\hat{\eta}^2 \qquad (7.59)\\
&\leq -\tilde{a} + \tilde{a}^3 + \beta_* < 0
\end{aligned}$$

where the last inequality holds by hypothesis.

Region 1:

$$\Omega_1^{\varepsilon} = \{(\Lambda, \hat{\xi}, \hat{\eta})|\left|\varepsilon\hat{\xi}\right| \geq \pi + 2\sqrt{2} + 0.1\} \cap \Omega^{\varepsilon}.$$

Our strategy for this region will be to show that the denominator of $m^{(\varepsilon)}$ is bounded away from zero, for sufficiently small, while the numerator goes to zero. We will show the denominator is bounded away from zero by showing that the imaginary part is bounded away from zero. Since $|\mathrm{Im}\,\lambda| \leq \pi$, in the denominator we have

$$|\mathrm{Im}(\lambda - \sigma_{\pm})| \geq \left|\varepsilon\hat{\xi}\right| - \pi - 2\frac{c_1}{c}\,|\mu|\,.$$

For the magnitute of μ,

$$\begin{aligned}
|\mu| = \Bigg(&(\sinh^2(\frac{\varepsilon\tilde{a}}{2})\cos^2(\frac{\varepsilon\hat{\xi}}{2}) - \cosh^2(\frac{\varepsilon\tilde{a}}{2})\sin^2(\frac{\varepsilon\hat{\xi}}{2}) - \sin^2(\frac{\varepsilon^2\hat{\eta}}{2}))^2 \\
&+4\sinh^2(\frac{\varepsilon\tilde{a}}{2})\cos^2(\frac{\varepsilon\hat{\xi}}{2})\cosh^2(\frac{\varepsilon\tilde{a}}{2})\sin^2(\frac{\varepsilon\hat{\xi}}{2})\Bigg)^{\frac{1}{4}}.
\end{aligned} \qquad (7.60)$$

We can bound this on the given region by noting,

$$\begin{aligned}
&\left|\sinh^2(\frac{\varepsilon\tilde{a}}{2})\cos^2(\frac{\varepsilon\hat{\xi}}{2}) - (\cosh^2(\frac{\varepsilon\tilde{a}}{2})\sin^2(\frac{\varepsilon\hat{\xi}}{2}) + \sin^2(\frac{\varepsilon^2\hat{\eta}}{2}))\right|^{\frac{1}{2}} \\
&\leq \left|\sinh(\frac{\varepsilon\tilde{a}}{2})\cos(\frac{\varepsilon\hat{\xi}}{2})\right| + (\cosh^2(\frac{\varepsilon\tilde{a}}{2})\sin^2(\frac{\varepsilon\hat{\xi}}{2}) + \sin^2(\frac{\varepsilon^2\hat{\eta}}{2}))^{\frac{1}{2}} \qquad (7.61)\\
&\leq \left|\sinh(\frac{\varepsilon\tilde{a}}{2})\cos(\frac{\varepsilon\hat{\xi}}{2})\right| + (\cosh^2(\frac{\varepsilon\tilde{a}}{2}) + \sin^2(\frac{\varepsilon^2\hat{\eta}}{2}))^{\frac{1}{2}},
\end{aligned}$$

and,

$$\sqrt{2\sinh(\frac{\varepsilon\tilde{a}}{2})\cos(\frac{\varepsilon\hat{\xi}}{2})\cosh(\frac{\varepsilon\tilde{a}}{2})\sin(\frac{\varepsilon\hat{\xi}}{2})} \leq \sqrt{2\left|\sinh(\frac{\varepsilon\tilde{a}}{2})\cosh(\frac{\varepsilon\tilde{a}}{2})\right|}. \tag{7.62}$$

Taking the limit as $\varepsilon \to 0$ we see that

$$\lim_{\varepsilon\to 0} 2\frac{c_1}{c}|\mu| = 2\sqrt{2}$$

on the region, so that for sufficiently small epsilon we may take $2\frac{c_1}{c}|\mu| < 2\sqrt{2}+0.05$, so that the denominator is bounded away from zero. For a sufficiently small ε, the numerator is bounded by

$$8\alpha\varepsilon^2|\sinh^2(\frac{\varepsilon}{2}(i\hat{\xi}-\tilde{a})) \leq \cosh^2(\frac{\varepsilon}{2}\tilde{a}) \leq 16\alpha\varepsilon^2.$$

Consequently $\sup_{\Omega_1^\varepsilon}|m^\varepsilon| \to 0$ as $\varepsilon \to 0$.

Region 2:

$$\Omega_3^\varepsilon = \{(\Lambda,\hat{\xi},\hat{\eta})|\varepsilon^{1-\delta} \leq \left|\varepsilon\hat{\xi}\right| \leq \pi + 2\sqrt{2}+0.1\} \cap \Omega^\varepsilon.$$

We will bound the denominator using the real part of $\lambda - \sigma_\pm$ (which we will compare with the real part of $\lambda - \mathcal{P}_\pm$), and the inequality

$$|\lambda - a|\,|\lambda - b| \geq \frac{1}{2}|b-a|\min(|\lambda-a|, |\lambda-b|).$$

In order to apply the inequality we use that

$$\left|\sinh(\frac{\varepsilon}{2}(i\hat{\xi}-\tilde{\alpha}))\right| = \left|\sinh^2(\frac{\varepsilon}{2}(i\hat{\xi}-\tilde{\alpha}))\right|^{\frac{1}{2}} \leq |\mu| + \left|\sin(\frac{\varepsilon^2}{2}\hat{\eta})\right|,$$

to write

$$\left|m^{(\varepsilon)}\right| \leq \frac{8\alpha\varepsilon^2 c^{-2}\left|\sin(\frac{\varepsilon}{2}(i\hat{\xi}-\tilde{\alpha}))\right|}{\min(|\lambda-\sigma_+|, |\lambda-\sigma_-|)}(1+\frac{\left|\sin(\frac{\varepsilon^2}{2}\hat{\eta})\right|}{|\mu|}).$$

We require that the term $\frac{\left|\sin(\frac{\varepsilon^2}{2}\hat{\eta})\right|}{|\mu|}$ remain bounded in the region for small ε. The size of μ is bounded below by

$$|\mu| \geq \left|\sinh^2(\frac{\varepsilon\tilde{a}}{2})\cos^2(\frac{\varepsilon\hat{\xi}}{2}) - \cosh^2(\frac{\varepsilon\tilde{a}}{2})\sin^2(\frac{\varepsilon\hat{\xi}}{2}) - \sin^2(\frac{\varepsilon^2\hat{\eta}}{2})\right|^{\frac{1}{2}}.$$

For sufficiently small ε and sufficiently large $\varepsilon\hat{\xi}$ the function $\sinh^2(\frac{\varepsilon\tilde{a}}{2})\cos^2(\frac{\varepsilon\hat{\xi}}{2})-\cosh^2(\frac{\varepsilon\tilde{a}}{2})\sin^2(\frac{\varepsilon\hat{\xi}}{2}) < 0$, consequently $\mu \geq \left|\sin^2(\frac{\varepsilon^2\hat{\eta}}{2})\right|$, and the desired bound holds. It remains to check that for a sufficiently small ε this condition holds for each $\varepsilon\hat{\xi}$ in the region. Note that roots of the function are located at $\frac{\varepsilon\hat{\xi}}{2} = \arctan(\tanh(\frac{\varepsilon\tilde{a}}{2})) \leq \frac{\varepsilon\tilde{a}}{2}$, since $\frac{\varepsilon\hat{\xi}}{2} \geq \varepsilon^{1-\delta}$, for a sufficiently small epsilon we have that $\frac{\varepsilon\hat{\xi}}{2} > \frac{\varepsilon\tilde{a}}{2}$.

For each $\varepsilon^2\hat{\eta} \in \mathbb{R}$ and $\left|\varepsilon\hat{\xi}\right| \leq \pi$ we have,

$$\operatorname{Re}\mathcal{P}_+ \leq \operatorname{Re}\sigma_- \leq \operatorname{Re}\sigma_+ \leq \operatorname{Re}\mathcal{P}_-, \tag{7.63}$$

so that

$$-\operatorname{Re}\sigma_+ \geq -\operatorname{Re}\sigma_- \geq -\operatorname{Re}\mathcal{P}_-. \tag{7.64}$$

From [17] we have on this region that

$$\operatorname{Re}(\lambda-\sigma_-) \geq \operatorname{Re}(\lambda-\mathcal{P}_-) \geq \varepsilon^{2-\delta}\tilde{a}\frac{\left|\varepsilon\hat{\xi}\right|}{25}. \tag{7.65}$$

Similarly $\varepsilon^2\hat{\eta} \in \mathbb{R}$ and $\pi \leq \left|\varepsilon\hat{\xi}\right| \leq \pi + 2\sqrt{2} + 0.1$ we have,

$$\operatorname{Re}\mathcal{P}_- \leq \operatorname{Re}\sigma_- \leq \operatorname{Re}\sigma_+ \leq \operatorname{Re}\mathcal{P}_+, \tag{7.66}$$

so that

$$-\operatorname{Re}\sigma_+ \geq -\operatorname{Re}\sigma_- \geq -\operatorname{Re}\mathcal{P}_+. \tag{7.67}$$

On this region we have that

$$\begin{aligned}\operatorname{Re}(\lambda-\mathcal{P}_-) \geq -\frac{1}{24}\varepsilon^3\beta_* + \varepsilon\tilde{a} + (\varepsilon\tilde{a} + \frac{\varepsilon^3}{24}(\tilde{a}^3-\tilde{a}) + O(\varepsilon^5))\cos(\frac{\varepsilon}{2}\hat{\xi})\\ = \varepsilon\tilde{a}(1+\cos(\frac{\varepsilon}{2}\hat{\xi})) + O(\varepsilon^3)\end{aligned} \tag{7.68}$$

Since $\frac{1}{2}(\pi+2\sqrt{2}+0.1) < \pi$ we have that $1+\cos(\frac{\varepsilon}{2}) \geq 1+\cos(\frac{1}{2}(\pi+2\sqrt{2}+0.1)) > 0$ on the region, and for a sufficiently small ε

$$\operatorname{Re}(\lambda-\mathcal{P}_-) \geq (1+\cos(\frac{1}{2}(\pi+2\sqrt{2}+0.1)))\varepsilon\tilde{a} \tag{7.69}$$

For a sufficiently small ε the lower bound in (7.65) is smaller than the lower bound in (7.69), and hence a lower bound for the denominator on the entire region. Hence on the region we have

$$\sup_{\Omega_2^\varepsilon} \left| m^{(\varepsilon)} \right| \le \frac{8\alpha}{c^2} \frac{25\varepsilon^2}{\varepsilon^{2-\delta}\tilde{a}\left|\varepsilon\hat{\xi}\right|} \left| \sinh(\frac{\varepsilon}{2}(i\hat{\xi} - \tilde{a})) \right|. \tag{7.70}$$

For the sinh term note that $\left|\sinh(\frac{\varepsilon}{2}(i\hat{\xi} - \tilde{a}))\right| \le \sqrt{\sin^2(\frac{\varepsilon}{2}\hat{\xi}) + \sinh^2(\frac{\varepsilon}{2}\tilde{a})}$. Since $\sin^2(x) \le x^2$ for all x and $\left|\sinh(\frac{\varepsilon}{2}\tilde{a})\right| \le \varepsilon$ for sufficiently small ε we have $\left|\sinh(\frac{\varepsilon}{2}(i\hat{\xi} - \tilde{a}))\right| \le \sqrt{\left|\varepsilon\frac{\hat{\xi}}{2}\right|^2 + \varepsilon^2} \le \left|\varepsilon\hat{\xi}\right|$, where the final inequality holds on this region since $\left|\varepsilon\hat{\xi}\right| \ge \varepsilon^{1-\delta} \ge \varepsilon$. Consequently we have

$$\sup_{\Omega_2^\varepsilon} m^{(\varepsilon)} \le \frac{200\alpha}{c^2}\varepsilon^\delta, \tag{7.71}$$

hence $\lim_{\varepsilon} \left| m^{(\varepsilon)} \right| = 0$ uniformly in this region.

We finish off by showing convergence to zero of $m^{(0)}$ in the appropriate regimes.

Region 1-2: ($\left|\varepsilon\hat{\xi}\right| \ge \varepsilon^{1-\delta}$) Recall by hypothesis that we have $-\tilde{a}+\tilde{a}^3 \le \operatorname{Re}\Lambda$

$$\begin{aligned} \operatorname{Re}(\nu - \nu^3 - \Lambda + 12\frac{c_2^2}{c_1^2}\hat{\eta}^2\frac{\bar{\nu}}{|\nu^2|}) &= -\tilde{a} + \tilde{a}^3 - 3\tilde{a}\hat{\xi}^2 - \operatorname{Re}\Lambda - 12\frac{\tilde{a}c_2^2}{c_1^2}\frac{\hat{\eta}^2}{|\nu|^2} \\ &\le -3\tilde{a}\hat{\xi}^2 \end{aligned} \tag{7.72}$$

In this region $\hat{\xi}$ is large for small ϵ, so that $|\nu| \le \left|\hat{\xi}\right| + \tilde{a} \le 2\left|\hat{\xi}\right|$, so

$$\sup_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} m^{(0)} \le 48\frac{\alpha}{c_1^2}\left|\hat{\xi}\right| \le 48\frac{\alpha}{c_1^2}\varepsilon^\delta \tag{7.73}$$

and $m^{(0)}$ converges to zero uniformly on the region as $\varepsilon \to 0$.

Region 3: ($\left|\varepsilon\hat{\xi}\right| \le \varepsilon^{1-\delta}, |\varepsilon\Lambda| \ge \varepsilon^\delta$) We split this into two pieces, when $|\hat{\eta}| < \varepsilon^{\delta-\frac{1}{2}}$ and when $|\hat{\eta}| \ge \varepsilon^{\delta-\frac{1}{2}}$. In both cases we will use that for a sufficiently small ε

When $|\hat{\eta}| < \varepsilon^{\delta-\frac{1}{2}}$ we have for a sufficiently small ε that

$$\begin{aligned} \left|\nu - \nu^3 - \Lambda + 12\frac{c_2^2}{c_1^2}\hat{\eta}^2\frac{1}{\nu}\right| &\ge |\Lambda| - |\nu| - |\nu|^3 - 12\frac{c_2^2}{\tilde{a}c_1^2}|\hat{\eta}|^2 \\ &\ge \varepsilon^{-1+\delta} - 2\varepsilon^{-\delta} - 8\varepsilon^{-3\delta} - 12\frac{c_2^2}{\tilde{a}c_1^2}\varepsilon^{-1+2\delta} \ge \frac{1}{2}\varepsilon^{-1+\delta}, \end{aligned} \tag{7.74}$$

so that

$$m^{(0)} \leq 48\frac{\alpha}{c_1^2}\varepsilon^{1-2\delta}. \tag{7.75}$$

Alternatively if $|\hat{\eta}| \geq \varepsilon^{\delta-\frac{1}{2}}$ we can estimate the denominator using the real part,

$$\begin{aligned}\operatorname{Re}(\nu - \nu^3 - \Lambda + 12\frac{c_2^2}{c_1^2}\hat{\eta}^2\frac{\bar{\nu}}{|\nu^2|}) &= -\tilde{a} + \tilde{a}^3 - 3\tilde{a}\hat{\xi}^2 - \operatorname{Re}\Lambda - 12\frac{\tilde{a}c_2^2}{c_1^2}\frac{\hat{\eta}^2}{|\nu|^2} \\ &\leq -12\frac{\tilde{a}c_2^2}{c_1^2}\frac{\hat{\eta}^2}{|\nu|^2}\end{aligned} \tag{7.76}$$

We then have that

$$\left|m^{(0)}\right| \leq \frac{2\alpha}{\tilde{a}c_2^2}\frac{|\nu|^3}{\hat{\eta}^2} \leq \frac{2\alpha}{\tilde{a}c_2^2}\varepsilon^{-3\delta}\varepsilon^{1-2\delta} = \frac{2\alpha}{\tilde{a}c_2^2}\varepsilon^{1-5\delta} \tag{7.77}$$

We see that the latter of these estimates is large for small ε, so

$$\sup_{\Omega_3^\varepsilon}\left|m^{(0)}\right| \leq 2\frac{\alpha}{\tilde{a}c_2^2}\varepsilon^{1-5\delta} \tag{7.78}$$

and $m^{()}$ converges to zero uniformly on the region as $\varepsilon \to 0$. □

Remark. As of now the proposition could not be fully generalized as there are problems getting good analytic estimates of the contribution of the transverse variable in the long wavelength, long time regime for $m^{(\varepsilon)}$. The statement has been fully generalized for $m^{(0)}$. Namely what remains to be proven is that $m^{(\varepsilon)}$ converges to $m^{(0)}$ on the following region:

$$\Omega_3^\varepsilon = \{(\Lambda, \hat{\xi}, \hat{\eta})| \left|\varepsilon\hat{\xi}\right| \leq \varepsilon^{1-\delta}, |\varepsilon\Lambda| > \varepsilon^\delta\} \cap \Omega^\varepsilon.$$

This is the reason for defining the region $\tilde{\Omega}^\varepsilon$ in the statement of the lemma.

Extending this lemma so that the convergence holds on this final region would show that the operator $m^{(\varepsilon)}$ converges either to zero or to the linearized KP-II equation as $\varepsilon \to 0$. We could use this, along with stability of the linearized KP-II equation in exponentially weighted spaces L_a^2, to obtain stability in ℓ_a^2 for sufficiently small energy for the first system in (7.15). For the second system in (7.15) we use that it is a constant coefficient equation whose spectrum has $\operatorname{Re}\lambda < 0$.

Chapter 8

Conclusion

In this thesis we extended the known results regarding the small-amplitude, long-wavelength limit of the FPU system to two dimensions. We provided a rigorous justification of the KP-II equation as an approximation in this limit, and studied the asymptotic stability of line solitary waves for the linearized two-dimensional FPU system.

Theorem 4.1 is our first major result, we justified the KP-II equation as the long-wavelength, small-amplitude limit of the two-dimensional FPU system on a square lattice. Our result states that if the initial conditions for a two-dimensional FPU α-model in strain coordinates is initially $\varepsilon^{\frac{5}{2}}$-close to a sufficiently smooth, and appropriately scaled, solution of the KP-II equation, it remains $\varepsilon^{\frac{5}{2}}$-close for timescales of $O(\varepsilon^{-3})$. While two-dimensional FPU systems had been rigorously, such as in [19] and [8], studied in this limit previously these results did not consider transverse perturbations. This result is a necessary step for study line solitary waves of the associated two-dimensional FPU system.

In our second major result, theorem 5.1, we justified the KP-II equation for the diagonal propagation of the two-dimensional FPU system. For this result we worked with the same system as in theorem 4.1, and proved that the KP-II equation is the long-wavelength, small-amplitude limit of this system in a propagation direction which is not the horizontal axis. We prove that, in the appropriate variables, if the system is initially $\varepsilon^{\frac{5}{2}}$-close to a collection of functions, which depend only on a solution of the KP-II equation and its derivatives, then it remains $\varepsilon^{\frac{5}{2}}$-close for time scales of $O(\varepsilon^{-3})$.

In our third result, theorem 6.1, we justify the cubic KP-II equation in the small-amplitude short-wavelength limit of a two-dimensional cubic FPU system. Our result states that if the initial conditions for a two-dimensional FPU β-model in strain coordinates is initially $\varepsilon^{\frac{3}{2}}$-close to a sufficiently smooth, and appropriately scaled, solution of the KP-II equation, it remains $\varepsilon^{\frac{3}{2}}$-close for timescales of $O(\varepsilon^{-3})$. The result is similar to that of the α-model, with the cubic nonlinearity changing the scaling used for the approximating functions, and the limiting equation replaced with a modified nonlinearity on the limiting KP-II equation.

In Chapter 7 we study the transverse stability of the line solitary waves in a two-dimensional FPU system. The stability of line-solitary waves would provide information about the long time dynamics of the system. We obtained rigorous results regarding the spectrum of the linearized operator of the FPU system from the spectrum of the linearized operator of the KP-II equation. In lemma 7.3 we established that the spectrum of the linearized FPU system had a negative real part, and while there are resonances present, the growth due to resonances is dominated by exponential decay. In lemma 7.4 we proved that the linearized FPU operator either converges to the linearized KP-II operator, or to zero, as long as the wave number is not too small. This is not a full generalization of the analogous one-dimensional result, and some more work is needed here to establish transverse stability of the line solitary waves.

The work on the KP-II limit can be extended by considering lattices with diagonal springs, as had been considered in other works, such as in [8]. Additionally in this thesis we only considered propagation along the horizontal axis or the main diagonal, but arbitrary propagation direction has not been considered. As with the one-dimensional case polyatomic models, ones with more complicated potentials can be considered.

As a future work, one can study if the proof of nonlinear stability of line solitary waves can be developed for small-amplitude solitary waves in two-dimensional FPU systems. Additional work can also be done on studying the resonant mode in order to establish asymptotic stability of line solitary waves in the two-dimensional FPU system. Additionally we did not study whether resonant solitary interactions, although a resonance did appear in the study of the FPU spectrum, are present in the two-dimensional FPU system, as they are in the KP-II equation.

Bibliography

[1] D. Bambusi and A. Maspero, "Birkhoff coordinates for the Toda lattice in the limit of infinitely many particles with an application to FPU," *Journal of Functional Analysis*, vol. 270, no. 5, pp. 1818–1887, 2016.

[2] D. Bambusi, "Nekhoroshev theorem for small amplitude solutions in nonlinear Schrödinger equations," *Mathematische Zeitschrift*, vol. 230, no. 2, pp. 345–387, 1999.

[3] D. Bambusi, A. Carati, A. Maiocchi, and A. Maspero, "Some analytic results on the FPU paradox," *Hamiltonian Partial Differential Equations and Applications*, vol. 75, pp. 235–254, 2015.

[4] J. T. Beale, "Exact solitary water waves with capillary ripples at infinity," *Comm. Pure Appl. Math.*, vol. 44, no. 2, pp. 211–257, 1991.

[5] G. P. Berman and F. M. Izrailev, "The Fermi–Pasta–Ulam problem: Fifty years of progress," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 15, no. 1, p. 015 104, Mar. 2005.

[6] J. Bourgain, "On the Cauchy problem for the Kadomstev-Petviashvili equation," *Geometric and Functional Analysis*, vol. 3, no. 4, pp. 315–341, 1993.

[7] Burtsev, S. P., "Damping of soliton oscillations in media with a negative dispersion law," *Zh. Èksper. Teoret. Fiz.*, vol. 88, no. 2, pp. 461–469, 1985.

[8] F. Chen and M. Herrmann, "KDV-like solitary waves in two-dimensional FPU-lattices," *Discrete and Continuous Dynamical Systems- Series A*, vol. 38, no. 5, pp. 2305–2332, 2018.

[9] E. Dumas and D. Pelinovsky, "Justification of the log–KdV Equation in Granular Chains: The Case of Precompression," *SIAM Journal on Mathematical Analysis*, vol. 46, no. 6, pp. 4075–4103, 2014.

[10] D. B. Duncan, J. C. Eilbeck, C. H. Walshaw, and V. E. Zakharov, "Solitary waves on a strongly anisotropic KP lattice," *Physics Letters A*, vol. 158, no. 3-4, pp. 107–111, 1991.

[11] T. E. Faver and H. J. Hupkes, "Micropteron traveling waves in diatomic Fermi–Pasta–Ulam–Tsingou lattices under the equal mass limit," *Physica D: Nonlinear Phenomena*, vol. 410, p. 132 538, 2020.

[12] T. E. Faver and J. D. Wright, "Exact diatomic fermi–pasta–ulam–tsingou solitary waves with optical band ripples at infinity," *SIAM Journal on Mathematical Analysis*, vol. 50, no. 1, pp. 182–250, 2018.

[13] E. Fermi, P. Pasta, S. Ulam, and M. Tsingou., "Studies of the nonlinear problems.," *No. LA-1940. Los Alamos Scientific Lab., N. Mex.*, no. May, 1955.

[14] G. Friesecke and R. L. Pego, "Solitary waves on FPU lattices: I. Qualitative properties, renormalization and continuum limit," *Nonlinearity*, vol. 12, no. 6, pp. 1601–1627, 1999.

[15] ——, "Solitary waves on FPU lattices: II. Linear implies nonlinear stability," *Nonlinearity*, vol. 15, no. 4, pp. 1343–1359, 2002.

[16] ——, "Solitary waves on Fermi–Pasta–Ulam lattices: III. Howland-type Floquet theory," *Nonlinearity*, vol. 17, pp. 207–227, 2004.

[17] ——, "Solitary waves on Fermi–Pasta–Ulam lattices: IV. Proof of stability at low energy," *Nonlinearity*, vol. 17, pp. 229–251, 2004.

[18] G. Friesecke and F. Theil, "Validity and failure of the Cauchy-Born hypothesis in a two-dimensional mass-spring lattice," *Journal of Nonlinear Science*, vol. 12, no. 5, pp. 445–478, 2002.

[19] G. Friesecke and K. Matthies, "Geometric solitary waves in a 2D mass-spring lattice," *Discrete and Continuous Dynamical Systems - Series B*, vol. 3, no. 1, pp. 105–114, 2003.

[20] J. Gaison, S. Moskow, J. D. Wright, and Q. Zhang, "Approximation of polyatomic FPU lattices by KdV equations," *Multiscale Modeling and Simulation*, vol. 12, no. 3, pp. 953–995, 2014.

[21] G. Gallavotti, Ed., *The Fermi-Pasta-Ulam Problem*, ser. Lecture Notes in Physics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008, vol. 728.

[22] T. Gallay and G. Schneider, "KP description of unidirectional long waves. The model case," *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, vol. 131, no. 4, pp. 885–898, 2001.

[23] M. Haragus, J. Li, and D. E. Pelinovsky, "Counting Unstable Eigenvalues in Hamiltonian Spectral Problems via Commuting Operators," *Communications in Mathematical Physics*, vol. 354, no. 1, pp. 247–268, 2017.

[24] A. Henrici and T. Kappeler, "Resonant normal form for even periodic FPU chains," *Journal of the European Mathematical Society*, vol. 11, no. 5, pp. 1025–1056, Sep. 2009.

[25] M. Herrmann and K. Matthies, "Asymptotic formulas for solitary waves in the high-energy limit of FPU-type chains," *Nonlinearity*, vol. 28, no. 8, pp. 2767–2789, 2015.

[26] ——, "Uniqueness of solitary waves in the high-energy limit of FPU-type chains," *Patterns of Dynamics*, vol. 205, pp. 3–15, 2018.

[27] ——, "Stability of high-energy solitary waves in Fermi-Pasta-Ulam-Tsingou chains," *Transactions of the American Mathematical Society*, vol. 372, no. 5, pp. 3425–3486, 2019.

[28] P. D. Hislop and I. M. Sigal, *Introduction to Spectral Theory*. 1996, vol. 113, pp. IX, 338.

[29] A. Hoffman and J. D. Wright, "Nanopteron solutions of diatomic Fermi-Pasta-Ulam-Tsingou lattices with small mass-ratio," *Physica D: Nonlinear Phenomena*, vol. 358, pp. 33–59, Nov. 2017.

[30] B. B. Kadomtsev and V. I. Petviashvili, "On the Stability of Solitary Waves in Weakly Dispersing Media," *Soviet Physics Doklady*, vol. 15, p. 539, Dec. 1970.

[31] A. Khan and D. E. Pelinovsky, "Long-time stability of small FPU solitary waves," *Discrete and Continuous Dynamical Systems- Series A*, vol. 37, no. 4, pp. 2065–2075, 2017.

[32] D. J. Korteweg and G. de Vries, "XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 39, no. 240, pp. 422–443, May 1895.

[33] P. D. Lax, "Integrals of nonlinear equations of evolution and solitary waves," *Communications on Pure and Applied Mathematics*, vol. 21, no. 5, pp. 467–490, 1968.

[34] R. Miura, "The korteweg–devries equation: A survey of results," *SIAM Review*, vol. 18, no. 3, pp. 412–459, 1976.

[35] T. Mizumachi, "Stability of line solitons for the KP-II equation in $\mathbb{R}^2$," *Memoirs of the American Mathematical Society*, vol. 238, no. 1125, pp. vii+95, 2015.

[36] ——, "Stability of line solitons for the KP-II equation in $\mathbb{R}^2$. II," *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, vol. 148, no. 1, pp. 149–198, 2018.

[37] T. Mizumachi and N. Tzvetkov, "Stability of the line soliton of the KP-II equation under periodic transverse perturbations," *Mathematische Annalen*, vol. 352, no. 3, pp. 659–690, 2012.

[38] T. Nishida, "A note on an existence of conditionally periodic oscillation in a one-dimensional anharmonic lattice," *Mem. Fac. Engrg. Kyoto Univ.*, vol. 33, pp. 27–34, 1971.

[39] D. E. Pelinovsky and G. Schneider, "The monoatomic FPU system as a limit of a diatomic FPU system," *Applied Mathematics Letters*, vol. 107, p. 106 387, 2020.

[40] J. Pöschel, "On Nekhoroshev Estimates for a Nonlinear Schrödinger Equation and a Theorem by Bambusi," *Nonlinearity*, vol. 12, pp. 1587–1600, 1999.

[41] B. Rink, "Proof of Nishida's conjecture on anharmonic lattices," *Communications in Mathematical Physics*, vol. 261, no. 3, pp. 613–627, 2006.

[42] J. C. Saut, "Remarks on the Generalized Kadomtsev-Petviashvili Equations," *Indiana University Mathematics Journal*, vol. 42, no. 3, pp. 1011–1026, 1993.

[43] G. Schneider and C. E. Wayne, "Counter-propagating Waves on Fluid Surfaces and the Continuum Limit of the Fermi-Pasta-Ulam Model," in *Equadiff 99*, World Scientific Publishing Company, Sep. 2000, pp. 390–404.

[44] ——, "The Long-Wave Limit for the Water Wave Problem I. The Case of Zero Surface Tension," *Communications on Pure and Applied Mathematics*, vol. 53, pp. 1475–1535, 2000.

[45] H. Takaoka, "Well-posedness for the Kadomtsev-Petviashvili II equation," *Advances in Differential Equations*, vol. 5, no. 10-12, pp. 1421–1443, 2000.

[46] N. Tzvetkov, *On the Cauchy problem for Kadomtsev-Petviashvili equation*, 7-8. 1999, vol. 24, pp. 1367–1397.

[47] S. Ukai, "Local solutions of the Kadomtsev-Petviashvili equation," *J. Fac. Sci. Univ. Tokyo*, vol. 36, pp. 193–209, 1989.

[48] N. J. Zabusky and M. D. Kruskal, "Interaction of "solitons" in a collisionless plasma and the recurrence of initial states," *Physical Review Letters*, vol. 15, no. 6, pp. 240–243, 1965.

[49] V. E. Zakharov, "Instability and nonlinear oscillations of solitons," *ZhETF Pisma Redaktsiiu*, vol. 22, p. 364, Oct. 1975.

[50] V. E. Zakharov and L. D. Faddeev, "The korteweg-de vries equation is a fully integrable hamiltonian system," *Funkcional. Anal. i Priložen.*, vol. 5, no. 4, pp. 18–27, 1971.