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A mysterious threshold for transverse instability of deep-water solitons

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Abstract

Properties of the linear eigenvalue problem associated to a hyperbolic non-linear Schrödinger equation are reviewed. The instability band of a deep-water soliton is shown to merge to the continuous spectrum of a linear Schrödinger operator. A new analytical approximation of the instability growth near a threshold is derived by means of a bifurcation theory of weakly localized wave functions. © 2001 Published by Elsevier Science B.V. on behalf of IMACS.

Keywords: Deep-water soliton; Threshold; Non-linear Schrödinger equation in two dimensions; Transverse instability

1. Introduction

This paper presents a solution of a challenging problem arising for the hyperbolic non-linear Schrödinger (NLS) equation

$$\mathrm{i}\Psi_t + \Psi_{xx} - \Psi_{yy} + 2|\Psi|^2\Psi = 0. \tag{1}$$

The hyperbolic NLS equation has been widely used in physical literature as a governing model for surface gravity-capillary waves in a deep fluid [1,2] and for lower hybrid resonant cones in a magnetically confined electrostatic plasma [3]. Problems related to existence and stability of a plane envelope-wave soliton were particularly important for further applications. One of such corner-stone problems was to study the dynamical behavior of a planar soliton initially disturbed by a small periodic transverse perturbation. Transverse instability of planar solitons was discovered for a finite instability band with respect to long transverse perturbations (see previous extensive reviews [4,5] and a recent contribution [6]). However, the existence and properties of the short-wave instability cut-off remained unclear. Several hypotheses were formulated on basis of numerical data [4,5] but a systematical mathematical solution of the problem has not yet been developed. In this paper, we present an asymptotic solution of Eq. (1) for

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the transverse instability of solitons near the instability threshold. Surprisingly, the new solution does not fit to any of several contradictory numerical data presented earlier.

The paper is organized as follows. Section 2 reviews the basic results devoted to the linear analysis of the transverse instability of a deep-water soliton. A linear decomposition method is developed in Section 3. Then, in Section 4, a new analytical solution of the problem on the instability threshold is presented. Other related results are described in Section 5 as consequences of the main method.

2. Formulation of the problem

It is straightforward to reduce the hyperbolic NLS Equation (1) to a linear eigenvalue problem for transverse instability of a plane soliton

$$(\mathcal{L}_1 - p^2)U = -\lambda W, \qquad (\mathcal{L}_0 - p^2)W = \lambda U. \tag{2}$$

The reduction is described by the following expansion at the plane soliton core

$$\Psi = [\Phi(x) + (U(x) + \mathrm{i}W(x))\,\mathrm{e}^{\mathrm{i}py+\lambda t} + (\bar{U}(x) - \mathrm{i}\bar{W}(x))\,\mathrm{e}^{-\mathrm{i}py+\bar{\lambda}t}]\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{3}$$

Here $\Phi(x) = \sqrt{\omega}\,\mathrm{sech}\left(\sqrt{\omega}x\right)$ is a plane soliton, $U(x)$ and $W(x)$ are perturbations at the plane soliton, λ is an instability growth rate, and p is the transverse wave number. The linear operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are

$$\mathcal{L}_0 = -\partial_x^2 + \omega - 2\omega\,\mathrm{sech}^2\left(\sqrt{\omega}x\right), \qquad \mathcal{L}_1 = -\partial_x^2 + \omega - 6\omega\,\mathrm{sech}^2\left(\sqrt{\omega}x\right). \tag{4}$$

The spectrum of the operator $\mathcal{L}_0\phi = \mu\phi$ consists of a single discrete mode

$$\phi_0 = \mathrm{sech}\left(\sqrt{\omega}x\right) \text{ at } \mu = 0$$

and a continuous spectrum for

$$\mu = \omega + k^2, \qquad \phi(x,k) = \frac{k + \mathrm{i}\sqrt{\omega}\tanh\left(\sqrt{\omega}x\right)}{k + \mathrm{i}\sqrt{\omega}}\,\mathrm{e}^{\mathrm{i}kx}.$$

The spectrum of the operator $\mathcal{L}_1\psi = \mu\psi$ consists of two isolated discrete modes:

$$\psi_{-1} = \mathrm{sech}^2\left(\sqrt{\omega}x\right) \text{ at } \mu = -3\omega,$$

$$\psi_0 = \tanh\left(\sqrt{\omega}x\right)\sinh\left(\sqrt{\omega}x\right) \text{ at } \mu = 0,$$

and a continuous spectrum for

$$\mu = \omega + k^2, \qquad \psi(x,k) = \frac{k^2 - 2\omega + 3\mathrm{i}k\sqrt{\omega}\tanh\left(\sqrt{\omega}x\right) + 3\omega\,\mathrm{sech}^2\left(\sqrt{\omega}x\right)}{\left(k + \mathrm{i}\sqrt{\omega}\right)\left(k + 2\mathrm{i}\sqrt{\omega}\right)}\,\mathrm{e}^{\mathrm{i}kx}.$$

The spectra of the operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are shown schematically in Fig. 1. The continuous spectrum of the coupled linear eigenvalue problem (2) has a gap for $p^2 < \omega$, which is located for $|\mathrm{Im}(\lambda)| < \omega - p^2$. In the opposite case, when $p^2 > \omega$, the continuous spectrum of (2) has no gap for imaginary values of λ. Neutral modes exist for $\lambda = p = 0$ while the other localized eigenstates of Eq. (2) are unknown analytically since the linear problem (2) is non-integrable.

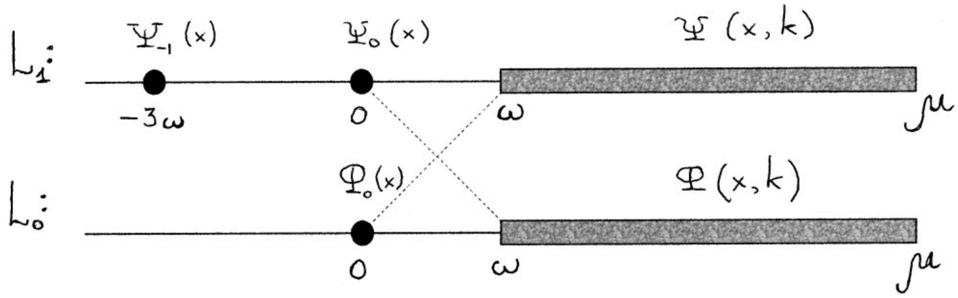

Fig. 1. The spectra of operators $\mathcal{L}_0$ and $\mathcal{L}_1$. The dotted lines display the connections of spectral data in the asymptotic solutions (14–15).

Zakharov and Rubenchik were the first who discovered the transverse instability of plane solitons [7] by perturbing the zero spectrum of the operators $\mathcal{L}_0$ and $\mathcal{L}_1$ due to small terms p^2 and λ in the linear eigenvalue problem (2). They found that the instability appears from the odd eigenstate $\psi_0(x)$ of the operator $\mathcal{L}_1$ if the transverse perturbation is long enough, i.e. $p \to 0$. The instability growth rate $\lambda = \lambda(p)$ was approximated according to the expansion

$$\lambda^2 = \frac{4\omega}{3}p^2 - \frac{4}{9}\left(\frac{\pi^2}{3} - 1\right)p^4 + \mathrm{O}(p^6). \tag{5}$$

A historical comment: the original paper [7] contains an arithmetic error in the second term of the approximation (5). This error was corrected implicitly in further papers [12,14], where the numerical constant was computed as $\lambda^2 \approx 4/3\omega p^2 - 1.02p^4$.

Two consequences follow from the asymptotic result of Zakharov and Rubenchik (see also similar results derived by Yajima [8]): (i) the focusing instability of a plane soliton is of bending type and (ii) the instability band is truncated at an instability cut-off, $p = p_{\mathrm{c}}$, with the following approximation for $p_{\mathrm{c}} \approx 1.15\sqrt{\omega}$. However, the limit $p \to p_{\mathrm{c}}$ cannot be clarified with the sane method.

Numerical computations of the curve $\lambda = \lambda(p)$ for the whole p-axis were first done by Cohen et al. [9] who solved a linear time-dependent problem associated to Eq. (1). We reproduce their result in Fig. 2(a) for $\gamma^{\tau}_{\mathrm{BF}} \sim \lambda$ and $Q^{1/2} = p/\sqrt{\omega}$. The curve clearly displays a jump of the non-zero instability growth rate as $p \to \sqrt{\omega}$ to zero as $p > \sqrt{\omega}$, where the (stable) continuous spectrum of Eq. (2) is located everywhere for imaginary part of λ. Thus, a first conjecture was formulated (see also the review [4]) that the instability disappears stepwise for the transverse wave number approaching the edge of the continuous spectrum at $p \to \sqrt{\omega}$.

Another numerical solution was developed by Saffman and Yuen [10] who solved the linear eigenvalue problem (2). Their results display a continuous graph of $\lambda = \lambda(p)$ up to the cut-off wave number $p_{\mathrm{c}} \approx 1.09\sqrt{\omega}$. The graph is reproduced in Fig. 2(b) (curve 1) for $\Omega^2/\gamma^4 \sim -\lambda^2$ and $\kappa^2/\gamma^2 = p/\sqrt{\omega}$. The instability is supported by odd eigenfunctions of Eq. (2). According to this conjecture (see also the book [2]), the instability band extends into the region of the continuous spectrum of the linear problem (2). Further development of the numerical method enabled one to compute the traveling transverse instability of even eigenstates of a plane soliton. The latter instability corresponds to the complex eigenvalues, $\lambda = \lambda_{\mathrm{r}} + \mathrm{i}\lambda_{\mathrm{i}}$ for $\lambda_{\mathrm{r}} > 0$ and $\lambda_{\mathrm{i}} \neq 0$ [11].

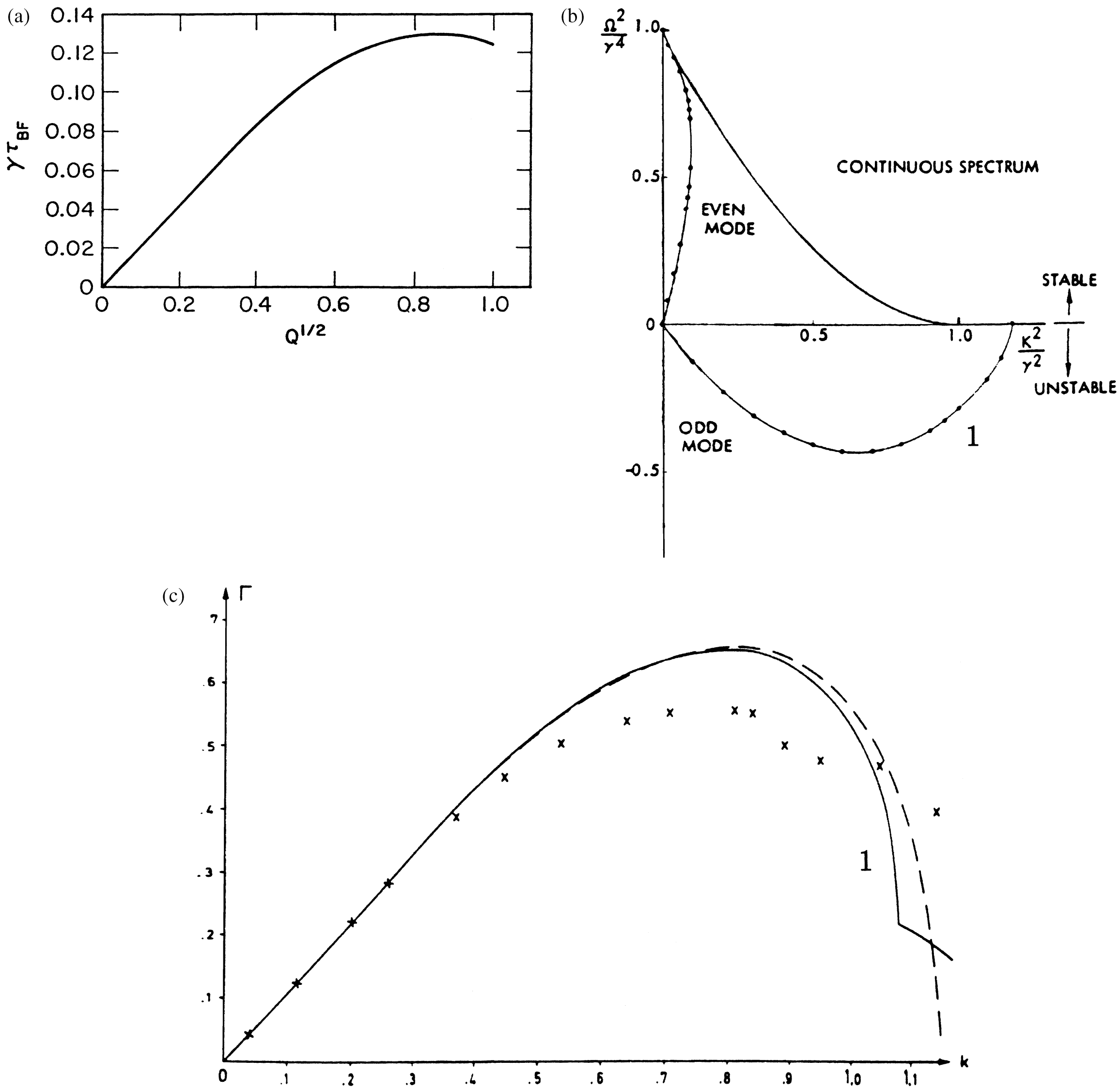

Fig. 2. The graph of the instability growth rate vs. the transverse wave number reproduced from [9] (a), [10] (b), and [12] (c). The correlation between the variables at the graphs and the variables λ and p is described in the text.

Revision of these numerical data and first analytical approximation for the instability cut-off were presented by Anderson et al. [12]. The growth rate was computed numerically by solving a linear time-dependent NLS Equation (1). No link between the odd eigenstate $\psi_0(x)$ of the operator $\mathcal{L}_1$ and the delocalized odd eigenstate $\phi(x, 0)$ of the operator $\mathcal{L}_0$ was found. Instead, the instability band was shown to extend to the region $p > \sqrt{\omega}$, where the instability growth rate bifurcated to the complex plane at $p \approx 1.08\sqrt{\omega}$. This feature resulted in a slope jump of the continuous graph $\lambda = \lambda(p)$ reproduced in Fig. 2(c) (curve 1) for $\Gamma \sim \lambda$ and $k = p/\sqrt{\omega}$. The authors developed a variational Ritz method for a weakly localized trial function, $f \sim e^{-\kappa|x|}\tanh(x)$, to explain the picture and conjectured that the graph has an intersection with $\mathrm{Re}(\lambda) = 0$ for larger value of $p > \sqrt{\omega}$. This conjecture appears to be the third hypothesis for the instability cut-off in the hyperbolic NLS equation (see also the review [5]).

The variational estimates were later improved by Laedke and Spatschek [13] who extended the variational method to the non-positively definite linear operators. Numerical evaluation of the variational principle applied to weakly localized trial functions revealed a good agreement with the second conjecture and the cut-off p_c was evaluated as $p_c \approx 1.08\sqrt{\omega}$. No complex eigenvalues were discovered by this method.

Numerical simulations of the non-linear model (1) were performed by Pereira et al. [14]. The plane envelope solitons were found to split into two-dimensional clusters during a bending (focusing) stage of the transverse instability. Isolated clusters were then seen to spread out and gradually decay during a dispersive (defocusing) stage. Indeed, no fully localized solitons exist for the hyperbolic NLS Equation (1) [15,16]. As a result, the break-up of a planar soliton leads to its disappearance.

The same behavior for non-linear dynamics of a bending soliton in the hyperbolic NLS equation was recently reproduced by Kivshar and Pelinovsky [6] (Chapter 5.4.2). The authors developed an analytical method based on modulation equations for transverse instability and found no stabilization of the instability at the self-focusing stage. Thus, the conclusion that a plane soliton is destroyed by the bending transverse instability was reconfirmed, while the linear properties of the instability were left contradictory. Here, we attend the linear instability problem and show that the graph $\lambda = \lambda(p)$ connects the cut-off root at $p = p_c = \sqrt{\omega}$.

3. Linear decomposition

Operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are standard Sturm–Liuville operators and, therefore, their spectra are complete with respect to the inner product defined by

$$\langle f|g\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, f^*(x)g(x).$$

The linear space of the coupled eigenvalue problem (2) can be spanned in many ways. For example, the $L^2(\mathcal{R})$ eigenfunctions of Eq. (2) can be decomposed through eigenfunctions of the linearized Schrödinger problem for $p = 0$ to follow deformations of the spectrum at $p \neq 0$ [17]. Here, we develop an alternative decomposition, through eigenfunctions of the uncoupled operators $\mathcal{L}_0$ and $\mathcal{L}_1$. This trick is motivated by the consideration of the asymptotic limit $\lambda \to 0$ at the instability threshold.

Provided $U(x)$, $W(x) \in L^2(\mathcal{R})$, solution of Eq. (2) can be expanded into the generalized Fourier decomposition

$$U(x) = \alpha_{-1}\psi_{-1}(x) + \alpha_0\psi_0(x) + \int_{-\infty}^{\infty} \frac{\alpha(k)\psi(x,k)\,\mathrm{d}k}{p^2 - (\omega + k^2)}, \tag{6}$$

$$W(x) = \beta_0\phi_0(x) + \int_{-\infty}^{\infty} \frac{\beta(k)\phi(x,k)\,\mathrm{d}k}{p^2 - (\omega + k^2)}, \tag{7}$$

where α_{-1}, α_0, β_0, $\alpha(k)$, and $\beta(k)$ are Fourier coefficients to be found. The differential system (2) can be equivalently rewritten in an integral representation

$$2\pi\alpha(k) = \lambda\left[\beta_0 K_0(k) + \int_{-\infty}^{\infty}\frac{\beta(k')K(k,k')\,\mathrm{d}k'}{p^2-(\omega+k'^2)}\right], \tag{8}$$

$$-2\pi\beta(k) = \lambda\left[\alpha_{-1}P_{-1}(k) + \alpha_0 P_0(k) + \int_{-\infty}^{\infty}\frac{\alpha(k')\bar{K}(k',k)\,\mathrm{d}k'}{p^2-(\omega+k'^2)}\right], \tag{9}$$

$$\frac{4(p^2+3\omega)\alpha_{-1}}{3\sqrt{\omega}} = \lambda\left[\frac{\pi\beta_0}{2\sqrt{\omega}} + \int_{-\infty}^{\infty}\frac{\beta(k)\bar{P}_{-1}(k)\,\mathrm{d}k}{p^2-(\omega+k^2)}\right], \tag{10}$$

$$\frac{2p^2\alpha_0}{3\sqrt{\omega}} = \lambda\int_{-\infty}^{\infty}\frac{\beta(k)\bar{P}_0(k)\,\mathrm{d}k}{p^2-(\omega+k^2)}, \tag{11}$$

$$-\frac{2p^2\beta_0}{\sqrt{\omega}} = \lambda\left[\frac{\pi\alpha_{-1}}{2\sqrt{\omega}} + \int_{-\infty}^{\infty}\frac{\alpha(k)\bar{K}_0(k)\,\mathrm{d}k}{p^2-(\omega+k^2)}\right]. \tag{12}$$

Here the integral elements are given explicitly

$$K_0(k) = -\frac{\pi\left(k+\mathrm{i}\sqrt{\omega}\right)}{2\sqrt{\omega}\left(k-2\mathrm{i}\sqrt{\omega}\right)}\,\mathrm{sech}\left(\frac{\pi k}{2\sqrt{\omega}}\right),$$

$$P_0(k) = -\frac{\pi\mathrm{i}\left(k+\mathrm{i}\sqrt{\omega}\right)}{2\omega}\,\mathrm{sech}\left(\frac{\pi k}{2\sqrt{\omega}}\right),$$

$$P_{-1}(k) = \frac{\pi k^2}{2\omega\left(k-\mathrm{i}\sqrt{\omega}\right)}\,\mathrm{cosech}\left(\frac{\pi k}{2\sqrt{\omega}}\right),$$

and

$$K(k,k') = \frac{2\pi k}{k-2\mathrm{i}\sqrt{\omega}}\delta(k-k') - \frac{\pi(k^2+3k'^2+4\omega)}{2\left(k'+\mathrm{i}\sqrt{\omega}\right)\left(k-\mathrm{i}\sqrt{\omega}\right)\left(k-2\mathrm{i}\sqrt{\omega}\right)}\,\mathrm{cosech}\left(\frac{\pi(k-k')}{2\sqrt{\omega}}\right).$$

The system (8)–(12) is valid generally for any λ. However, it is particularly important that the case $\lambda = 0$ decouples the system with the spectrum given in Fig. 1. This fact enables us to simplify the system (8)–(12) in the asymptotic limit $\lambda \to 0$ and study the deformation of the spectra in the two important cases: (i) $p \to p_{\mathrm{c}} = \sqrt{\omega}$ (Section 4) and (ii) $p \to 0$ (Section 5). The following formulas will be especially useful for the next section:

$$\lim_{k\to 0} K_0(k) = \frac{\pi}{4\sqrt{\omega}}, \qquad \lim_{k\to 0} P_0(k) = \frac{\pi}{2\sqrt{\omega}}, \qquad \lim_{k\to 0} P_{-1}(k) = \lim_{k\to 0}\lim_{k'\to 0} K(k,k') = 0. \tag{13}$$

4. Asymptotic approximation for the instability cut-off

We present here the derivation of an asymptotic reduction of Eqs. (8)–(12) in the limit $\lambda \to 0$ and $p \to p_{\mathrm{c}} = \sqrt{\omega}$. It includes two asymptotic results: (i) the cut-off of the graph $\lambda = \lambda(p)$ for odd eigenfunctions and (ii) the cut-off of the graph $\lambda = \lambda(p)$ for even eigenfunctions. The first result is associated to the transverse instability of a planar soliton described in Section 2. The second result connects with the traveling (oscillatory) instabilities found in [11]. These two results are formulated in Proposition 1 and proved below.

4.1. Main results

Proposition 1.

1. *The linear eigenvalue problem* (2) *has a localized odd solution as* $p \to p_c = \sqrt{\omega}$ *and the eigenvalue* λ *is given asymptotically as*

$$\lambda^2 = \frac{16\omega p_c}{3\pi^2}\sqrt{p_c^2 - p^2} + O(p_c - p). \tag{14}$$

2. *The linear eigenvalne problem* (2) *has a localized even solution as* $p \to p_c = \sqrt{\omega}$ *and the eigenvalue* λ *is given asymptotically as*

$$\lambda^2 = \frac{64\omega p_c}{\pi^2}\sqrt{p_c^2 - p^2} + O(p_c - p). \tag{15}$$

Proof. The integral kernels in Eqs. (8)–(12) have a resonance for $p^2 \geq \omega$. It was shown in [17] that bifurcations of new eigenvalues may occur near edges of the continuous spectrum at the resonance. Therefore, we can assume that $p^2 = \omega - \kappa^2$ and $\lambda^2 \sim O(\kappa)$, where $\kappa \ll \sqrt{\omega}$. Computing the singular contribution of the integrals in the limit $\kappa \to 0$, the system (8)–(12) reduces to the algebraic equations

$$2\pi\alpha(0) = \lambda K_0(0)\beta_0, \qquad -2\pi\beta(0) = \lambda P_0(0)\alpha_0,$$

$$\frac{16}{3}\sqrt{\omega}\alpha_{-1} = \frac{\pi\lambda}{2\sqrt{\omega}}\beta_0, \qquad \frac{2}{3}\sqrt{\omega}\alpha_0 = -\frac{\pi\lambda\bar{P}_0(0)}{|\kappa|}\beta(0),$$

$$-2\sqrt{\omega}\beta_0 = \lambda\left[\frac{\pi}{2\sqrt{\omega}}\alpha_{-1} - \frac{\pi\bar{K}_0(0)}{|\kappa|}a(0)\right].$$

In the limit $\lambda \to 0$, this system decouples into two subsystems: for odd eigenfunctions described by the coefficients α_0 and $\beta(0)$ and for even eigenfunctions described by $\alpha(0)$ and β_0 (see doted lines in Fig. 1). The coefficient α_{-1} jumps beyond the leading order of the system. As a result, the algebraic equations above reduces with the help of Eq. (13) to Eqs. (14) and (15). □

4.2. Discussions

The transverse instability described in Section 2 is the instability with respect to odd eigenfunctions. Therefore, the first result (14) corresponds to the asymptotic behavior of the graph $\lambda = \lambda(p)$ previously shown in Fig. 2(a)–(c). It is final, therefore, to conclude that the graph links two odd eigenstates at $p = 0[U = \psi_0(x), W = 0]$ and $p = p_c = \sqrt{\omega}[U = 0, W = \phi(x, 0)]$, where the short-wave cut-off wave number p_c coincides with the edge of the continuous spectrum (so that the corresponding eigenfunction $\phi(x, 0)$ is delocalized). As a result, the asymptotic approximation of the instability growth rate beyond the cut-off is unusually steep, $\lambda \sim (p_c - p)^{1/4}$. If the cut-off eigenfunction would be localized, the asymptotic approximation would be smoother, $\lambda \sim (p_c - p)^{1/2}$ as shown for other examples (see [6]). The steepness of the instability growth rate and the delocalization of the corresponding eigenfunction explain why the problem of constructing the graph $\lambda = \lambda(p)$ was difficult numerically. Either a jump of the graph at $p \to \sqrt{\omega}$ (see Fig. 2(a)) or a continuation of the graph to the region $p > \sqrt{\omega}$ (see Fig. 2(b) and (c)) were found by numerical routines. Thus, our analytical studies based on direct asymptotic decomposition complete the problem and improve the inaccurate numerical results.

Additional (complex or oscillatory) transverse instability of a planar soliton was also detected numerically for $p < \sqrt{\omega}$ [11], where the instability eigenvalue $\lambda = \lambda_r + i\lambda_i$ is shown in Figs. 3 and 4 in [11]. The other asymptotic result (15) is connected to this instability in the limit $p \to p_c = \sqrt{\omega}$. Combining numerical data and the asymptotic result, one can recover the whole pattern for the oscillatory instability.

At $p > 0$, two even localized eigenstates appear in the linear problem (2) with $\lambda_r = 0$ and $\lambda_i > 0$, one detaches from the continuous spectrum at $\mathrm{Im}(\lambda) = \omega$ and the other is a perturbed even eigenstate $\phi_0(x)$ (see Fig. 2(b) for even mode). These two eigenvalues move toward each other as the transverse wave number p grows. After coalescence at $p = p_{osc} \approx 0.1\sqrt{\omega}$, they split off the imaginary axis, which is a typical Hamiltonian–Hopf bifurcation leading to an oscillatory instability. Then, the imaginary part λ_i disappears somehow at $p \approx 0.7\sqrt{\omega}$, while the real part λ_r survives and matches the asymptotic result (15) as $p \to \sqrt{\omega}$. Further details in the intermediate range of p are difficult to find both from analytical and numerical treatments.

5. Related results

In this section, we use the system (8)–(12) for developing a standard perturbation theory of localized eigenstates. In particular, the perturbation series for the even and odd neutral eigenstates $\phi_0(x)$ and $\psi_0(x)$ enables us to recover the results of Zakharov and Rubenchik [7] in the limit $p \to 0$ while the perturbation series of the even eigenstate $\psi_{-1}(x)$ reproduces the result of Janssen and Rasmussen [18] in the limit $p^2 + 3\omega \to 0$. The latter expansion describes a short-wave asymptotic approximation for the instability growth rate of the transverse instability of a plane envelope soliton in the elliptic NLS equation. The corresponding linear eigenvalue problem differs from Eq. (2) by a simple replacement $p^2 \to -p^2$. As a result, the eigenstate $U = \psi_{-1}(x)$, $W = 0$ corresponds to an instability band cut-off at $p = \sqrt{3\omega}$. By comparing the known results of direct asymptotic expansions [7,18] and the perturbation theory within the system (8)–(12), we deduce the exact numerical values of two auxiliary integrals (see Proposition 2), which are not included in the reference table of integrals [19]. Direct computation of these integrals is also provided.

Proposition 2. *The following integrals have exact numerical values*

$$\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)} \mathrm{sech}^2(\pi x) = \frac{\pi^2 - 8}{2\pi}, \tag{16}$$

$$\int_0^\infty \frac{x^4 \mathrm{d}x}{(1+x^2)(4+x^2)} \mathrm{cosech}^2\left(\frac{\pi x}{2}\right) = \frac{5\pi^2 - 48}{18\pi}. \tag{17}$$

Proof. The perturbation series for the eigenstate $\psi_0(x)$ is derived from Eqs. (9) and (11) at the leading order as $p \to 0$

$$\beta(k) = -\frac{\lambda P_0(k)}{2\pi}\alpha_0, \qquad \alpha_0 = -\frac{3\lambda\sqrt{\omega}}{2p^2}\int_{-\infty}^{\infty}\frac{\beta(k)\bar{P}_0(k)\,\mathrm{d}k}{(\omega+k^2)}.$$

Using a simple result, $\int_{-\infty}^{\infty}\mathrm{sech}^2(x)\,\mathrm{d}x = 2$, one can reduce the closed system to the expression, $\lambda^2 = 4/3\omega p^2 + \mathrm{O}(p^4)$, which is in agreement with Eq. (5).

The perturbation series for the eigenstate $\phi_0(x)$ is derived from Eqs. (8), (10), and (12) at the leading order as $p \to 0$

$$\alpha(k) = \frac{\lambda K_0(k)}{2\pi}\beta_0, \qquad \alpha_{-1} = \frac{\pi\lambda}{8\omega}\beta_0,$$

and

$$-\frac{2p^2\beta_0}{\sqrt{\omega}} = \lambda\left[\frac{\pi\alpha_{-1}}{2\sqrt{\omega}} - \int_{-\infty}^{\infty}\frac{\alpha(k)\bar{K}_0(k)\,\mathrm{d}k}{(\omega+k^2)}\right].$$

This closed system reduces to the known asymptotic approximation [7], $\lambda^2 = -4\omega p^2 + \mathrm{O}(p^4)$, provided the integral (16) is valid.

The perturbation series for the eigenstate $\psi_{-1}(x)$ is derived from Eqs. (9), (10) and (12) at the leading order as $p^2 + 3\omega \to 0$

$$\beta(k) = -\frac{\lambda P_{-1}(k)}{2\pi}\alpha_{-1}, \qquad \beta_0 = \frac{\pi\lambda}{12\omega}\alpha_{-1},$$

and

$$\frac{4(p^2+3\omega)\alpha_{-1}}{3\sqrt{\omega}} = \lambda\left[\frac{\pi\beta_0}{2\sqrt{\omega}} - \int_{-\infty}^{\infty}\frac{\beta(k)\bar{P}_{-1}(k)\,\mathrm{d}k}{(4\omega+k^2)}\right].$$

Again, this closed system reduces to the known asymptotic approximation [18], $\lambda^2 = 12\omega(p^2+3\omega)/(\pi^2-6)$, provided the integral (17) is valid.

Thus, the integrals (16) and (17) provide a missing link between the perturbation theory based on the system (8)–(12) and the known results of [7,18]. We prove Eqs. (16) and (17) by a direct method. First, the integrals can be rewritten equivalently as

$$\int_0^{\infty}\frac{\mathrm{d}x}{(1+x^2)}\,\mathrm{sech}^2(\pi, x) = \frac{1}{\pi}\left(1 - \frac{1}{\pi}\mathcal{I}_1''(1)\right),$$

$$\int_0^{\infty}\frac{x^4\mathrm{d}x}{(1+x^2)(4+x^2)}\,\mathrm{cosech}^2\left(\frac{\pi x}{2}\right) = \frac{\pi^2}{3}\left(\mathcal{I}_2''\left(\frac{1}{2}\right) - 8\mathcal{I}_2''(1)\right),$$

where $\mathcal{I}_j''(a) = \mathrm{d}^2\mathcal{I}_j(a)/\mathrm{d}a^2$, and $\mathcal{I}_1(a)$ and $\mathcal{I}_2(a)$ are given by

$$\mathcal{I}_1(a) = \int_0^{\infty}\frac{\mathrm{d}x}{1+x^2}\ln(1+\mathrm{e}^{-2\pi a x}) = -\pi\left[\ln\Gamma(2a) - \Gamma\ln(a) + a(1-\ln a) + \left(\frac{1}{2}-2a\right)\ln 2\right].$$

$$\mathcal{I}_2(a) = \int_0^\infty \frac{\mathrm{d}x}{1+x^2}\ln(1-\mathrm{e}^{-2\pi a x}) = \pi\left[\frac{1}{2}\ln(2\pi a) + a(\ln a - 1) - \ln\Gamma(a+1)\right].$$

Here we have used the integrals (4.319) from [19]. Evaluating the second derivative with respect to the implicit parameter a, one can derive exact expressions

$$\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)}\,\mathrm{sech}^2(\pi x) = \frac{1}{\pi}(4\psi'(2) - \psi'(1)),$$

$$\int_0^\infty \frac{x^4\,\mathrm{d}x}{(1+x^2)(4+x^2)}\,\mathrm{cosech}^2\left(\frac{\pi x}{2}\right) = \frac{1}{3\pi}\left(8\psi'(1) - \psi'\left(\frac{1}{2}\right) - 8\right),$$

where $\psi(a) = \frac{\mathrm{d}}{\mathrm{d}a}\ln\Gamma(a)$ is the ψ function defined in Chapter 8.36 of [19]. By using particular values of $\psi(a)$ from Eq. (8.366) of [19], one can reduce the exact expressions above to Eqs. (16) and (17). □

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