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Travelling kinks in discrete ϕ^4 models

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Abstract
In recent years, three exceptional discretizations of the ϕ^4 theory have been discovered (by Speight and Ward, Bender and Tovbis, and Kevrekidis) which support translationally invariant kinks, i.e. families of stationary kinks centred at arbitrary points between the lattice sites. It has been suggested that the translationally invariant stationary kinks may persist as *sliding* kinks, i.e. discrete kinks travelling at nonzero velocities without experiencing any radiation damping. The purpose of this study is to check whether this is indeed the case. By computing the Stokes constants in beyond-all-order asymptotic expansions, we prove that the three exceptional discretizations do not support sliding kinks for most values of the velocity—just like the standard, one-site discretization. There are, however, isolated values of velocity for which radiationless kink propagation becomes possible. There is one such value for the discretization of Speight and Ward and three *sliding velocities* for the model of Kevrekidis.

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1. Introduction

Spatially discretized partial differential equations (or, equivalently, chains of coupled ordinary differential equations) have attracted considerable attention recently. One of the issues that has been vigorously debated and that will concern us in this paper, is whether discrete systems can support solitary waves travelling without losing energy to resonant radiation and decelerating

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as a result. We address this issue for one of the prototype models of nonlinear physics, the ϕ^4-theory:

$$u_{tt} = u_{xx} + \tfrac{1}{2}u(1-u^2). \tag{1.1}$$

The ϕ^4-equation (1.1) is Lorentz-invariant and so the existence of the travelling kink

$$u(x,t) = \tanh\frac{x-ct-s}{2\sqrt{1-c^2}}, \tag{1.2}$$

where $|c| < 1$ and $s \in \mathbb{R}$, is an immediate consequence of the existence of the stationary kink for $c = 0$. On the other hand, if we discretize equation (1.1) in x,

$$\ddot{u}_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + f(u_{n-1}, u_n, u_{n+1}), \tag{1.3}$$

the translation and Lorentz invariances are lost and the existence of the travelling kink (and even of an arbitrarily centred stationary one) becomes a nontrivial matter. In equation (1.3), $u_n \in \mathbb{R}$, $n \in \mathbb{Z}$, $t \in \mathbb{R}$, h is the lattice spacing and the nonlinearity $f(u_{n-1}, u_n, u_{n+1})$ satisfies the continuity condition

$$f(u,u,u) = \tfrac{1}{2}u(1-u^2). \tag{1.4}$$

We restrict ourselves to symmetric discretizations, i.e.

$$f(u_{n-1}, u_n, u_{n+1}) = f(u_{n+1}, u_n, u_{n-1}). \tag{1.5}$$

Equation (1.1) results from (1.3) in the continuum limit, where $u_n(t) = u(x_n, t)$, $x_n = nh$ and $h \to 0$. In this limit, the truncation error of the Taylor series is $\mathcal{O}(h^2)$. We shall be concerned with monotonic kink solutions of (1.3): $u_{n+1}(t) \geqslant u_n(t)$ for all $n \in \mathbb{Z}$. As $h \to 0$, such monotonic discrete kinks approach the continuous kink (1.2).

The most common, one-site discretization of the nonlinearity function is given by

$$f(u_{n-1}, u_n, u_{n+1}) = \tfrac{1}{2}u_n(1-u_n^2). \tag{1.6}$$

It is a well-established fact [L88], however, that the discrete Klein–Gordon equation (1.3), (1.6) admits only a countable set of stationary monotonic kinks with the boundary conditions

$$\lim_{n\to-\infty} u_n(t) = -1, \qquad \lim_{n\to+\infty} u_n(t) = +1. \tag{1.7}$$

Physically, this fact is related to the presence of the Peierls–Nabarro barrier, an effective potential periodic with the spacing of the lattice. Half of the stationary kinks are centred at the minima (the on-site kinks) and the other half (the off-site kinks) at the maxima of the Peierls–Nabarro potential. There are no continuous families of stationary discrete kinks of the form $u_n = u(n-s)$, with s a free parameter, which would interpolate between the two solutions. In an abuse of terminology, we will be calling such families 'translationally invariant kinks'—although, in the first place, translation invariance is a property of an equation rather than a solution, and in the second, all lattice equations are of course *not* translationally invariant. As for propagating waves, of special importance are kinks moving at constant speed and without the emission of radiation. We will be referring to such kinks, i.e. solutions of the form $u_n = u(n-ct-s)$ where $u(\xi)$ is a monotonically growing function satisfying the boundary conditions (1.7), as *sliding kinks*, to emphasize the fact that they do not experience any radiative friction. Being an obstacle to the 'translational invariance' of static kinks, the Peierls–Nabarro barrier is also detrimental to the existence of sliding kinks—at least for small c (see reviews in [S03] and [IJ05]).

In an attempt to find a discrete model with 'translationally invariant' and sliding kinks, Speight and Ward [SW94, S97] considered a Hamiltonian discretization of the form

$$f(u_{n-1}, u_n, u_{n+1}) = \frac{1}{12}(2u_n + u_{n+1})\left(1 - \frac{u_n^2 + u_n u_{n+1} + u_{n+1}^2}{3}\right) + \frac{1}{12}(2u_n + u_{n-1})\left(1 - \frac{u_n^2 + u_n u_{n-1} + u_{n-1}^2}{3}\right). \tag{1.8}$$

In the static limit, the corresponding energy admits a topological lower bound which is saturated by a first- (rather than second-) order difference equation. This equation is readily shown to have a one-parameter continuous family of stationary kink solutions $u_n = u(n - s)$ for $0 \leqslant h \leqslant 2$ (see proposition 1 in [S97]). The parameter s of the family defines the position of the kink relative to the lattice. Since all members of the family have the same (lowest attainable) energy, the stationary kink experiences no Peierls–Nabarro barrier. As for travelling kinks, Speight and Ward's numerical simulations revealed that although moving kinks in this model do lose energy to Cherenkov radiation and decelerate as a result, this happens at a slower rate than a similar process in equation (1.6) (see figures 4 and 5 in [S97]).

Another line of attack was chosen by Bender and Tovbis [BT97] who proposed a different discretization supporting a continuous family of arbitrarily centred stationary kinks:

$$f(u_{n-1}, u_n, u_{n+1}) = \tfrac{1}{4}(u_{n+1} + u_{n-1})(1 - u_n^2). \tag{1.9}$$

In this case, the family arises due to the suppression of the stationary kink's resonant radiation. In fact, the family of stationary kinks can be found explicitly as

$$u_n(t) = \tanh[a(n - s)], \tag{1.10}$$

where $a = \operatorname{arcsinh}(h/2)$ for all $h \in \mathbb{R}$. (The solution (1.10) coincides with the stationary dark soliton of the repulsive Ablowitz–Ladik equation [HA93].)

Finally, the nonlinearity

$$f(u_{n-1}, u_n, u_{n+1}) = \tfrac{1}{8}(u_{n+1} + u_{n-1})(2 - u_{n+1}^2 - u_{n-1}^2) \tag{1.11}$$

was introduced by Kevrekidis [K03], who demonstrated the existence of a two-point invariant and hence a first-order difference equation associated with the stationary equation. Consequently, the discretization (1.11) also supports a continuous family of stationary kinks for all $h \in [0, h_0]$ with some $h_0 > 0$. (For general discussion, see [S99, BOP05, DKY05a].) A relevant property of the model (1.11), which is related to the existence of a two-point invariant [K03] and indicates some additional underlying symmetry, is the conservation of momentum. (See also [DKY05b].)

Since the reasons for the nonexistence of 'translationally invariant' kinks and sliding kinks are apparently related (the breaking of symmetries of the underlying continuum theory or, speaking physically, the presence of the Peierls–Nabarro barrier), the availability of 'translation-invariant' stationary kinks in the models (1.8), (1.9) and (1.11) suggests that they might have sliding kinks as well. It is the purpose of the present study to find out whether this is indeed the case. We shall analyse the persistence of continuous families of stationary kinks $u_n = u(n - s)$ for nonzero velocities; in other words, examine the existence of solutions of the form $u(n - ct - s)$ where $u(z)$ is a monotonically growing function satisfying (1.7) and $c \neq 0$. We develop an accurate numerical test in the limit $h \to 0$ which shows whether or not standing and travelling kinks of the discrete ϕ^4 model (1.3) bifurcate from the exact kink solutions (1.2) of its continuous counterpart (1.1). The analysis of this bifurcation poses a singular problem in perturbation theory which can be analysed using two (inner and outer) matched asymptotic scales on the complex plane [TTJ98, T00a]. In particular, the nonvanishing of the Stokes

constant in the inner asymptotic equation serves as a sufficient condition for the nonexistence of continuous solutions of the difference equations [TTJ98].

Our test will be based on computing the Stokes constant for the differential–difference equation underlying the lattice system. We will examine all four discretizations of the ϕ^4 theory mentioned above, i.e. equations (1.6), (1.8), (1.9) and (1.11). Since translationally invariant stationary kinks $u_n = u(n - s)$ do exist for the three exceptional nonlinearities (1.8), (1.9) and (1.11), the Stokes constant is *a priori* vanishing for $c = 0$ in these three cases. However, we will show that in all three cases the Stokes constant acquires a nonzero value as soon as c deviates from zero. It remains nonzero for all c except a few isolated values which define the particular velocities of the sliding kinks in the corresponding model. There is one such isolated zero of the Stokes constant for the nonlinearity (1.8) and three *sliding velocites* for the discretization (1.11). Consequently, the main conclusion of this work is that the sliding kinks, i.e. kinks travelling at a constant speed without the emission of radiation, can occur only at particular values of the velocity. The sliding velocities are, of course, functions of the discretization spacing h, so that sliding kinks arise along continuous curves on the (c, h)-plane.

We conclude this introduction with a remark on a convention adopted in the remainder of this paper—namely, that the linear part of the function $f(u_{n-1}, u_n, u_{n+1})$ in (1.3) can always be fixed to $(1/2)u_n$ without loss of generality. Indeed, the most general function satisfying (1.4) and (1.5) is $f = [(1/2) - 2a]u_n + a(u_{n+1} + u_{n-1})$ + cubic terms, where a is arbitrary. Since h^2 in (1.3) is also a free parameter, we can always make a replacement $h \to \tilde{h}$ such that $1/h^2 + a = 1/\tilde{h}^2$. This gives

$$f(u_{n-1}, u_n, u_{n+1}) = \tfrac{1}{2}u_n - Q(u_{n-1}, u_n, u_{n+1}), \tag{1.12}$$

where Q is a homogeneous polynomial of degree 3 which is independent of the parameter h.

The outline of this paper is as follows. In the next section (section 2) we review the construction of the outer and inner asymptotic solutions in the limit $h \to 0$. Section 3 contains details of the numerical computation of the Stokes constants while the last section (section 4) summarizes the results of our work.

2. Inner and outer asymptotic expansions in the limit $h \to 0$

We are looking for a sliding-kink solution of the discrete ϕ^4 models (1.3) in the form

$$u_n(t) = \phi(z), \qquad z = h(n - s) - ct, \tag{2.1}$$

where $\phi(z)$ is assumed to be a twice differentiable function of $z \in \mathbb{R}$ that satisfies the differential advance–delay equation

$$c^2\phi''(z) = \frac{\phi(z+h) - 2\phi(z) + \phi(z-h)}{h^2} + \frac{1}{2}\phi(z) - Q\left(\phi(z-h), \phi(z), \phi(z+h)\right), \tag{2.2}$$

with the boundary conditions $\phi(z) \to \pm 1$ as $z \to \pm\infty$. The velocity c is assumed to be smaller than 1 in modulus. If a solution to this boundary-value problem (i.e. a heteroclinic orbit) exists, then the parameter s is arbitrary due to the translation invariance of the advance–delay equation (2.2). The scaling parameter h (which stands for the lattice step-size) can be used to reduce equation (2.2) to a singularly perturbed differential equation as $h \to 0$ [TTJ98]. Formal asymptotic solutions of the problem (2.2) can be constructed at the inner and outer asymptotic scales. The formal series represent convergent asymptotic solutions of the singular perturbation problem only if the Stokes constants are all zero [T00a].

Asymptotic analysis beyond all orders of perturbation theory was pioneered by Kruskal and Segur [KS91] and has been employed by many authors. It was extended by Pomeau *et al* [PRG88] to allow the computation of radiation coefficients from the Borel summation of series

rather than from the numerical solution of differential equations. Essentially the same method has been applied to different problems by Grimshaw and Joshi [GJ95, G95] and Tovbis and collaborators [TTJ98, T00a, T00b, TP05]. In this paper, we shall work with formal inner and outer asymptotic series for the problem (2.2) without attempting rigorous analysis of their asymptoticity.

2.1. Outer asymptotic series

Assuming that the solution $\phi(z)$ is a real analytic function of z, we consider the Taylor series for the second difference in a strip $\mathcal{D}_\delta = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \delta\}$, where $\delta > 0$:

$$\phi(z+h) - 2\phi(z) + \phi(z-h) = h^2\phi''(z) + \sum_{n=2}^{\infty} h^{2n} \frac{2}{(2n)!}\phi^{(2n)}(z). \tag{2.3}$$

Since the cubic polynomial $Q(u_{n-1}, u_n, u_{n+1})$ satisfies the continuity and symmetry relations (1.4) and (1.5), the nonlinearity of (2.2) can also be expanded in a Taylor series in the same strip:

$$Q\left(\phi(z-h), \phi(z), \phi(z+h)\right) = \frac{1}{2}\phi^3(z) + \sum_{n=1}^{\infty} h^{2n} Q_{2n}\left(\phi, (\phi')^2, \ldots, \phi^{(2n)}\right), \tag{2.4}$$

where the coefficients Q_{2n} depend on even derivatives and even powers of odd derivatives of $\phi(z)$ and also $Q_{2n}(\phi, 0, \ldots, 0) = 0$. The differential advance–delay equation (2.2) can thus be written as

$$(1-c^2)\phi'' + \frac{1}{2}\phi(1-\phi^2) + \sum_{n=1}^{\infty} h^{2n}\left(\frac{2}{(2n+2)!}\phi^{(2n+2)} - Q_{2n}(\phi, (\phi')^2, \ldots, \phi^{(2n)})\right) = 0. \tag{2.5}$$

For $h = 0$, equation (2.5) becomes the travelling wave reduction of the continuous model (1.1), with the explicit solution

$$\phi_0(z) = \tanh \xi; \qquad \xi = \frac{z}{2\sqrt{1-c^2}}, \quad |c| < 1. \tag{2.6}$$

We will search for solutions of equation (2.5) of the form

$$\hat{\phi}(z) = \phi_0(z) + \sum_{n=1}^{\infty} h^{2n}\phi_{2n}(z). \tag{2.7}$$

Substituting the expansion (2.7) into (2.5) we get, at order h^{2n},

$$\mathcal{L}\phi_{2n} = H_{2n},$$

where the linearized operator $\mathcal{L}$ is given by

$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + 4 - 6\,\mathrm{sech}^2\,\xi,$$

and H_{2n} are polynomials in $\phi_0, \phi_2, \ldots, \phi_{2n-2}$ and their derivatives. The kernel of $\mathcal{L}$ is one-dimensional and spanned by an even eigenfunction $y_0 = \mathrm{sech}^2\,\xi$. The rest of the spectrum of $\mathcal{L}$ is positive. It is not difficult to prove by induction that if $\phi_{2k}(z)$ are all odd in z for $k = 0, 1, \ldots, n-1$, the nonhomogeneous term H_{2n} is also odd in z and hence, by the Fredholm alternative, there exists a unique odd bounded solution $\phi_{2n}(z)$ for $z \in \mathbb{R}$. Moreover, since H_{2n} decays to zero exponentially fast as $|z| \to \infty$, the function $\phi_{2n}(z)$ is also exponentially decaying for any $n \geqslant 1$. The perturbation $\phi_2(z)$, in particular, satisfies the nonhomogeneous equation

$$\mathcal{L}\phi_2 = -\tfrac{1}{3}[\phi_0^{(\mathrm{iv})} + \alpha\phi_0'' + \beta\phi_0^2\phi_0'' + \gamma\phi_0(\phi_0')^2], \tag{2.8}$$

where the numerical coefficients depend on whether the nonlinearity function f is given by (1.6), (1.8), (1.9) or (1.11):

One-site (1.6):	$\alpha = \beta = \gamma = 0,$
Speight–Ward (1.8):	$\alpha = 1, \quad \beta = \gamma = -4,$
Bender–Tovbis (1.9):	$\alpha = 3, \quad \beta = -3, \quad \gamma = 0,$
Kevrekidis (1.11):	$\alpha = 3, \quad \beta = -9, \quad \gamma = -6.$

The odd bounded solution $\phi_2(z)$ of the nonhomogeneous equation (2.8) is

$$\phi_2(z) = \mathcal{A} \tanh \xi \operatorname{sech}^2 \xi + \mathcal{B}\xi \operatorname{sech}^2 \xi, \tag{2.9}$$

where

$$\mathcal{A} = \frac{(1-c^2)(\gamma + 2\beta) + 6}{72(1-c^2)^2}, \qquad \mathcal{B} = -\frac{(1-c^2)(\alpha+\beta)+1}{24(1-c^2)^2}.$$

The hat in the series (2.7) indicates that the series is formal, i.e. it may or may not converge [TTJ98, T00a], depending on the choice of c and Q in equation (2.2). We shall be referring to (2.7) as the outer asymptotic expansion.

2.2. *Inner asymptotic series*

The leading-order term (2.6) of the outer expansion (2.5) has poles at $\xi = (\pi \mathrm{i}/2)(1+2n)$, where $n \in \mathbb{Z}$. We apply the scaling transformation

$$z = h\zeta + \mathrm{i}\pi\sqrt{1-c^2}, \qquad \phi(z) = \frac{1}{h}\psi(\zeta) \tag{2.10}$$

to equation (2.2) in order to study the convergence of the formal asymptotic solution (2.7) near the pole $\xi = (\pi \mathrm{i}/2)$ (see [TTJ98, T00a]). This yields the following differential advance–delay equation for $\psi(\zeta)$:

$$c^2\psi''(\zeta) = \psi(\zeta+1) - 2\psi(\zeta) + \psi(\zeta-1) - Q(\psi(\zeta-1), \psi(\zeta), \psi(\zeta+1)) + \frac{h^2}{2}\psi(\zeta). \tag{2.11}$$

The following are the cubic functions Q for each of the four discretizations that we deal with in this paper:

One-site (1.6):	$Q = \frac{1}{2}\psi^3(\zeta),$
Speight–Ward (1.8):	$Q = \frac{1}{36}[\psi^3(\zeta+1) + 3\psi^2(\zeta+1)\psi(\zeta) + 3\psi(\zeta+1)\psi^2(\zeta) + 4\psi^3(\zeta) + 3\psi(\zeta-1)\psi^2(\zeta) + 3\psi^2(\zeta-1)\psi(\zeta) + \psi^3(\zeta-1)],$
Bender–Tovbis (1.9):	$Q = \frac{1}{4}\psi^2(\zeta)[\psi(\zeta+1) + \psi(\zeta-1)],$
Kevrekidis (1.11):	$Q = \frac{1}{8}[\psi^3(\zeta+1) + \psi^2(\zeta+1)\psi(\zeta-1) + \psi(\zeta+1)\psi^2(\zeta-1) + \psi^3(\zeta-1)].$

We note that the heteroclinic orbit becomes small as $h \to 0$ under the normalization (2.10): if $\phi(z) \to \pm 1$ as $z \to \pm\infty$, then $\psi(\zeta) \to \pm h$ as $\operatorname{Re}\zeta \to \pm\infty$. The formal asymptotic series (2.7) in the new variables (2.10) becomes a new formal series

$$\hat{\psi}(\zeta) = \hat{\psi}_0(\zeta) + \sum_{n=1}^{\infty} h^n \hat{\psi}_n(\zeta), \tag{2.12}$$

where each term $\hat{\psi}_n(\zeta)$ can be expanded in a formal series in descending powers of ζ. In particular, the leading-order function $\hat{\psi}_0(\zeta)$ has the general form

$$\hat{\psi}_0(\zeta) = \sum_{n=0}^{\infty} \frac{a_{2n}}{\zeta^{2n+1}}. \tag{2.13}$$

By comparing the series (2.12) and (2.13) with the solutions (2.6) and (2.9) in variables (2.10), we note the correspondence

$$a_0 = 2\sqrt{1-c^2}, \qquad a_2 = -\frac{(1-c^2)(\gamma+2\beta)+6}{9\sqrt{1-c^2}}.$$

We shall be referring to (2.12) as the inner asymptotic expansion. The odd powers of h in the inner asymptotic expansion (2.12) appear due to the matching conditions with the outer asymptotic expansion (2.7) under the scaling (2.10), as well as due to the nonzero boundary conditions for the heteroclinic orbits $\psi(\zeta) \to \pm h$ as $\operatorname{Re}\zeta \to \pm\infty$.

2.3. *Leading-order problem for an inner solution*

Convergence of the formal inverse-power series (2.13) for the leading-order solution $\hat{\psi}_0(\zeta)$ depends on the values of the Stokes constants [T00a]. Computation of the Stokes constants is based on Borel–Laplace transforms of the inner equation (2.11) [TTJ98]. Assuming continuity in h, we study the leading-order solution $\psi_0(\zeta) = \lim_{h\to 0}\psi(\zeta)$ of the truncated inner equation

$$c^2\psi_0''(\zeta) = \psi_0(\zeta+1) - 2\psi_0(\zeta) + \psi_0(\zeta-1) - Q\left(\psi_0(\zeta-1), \psi_0(\zeta), \psi_0(\zeta+1)\right). \tag{2.14}$$

By substituting the series (2.13) into equation (2.14), one can derive a recurrence relation between the coefficients in the set $\{a_n\}_{n=0}^{\infty}$. The Stokes constants can be computed from the asymptotic behaviour of the coefficients a_n for large n. Alternatively, the leading-order solution $\psi_0(\zeta)$ and the Stokes constants can be defined using the Borel–Laplace transform:

$$\psi_0(\zeta) = \int_{\gamma} V_0(p)\mathrm{e}^{-p\zeta}\,\mathrm{d}p. \tag{2.15}$$

The choice of the contour of integration γ determines the domain of $\psi_0(\zeta)$ in the complex ζ-plane. We define two solutions $\psi_0^{(s)}(\zeta)$ and $\psi_0^{(u)}(\zeta)$, which lie on the stable and unstable manifolds, respectively, such that

$$\lim_{\operatorname{Re}\zeta\to+\infty} \psi_0^{(s)}(\zeta) = 0, \qquad \lim_{\operatorname{Re}\zeta\to-\infty} \psi_0^{(u)}(\zeta) = 0. \tag{2.16}$$

We note that the stable and unstable solutions tend to the stationary point at the origin, since the heteroclinic orbits connect the stationary points at $\psi = \pm h$ which move to the origin as $h \to 0$. The three stationary points coalesce to become a degenerate stationary point at the origin within the truncated inner equation (2.14).

The Borel–Laplace transform (2.15) produces the stable solution $\psi_0^{(s)}(\zeta)$ when the contour of integration γ_s lies in the first quadrant of the complex p-plane and extends from $p = 0$ to $p = \infty$. Similarly, it produces the unstable solution $\psi_0^{(u)}(\zeta)$ when the contour of integration γ_u lies in the second quadrant. We choose the integration contours in such a way that $\arg p \to \pi/2$ as $p \to \infty$, so that the solutions $\psi_0^{(s)}(\zeta)$ and $\psi_0^{(u)}(\zeta)$ are defined by (2.15) for all complex ζ with $\operatorname{Im}\zeta < 0$.

The Borel transform $V_0(p)$ satisfies the following integral equation, which follows from (2.14) and (2.15):

$$\left(4\sinh^2\frac{p}{2} - c^2p^2\right)V_0(p) = \hat{Q}[V_0(p)]. \tag{2.17}$$

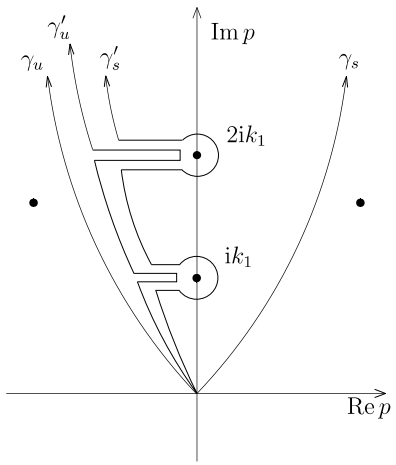

Figure 1. Contours of integration for the stable and unstable solutions $\psi_0^{(s)}$ and $\psi_0^{(u)}$. The curves γ_s' and γ_u' are deformations of the contours γ_s and γ_u, respectively.

Here, $\hat{Q}[V(p)]$ denotes a double convolution of $V(p)$ with itself (in this case, the hat is used to denote an operator). We list below the convolutions $\hat{Q}[V(p)]$ for each of the four models under consideration:

One-site (1.6): $2\hat{Q} = V(p) * V(p) * V(p)$,

Speight–Ward (1.8): $18\hat{Q} = \cosh p[V(p) * V(p) * V(p)] + 3\{\cosh p[V(p) * V(p)]\} * V(p) + 3[\cosh pV(p)] * V(p) * V(p) + 2V(p) * V(p) * V(p)$,

Bender–Tovbis (1.9): $2\hat{Q} = [\cosh pV(p)] * V(p) * V(p)$,

Kevrekidis (1.11): $4\hat{Q} = \cosh p[V(p) * V(p) * V(p)] + [\cosh pV(p)] * [\mathrm{e}^p V(p)] * [\mathrm{e}^{-p} V(p)]$,

where the asterisk $*$ denotes the convolution integral for the Borel–Laplace transform:

$$V(p) * W(p) = \int_0^p V(p - p_1) W(p_1) \mathrm{d}p_1,$$

and the integration is performed from the origin to the point p on the complex plane, along the contour γ. The inverse power series (2.13) for the limiting solution $\psi_0(\zeta)$ becomes the following power series for the Borel transform $V_0(p)$:

$$\hat{V}_0(p) = \sum_{n=0}^{\infty} v_{2n} p^{2n}, \qquad v_{2n} = \frac{a_{2n}}{(2n)!}, \tag{2.18}$$

where $v_0 = a_0 = 2\sqrt{1 - c^2}$. The hat denotes a formal series which might only converge for some values of p. The virtue of the integral form (2.17) is that the limiting behaviour of v_n for large n can be related to singularities of $V_0(p)$, which in turn correspond to the oscillatory tails of $\psi_0(\zeta)$.

If the sliding kink exists, the inverse-power series for $\psi_0(\zeta)$ will converge for all $\zeta \in \mathbb{C}$ such that $\mathrm{Im}\, \zeta < 0$. This implies that the stable and unstable solutions $\psi_0^{(s)}(\zeta)$ and $\psi_0^{(u)}(\zeta)$ coincide, i.e. that the contour γ_s in the right half of the complex p-plane can be continously deformed to the contour γ_u in the left half-plane (see figure 1). If, however, there are any singularities between the two contours, then a continuous deformation is possible only if the

residues are zero. The residues are proportional to the values of the Stokes constants. When the Stokes constants are nonzero, the formal power series (2.18) for the solution $V_0(p)$ of the integral equation (2.17) diverges for some values of p in the sector between the contours γ_s and γ_u.

2.4. Stokes constants

The Borel transform $V_0(p)$ is singular near the points in the p-plane where the coefficient in front of $V_0(p)$ on the left-hand side of the integral equation (2.17) vanishes [T00a], except for the point $p = 0$ where the right-hand side is also zero. That is, singularities occur when $(2/p)\sinh(p/2) = \pm c$. The location of these singularities is important because the stable and unstable solutions are not, in fact, uniquely defined by (2.16); different solutions are generated depending on where the contours lie relative to the singularities of $V_0(p)$ with $\operatorname{Re} p \neq 0$. Exploiting this nonuniqueness, we wish to choose the contours γ_s and γ_u to lie above all the singularities with nonzero real part; this will minimize the number of singularities between the stable and unstable solutions.

It is not difficult to show that the contour γ_s extending from 0 to ∞ can be chosen in such a way, i.e. so that there are no singularities between it and the imaginary axis. Indeed, assume, for definiteness, that $c > 0$. Let $(2n_c - 1)$ be the number of positive roots of the equation $\sin q = cq$ and denote the real and imaginary parts of $p/2$ by κ and q: $p/2 = \kappa + \mathrm{i}q$. In the (κ, q)-plane, consider a rectangular region $\mathcal{D}$ bounded by the horizontal segments $q = 2\pi n$ and $q = \epsilon$ at the top and bottom, and vertical segments $\kappa = -\epsilon$ and $\kappa = \epsilon$ on the left and right. Here n is any positive integer greater than n_c and $\epsilon > 0$ is taken to be small. Using the argument principle, we can count the number of (complex) roots of the equation $\sinh(p/2) = c(p/2)$ in the region $\mathcal{D}$. We have

$$\tan\arg\frac{p}{2} = \frac{\cosh\kappa\sin q - cq}{\sinh\kappa\cos q - c\kappa}.$$

On the right lateral side, where $\kappa = \epsilon$, this becomes

$$\tan\arg\frac{p}{2} \approx \frac{1}{\epsilon}\frac{\sin q - cq}{\cos q - c}. \tag{2.19}$$

As we move from $q = \epsilon$ to $q = 2\pi n$, the numerator in (2.19) will change sign $(2n_c - 1)$ times. In a similar way, moving down along the left side there will be $(2n_c - 1)$ more zero crossings, while no zero crossings will occur along the horizontal segments. This means that the argument can change by no more than $(4n_c - 2)\pi$ and hence there are at most $(2n_c - 1)$ roots in the region $\mathcal{D}$, no matter how large n is. Similarly, we can show that the equation $\sinh(p/2) = -c(p/2)$ has no more than $2n_c$ roots in the region $\mathcal{D}$, if $2n_c$ is the number of positive roots of $\sin q = -cq$. The upshot is that for any finite c, there are only a finite number of singularities with small real parts; the singularities cannot accumulate to the imaginary axis. For $c \neq 0$, the singularities with nonzero real parts lie on the curves

$$q = \pm\sqrt{\frac{1}{c^2}\cosh^2\kappa - \kappa^2\coth^2\kappa} \quad \to \quad \pm\frac{1}{c}\cosh\kappa \qquad \text{as } |\kappa| \to \infty.$$

Accordingly, in order for the integration contours γ_s and γ_u to lie above these singularities, they must be curvilinear (and not just rays) as shown in figure 1.

Having chosen the contours γ_s and γ_u to lie above the singularities in the first and second quadrants, respectively, the only singularities of $V_0(p)$ that determine whether the stable solution $\psi_0^{(s)}(\zeta)$ can be continuously transformed into the unstable solution $\psi_0^{(u)}(\zeta)$ are those

at nonzero pure imaginary values of p. We will be referring to these values as resonances. The set of resonances $\mathcal{R}_c$ is defined by the transcendental equation

$$\mathcal{R}_c = \left\{ p = \mathrm{i}k, k \in \mathbb{R}_+ : \frac{2}{k} \sin \frac{k}{2} = \pm c \right\}. \tag{2.20}$$

When $c = 0$, the set $\mathcal{R}_0$ is infinite-dimensional and can be described explicitly:

$$\mathcal{R}_0 = \{p = 2\pi n\mathrm{i}, n \in \mathbb{N}\}.$$

Let $p_1 = \mathrm{i}k_1$ be the smallest imaginary root in the set $\mathcal{R}_c$. It is clear from (2.20) that $0 < k_1 < 2\pi$ for $c \in (0, 1)$, so that $k_1 \to 2\pi$ as $c \to 0^+$ and $k_1 \to 0$ as $c \to 1^-$. The set of resonances $\mathcal{R}_c$ is finite-dimensional for $c \in (0, 1)$ and it consists of only one root $p_1 = \mathrm{i}k_1$ for $c \in (c_1, 1)$, where $c_1 \approx 0.22$.

Due to the resonances, a function $\psi_0(\zeta)$ that satisfies the truncated inner equation (2.14) may have oscillatory tails as $|\mathrm{Re}\,\zeta| \to \infty$. Adding the solutions of equation (2.14) linearized about $\hat{\psi}_0(\zeta)$, the general bounded solution of (2.14) in the limit $|\zeta| \to \infty$ can be represented as [TTJ98]

$$\psi_0(\zeta) = \hat{\psi}_0(\zeta) + \sum_{\mathrm{i}k_n \in \mathcal{R}_c} \alpha_n \hat{\varphi}_n(\zeta) \mathrm{e}^{-\mathrm{i}k_n\zeta} + \text{multiple harmonics}. \tag{2.21}$$

Here, $\hat{\psi}_0(\zeta)$ is given by the power series (2.13); α_n are coefficients which we will be referring to as amplitudes in what follows; $k_n > 0$ are roots of (2.20) for $p = \mathrm{i}k_n$ and the functions $\hat{\varphi}_n(\zeta)\mathrm{e}^{-\mathrm{i}k_n\zeta}$, $n \geqslant 1$, satisfy the linearized truncated inner equation (2.14). In particular, the equation for the leading-order term $\hat{\varphi}_1(\zeta)$ is

$$\begin{aligned} &\mathrm{e}^{-\mathrm{i}k_1}\hat{\varphi}_1(\zeta+1) + (c^2k_1^2 - 2)\hat{\varphi}_1(\zeta) + \mathrm{e}^{\mathrm{i}k_1}\hat{\varphi}_1(\zeta-1) + 2\mathrm{i}c^2k_1\hat{\varphi}_1'(\zeta) - c^2\hat{\varphi}_1''(\zeta) \\ &\qquad = D_1 Q\hat{\varphi}_1(\zeta-1) + D_2 Q\,\hat{\varphi}_1(\zeta) + D_3 Q\,\hat{\varphi}_1(\zeta+1), \end{aligned} \tag{2.22}$$

where $D_{1,2,3}Q$ are the partial derivatives of $Q(\psi_0(\zeta-1), \psi_0(\zeta), \psi_0(\zeta+1))$ with respect to its first, second and third argument, respectively, evaluated at $\psi_0 = \hat{\psi}_0(\zeta)$.

If the amplitude α_n is nonzero for some n, the formal power series (2.13) does not converge because the solution (2.21) does not decay as $|\mathrm{Re}\,\zeta| \to \infty$. The amplitudes α_n are proportional to the Stokes constants computed for the formal power series (2.13). Each oscillatory term in the sum (2.21) becomes exponentially small in h when we transform from ζ to z using the transformation (2.10). Since $p_1 = \mathrm{i}k_1$ is the element of $\mathcal{R}_c$ with the smallest imaginary part, it follows that the $n = 1$ term dominates the sum in (2.21) when the transformation (2.10) is made (unless $\alpha_1 = 0$). Furthermore, when $c \in (c_1, 1)$, where $c_1 \approx 0.22$, it is the *only* term in the sum since the resonant set $\mathcal{R}_c$ consists of just the one root $p_1 = \mathrm{i}k_1$. We shall, therefore, only be concerned with the leading-order Stokes constant, which multiplies the function $\hat{\varphi}_1(\zeta)$.

If $\hat{\psi}_0(\zeta)$ is given by the power series (2.13), the solution of the linearized equation (2.22) can also be represented by a formal power series:

$$\hat{\varphi}_1(\zeta) = \zeta^r \sum_{\ell=0}^{\infty} b_\ell \zeta^{-\ell}, \tag{2.23}$$

where we can set $b_0 = 1$ due to the linearity of (2.22). Substituting (2.13) and (2.23) into (2.22) and using (2.20), the coefficient b_1 can be determined from

$$\begin{aligned} &2\mathrm{i}r\zeta^{r-1}(c^2k_1 - \sin k_1) + \zeta^{r-2}[r(r-1)(\cos k_1 - c^2) \\ &\qquad + 2\mathrm{i}b_1(r-1)(c^2k_1 - \sin k_1) - 6(1-c^2)] + \mathcal{O}(\zeta^{r-3}) = 0. \end{aligned}$$

In this equation, the coefficient of each power of ζ should be set to zero. In order to set the coefficent in front of the first term to zero in the situation where $c \neq 0$, we must choose $r = 0$. The second term then gives

$$b_1 = \frac{3\mathrm{i}(1-c^2)}{c^2 k_1 - \sin k_1},$$

after which all the other coefficients $b_2, b_3, \ldots$, can be computed recursively. On the other hand, in the situation with $c = 0$, we have $k_1 = 2\pi$ and the coefficient in front of ζ^{r-1} is zero regardless of the value of r. Setting the coefficient in front of ζ^{r-2} to zero requires that we choose either $r = 3$ or $r = -2$, and hence we have two different descending-power series, one starting with ζ^3 and the other one with ζ^{-2}. We shall focus on the former as it dominates the latter in the limit $\zeta \to \infty$. Again, the succeeding terms in (2.23) are determined recursively.

Thus, we have established that the leading-order oscillatory term in the expansion (2.21) behaves as

$$\begin{aligned} &\alpha_1 \left[1 + \frac{b_1}{\zeta} + \mathcal{O}\left(\frac{1}{\zeta^2}\right)\right] \mathrm{e}^{-\mathrm{i}k_1\zeta} && \text{for } c \neq 0 \quad \text{and} \\ &\alpha_1 \left[\zeta^3 + b_1\zeta^2 + \mathcal{O}(\zeta^1)\right] \mathrm{e}^{-2\pi\mathrm{i}\zeta} && \text{for } c = 0. \end{aligned} \tag{2.24}$$

For $c \neq 0$, the two leading order terms in the expression above are generated by, respectively, a simple pole and a logarithmic singularity of the Borel transform $V_0(p)$ at $p = \mathrm{i}k_1$. For $c = 0$ they are generated by a quadruple pole of $V_0(p)$ at $p = 2\pi\mathrm{i}$. From the fact that V_0 is an even function of p, we deduce the structure of this function near the poles:

$$V_0(p) \to \begin{cases} \dfrac{k_1^2 K_1(c)}{p^2 + k_1^2} - \sigma(c)\ln(p^2 + k_1^2) + \cdots & (\text{for } c \neq 0), \\[2ex] \dfrac{6(2\pi)^8 S_1}{\left(p^2 + 4\pi^2\right)^4} + \dfrac{2(2\pi)^6 \rho}{\left(p^2 + 4\pi^2\right)^3} + \cdots & (\text{for } c = 0), \end{cases} \tag{2.25}$$

as $p \to \pm\mathrm{i}k_1$. Here $K_1(c)$ and S_1 are the leading-order Stokes constants for $c \neq 0$ and $c = 0$, respectively; $\sigma(c)$ and ρ are independent of p, and $\cdots$ stands for terms with even slower growth as $p \to \mathrm{i}k_1$.

To show that these singularities do indeed give rise to the oscillatory tails in (2.24), we compare the two integrals $\psi_0^{(s)}$ and $\psi_0^{(u)}$ for a given value of ζ. To this end, we deform the paths of integration γ_s and γ_u to γ_s' and γ_u', respectively, without crossing any singularities. This is illustrated in figure 1. There are two contributions to the difference $\psi_0^{(s)}(\zeta) - \psi_0^{(u)}(\zeta)$. The first comes from integrating around the pole at $p = \mathrm{i}k_1$, and is equal to $2\pi\mathrm{i}$ times the residue of the function $V_0(p)\mathrm{e}^{-p\zeta}$ at $p = \mathrm{i}k_1$, determined from (2.25). The second contribution arises because the *integrand* increases as the singularity is encircled, since it is a branch point of the logarithm. Since $\ln z$ can be written as $\ln|z| + \mathrm{i}\arg z$, where $z = p - \mathrm{i}k_1$, we see that $V_0(p)$ increases by $-2\pi\mathrm{i}\sigma(c)$ as the branch point $p = \mathrm{i}k_1$ is encircled in the $c \neq 0$ case. Therefore, the difference in the integrand of (2.15) along the paths γ_s' and γ_u' is $-2\pi\mathrm{i}\sigma(c)\mathrm{e}^{-p\zeta}$, which must be integrated along the path of integration from $p = \mathrm{i}k_1$ to infinity, to give $-2\pi\mathrm{i}\sigma(c)\mathrm{e}^{-\mathrm{i}k_1\zeta}/\zeta$. (We have considered the integration on a Riemann surface in order to account for branch points.) Adding together the two contributions discussed above, we have

$$\psi_0^{(s)}(\zeta) - \psi_0^{(u)}(\zeta) = \begin{cases} \left[\pi k_1 K_1(c) - \dfrac{2\pi\mathrm{i}\sigma(c)}{\zeta} + \mathcal{O}\left(\dfrac{1}{\zeta^2}\right)\right] \mathrm{e}^{-\mathrm{i}k_1\zeta} & \text{for } c \neq 0, \\[2ex] -\dfrac{1}{128}\big[16\pi^3\mathrm{i}S_1\zeta^3 \\ \qquad + (192\pi^4 S_1 + \rho)\zeta^2 + \mathcal{O}(\zeta^1)\big]\mathrm{e}^{-2\pi\mathrm{i}\zeta} & \text{for } c = 0. \end{cases} \tag{2.26}$$

If we take the limit $\operatorname{Re}\zeta \to -\infty$, then the unstable solution $\psi_0^{(u)}(\zeta)$ decays to zero as a power law, according to the expansion (2.13). Thus, the stable solution $\psi_0^{(s)}(\zeta)$ has the oscillatory tail given by the representation (2.21) with the amplitude factor

$$\alpha_1 = \begin{cases} \pi k_1 K_1(c) & \text{for } c \neq 0, \\ -\dfrac{\mathrm{i}\pi^3}{8} S_1 & \text{for } c = 0. \end{cases} \tag{2.27}$$

Similarly, if we take the limit $\operatorname{Re}\zeta \to +\infty$, then the stable solution $\psi_0^{(s)}(\zeta)$ decays to zero, while the unstable solution $\psi_0^{(u)}(\zeta)$ has the representation (2.21) with the amplitude factor given by the negative of expression (2.27). By comparing the other terms on the right-hand side of (2.26) to the corresponding terms in (2.24), $\sigma(c)$ and ρ can be uniquely determined.

We now match the leading-order singular behaviour of $V_0(p)$ near $p = \pm\mathrm{i}k_1$, given by (2.25), to the formal power series (2.18). Expanding the expressions in (2.25) as power series gives us

$$V_0(p) \to \begin{cases} K_1(c) - \sigma(c)\ln(k_1^2) + \cdots \\ \qquad + \displaystyle\sum_{n=1}^{\infty} (-1)^n k_1^{-2n} \left(K_1(c) + \frac{\sigma(c)}{n} + \cdots \right) p^{2n} & \text{for } c \neq 0, \\ \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)}{(2\pi)^{2n}} [(n+3)S_1 + \rho + \cdots] p^{2n} & \text{for } c = 0, \end{cases} \tag{2.28}$$

as $p \to \pm\mathrm{i}k_1$. These series converge for all $|p| < k_1$; in particular, they are valid for $p \to \pm\mathrm{i}k_1$, provided $|p| < k_1$. Hence we can replace (2.25) with (2.28) in this neighbourhood. In (2.28), the dots '...' stand for coefficients of the expansion of terms with a slower growth as $p \to \pm\mathrm{i}k_1$ which were dropped in (2.25). The discarded terms would modify the coefficients of the power series (2.28); however, there are terms which would not be affected by these modifications, namely terms with large n. For example, the coefficients proportional to $\sigma(c)$ and ρ in (2.28) are a factor of n smaller than those proportional to $K_1(c)$ and S_1; the discarded coefficients would be even smaller. Therefore the leading singular behaviour of $V_0(p)$ as $p \to \pm\mathrm{i}k_1$ is determined just by the large-n coefficients of the power series (2.28), and hence only the large-n coefficients should be matched to the coefficients of the expansion (2.18). This gives the Stokes constant as a limit of the coefficients v_{2n} of the series (2.18):

$$\begin{aligned} K_1(c) &= \lim_{n\to\infty} (-1)^n k_1^{2n} v_{2n} && \text{for } c \neq 0, \\ S_1 &= \lim_{n\to\infty} \frac{(-1)^n (2\pi)^{2n} v_{2n}}{(n+3)(n+2)(n+1)} && \text{for } c = 0. \end{aligned} \tag{2.29}$$

This formula is used in the next section for numerical computations of the leading-order Stokes constant $K_1(c)$ for $c \neq 0$.

Note that, since (2.18) matches (2.28) in the limit $n \to \infty$, the formal power series $\hat{V}_0(p)$ also has a radius of convergence k_1. However, the formal inverse-power series $\hat{\psi}_0(\zeta)$ diverges for all ζ unless $\hat{V}_0(p)$ converges everywhere (which requires that all the Stokes constants be zero).

Next, we note that as $c \to 0$, the Stokes constant $K_1(c)$ does not tend to S_1, its value at $c = 0$. This discontinuity is due to the fact that, as $c \to 0$, pairs of simple roots in the resonant set $\mathcal{R}_c$ coalesce (e.g. $\mathrm{i}k_1$ coalesces with $\mathrm{i}k_2$ at $2\pi\mathrm{i}$, and so on). As a result, all roots are double and the representation of $\hat{\varphi}_1(\zeta)$ is discontinuous at $c = 0$, with the power degree r of the prefactor in (2.23) jumping from $r = 0$ for $c \neq 0$ to $r = 3$ for $c = 0$. In particular, in exceptional models, i.e. discrete models with continuous families of stationary kinks (such

as (1.8), (1.9) and (1.11)) the constant S_1 is *a priori* zero while the limit of $K_1(c)$ as $c \to 0$ may be nonvanishing. In fact, numerical computations of the top limit in (2.29) indicate that the Stokes constant blows up as $c \to 0$. Renormalization of $K_1(c)$ for small c is, however, a nontrivial asymptotic problem which is beyond the scope of our current investigation.

For $c \in (c_1, 1)$, where $c_1 \approx 0.22$, the resonant set $\mathcal{R}_c$ contains only one root $\mathrm{i}k_1$ and, therefore, there is just one Stokes constant $K_1(c)$, which completely determines the convergence of the formal power series for $\hat{\psi}_0(\zeta)$. If $K_1(c_0) = 0$ at some point $c_0 \in (c_1, 1)$, the stable and unstable solutions $\psi_0^{(s)}(\zeta)$ and $\psi_0^{(u)}(\zeta)$ coincide to leading order. Arguments based on the implicit function theorem (see [TP05]) reveal a heteroclinic bifurcation which occurs on crossing a smooth curve $c = c_*(h)$ on the (c, h)-plane, with $c_*(0) = c_0$.

On the other hand, for $c \in (0, c_1)$ the resonant set $\mathcal{R}_c$ contains more than one root. If $K_1(c_0) = 0$ for some $c_0 \in (0, c_1)$, this alone is not sufficient for the convergence of the formal power series $\hat{\psi}_0(\zeta)$. The higher-order Stokes constants $K_2(c), K_3(c), \ldots$, must be introduced and computed from the asymptotic behaviour of the power series $\hat{V}_0(p)$.

As we shall show in the next section, the function $K_1(c)$ does have zeros in the case of the discretizations (1.8) and (1.11). All these zeroes are 'safe'; that is, all c_0 values lie in the interval $(c_1, 1)$, so that the higher-order Stokes constants do not have to be computed.

3. Numerical computations of the Stokes constant

In this section, we report on the numerical computation of the Stokes constants $K_1(c)$ for the four different discretizations of the ϕ^4 model (1.3) under consideration. Our numerical method uses the expression (2.29) of the Stokes constant in terms of the coefficients of the formal power series solution (2.18). First, we obtain the recurrence relation for the coefficients in the set $\{v_n\}_{n=0}^{\infty}$ by substituting the power series expansion (2.18) into the limiting integral equation (2.17) and using the convolution formula

$$p^n * p^m = \frac{n!\, m!}{(n+m+1)!} p^{n+m+1}. \tag{3.1}$$

After that, we compute the asymptotic behaviour of these coefficients as $n \to \infty$ and evaluate the limit (2.29) numerically for a fixed value of $c \neq 0$.

In order to calculate the Stokes constants for the four models in a uniform way, we write a general symmetric homogeneous cubic polynomial $Q(u_{n-1}, u_n, u_{n+1})$ as

$$Q = \sum_{\alpha=-1}^{1} \sum_{\beta=\alpha}^{1} \sum_{\gamma=\beta}^{1} a_{\alpha,\beta,\gamma} u_{n+\alpha} u_{n+\beta} u_{n+\gamma}, \tag{3.2}$$

where $a_{\alpha,\beta,\gamma}$ are numerical coefficients, with $\alpha, \beta, \gamma \in \{-1, 0, 1\}$ and $\alpha \leqslant \beta \leqslant \gamma$. The symmetry implies that $a_{\alpha,\beta,\gamma} = a_{-\gamma,-\beta,-\alpha}$ and therefore it is sufficient to specify just six coefficients. The values of these coefficients for the four nonlinearities in question are given in table 1.

By applying the Borel–Laplace transform (2.15) to equation (2.14) with Q as in (3.2), we obtain the corresponding cubic convolution function $\hat{Q}[V(p)]$ on the right-hand side of the integral equation (2.17):

$$\hat{Q}[V(p)] = \sum_{\alpha=-1}^{1} \sum_{\beta=\alpha}^{1} \sum_{\gamma=\beta}^{1} a_{\alpha,\beta,\gamma} \mathrm{e}^{\alpha p} V(p) * \mathrm{e}^{\beta p} V(p) * \mathrm{e}^{\gamma p} V(p). \tag{3.3}$$

Table 1. The coefficients $a_{\alpha,\beta,\gamma} = a_{-\gamma,-\beta,-\alpha}$ of the cubic polynomial (3.2) for the four models under consideration.

Model	$a_{1,1,1}$	$a_{0,0,0}$	$a_{0,1,1}$	$a_{0,0,1}$	$a_{-1,1,1}$	$a_{-1,0,1}$
One-site (1.6)	0	1/2	0	0	0	0
Speight–Ward (1.8)	1/36	1/9	1/12	1/12	0	0
Bender–Tovbis (1.9)	0	0	0	1/4	0	0
Kevrekidis (1.11)	1/8	0	0	0	1/8	0

To derive the recurrence formula for the coefficents v_{2n} in (2.18), it will be more convenient to consider the power series expansion which consists of both even and odd powers of p:

$$\hat{V}_0(p) = \sum_{n=0}^{\infty} v_n p^n. \tag{3.4}$$

We now substitute the series (3.4) into (2.17) with $\hat{Q}[V(p)]$ given by (3.3) and use the convolution formula (3.1). Equating the coefficients of p^{n+2} where $n = 0, 1, 2, \ldots$, in the resulting equation, we find that

$$\sum_{i=0}^{[n/2]} \frac{2}{(2i+2)!} v_{n-2i} - c^2 v_n = \sum_{\alpha=-1}^{1} \sum_{\beta=\alpha}^{1} \sum_{\gamma=\beta}^{1} \frac{a_{\alpha,\beta,\gamma}}{(n+2)(n+1)} \left\{ \sum_{i=0}^{n} \left(\sum_{k=0}^{n-i} \frac{\alpha^k}{k!} v_{n-i-k} \right) \right.$$
$$\left. \times \left[\sum_{j=0}^{i} \left(\sum_{l=0}^{j} \frac{\beta^l}{l!} v_{j-l} \right) \left(\sum_{m=0}^{i-j} \frac{\gamma^m}{m!} v_{i-j-m} \right) \frac{j!(i-j)!}{i!} \right] \frac{i!(n-i)!}{n!} \right\}, \tag{3.5}$$

where $[n/2]$ is the integer part of $n/2$ and $0^0 = 1$. Equation (3.5) is a recurrence relation between the coefficients $\{v_n\}_{n=0}^{\infty}$. Solving equation (3.5) with $n = 0$, we get $v_0 = 2\sqrt{1-c^2}$. Note that this result is independent of the choice of $a_{\alpha,\beta,\gamma}$, i.e. independent of the model. Letting $v_1 = 0$ and making use of the symmetry of Q, one can show by induction that the coefficients of all odd powers in (3.4) are zero (as we concluded previously on the basis that the outer expansion is odd in z).

To prevent overflow or underflow when evaluating the recurrence relation numerically, we shall work with the normalized coefficients

$$w_n = (-1)^n k_1^{2n} v_{2n}$$

so that the Stokes constant (2.29) for $c \neq 0$ is given by

$$K_1(c) = \lim_{n\to\infty} w_n. \tag{3.6}$$

Reformulating (3.5) in terms of w_n, we use the relation (3.6) to compute K_1 numerically. We truncate the sums involving $1/(2i+2)!$, $1/l!$ and $1/m!$ when these factors become smaller than 10^{-50} and evaluate the sums involving the combinatorial factors in two halves. In the first, the summation index increases from zero to the halfway point, and in the second it decreases from its maximum. This ensures that the combinatorial factors are always decreasing from one step to the next so that they can be accurately determined recursively. We also truncate these sums when the combinatorial factors fall below 10^{-50}.

These expedients result in a numerical routine fast enough to allow for evaluation of the recurrence relation up to very large n; this is essential given the slow convergence of w_n to a constant. Matching (2.18) to (2.28) yields

$$v_{2n} \to (-1)^n k_1^{-2n} [K_1(c) + \sigma(c)/n] \qquad \text{as } n \to \infty,$$

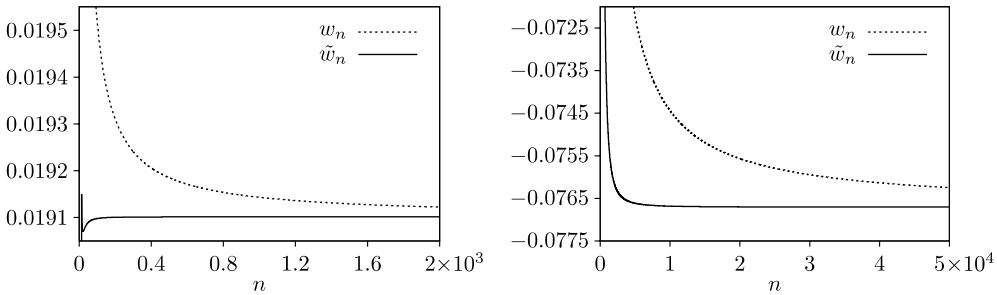

Figure 2. Convergence of the sequence w_n and its accelerated counterpart $\tilde{w}_n$ to the Stokes constant $K_1(c)$. The dashed line depicts the values of w_n and the solid line marks the accelerated sequence $\tilde{w}_n$. Left: the Kevrekidis discretization (1.11) with $c = 0.5$. Right: Speight–Ward discretization (1.8) with $c = 0.005$.

therefore, the rate at which w_n converges to $K_1(c)$ is of order $1/n$:

$$w_n = K_1(c) + \frac{\sigma(c)}{n} + \frac{\tilde{\sigma}(c)}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right). \tag{3.7}$$

Although the convergence of w_n to $K_1(c)$ is extremely slow, we can accelerate the process using (3.7). Defining

$$\tilde{w}_n \equiv w_n + n(w_n - w_{n-1}),$$

we get

$$\tilde{w}_n = K_1(c) - \frac{\sigma(c) + \tilde{\sigma}(c)}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right).$$

The convergence of the sequence $\tilde{w}_n$ is much faster than that of w_n; see figure 2. The relative error

$$E(n) = \frac{\sigma(c) + \tilde{\sigma}(c)}{n^2} \frac{1}{K_1(c)}$$

can be written as

$$E(n) = \frac{n}{2} \frac{\tilde{w}_n - \tilde{w}_{n-1}}{\tilde{w}_n}$$

plus terms of order $1/n^4$. This gives an empirical criterion for the termination of the process. We continued our computations until $E(n)$ reached a value smaller than 10^{-3}, i.e. until the percentage error dropped below 0.1%. For $c > 0.5$, the value of n to which we have to compute in order to achieve this accuracy is less than 100, increasing for smaller values of c to approximately 5000 for $c = 0.005$. Consequently, the above numerical algorithm is not suited to the study of the $c \to 0$ limit and would have to be modified for that purpose.

Figure 3 displays the Stokes constant computed using the above numerical procedure, for the four models of table 1. We see that, in all four models, the Stokes constant $K_1(c)$ vanishes almost nowhere in the region $c \neq 0$. There are, however, several isolated zeros: $K_1(c) = 0$ for $c_0 \approx 0.45$ in the case of the Speight–Ward nonlinearity (1.8) and for $c_0 \approx 0.37$, 0.63 and 0.83 in the case of the Kevrekidis discretization (1.11). Importantly, all of these lie in the region $(c_1, 1)$ where the resonance set (2.20) consists of only one value, $p_1 = \mathrm{i}k_1$. (Here $c_1 \approx 0.22$.) Therefore, there is a sliding kink in the $h \to 0$ limit for each of these isolated values of velocity. Furthermore, strong parallels between our current setting and that of solitons of the fifth-order KdV equation [TP05] suggest that sliding kinks should exist

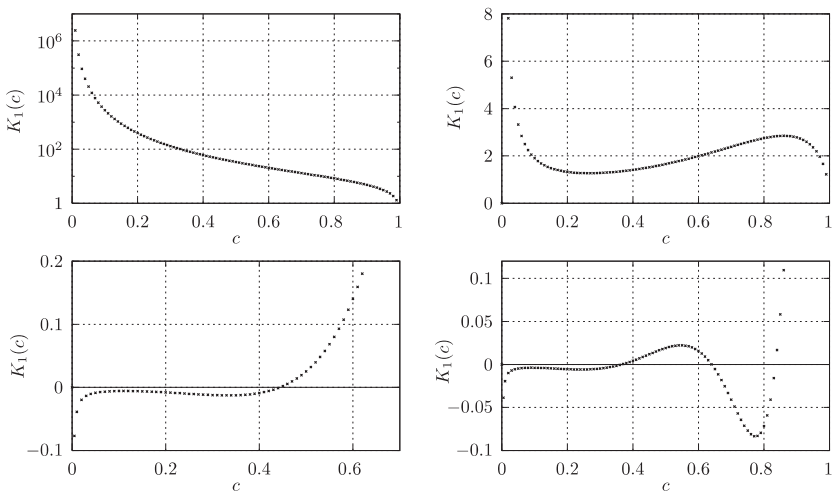

Figure 3. The Stokes constant K_1 as a function of the kink's speed c for the four discretizations of the ϕ^4 model. Clockwise from the top left: one-site (1.6); Bender–Tovbis (1.9); Kevrekidis (1.11); Speight–Ward (1.8).

along a curve on the (c, h) plane emanating from each of the points $(c_0, 0)$. In other words, we conjecture that there is a radiationless kink travelling with a certain particular speed $c_*(h)$ for each h in the case of the Speight–Ward nonlinearity, and that there are three such velocities (for each h) in the case of the Kevrekidis model. For small h, $c_*(h)$ should be close to the above values c_0.

In order to verify the existence of kinks sliding at these isolated velocities by an independent method, we solved the differential advance–delay equation (2.2) numerically. The infinite line was approximated by an interval of length $2L = 200$, with the antiperiodic boundary conditions $\phi(-L) = -\phi(L)$. We made use of Newton's iteration with an eighth-order finite-difference approximation of the second derivative; the step size was chosen to be $h/10$. The continuum solution (1.2) was used as an initial guess.

If we find a solution to the advance–delay equation with $\phi(z)$ decaying to a constant for large positive and negative z, then we regard this solution as (a numerical approximation to) a radiationless travelling kink. We were able to tune c for a fixed value of h so that the radiation was reduced to the order of 10^{-12}, whereupon the finite accuracy of our numerical scheme prevented any further reduction. To make sure that the radiation does vanish rather than reaching a local minimum but remaining nonzero, we plot the average magnitude of the radiation near the ends of the interval as a function of c, for fixed h. This is defined as the average of $[\phi(z) - \overline{\phi}]^2$ over the last 20 units of the interval, where $\overline{\phi}$ is the average value of $\phi(z)$ over these last 20 units. The results are shown in figure 4. Note the straight-line behaviour of the graphs near the isolated values of c; this indicates that the coefficient of the sinusoid superimposed over the kink's flat asymptote crosses through zero (rather than attaining a small but nonzero minimum). The suppression of radiation at the isolated points is thereby verified.

Finally, the last question that we would like to address here is whether the intensity of the radiation from the moving discrete kink depends on the type of discretization. More specifically, we would like to know whether the choice of one of the exceptional discretizations (which, by definition, support translationally invariant *stationary* kinks) serves to reduce the radiation from the *moving* kinks. Speight and Ward have already given an affirmative answer

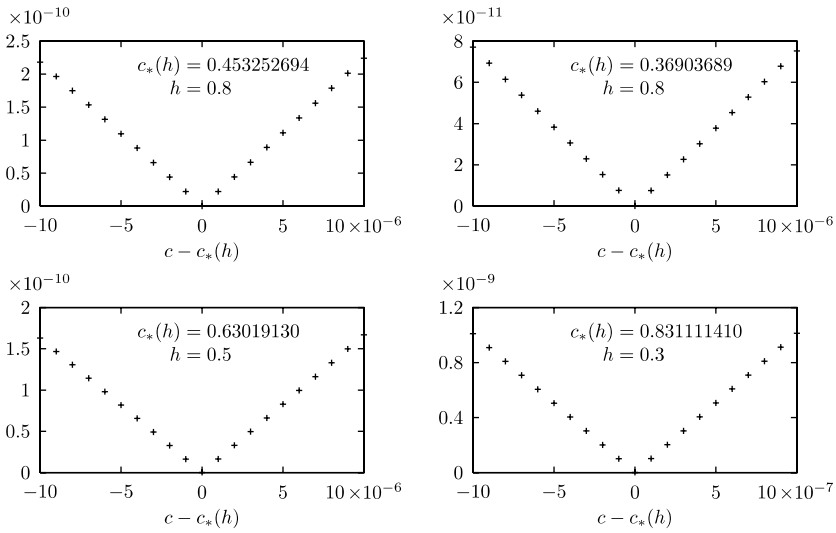

Figure 4. The numerical evidence for the disappearance of radiation at isolated values of $c = c_*$ in Speight and Ward's model (top left panel) and Kevrekidis' model (three other panels). The magnitude of the oscillatory tails, as defined in the text, is plotted as a function of c for a fixed value of h. The minimum radiation is attained at the value $c_*(h)$ which is found numerically. (This $c_*(h)$ is of course slightly different from the value $c_*(0)$ for which the Stokes constant vanishes.)

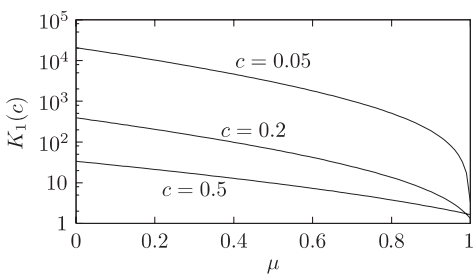

Figure 5. The Stokes constant as a function of μ in the model (3.8). Note the logarithmic scale of the vertical axis.

for their exceptional discretization; here we consider the one-parameter nonlinearity

$$Q = \frac{(1-\mu)}{2} u_n^3 + \frac{\mu}{4} u_n^2 (u_{n+1} + u_{n-1}), \tag{3.8}$$

which interpolates between the one-site nonlinearity (1.6) (for which $\mu = 0$) and the exceptional discretization (1.9) of Bender and Tovbis (for which $\mu = 1$). Figure 5 shows the Stokes constant for the model (3.8), as μ changes from 0 to 1 for fixed values of c. The Stokes constant is indeed seen to be drastically reduced as μ approaches 1—that is, in the limit of the exceptional discretization. (It nonetheless remains nonzero, of course, unless $c = 0$.)

4. Concluding remarks

In this paper we have investigated the existence of sliding kinks—i.e. discrete kinks travelling at a constant velocity over a flat background, without emitting any radiation—in four

discrete versions of the quartic-coupling theory. One of these models is the most common, one-site, discretization. As the overwhelming majority of discrete ϕ^4-equations, it does support travelling kinks, but these kinks radiate and decelerate as a result. The other three discretizations we considered are all exceptional in the sense that they all support one-parameter continuous families of stationary kinks where the free parameter defines the position of the kink relative to the lattice. This property is clearly nongeneric; the translation invariance of the continuous ϕ^4-theory is broken by the discretization and hence in generic disretizations kinks may only be centred at a site or midway between two sites. Since the nonexistence of 'translationally-invariant' and sliding kinks in the generic models can be ascribed to similar factors, namely, the breaking of the translation and Lorentz invariances, it was hoped that the exceptional discretizations might turn out to be equally exceptional from the point of view of sliding kinks. Our approach was based on the computation of the Stokes constants associated with the putative sliding kink in a given equation.

The main conclusion of our work is that sliding kinks do exist in the discrete ϕ^4 theories but only with special, isolated, velocities (which of course depend on h). There is one such velocity in the exceptional model of Speight and Ward and three different sliding velocities in the discretization of Kevrekidis. It is natural to expect that the sliding kinks should play the role of attractors similarly to the fronts moving with 'stable velocities' in dissipative systems; that is, radiating travelling kinks should evolve into kinks travelling with the sliding velocities—if there are such velocities in the system. Not every discretization supports sliding kinks, of course; in particular, no sliding velocities arise for the generic, one-site, nonlinearity and even for the exceptional discretization of Bender and Tovbis.

One natural way of trying to construct the sliding kinks is via power series expansions in powers of c^2; for the exceptional discretizations, this construction can be carried out to any order. This approach was pursued in the recent work of Ablowitz and Musslimani [AM03]. Our results indicate, however, that these power series will not converge and exponentially small terms (terms lying beyond all orders of the power expansion) emerge because of the singular behaviour of the Stokes constant $K_1(c)$ as $c \to 0$. Detailed studies of this singular limit will be presented elsewhere.

The exceptional discretizations have richer underlying symmetries than generic nonlinearities but the 'translation invariance' of the stationary kink alone does not automatically guarantee the existence of the sliding velocities. The exact relation between the 'translational invariance' and mobility of kinks is still to be clarified; at this stage it is worth mentioning that the Stokes constant associated with (and hence the intensity of radiation from) a moving kink is several orders of magnitude smaller in exceptional models than in generic discretizations.

Finally, it is instructive to point out some parallels with an earlier work of Flach *et al* [FZK99] who also studied the phenomenon of kink sliding in Klein–Gordon lattices. In the scheme of [FZK99], one postulates an analytic expression for the sliding kink, $u_n(t) = \phi(n-ct-s)$, with some explicit function $\phi(z)$ and then reconstructs the Klein–Gordon nonlinearity for which this is an exact solution. Our present conclusions are in agreement with the results of these authors who observed that for a given h, the kink may only slide at a particular, isolated, velocity. The two approaches, ours and that of [FZK99], are reciprocal; while we examine the existence of sliding kinks for particular discretizations of the ϕ^4-theory, with fixed parameters independent of the kink's velocity, in the 'inverse method' of [FZK99] one assumes an explicit solution of a particular form but does not have any control over the resulting nonlinearities. Consequently, the discrete Klein–Gordon models generated by the 'inverse method' are not discretizations of the ϕ^4-theory and do include explicit dependence on the velocity of the sliding kink.

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