Stability analysis of embedded solitons in the generalized third-order nonlinear Schrödinger equation

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We study the generalized third-order nonlinear Schrödinger (NLS) equation which admits a one-parameter family of single-hump embedded solitons. Analyzing the spectrum of the linearization operator near the embedded soliton, we show that there exists a resonance pole in the left half-plane of the spectral parameter, which explains linear stability, rather than nonlinear semistability, of embedded solitons. Using exponentially weighted spaces, we approximate the resonance pole both analytically and numerically. We confirm in a near-integrable asymptotic limit that the resonance pole gives precisely the linear decay rate of parameters of the embedded soliton. Using conserved quantities, we qualitatively characterize the stable dynamics of embedded solitons. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1929587]

Embedded solitons are solitary waves that reside inside the continuous spectrum of a nonlinear-wave system. Even though these solitons are in resonance with the continuous spectrum and thus have a tendency to shed radiation, they can still be semistable, or more surprisingly, fully stable. Full stability of embedded solitons is understood poorly so far. This article addresses stability of embedded solitons in a generalized third-order NLS equation.

I. INTRODUCTION

Embedded solitons are nonlinear localized states residing inside the continuous spectrum of the underlying nonlinear wave system.[10,34] The existence of embedded solitons is a little surprising, as inside the continuous spectrum, nonlocal solitons with continuous-wave tails, rather than localized embedded solitons, are normally anticipated.[3] So far, embedded solitons have been reported in a wide array of nonlinear-wave systems such as the fifth-order Korteweg–de Vries (KdV) equations,[5,6,21,31] the fourth-order nonlinear Schrödinger (NLS) equations,[4,13] the coupled KdV equations,[16] the second-harmonic-generation system,[34] massive Thirring model,[7,8] three-wave interaction system,[9] the third-order NLS equations,[12,17,18,23,32,33] the coupled Bragg-grating system,[22] and the discrete fourth-order NLS equation.[14] Recently, moving discrete breathers were linked to embedded solitons as well.[28]

What makes embedded solitons even more surprising is their unusual dynamical properties. Because embedded solitons are in resonance with the continuous spectrum, one tends to expect that these solitons will continuously leak energy through continuous-wave radiation and thus break up eventually. This is not true, however, it has been shown that many embedded solitons are actually nonlinearly semistable, i.e., if the initial perturbation increases the energy of the embedded soliton, the soliton tends to persist; on the other hand, if the initial perturbation decreases the energy of the embedded soliton, the soliton disappears. This semistability holds not only for isolated embedded solitons,[27,30,31,34] but also for continuous families of embedded solitons.[33] Even more interestingly, it was discovered for the generalized third-order NLS equation[17,32] and the coupled Bragg-grating system[22] that embedded solitons can be fully stable as well, no matter what initial perturbations are imposed.

Full stability of embedded solitons is not yet well understood either mathematically or physically. Even though these solitons have a natural tendency to leak energy due to resonance with the continuous spectrum, they can still resist that tendency and remain robust. Motivated by numerical simulations by Gromov *et al.*,[17] Yang[32] constructed a soliton perturbation theory and analytically proved the full stability of embedded solitons in the asymptotic limit where the third-order NLS equation is close to the integrable Hirota equation.[20] This work leads to the following general question: *How do we explain the stability of embedded solitons within the framework of a general third-order NLS equation?*

In this paper, we study the stability of embedded solitons in the generalized third-order NLS equation where a continuous family of single-hump embedded solitons exists in an analytical form. First, we study the spectrum of the linearization operator near the embedded soliton and show that there exists a resonance pole in the left half-plane of the spectral parameter. Resonant poles correspond to spatially unbounded eigenfunctions, which become decaying in exponentially weighted spaces.[26] Such eigenfunctions represent radiative modes of the continuous spectrum that explain the exponential decay of perturbations of embedded solitons in time evolution.[11,24]

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We approximate the resonance pole analytically in the asymptotic limit where the third-order NLS equation is close to the Hirota equation. We also compute the resonance pole numerically for a general set of parameter values. We found that the asymptotic formulas and numerical values agree very well. In addition, the resonance pole gives precisely the linear decay rate of parameters of embedded solitons computed in Ref. 32 for the perturbed Hirota equation. Finally, using conserved quantities, we qualitatively derive the normal-form equations for nonlinear dynamics of small perturbations of embedded solitons. These dynamical equations establish the linear and nonlinear stability of embedded solitons within the framework of the general third-order NLS equation.

The paper is structured as follows. Section II presents transformation of the generalized third-order NLS equation to a normalized form, from which time-independent embedded solitons and the associated linearization operator are introduced. Section III describes the spectrum of the linearization operator, which includes the essential spectrum and the nontrivial kernel. The adjoint linearization operator is studied in Sec. IV. Exponentially weighted spaces are introduced in Sec. V. The resonant pole of the linearization operator is investigated in Sec. VI with perturbation methods and numerical computations. Reduced equations for stable dynamics of embedded solitons are derived from conserved quantities and studied in Sec. VII. Section VIII concludes the paper.

II. FORMULATION OF THE PROBLEM

We consider a generalized third-order NLS equation in the form,

$$iu_t + iu_{xxx} + \alpha|u|^2u + i\beta|u|^2u_x + i\gamma(|u|^2)_x u = 0, \tag{2.1}$$

where α,β,γ are real-valued parameters, and $u(x,t)$ is a complex-valued function. This equation has been used to model ultra-short pulses in optical fibers.[2,19] Normally, Eq. (2.1) also contains the second-order derivative term. However, once the third-order derivative term is included, the second-order derivative term can be removed by a gauge transformation.[29,32] When $\alpha=\gamma=0$, this equation is referred to as the Hirota equation or the complex modified KdV equation, which is integrable with the inverse scattering transform method.[1,20] The Hirota equation has a *two-parameter* family of embedded solitons, which are in resonance with dispersive waves of the linear third-order equation $u_t+u_{xxx}=0$. This family of embedded solitons is expressed explicitly as

$$u(x,t) = r\,\mathrm{sech}(r\kappa z)e^{ikx+i\lambda t}, \quad z = x - vt, \tag{2.2}$$

where (k,r) are arbitrary real parameters, $\kappa=\sqrt{\beta/6}$, and (v,λ) are expressed in terms of k, r, and κ as

$$\lambda = k(k^2 - 3\kappa^2r^2), \quad v = \kappa^2r^2 - 3k^2. \tag{2.3}$$

When α and γ are small compared to β, the family of embedded solitons (2.2) and (2.3) is destroyed by the oscillatory nondecaying tails at infinity,[12,32] unless the parameters k and κ are fixed as

$$k = -\frac{\alpha}{2\gamma}, \quad \kappa = \sqrt{\frac{\beta+2\gamma}{6}}, \tag{2.4}$$

while the parameter r remains arbitrary. This *one-parameter* family of embedded solitons (2.2)–(2.4) gives an exact solution of the third-order NLS Eq. (2.1) for arbitrary values of α, β, and γ as long as $\beta+2\gamma>0$.[18] It was shown with the soliton perturbation theory[32] that the one-parameter family of embedded solitons with $0<|\alpha|+|\gamma|\ll\beta$ is linearly and nonlinearly stable in the time evolution of arbitrary localized initial data. These analytical results explained previous numerical computations of the generalized third-order NLS Eq. (2.1).[17]

We note that the family of embedded solitons (2.2) does not exist in the third-order NLS Eq. (2.1) with $\beta=\gamma=0$.[5,15] Although multihumped embedded solitons still exist in this case,[5,33] we focus on the single-humped embedded solitons and consider the general third-order NLS Eq. (2.1) under the constraint $|\beta|+|\gamma|\neq 0$.

For the convenience of analysis, we first transform the third-order NLS Eq. (2.1) with the following substitution:

$$u(x,t) = rU(X,T)e^{ikx+i\lambda t}, \quad X = r\kappa(x-vt), \quad T = r^3\kappa^3t, \tag{2.5}$$

where (λ,v) are defined by (2.3), (k,κ) defined by (2.4), and r is arbitrary. The function $U(x,t)$ solves the normalized third-order NLS equation:

$$iU_T + i(U_{XXX} - U_X + 6|U|^2U_X) + \mu(U_{XX} - U + 2|U|^2U) = i\nu(U_X\bar{U} - U\bar{U}_X)U, \tag{2.6}$$

where

$$\mu = \frac{3\alpha}{2\gamma\kappa r}, \quad \nu = \frac{6\gamma}{\beta+2\gamma}. \tag{2.7}$$

When (α,β,γ) are fixed parameters of the general Eq. (2.1), the normalized NLS Eq. (2.6) has the free parameter μ and fixed parameter ν.

When $\nu=0$, the normalized Eq. (2.6) has the Hamiltonian structure $i\dot{U}=\delta_{\bar{U}}H$ associated with the Hamiltonian:

$$H = \frac{i}{2}\int_{-\infty}^{\infty}(\bar{U}_XU_{XX} - U_X\bar{U}_{XX} + \bar{U}U_X - U\bar{U}_X + 3|U|^2(\bar{U}_XU - \bar{U}U_x))dX + \mu\int_{-\infty}^{\infty}(|U_X|^2 - |U|^4 + |U|^2)dX. \tag{2.8}$$

When $\nu\neq 0$, Eq. (2.6) does not have a Hamiltonian structure, even though it still has two conserved quantities (see Sec. VII).

Under the normalization (2.5), the embedded solitons (2.2) reduce to the time-independent localized solution

$$U=\Phi(X)=\operatorname{sech} X. \tag{2.9}$$

Due to gauge and translation invariances, the function $\Phi(X)$ generates a two-parameter orbit of time-independent solutions: $U=\Phi(X-X_0)e^{i\theta_0}$, where X_0 and θ_0 are real constants. Now we consider the linearization of the nonlinear Eq. (2.6) around the embedded soliton (2.9). For this purpose, we write

$$U=\Phi(X)+(V(X)+iW(X))e^{\lambda t}+(\bar{V}(X)+i\bar{W}(X))e^{\bar{\lambda} t}, \tag{2.10}$$

where V and W are infinitesimal perturbations, and the overbar represents complex conjugation. Neglecting the quadratic terms in V and W, we get the linear eigenvalue problem,

$$\mathcal{L}_\nu \mathbf{v}=\lambda\mathbf{v}, \quad \mathbf{v}=(V,W)^T, \tag{2.11}$$

where the linearization operator $\mathcal{L}_\nu=\mathcal{L}_0+2\nu\mathcal{L}_1$ with

$$\mathcal{L}_0=\begin{bmatrix} -\dfrac{d^3}{dX^3}+\dfrac{d}{dX}-6\Phi^2(X)\dfrac{d}{dX}-12\Phi(X)\Phi'(X) & \mu\left(-\dfrac{d^2}{dX^2}+1-2\Phi^2(X)\right) \\ \mu\left(\dfrac{d^2}{dX^2}-1+6\Phi^2(X)\right) & -\dfrac{d^3}{dX^3}+\dfrac{d}{dX}-6\Phi^2(X)\dfrac{d}{dX} \end{bmatrix} \tag{2.12}$$

$$\mathcal{L}_1=\begin{bmatrix} 0 & 0 \\ 0 & \Phi^2(X)\dfrac{d}{dX}-\Phi(X)\Phi'(X) \end{bmatrix}, \tag{2.13}$$

and the superscript T represents the transpose of a matrix. When $\mu=0$, the operator $\mathcal{L}_\nu$ decouples as

$$\mathcal{L}_\nu=\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \tag{2.14}$$

where

$$M_1=-\frac{d^3}{dX^3}+\frac{d}{dX}-6\Phi^2(X)\frac{d}{dX}-12\Phi(X)\Phi'(X),$$

$$M_2=-\frac{d^3}{dX^3}+\frac{d}{dX}-6\Phi^2(X)\frac{d}{dX}+2\nu\left(\Phi^2(X)\frac{d}{dX}-\Phi(X)\Phi'(X)\right).$$

We shall study the spectrum of the linearization operator $\mathcal{L}_\nu$ in the $L^2(\mathbb{R},\mathbb{C}^2)$ space, as well as in the exponentially weighted space:

$$L_a^2=\left\{\mathbf{v}(X):e^{aX}\mathbf{v}\in L^2(\mathbb{R},\mathbb{C}^2)\right\}, \quad a>0. \tag{2.15}$$

Here $\mathbb{R}$ and $\mathbb{C}$ are the real and complex sets, respectively, and the functional space $L^2(\mathbb{R},\mathbb{C}^2)$ is the space of square-integrable functions $\mathbf{f}(X)$ from $\mathbb{R}$ to $\mathbb{C}^2$. The exponentially weighted space has been used in the linearized KdV equation by Pego and Weinstein[26] in the proof of asymptotical orbital stability of KdV solitons. Here we use the space L_a^2 in order to shift the essential spectrum away from the origin $\lambda=0$, such that the essential spectrum is no longer in resonance with the nonempty kernel of $\mathcal{L}_\nu$. The resonance between the essential spectrum and the kernel of $\mathcal{L}_\nu$ in L^2 is the standard feature of KdV solitons[25] and also embedded solitons in dispersive wave equations.[27,30]

In the Hamiltonian case $\nu=0$, the generalized kernel of $\mathcal{L}_0$ in $L^2(\mathbb{R},\mathbb{C}^2)$ is spanned by two eigenfunctions and two generalized eigenfunctions. In what follows, we will show that when $\nu\neq 0$, one of the two generalized eigenfunctions of $\mathcal{L}_\nu$ is no longer in $L^2(\mathbb{R},\mathbb{C}^2)$. Using the exponentially weighted space L_a^2, we will compute for $\nu\neq 0$ the splitting of a double zero eigenvalue into a simple zero eigenvalue and a nonzero negative eigenvalue, which results in the linear stability of embedded solitons. This new discrete eigenvalue in L_a^2 corresponds to a resonant pole in L^2 with exponentially growing eigenfunctions in space. Similar results on appearance of resonant poles and their role in stability of solitary waves were discovered with the Evans function in the exponentially weighted spaces for the Kawahara equation,[11,24] which is the KdV equation with a singular dissipative perturbation.

III. SPECTRUM OF OPERATOR $\mathcal{L}_\nu$

The essential spectrum of $\mathcal{L}_\nu$ is defined in the limit $|X|\to\infty$, where $\Phi(X)$ decays to zero. In this limit, substituting the Fourier modes

$$\mathbf{v}_\pm\to\begin{pmatrix}1\\ \pm i\end{pmatrix}e^{iKX}, \quad \text{as} \quad |X|\to\infty, \tag{3.1}$$

into the eigenvalue problem (2.11) with $K\in\mathbb{R}$, we find that the essential spectrum is located on the two curves:

$$S^\pm=\left\{\lambda\in\mathbb{C}:\lambda=i(1+K^2)(K\pm\mu),K\in\mathbb{R}\right\}. \tag{3.2}$$

The two branches of the continuous spectrum overlap on the entire $i\mathbb{R}$ axis. At $\lambda=0$, the two branches are in resonance with the nonempty kernel of $\mathcal{L}_\nu$.

The kernel of $\mathcal{L}_\nu$ is at least two-dimensional, since it contains the two eigenfunctions

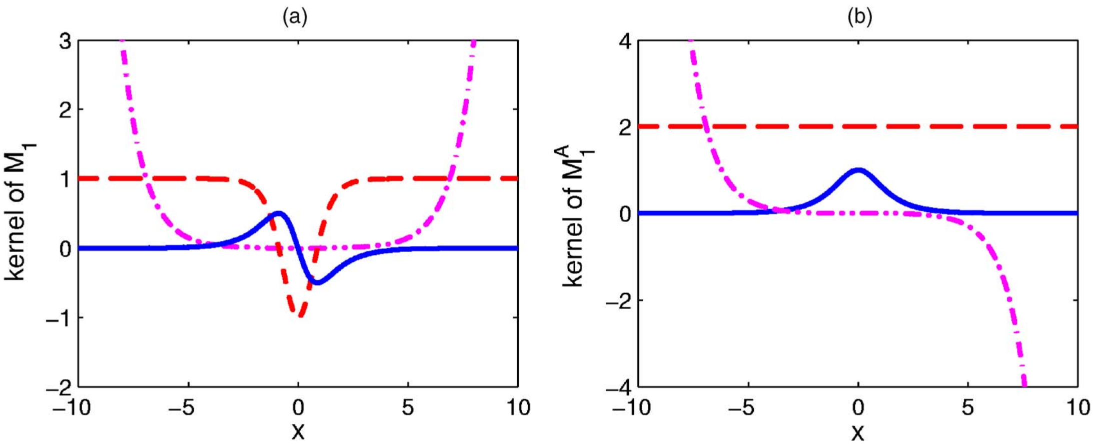

FIG. 1. Kernels of operators M_1 and M_1^A, shown in (a) and (b), respectively.

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ \Phi(X) \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} \Phi'(X) \\ 0 \end{pmatrix}, \tag{3.3}$$

which are generated by the phase and space translations of the time independent solutions $U=\Phi(X-\xi_0)e^{i\theta_0}$. The kernel of $\mathcal{L}_\nu$ also contains other functions which are not in L^2. Since these functions, especially those which are bounded at infinity, play an important role in the stability analysis of embedded solitons,[27,30] we will study them in more detail here.

When $\mu=0$, the operator $\mathcal{L}_\nu$ is decoupled as (2.14). The three functions in the kernel of M_1 are plotted in Fig. 1(a) (note that operator M_1 is independent of ν). The localized function is $\Phi'(X)$, the bounded function is $1-2\Phi^2(X)$, while the third function is unbounded. The other operator M_2 depends on ν and the functions in its kernel are plotted in Figs. 2(a) and 2(c) at $\nu=0$ and $\nu=1$, respectively. When $\nu=0$, the localized function is $\Phi(X)$, the bounded solution is a constant, and the third solution is unbounded. When $\nu=1$, the localized function $\Phi(X)$ persists, but the bounded solution disappears. The same property holds for all $\nu\neq 0$ except for $\nu=1.5$, when the bounded solution appears again. The case $\mu=0$ and $\nu=1.5$ is also special as it corresponds to another integrable equation called the Sasa–Satsuma equation.[29]

We have numerically investigated functions in the kernel of $\mathcal{L}_\nu$ in the general case of $\mu\neq 0$ and $\nu\neq 0$. For a general set

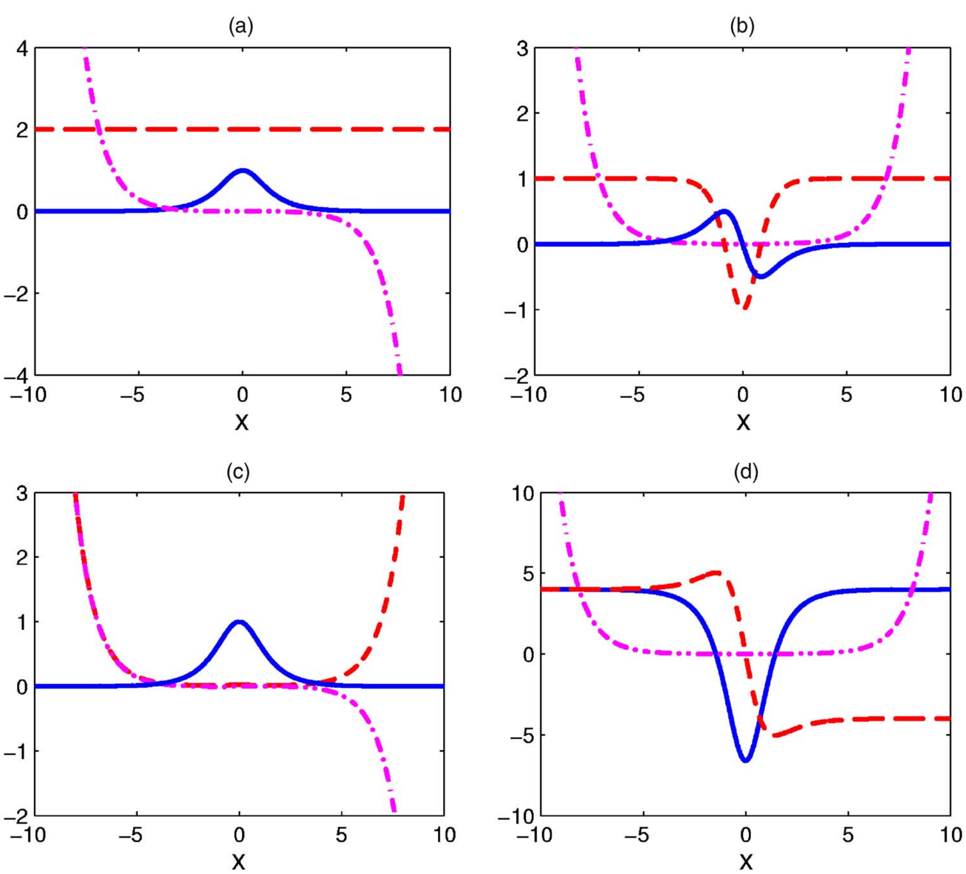

FIG. 2. (a) and (c) Kernels of the operator M_2 with $\nu=0$ and $\nu=1$, respectively; (b) and (d) kernels of the operator M_2^A with $\nu=0$ and $\nu=1$, respectively.

of parameters, the kernel of $\mathcal{L}_\nu$ contains two decaying eigenfunctions (3.3), one bounded eigenfunction with oscillatory tails at infinity $\mathbf{v}=(v_1,v_2)^T$, where $v_1(X)$ is even and $v_2(X)$ is odd, and three unbounded solutions.

Next, we discuss the generalized eigenfunctions of operator $\mathcal{L}_\nu$ at zero eigenvalue, which are solutions of the inhomogeneous equations:

$$\mathcal{L}_\nu \mathbf{w}_1 = \mu \mathbf{v}_1 - \mathbf{v}_2, \quad \mathcal{L}_\nu \mathbf{w}_2 = \mathbf{v}_1 + \mu \mathbf{v}_2. \tag{3.4}$$

In the integrable case where $\nu=0$, the generalized eigenfunctions are given explicitly as

$$\mathbf{w}_1^{(0)} = \frac{1}{2}\begin{pmatrix} \Phi(X) + X\Phi'(X) \\ 0 \end{pmatrix}, \quad \mathbf{w}_2^{(0)} = -\frac{1}{2}\begin{pmatrix} 0 \\ X\Phi(X) \end{pmatrix}, \tag{3.5}$$

which motivates the use of linear combinations of $\{\mathbf{v}_1,\mathbf{v}_2\}$ in the right-hand-sides of Eq. (3.4). The two solutions (3.5) are related to the derivatives of the general two-parameter family (2.2) with respect to parameters r and k, respectively. The first solution $\mathbf{w}_1^{(0)}(X)$ persists in the general case $\nu\neq 0$, such that $\mathbf{w}_1=\mathbf{w}_1^{(0)}(X)$, because parameter r is free. The second solution $\mathbf{w}_2^{(0)}(X)$, however, does not persist, since parameter k is fixed when $\nu\neq 0$. Due to the resonance between the essential spectrum and the kernel of $\mathcal{L}_\nu$, the second solution $\mathbf{w}_2(X)$ generally contains oscillatory tails with nonzero amplitudes at infinity. By preserving the symmetry of the solution $\mathbf{w}_2^{(0)}(X)$ for $\nu\neq 0$, we represent the solution of the second equation in (3.4) as follows:

$$\mathbf{w}_2 = \begin{pmatrix} \xi(X) \\ \eta(X) \end{pmatrix}, \quad \xi(-X)=\xi(X), \eta(-X) = -\eta(X). \tag{3.6}$$

When $\nu=0$, we have $\xi=0$ and $\eta=-\frac{1}{2}X\Phi(X)$. We can show numerically that there exists a solution of the second equation in (3.4) in the form (3.6), where $\xi(X)$ and $\eta(X)$ are not decaying at infinity for a general value of $\nu\neq 0$.

IV. THE ADJOINT OPERATOR

The adjoint operator $\mathcal{L}_\nu^A$ takes the explicit form $\mathcal{L}_\nu^A=\mathcal{L}_0^A+2\nu\mathcal{L}_1^A$, where

$$\mathcal{L}_0^A = \begin{bmatrix} \frac{d^3}{dX^3} - \frac{d}{dX} + 6\Phi^2(X)\frac{d}{dX} & \mu\left(\frac{d^2}{dX^2} - 1 + 6\Phi^2(X)\right) \\ \mu\left(-\frac{d^2}{dX^2} + 1 - 2\Phi^2(X)\right) & \frac{d^3}{dX^3} - \frac{d}{dX} + 6\Phi^2(X)\frac{d}{dX} + 12\Phi(X)\Phi'(X) \end{bmatrix} \tag{4.1}$$

and

$$\mathcal{L}_1^A = -\begin{bmatrix} 0 & 0 \\ 0 & \Phi^2(X)\frac{d}{dX} + 3\Phi(X)\Phi'(X) \end{bmatrix}. \tag{4.2}$$

In the integrable case where $\nu=0$, the linearization operator $\mathcal{L}_0$ takes the form $\mathcal{L}_0=\mathcal{J}\mathcal{H}_0$, with

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\mathcal{H}_0 = \begin{bmatrix} \mu\left(-\frac{d^2}{dX^2} + 1 - 2\Phi^2(X)\right) & \frac{d^3}{dX^3} - \frac{d}{dX} + 6\Phi^2(X)\frac{d}{dX} + 12\Phi(X)\Phi'(X) \\ -\frac{d^3}{dX^3} + \frac{d}{dX} - 6\Phi^2(X)\frac{d}{dX} & \mu\left(-\frac{d^2}{dX^2} + 1 - 6\Phi^2(X)\right) \end{bmatrix}.$$

As a result, the eigenfunctions of the adjoint operator $\mathcal{L}_0^A=-\mathcal{H}_0\mathcal{J}$ are related to eigenfunctions of the operator $\mathcal{L}_0$. The kernel of the adjoint operator $\mathcal{L}_0^A$ includes two eigenfunctions:

$$\tilde{\mathbf{v}}_1^{(0)} = \begin{pmatrix} \Phi(X) \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{v}}_2^{(0)} = \begin{pmatrix} 0 \\ \Phi'(X) \end{pmatrix}. \tag{4.3}$$

It is seen from the explicit formulas (4.1) and (4.2) that the first eigenfunction $\tilde{\mathbf{v}}_1^{(0)}(X)$ persists for any $\nu\neq 0$, such that $\mathbf{v}_1^A=\tilde{\mathbf{v}}_1^{(0)}(X)$. The second eigenfunction $\tilde{\mathbf{v}}_2^{(0)}(X)$ is, however, destroyed in the parameter continuation, such that the eigenfunction $\mathbf{v}_2^A$ needs to be studied separately.

We use the fact that the adjoint equation $\mathcal{L}_\nu^A\mathbf{v}^A=0$ admits a reduction. Looking for solutions in the form

$$\mathbf{v}^A = \begin{pmatrix} \mu\zeta(X) \\ -\zeta'(X) \end{pmatrix}, \tag{4.4}$$

we find that $\zeta(X)$ satisfies a fourth-order scalar equation:

$$\mathcal{S}_\nu \zeta = 0, \tag{4.5}$$

where $\mathcal{S}_\nu = \mathcal{S}_0 - 2\nu\mathcal{S}_1$, and

$$\mathcal{S}_0 = \frac{d^4}{dX^4} - \frac{d^2}{dX^2} + 6\Phi^2(X)\frac{d^2}{dX^2} + 12\Phi(X)\Phi'(X)\frac{d}{dX} - \mu^2\left(-\frac{d^2}{dX^2} + 1 - 2\Phi^2(X)\right), \tag{4.6}$$

$$\mathcal{S}_1 = \Phi^2(X)\frac{d^2}{dX^2} + 3\Phi(X)\Phi'(X)\frac{d}{dX}. \tag{4.7}$$

Thus we can use the fourth-order Eq. (4.5) to obtain four of the six functions in the kernel of the adjoint operator $\mathcal{L}_\nu^A$. This fact is convenient both for analysis and numerical computations. The remaining two solutions are a decaying solution $\mathbf{v}_1^A = \tilde{\mathbf{v}}_1^{(0)}(X)$ as given in Eq. (4.3), and an unbounded solution which we are not interested in. When $\nu=0$, the homogeneous Eq. (4.5) has a decaying solution $\zeta = \Phi(X)$, which corresponds to $\tilde{\mathbf{v}}_2^{(0)}(X)$. However, this decaying solution does not persist in the general nonintegrable case $\nu \neq 0$ due to a resonance with the essential spectrum of $\mathcal{L}_\nu^A$ at $\lambda=0$. The second eigenfunction $\mathbf{v}_2^A(X)$ generally contains oscillatory tails at infinity.

Eigenfunctions in the kernel of $\mathcal{L}_\nu^A$ can be studied numerically for $\nu \neq 0$. When $\mu=0$, the operator $\mathcal{L}_\nu^A$ is decomposed as follows:

$$\mathcal{L}_\nu^A = \begin{bmatrix} M_1^A & 0 \\ 0 & M_2^A \end{bmatrix}, \tag{4.8}$$

where

$$M_1^A = \frac{d^3}{dX^3} - \frac{d}{dX} + 6\Phi^2(X)\frac{d}{dX},$$

$$M_2^A = \frac{d^3}{dX^3} - \frac{d}{dX} + 6\Phi^2(X)\frac{d}{dX} + 12\Phi(X)\Phi'(X) - 2\nu\left(\Phi^2(X)\frac{d}{dX} + 3\Phi(X)\Phi'(X)\right).$$

Functions in the kernel of M_1^A are plotted in Fig. 1(b). The decaying solution is $\Phi(X)$ which corresponds to the eigenfunction $\mathbf{v}_1^A(X)$, the bounded solution is a constant, and the third solution is unbounded. The other operator M_2^A is dependent on ν and the functions in its kernel are plotted in Figs. 2(b) and 2(d) for $\nu=0$ and $\nu=1$, respectively. When $\nu=0$, the decaying solution is $\Phi'(X)$ which corresponds to $\tilde{\mathbf{v}}_2^{(0)}(X)$, the bounded solution is $1-2\Phi^2(X)$, while the third solution is unbounded. When $\nu=1$, the decaying solution disappears and two bounded solutions arise, where one solution is symmetric and the other one is anti-symmetric. This property holds for all $\nu \neq 0$ except for $\nu=1.5$, when the decaying solution appears again, as the case $\mu=0$ and $\nu=1.5$ corresponds to the integrable Sasa–Satsuma equation.[29]

In the general case of $\mu \neq 0$ and $\nu \neq 0$, we have shown numerically that the kernel of $\mathcal{L}_\nu^A$ includes one decaying eigenfunction $\mathbf{v}_1^A$, three bounded and two unbounded solutions. The bounded solutions have oscillatory tails at infinity, and can be represented as $\mathbf{v}^A = (v_1^A, v_2^A)$, where $v_1^A(X)$ is even for two solutions and odd for the third solution, while $v_2^A(X)$ has the opposite symmetry.

We have shown above that the decaying eigenfunction $\mathbf{v}_1^A$ of $\mathcal{L}_\nu^A$ persists and remains in L^2. We would expect that the eigenfunction $\mathbf{v}_1^A$ has a decaying generalized eigenfunction $\mathbf{w}_1^A$ such that $\mathcal{L}_\nu^A \mathbf{w}_1^A = \mathbf{v}_1^A$. This is certainly true when $\mu=0$ where the decomposition (4.8) holds. In this case, the problem reduces to the scalar equation $M_1^A w_1^A = \Phi$ which does have a localized solution $w_1^A = \frac{1}{2}X\Phi(X)$. In the general case $\mu \neq 0$, the existence of a decaying generalized eigenfunction $\mathbf{w}_1^A$ is yet to be confirmed.

In order to study persistence of the generalized eigenvector $\mathbf{w}_2$ in the generalized kernel of $\mathcal{L}_\nu$, one should define the inner product in $L^2(\mathbb{R},\mathbb{C}^2)$:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{\mathbb{R}} (f_1 \bar{g}_1 + f_2 \bar{g}_2) dX. \tag{4.9}$$

Assume that the eigenfunctions $\mathbf{w}_2(X)$ and $\mathbf{v}_2^A(X)$ are represented by (3.6) and (4.4) with nondecaying eigenfunctions $\xi(X)$, $\eta(X)$, and $\zeta(X)$. If $\zeta(X)$ is even function of X (by parameter continuation from $\zeta=\Phi(X)$ at $\nu=0$), the inner product $\langle \mathbf{v}_2^A, \mathbf{w}_2 \rangle$ generally diverges, which indicates that the generalized eigenfunction $\mathbf{w}_2$ should not be considered in space L^2. Therefore, we study the generalized eigenfunction in the exponentially weighted space L_a^2 for $a>0$.

V. EIGENFUNCTIONS IN EXPONENTIALLY WEIGHTED SPACES

The eigenfunctions $\mathbf{w}_2$ and $\mathbf{v}_2^A$, which are decaying for $\nu=0$, do not persist in the L^2 space when $\nu \neq 0$, since they contain oscillatory tails at infinity. Therefore, here we consider generalized kernels of $\mathcal{L}_\nu$ and $\mathcal{L}_\nu^A$ in the exponentially weighted spaces. The exponentially weighted space shifts the essential spectrum from the origin, where the generalized kernels of linearization operators reside.[25,26] In the space L_a^2, the essential spectrum of $\mathcal{L}_\nu$ transforms as follows:

$$S_a^{\pm} = \{\lambda \in \mathbb{C} : \lambda = \lambda_{\pm}(K) = (1-(iK-a)^2)(iK \pm i\mu - a), K \in \mathbb{R}\}. \tag{5.1}$$

The curves $S_a^{\pm}$ are shown in Figs. 3(a) and 3(b) for $\mu=1$ and $\mu=2$, where $a=0.1$ (solid curve) and $a=0.3$ (dashed curve). It is easy to check that $\mathrm{Re}\,\lambda_{\pm}(K)$ has maximum at $K = \mp\mu/3$, $a>0$, where

$$\mathrm{Re}\,\lambda_{\pm}(K)\Big|_{K=\mp\frac{\mu}{3}} = a\left(a^2 - 1 + \frac{\mu^2}{3}\right).$$

Therefore, the curves $S_a^{\pm}$ are located in the left half-plane of λ if and only if $|\mu| < \sqrt{3}$ and $0 < a < \sqrt{1-\mu^2/3}$. The transition from $|\mu| < \sqrt{3}$ to $|\mu| > \sqrt{3}$ is clear from Figs. 3(a) and 3(b).

First we consider the generalized kernel of $\mathcal{L}_\nu$. We have shown that the eigenfunctions $\mathbf{v}_1$, $\mathbf{v}_2$, and generalized eigenfunction $\mathbf{w}_1$ as defined in (3.3) and (3.4) persist in L^2, but $\mathbf{w}_2$ does not. Taking into account the homogeneous solutions of the operator $\mathcal{L}_\nu$ at infinity for $\lambda=0$, we consider the Sommerfeld radiation boundary condition for $\mathbf{w}_2(X)$:

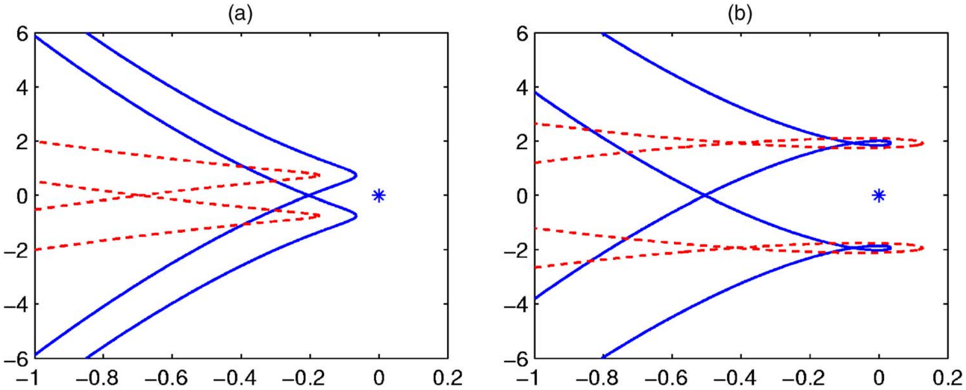

FIG. 3. Essential spectrum and kernel of $\mathcal{L}_\nu$ on the complex plane λ in exponentially weighted space L_a^2 for (a) $\mu=1$ and (b) $\mu=2$, where $a=0.1$ (solid curve) and $a=0.3$ (dashed curve).

$$\mathbf{w}(X) \to \mathbf{a}_1^+ e^{-X}, \quad \text{as} \quad X \to +\infty,$$

$$\mathbf{w}(X) \to \mathbf{a}_1^- e^{X} + \mathbf{a}_2^+ e^{i\mu X} + \mathbf{a}_2^- e^{-i\mu X}, \quad \text{as} \quad X \to -\infty, \tag{5.2}$$

where $\mathbf{a}_{1,2}^{\pm}$ are some vectors. If $\mathbf{w}_2(X)$ satisfies (5.2), then $\mathbf{w}_2$ belongs to the exponentially weighted space L_a^2 with $0<a<1$. Recall that $\mathbf{w}_2(X)$ satisfies the second inhomogeneous equation in (3.4), which admits an inhomogeneous solution in the form (3.6). The components (ξ,η) are oscillatory at infinity when $\nu\neq 0$, with $\xi(X)$ even and $\eta(X)$ odd. On the other hand, it follows from our numerical results that the homogeneous part of (3.4) also admits a single bounded solution with the same symmetry. Since two oscillatory functions $e^{\pm i\mu X}$ as $X\to+\infty$ cannot be removed with a single bounded homogeneous solution, oscillatory tails in $\mathbf{w}_2(X)$ always appear in both infinities and $\mathbf{w}_2(X)$ cannot satisfy the Sommerfeld radiation boundary condition (5.2). This result indicates that the double zero eigenvalue associated with the subspace $\{\mathbf{v}_1+\mu\mathbf{v}_2,\mathbf{w}_2^{(0)}\}$ for $\nu=0$ splits in the weighted space L_a^2 as $\nu\neq 0$.

Next we consider the kernel of $\mathcal{L}_\nu^A$. We have shown that the eigenfunction $\mathbf{v}_1^A$ persists in L^2 space, but $\mathbf{v}_2^A$ does not. Since the eigenfunctions of $\mathcal{L}_\nu$ are considered in the exponentially weighted space L_a^2, the adjoint eigenfunctions of $\mathcal{L}_\nu^A$ should be considered in the adjoint weighted space L_{-a}^2 so that the inner product (4.9) between eigenfunctions and adjoint eigenfunctions is independent of the artificial parameter a. Taking into account the homogeneous solutions of the operator $\mathcal{L}_\nu^A$ at infinity for $\lambda=0$, we consider the adjoint Sommerfeld radiation boundary condition for $\mathbf{v}_2^A(X)$:

$$\mathbf{v}^A(X) \to \mathbf{b}_1^- e^{X}, \quad \text{as} \quad X \to -\infty,$$

$$\mathbf{v}^A(X) \to \mathbf{b}_1^+ e^{-X} + \mathbf{b}_2^+ e^{i\mu X} + \mathbf{b}_2^- e^{-i\mu X}, \quad \text{as} \quad X \to +\infty, \tag{5.3}$$

where $\mathbf{b}_{1,2}^{\pm}$ are some vectors. If $\mathbf{v}_2^A(X)$ satisfies (5.3), then $\mathbf{v}_2^A$ belongs to the exponentially weighted space L_{-a}^2 with $0<a<1$. It follows from our numerical results that the kernel of $\mathcal{L}_\nu^A$ contains three bounded solutions for $\nu\neq 0$, two of which have the same symmetry, while the third has the opposite symmetry. An appropriate linear combination of these bounded solutions can satisfy the adjoint Sommerfeld conditions (5.3). For instance, in the case $\mu=0$, by subtracting the two bounded solutions in the kernel of operator M_2^A [see Fig. 2(d)], the resulting eigenfunction $\mathbf{v}_2^A$ satisfies the condition (5.3). Therefore, the adjoint eigenfunction $\mathbf{v}_2(X)$ persists in the exponentially weighted space L_{-a}^2 where $0<a<1$.

VI. SPLITTING OF THE ZERO EIGENVALUE FOR SMALL $\nu\neq 0$

When $\nu=0$, the generalized kernel of $\mathcal{L}_0$ contains two functions $\{\mathbf{v}_1+\mu\mathbf{v}_2,\mathbf{w}_2^{(0)}\}$, where $\mathbf{v}_{1,2}(X)$ and $\mathbf{w}_2^{(0)}(X)$ are given by (3.3) and (3.5). These functions correspond to a double zero eigenvalue in the operator $\mathcal{L}_0$. When $\nu\neq 0$, the generalized eigenfunction $\mathbf{w}_2^{(0)}(X)$ does not persist in the L_a^2 space ($a>0$). Here we show that this double zero eigenvalue splits into a simple zero eigenvalue and a real negative eigenvalue $\lambda=\lambda_0(\nu)<0$ with $\lambda_0(0)=0$ in the L_a^2 space. Our analysis is based on a perturbation theory applied for $\nu\ll 1$.

When $\nu\ll 1$, we expand the solution to equation $\mathcal{L}_\nu\mathbf{v}=\lambda_{\mathbf{v}}$ into a perturbation series in powers of ν:

$$\mathbf{v} = \mathbf{v}^{(0)}(X) + \sum_{k=1}^{\infty}(2\nu)^k \mathbf{v}^{(k)}(X), \quad \lambda = \sum_{k=1}^{\infty}(2\nu)^k\lambda_k, \quad \mathbf{v}\in L_a^2, \tag{6.1}$$

where $\mathbf{v}^{(0)}=\mathbf{v}_1(X)+\mu\mathbf{v}_2(X)$, and $\mathbf{v}^{(k)}(X)$, $k=1, 2, 3, 4$ solve the inhomogeneous equations:

$$\mathcal{L}_0\mathbf{v}^{(1)} = -\mathcal{L}_1\mathbf{v}^{(0)} + \lambda_1\mathbf{v}^{(0)}, \tag{6.2}$$

$$\mathcal{L}_0\mathbf{v}^{(2)} = -\mathcal{L}_1\mathbf{v}^{(1)} + \lambda_1\mathbf{v}^{(1)} + \lambda_2\mathbf{v}^{(0)}, \tag{6.3}$$

$$\mathcal{L}_0\mathbf{v}^{(3)} = -\mathcal{L}_1\mathbf{v}^{(2)} + \lambda_1\mathbf{v}^{(2)} + \lambda_2\mathbf{v}^{(1)} + \lambda_3\mathbf{v}^{(0)}, \tag{6.4}$$

$$\mathcal{L}_0\mathbf{v}^{(4)} = -\mathcal{L}_1\mathbf{v}^{(3)} + \lambda_1\mathbf{v}^{(3)} + \lambda_2\mathbf{v}^{(2)} + \lambda_3\mathbf{v}^{(1)} + \lambda_4\mathbf{v}^{(0)}. \tag{6.5}$$

Decaying solutions of the inhomogeneous Eqs. (6.2)–(6.5) exist in L_a^2 only if the right-hand-side functions are orthogonal to eigenfunctions in the kernel of $\mathcal{L}_0^A$, which are vectors $(\Phi,0)^T$ and $(0,\Phi')^T$. The orthogonality with respect to $(\Phi,0)^T$ is always satisfied for the series (6.1), thus we consider only the orthogonality with respect to $(0,\Phi')^T$. Since $\mathbf{v}^{(0)}$ can always be added to $\mathbf{v}^{(k)}$ for $k\geq 1$ but it coincides with the leading-order term of (6.1), we assume that the solution $\mathbf{v}^{(k)}$ for $k\geq 1$ does not include the homogeneous solution $\mathbf{v}^{(0)}$.

It is remarkable that $\mathcal{L}_1\mathbf{v}^{(0)}=\mathbf{0}$, such that the orthogonality condition is always satisfied for $\mathbf{v}^{(1)}(X)$, with the exact solution:

$$\mathbf{v}^{(1)} = \lambda_1 \mathbf{w}_2^{(0)}(X), \tag{6.6}$$

where

$$\mathbf{w}_2^{(0)} = \begin{pmatrix} 0 \\ \eta_0(X) \end{pmatrix}, \quad \eta_0 = -\tfrac{1}{2} X\Phi(X). \tag{6.7}$$

The orthogonality condition for $\mathbf{v}^{(2)}(X)$ takes the form:

$$\lambda_1^2(\Phi', \eta_0) + \lambda_1(\Phi^2\Phi'' + 3\Phi(\Phi')^2, \eta_0) = 0. \tag{6.8}$$

Due to the symmetry of $\Phi(X)$, the condition is satisfied if and only if $\lambda_1=0$, such that

$$\mathbf{v}^{(1)} = \mathbf{0}, \quad \mathbf{v}^{(2)} = \lambda_2 \mathbf{w}_2^{(0)}(X). \tag{6.9}$$

The orthogonality condition for $\mathbf{v}^{(3)}(X)$ is empty, such that the bounded solution exists in the form:

$$\mathbf{v}^{(3)} = \lambda_2 \begin{pmatrix} \xi_1(X) \\ \eta_1(X) \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ \eta_0(X) \end{pmatrix}. \tag{6.10}$$

The vector $(\xi_1, \eta_1)^T$ is defined by the inhomogeneous Eq. (6.4). It can be shown explicitly that the problem (6.4) is simplified with the solution,

$$\xi_1 = \frac{1}{\mu}\left(\eta_1'(X) + \frac{1}{8}\Phi(X) \right), \tag{6.11}$$

where $\eta_1(X)$ solves the scalar inhomogeneous equation:

$$\mathcal{S}_0 \eta_1 = -\tfrac{3}{2}\Phi^2(X)\Phi'(X), \tag{6.12}$$

and $\mathcal{S}_0$ is defined by (4.6). It is understood that the solution $\eta_1(X)$ of the scalar Eq. (6.12) satisfies the Sommerfeld radiation conditions (5.2), such that $\lim_{X\to+\infty}\eta_1(X)=0$.

The orthogonality condition for $\mathbf{v}^{(4)}(X)$ gives finally a nontrivial equation:

$$\lambda_2^2(\Phi', \eta_0) + \lambda_2(\Phi^2\Phi'' + 3\Phi(\Phi')^2, \eta_1) = 0, \tag{6.13}$$

which admits two roots for λ_2. The zero root corresponds to the eigenfunction $\mathbf{v}^{(0)}=\mathbf{v}_1(X)+\mu\mathbf{v}_2(X)$, while the nonzero root corresponds to the new eigenfunction $\mathbf{v}(X)$, defined by the perturbation series expansion (6.1) for $\lambda=\lambda_0(\nu)\neq 0$. At the leading order, the nonzero eigenvalue $\lambda=\lambda_0(\nu)$ is computed as:

$$\lambda_0(\nu) = -8\nu^2(\Phi^2\Phi'' + 3\Phi(\Phi')^2, \eta_1) + \mathrm{O}(\nu^3), \tag{6.14}$$

where we have used the numerical value $(\Phi', \eta_0)=1/2$.

We can further show that the splitting of the double zero eigenvalue in L_a^2 occurs exactly when the adjoint eigenfunction $\mathbf{v}_2^A \in L_{-a}^2$ becomes nonorthogonal to the eigenfunction $\mathbf{v}_1+\mu\mathbf{v}_2 \in L_a^2$. The inner product between these two functions is

$$(\mathbf{v}_2^A, \mathbf{v}_1 + \mu\mathbf{v}_2) = (1+\mu^2)(\Phi', \zeta), \tag{6.15}$$

where the solution form (4.4) for $\mathbf{v}_2$ is utilized. To further calculate this inner product, we note from (4.6) that $\mathcal{S}_0^A=\mathcal{S}_0$ and

$$\mathcal{S}_0(X\Phi(X)) = 2(1+\mu^2)\Phi'(X), \tag{6.16}$$

such that

$$(1+\mu^2)(\Phi', \zeta) = \tfrac{1}{2}(X\Phi, \mathcal{S}_0\zeta) = \nu(X\Phi, \mathcal{S}_1\zeta) = 3\nu(\Phi^2\Phi', \zeta). \tag{6.17}$$

Using the perturbation series expansions,

$$\zeta = \Phi(X) + \sum_{k=1}^{\infty} (2\nu)^k \zeta_k(X), \tag{6.18}$$

we find that $\zeta_1(X)$ is a solution of the inhomogeneous problem:

$$\mathcal{S}_0\zeta_1 = \Phi^2\Phi'' + 3\Phi(\Phi')^2, \tag{6.19}$$

subject to the adjoint Sommerfeld radiation conditions (5.3), such that $\lim_{X\to-\infty}\zeta_1(X)=0$. As a result, direct computations show that

$$(\Phi^2\Phi'' + 3\Phi(\Phi')^2, \eta_1) = (\zeta_1, \mathcal{S}_0\eta_1) = -\tfrac{3}{2}(\Phi^2\Phi', \zeta_1) \tag{6.20}$$

and

$$\lambda_0(\nu) = 2(\mathbf{v}_2^A, \mathbf{v}_1 + \mu\mathbf{v}_2) + \mathrm{O}(\nu^3). \tag{6.21}$$

Therefore, the eigenvalue $\lambda=\lambda_0(\nu)$ in the weighted space L_a^2 is negative for small $\nu\neq 0$ if and only if

$$(\mathbf{v}_2^A, \mathbf{v}_1 + \mu\mathbf{v}_2) < 0. \tag{6.22}$$

We show that $\lambda_0(\nu)<0$ for $\nu\neq 0$ in the particular case $\mu=0$. When $\mu=0$, there exists an analytical solution of the inhomogeneous Eq. (6.12) satisfying the Sommerfeld radiation condition:

$$\eta_1(X) = \frac{1}{8}\left(\frac{\pi}{2} - \int_0^X \Phi(X)dX \right). \tag{6.23}$$

As a result, the integral in the inner product of (6.14) can be evaluated analytically as follows:

$$(\Phi^2\Phi'' + 3\Phi(\Phi')^2, \eta_1) = \frac{\pi}{16}(\Phi^2\Phi'' + 3\Phi(\Phi')^2, 1) = \frac{\pi^2}{128}. \tag{6.24}$$

The leading-order asymptotic result given by (6.14) and (6.24) is shown in Fig. 4(a) as the dashed curve. It agrees with the solid curve which is the function $\lambda_0(\nu)$ obtained directly by a numerical shooting method for $\mu=0$. The shooting method is based on the observation that when $\mathrm{Re}(\lambda)<0$, the solutions $e^{\delta_i X}$, $i=1, 2, 3$ to the eigenvalue equation $M_2 Y=\lambda Y$ as $|X|\to\infty$ are such that $\delta_1<\delta_2<0<\delta_3$. Thus, based on similar numerical studies in Ref. 11, we require the underlying eigenfunction to decay as $e^{\delta_1 X}$ as $X\to+\infty$ and to grow as $e^{\delta_2 X}$ as $X\to-\infty$. Figure 4(b) shows the eigenfunction of the operator M_2 which corresponds to the resonant pole $\lambda_0(\nu)$ for $\nu=0.5$ and $\mu=0$. The eigenfunction decays exponentially in the limit $X\to+\infty$ but it diverges exponentially in the limit $X\to-\infty$. Since the divergence is slow for small values of $\lambda_0(\nu)$, the eigenfunction exists in the exponentially weighted space L_a for sufficiently large a, such that $0<a_*<a<1$. In fact, $a_*=-\delta_2$.

It follows from the theory of eigenvalues in exponentially weighted spaces[26] that the location of eigenvalues is

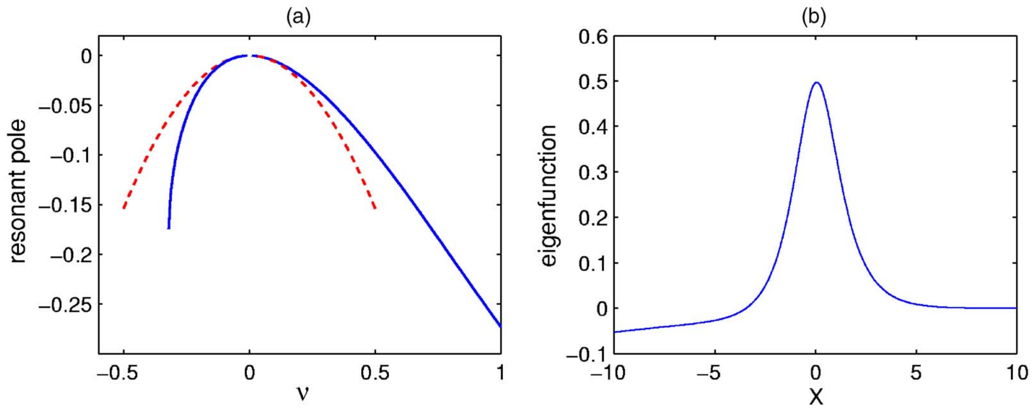

FIG. 4. (a) Dependence of the resonance pole of operator M_2 on ν for $\mu=0$. Solid lines: Numerical values; dashed lines: Analytical formulas (6.14) and (6.24); (b) the corresponding eigenfunction of M_2 for the resonance pole at $\nu=0.5$.

independent of the weight parameter a. When $a>a_*$ for some $a_*>0$, the nonzero eigenvalue $\lambda=\lambda_0(\nu)$ of the spectrum of $\mathcal{L}_\nu$ in L_a^2 is located to the right of the essential spectrum $S_a^\pm$ defined in (5.1). When a decreases, the essential spectrum $S_a^\pm$ crosses the eigenvalue $\lambda=\lambda_0(\nu)$ at some $a=a_*$, which remains on the left of $S_a^\pm$ for $0\leq a<a_*$. It also follows from the theory of exponentially weighted spaces[26] that this crossing results in the exponential growth of the eigenfunction $\mathbf{v}(X)$ at infinity in L_a^2 for $0\leq a<a_*$. As a result, the eigenvalue $\lambda=\lambda_0(\nu)$ in L_a^2, for $a>a_*$ corresponds to a resonant pole in L^2 for $a=0$. Similar splitting of the zero eigenvalue occurs in the Kawahara equation.[11,24]

Lastly, we relate the resonant pole $\lambda_0(\nu)$ to the linear exponential decay of parameters of the embedded soliton obtained in Ref. 32 for a perturbed Hirota equations (corresponding to the limit $\nu\to 0$ or equivalently $|\alpha|+|\gamma|\to 0$). For simplicity, we consider the case $\mu=0$ or equivalently $\alpha=0$. In this case, the resonant pole is approximated by Eqs. (6.14) and (6.24), which are rewritten in the original variables (2.5) as follows:

$$\lambda_0(\gamma) = -\frac{\pi^2}{16}\gamma^2 r^3 + \mathrm{O}(\gamma^3). \tag{6.25}$$

On the other hand, dynamical equations in Ref. 32 show that when an embedded soliton is perturbed, the parameter k decays exponentially with the decay rate:

$$\lambda_{\mathrm{decay}} = -\frac{\pi^2}{16}\gamma^2 r^3 + \mathrm{O}(\gamma^3), \tag{6.26}$$

which matches the approximation (6.25). In the general case where γ is not small and $\alpha\neq 0$, the qualitative analysis in the next section will show that parameter k of the embedded soliton decays exponentially as well.

VII. STABLE DYNAMICS OF EMBEDDED SOLITONS

The (stable) resonant pole of $\mathcal{L}_\nu$ in $L^2(\mathbb{R},\mathbb{C}^2)$ explains linear stability of embedded solitons in the generalized third-order NLS Eq. (2.1) with $\beta+2\gamma>0$, discovered in Refs. 17 and 32. We show that the same conclusion can be recovered with a qualitative method based on conserved quantities of the NLS Eq. (2.1). Moreover, the method allows us to illustrate the stable dynamics of the time-dependent solutions near the one-parameter family of embedded solitons (2.2)–(2.4). For clarity of presentation, we will be working with the original NLS Eq. (2.1) for $u(x,t)$ before renormalization (2.5). We shall use the two conserved quantities of the NLS Eq. (2.1):

$$I_1(u) = \int_{-\infty}^{\infty} |u|^2 dx \equiv \int_{-\infty}^{\infty} J_1(u) dx \tag{7.1}$$

and

$$I_2(u) = \int_{-\infty}^{\infty}\left(\frac{i\alpha}{2}(u\bar{u}_x - u_x\bar{u}) + \gamma|u_x|^2 - \frac{\gamma(\beta+2\gamma)}{6}|u|^4\right)dx \equiv \int_{-\infty}^{\infty} J_2(u)dx. \tag{7.2}$$

Let us consider the family of embedded solitons:

$$u_s(x,t) = r\,\mathrm{sech}(r\kappa z)e^{ikx+i\lambda t}, \quad z = x - \upsilon t, \tag{7.3}$$

where parameters (λ,υ) are given in terms of parameters (r,k) by (2.3) and parameter κ is given by (2.4). If $k=k_0=-\alpha/2\gamma$, the above function is the exact embedded soliton which propagates without change of shape. If $k\neq k_0$, this function is not an exact solution anymore (unless $\alpha=\gamma=0$) and dynamics of embedded solitons shed continuous-wave radiation. We consider dynamics of the embedded soliton for small values of $(k-k_0)$. Since the group velocity of linear waves is $\upsilon_g(k)=-3k^2$ which is less than the speed υ of the embedded soliton, the continuous-wave radiation moves to the left side of the embedded soliton, and the radiation part $u_c(x,t)$ of the solution should satisfy the Sommerfeld radiation boundary condition:

$$u_c(x,t)\to 0, \quad \text{as} \quad x\to+\infty,$$

$$u_c(x,t)\to h_\infty(r)(k-k_0)e^{ik_0x+i\lambda_0 t}, \quad \text{as} \quad x\to-\infty, \tag{7.4}$$

where $|x|/t\to\upsilon_r$ as $t\to\infty$ and $\upsilon_r=\upsilon_g(k_0)-\upsilon=-\kappa^2 r^2$.

We approximate the solution $u(x,t)$ by the two parts: the embedded soliton $u_s(x,t)$, given by (7.3) with two slowly varying parameters $r(t)$ and $k(t)$, and the continuous-wave radiation part $u_c(x,t)$, given by (7.4). We substitute the approximation,

$$u(x,t) \approx u_s(x,t) + u_c(x,t), \tag{7.5}$$

into the two conserved quantities $I_1(u)$ and $I_2(u)$ and derive the reduced equations:

$$\frac{d}{dt}I_k(u_s) \approx -\frac{d}{dt}I_k(u_c) \approx -|v_r| \lim_{x\to-\infty} J_k(u_c), \quad k=1,2. \tag{7.6}$$

Simple calculations show that

$$I_1(u_s) = \frac{2r}{\kappa}, \quad I_2(u_s) = \frac{2r(\alpha k + \gamma k^2)}{\kappa} - \frac{\gamma(\beta + 2\gamma)r^3}{9\kappa} \tag{7.7}$$

and

$$\lim_{x\to-\infty} J_1(u_c) = |h_\infty|^2 (k-k_0)^2,$$
$$\lim_{x\to-\infty} J_2(u_c) = (\alpha k_0 + \gamma k_0^2)|h_\infty|^2 (k-k_0)^2. \tag{7.8}$$

As a result, the reduced Eqs. (7.6) are equivalent to the system of two equations for $r(t)$ and $k(t)$:

$$\frac{dr}{dt} = -\frac{1}{2}|h_\infty|^2 \kappa^3 r^2 (k-k_0)^2 \tag{7.9}$$

$$\frac{dk}{dt} = -\frac{1}{4}|h_\infty|^2 \kappa^5 r^3 (k-k_0). \tag{7.10}$$

It is important to note that the second Eq. (7.10) is linear in $(k-k_0)$. This is because the second conservation law I_2 depends on k through the combination $\alpha k + \gamma k^2$, whose time derivative is proportional to $k-k_0$. Hence a factor of $k-k_0$ is canceled out from Eq. (7.6). The method of conserved quantities gives an approximation for the resonant pole studied in the previous section as

$$\lambda_0 \approx -\tfrac{1}{4}|h_\infty|^2 \kappa^5 r^3, \tag{7.11}$$

where r is an arbitrary parameter of the one-parameter family of embedded solitons. This formula can be compared with the asymptotic approximation (6.25) for $\alpha=0$, from which the approximation for $|h_\infty|^2$ can be found in terms of γ. We note, however, that the qualitative method above produces the formula (7.11) without assumptions of small γ.

Dynamical Eqs. (7.9) and (7.10) for soliton parameters (r,k) admit a continuous family of equilibrium points where $k=k_0$ and r is arbitrary. It is easy to verify that these equilibrium points (and the corresponding embedded solitons) are asymptotically stable, such that

$$\lim_{t\to+\infty} r(t) = r_\infty, \quad \lim_{t\to+\infty} k(t) = k_0, \tag{7.12}$$

where the value of r_∞ is determined uniquely from the initial condition $(r(0),k(0))$. The family of trajectories on the phase plane are hyperbolas:

$$r^2 = r_\infty^2 + \frac{2}{\kappa^2}(k-k_0)^2. \tag{7.13}$$

We note that similar dynamical equations and trajectories of solutions were obtained in Ref. 32 by a soliton perturbation theory, applied to the generalized third-order NLS Eq. (2.1) in the limit $|\alpha|+|\gamma|\to 0$.

VIII. SUMMARY

In this paper, we have studied the stability of embedded solitons in the generalized third-order NLS equation. We have shown that the linearization operator associated with the embedded soliton admits a resonance pole in the left half-plane of the spectral parameter, which explains the linear stability, rather than nonlinear semistability, of these embedded solitons. We have approximated the resonance pole both analytically and numerically, and revealed that the resonance pole gives precisely the linear decay rate of parameters of the embedded soliton. In addition, using conserved quantities, we have derived normal forms for embedded solitons under internal perturbations. These normal forms establish the stable dynamics of embedded solitons in the generalized third-order NLS equation.

Results of this paper shed much light on the understanding of stable embedded solitons. However, more work needs to be done from the point of rigorous analysis. In particular, derivation of the normal forms from conservation laws is only qualitatively valid. Additionally, the connection between the resonance pole in the linearization operator and the stable dynamics of embedded solitons was established only for the generalized third-order NLS equation in the asymptotic limit of Hirota equation, where the resonance pole is small. It is highly desirable to extend this analysis to a more general third-order NLS equation.

ACKNOWLEDGMENTS

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