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One-parameter localized traveling waves in nonlinear Schrödinger lattices

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Abstract

We address the existence of traveling single-humped localized solutions in the spatially discrete nonlinear Schrödinger (NLS) equation. A mathematical technique is developed for analysis of persistence of these solutions from a certain limit in which the dispersion relation of linear waves contains a triple zero. The technique is based on using the Implicit Function Theorem for solution of an appropriate differential advance–delay equation in exponentially weighted spaces. The resulting Melnikov calculation relies on a number of assumptions on the spectrum of the linearization around the pulse, which are checked numerically. We apply the technique to the so-called Salerno model and the translationally invariant discrete NLS equation with a cubic nonlinearity. We show that the traveling solutions terminate in the Salerno model whereas they generally persist in the translationally invariant NLS lattice as a one-parameter family of solutions. These results are found to be in a close correspondence with numerical approximations of traveling solutions with zero radiation tails. Analysis of persistence also predicts the spectral stability of the one-parameter family of traveling solutions under time evolution of the discrete NLS equation.
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1. Introduction

In recent years there has been considerable interest in finding so-called intrinsic localized modes of nonlinear lattice equations in one spatial dimension; see e.g. the focus issue [7]. A particularly delicate question is whether such excitations can be made to move without shedding any radiation. The general answer is that they cannot, due to the presence of the so-called Peierls–Nabarro barrier, which comes about because of the loss of spatial translation symmetry, and the consequent existence of localized modes only for certain fixed locations on the lattice. For example, in the context of discrete nonlinear Schrödinger equations with pure cubic onsite nonlinearity, it is known that site centered localized modes are always stable and intersite localized modes are always unstable [12]. These intersite modes are only stabilized in the continuum limit, therefore excluding the possibility of genuine traveling localized solitary waves as traveling waves would quickly become pinned to a lattice site. However, if we are not restricted to purely onsite cubic terms but are instead free to choose more general discretizations of the nonlinear term in the NLS equation then both intersite and onsite localized modes can be neutrally stable leading to the possibility of finding truly localized traveling waves [17]. Such translationally invariant lattices have also been found in the presence of saturable nonlinearity [14], where moving localized modes were approximated numerically, for a discrete set of wave speeds that were sufficiently large. Such solutions have also been prescribed in the saturable model in the small amplitude limit, by computing the Stokes constant associated with the beyond all orders expansions for splitting of stable and unstable separatrices [15].

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The present paper addresses models that reduce in the continuum limit to the usual one-dimensional nonlinear Schrödinger equation with pure cubic nonlinearity. Specifically we consider

$$i\dot{u}_n + \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + f(u_{n-1}, u_n, u_{n+1}) = 0, \quad n \in \mathbb{Z}, t \in \mathbb{R}, \tag{1.1}$$

where h is the lattice spacing and $f(u_{n-1}, u_n, u_{n-1})$ is represented by the ten-parameter family

$$\begin{aligned}
f &= \alpha_1 |u_n|^2 u_n + \alpha_2 |u_n|^2 (u_{n+1} + u_{n-1}) + \alpha_3 u_n^2 (\bar{u}_{n+1} + \bar{u}_{n-1}) + \alpha_4 (|u_{n+1}|^2 + |u_{n-1}|^2) u_n \\
&\quad + \alpha_5 (\bar{u}_{n+1} u_{n-1} + u_{n+1} \bar{u}_{n-1}) u_n + \alpha_6 (u_{n+1}^2 + u_{n-1}^2) \bar{u}_n + \alpha_7 u_{n+1} u_{n-1} \bar{u}_n \\
&\quad + \alpha_8 (|u_{n+1}|^2 u_{n+1} + |u_{n-1}|^2 u_{n-1}) + \alpha_9 (u_{n+1}^2 \bar{u}_{n-1} + u_{n-1}^2 \bar{u}_{n+1}) + \alpha_{10} (|u_{n+1}|^2 u_{n-1} + |u_{n-1}|^2 u_{n+1}).
\end{aligned} \tag{1.2}$$

Note in particular that f contains all possible cubic terms that reduce to $u|u|^2$ in the continuum limit, while retaining spatial reversibility under $n \to -n$ and gauge invariance under $u_n \to e^{i\theta} u_n$ for any $\theta \in \mathbb{R}$.

When $\alpha_1 = 2(1 - \alpha_2)$, $\alpha_2 \in \mathbb{R}$, and $\alpha_j = 0$ for $3 \le j \le 10$, the nonlinear function (1.2) reduces to the Salerno model [21]

$$f = 2(1 - \alpha_2)|u_n|^2 u_n + \alpha_2 |u_n|^2 (u_{n+1} + u_{n-1}), \tag{1.3}$$

which is a linear interpolation between the cubic dNLS model ($\alpha_2 = 0$) and the Ablowitz–Ladik (AL) model ($\alpha_2 = 1$). Stationary solutions of the Salerno model (1.3) were reviewed in [6,9].

Another interesting model is defined by the nonlinear function (1.2) with

$$\alpha_1 = \alpha_4 + \alpha_6, \qquad \alpha_5 = \alpha_6, \qquad \alpha_7 = \alpha_4 - \alpha_6, \qquad \alpha_{10} = \alpha_8 - \alpha_9, \tag{1.4}$$

where parameters $(\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_8, \alpha_0)$ are arbitrary. This model was derived in [5,17] from the condition that the momentum $M = i \sum_{n \in \mathbb{Z}} \bar{u}_n u_{n+1}$ is conserved in the time evolution of the discrete NLS equation (1.1). Stationary solutions of the model (1.4) were reviewed in [17] where it was found that the single-humped localized solution $u_n(t) = \phi_n e^{i\omega t}$ with $\phi_n : \mathbb{Z} \mapsto \mathbb{R}$ can be interpolated into a continuous (translationally invariant) function $\phi(z) : \mathbb{R} \mapsto C(\mathbb{R})$ with $\phi(hn) = \phi_n$ for any $\omega > 0$ and sufficiently small h. The continuous function $\phi(z)$ solves a relevant advance–delay equation and it represents a continuous deformation between onsite and intersite localized modes.

The purpose of this paper is to consider existence of traveling solutions of the form

$$u_n(t) = \phi(hn - 2ct) e^{i\omega t}, \quad \phi : \mathbb{R} \mapsto \mathbb{C}, \tag{1.5}$$

where ω is temporal frequency and c the velocity of traveling solutions, in particular we are interested in traveling localized modes for which $\phi(z) \to 0$ as $z = hn - 2ct \to \pm\infty$. Direct substitution of (1.5) into the discrete NLS equation (1.1) shows that the function $\phi(z)$ solves the differential advance–delay equation

$$2ic\phi'(z) = \frac{\phi(z + h) - 2\phi(z) + \phi(z - h)}{h^2} - \omega\phi(z) + f(\phi(z - h), \phi(z), \phi(z + h)), \quad z \in \mathbb{R}. \tag{1.6}$$

Hence we are looking for localized solutions $\phi \in H^1(\mathbb{R})$ of the differential advance–delay equation (1.6) that are single-humped, similar to the sech solitons of the continuous NLS equation. Besides parameters of the nonlinear function f and the lattice spacing parameter h, the solution $\phi(z)$ of the differential advance–delay equation (1.6) has two "internal" parameters ω and c. It is convenient to replace (ω, c) by new parameters (κ, β) according to the parametrization

$$\omega = \frac{2}{h}\beta c + \frac{2}{h^2}(\cos\beta \cosh(\kappa) - 1), \qquad c = \frac{1}{h\kappa}\sin\beta \sinh(\kappa) \tag{1.7}$$

and to transform the variables $(z, \phi(z))$ to new variables $(Z, \Phi(Z))$, where

$$\phi(z) = \frac{1}{h}\Phi(Z) e^{i\beta Z}, \quad Z = \frac{z}{h}. \tag{1.8}$$

As a result, the new function $\Phi(Z)$ satisfies the differential advance–delay equation

$$2i\sin\beta \frac{\sinh\kappa}{\kappa} \frac{d\Phi}{dZ} + 2\cos\beta \cosh\kappa \, \Phi = \Phi_+ e^{i\beta} + \Phi_- e^{-i\beta} + f(\Phi_- e^{-i\beta}, \Phi, \Phi_+ e^{i\beta}), \quad Z \in \mathbb{R}, \tag{1.9}$$

where the lattice spacing h has been scaled out and $\Phi_\pm = \Phi(Z \pm 1)$. Bifurcations of traveling wave solutions in the differential advance–delay equation (1.9) at the point $\kappa = 0$ and $\beta = \frac{\pi}{2}$ were studied in [17,18], where a third-order ODE was obtained as a normal form reduction (see also the related paper [10]). The third-order equation is derived in Appendix A for reader's convenience by using a formal expansion of the solution $\Phi(Z)$ in powers of κ along the line $\beta = \frac{\pi}{2}$. The relevant third-order ODE has a local

cubic term $|\Phi|^2\Phi$, unless parameters of the nonlinear function (1.2) satisfy the constraint

$$\alpha_1 + 2\alpha_4 - 2\alpha_5 - 2\alpha_6 + \alpha_7 = 0. \tag{1.10}$$

Such a third-order ODE with a local cubic term $|\Phi|^2\Phi$ does not support existence of single-humped localized solutions, due to the presence of oscillatory tails [8]. Hence we should expect that traveling single-humped localized solutions of the differential advance–delay equation (1.9) exist near the point $\kappa = 0$ and $\beta = \frac{\pi}{2}$ only if the constraint (1.10) is satisfied. Under this condition, the third-order ODE has two cubic terms with first-order derivatives, namely $|\Phi|^2\Phi'$ and $\Phi^2\bar{\Phi}'$, which generally do support the existence of non-trivial single-humped localized solutions [17].[1] Furthermore, according to the review in [17], if $\alpha_2 + 2\alpha_8 - 2\alpha_9 \neq 0$ and

$$\text{either} \quad \alpha_3 - \alpha_8 - \alpha_9 + \alpha_{10} = 0 \quad \text{or} \quad \alpha_2 + 3\alpha_3 - \alpha_8 - 5\alpha_9 + 3\alpha_{10} = 0, \tag{1.11}$$

the relevant third-order ODE reduces to the integrable Hirota or Sasa–Satsuma equations respectively which admit two-parameter families of traveling solutions in (κ, β). If the constraints (1.11) are violated but $\alpha_2 - \alpha_3 + 3\alpha_8 - \alpha_9 - \alpha_{10} > 0$, the third-order ODE has a one-parameter family of solutions along the line $\beta = \frac{\pi}{2}$ for small $\kappa > 0$. Persistence of two-parameter and one-parameter families of solutions beyond the third-order ODE was left open in [17].

A similar problem of persistence of two-parameter family of traveling solution of the AL lattice was considered recently in [2]. The authors applied the necessary condition for persistence of homoclinic orbits given by the Melnikov integral to the Salerno model (1.3) and other reversible perturbations of the AL model and found that the Melnikov integrals were identically zero to leading-order approximation. As a result, this method failed to settle the persistence question for the two-parameter family of traveling solutions of the AL lattice extended into a general discrete NLS equations (1.1) and (1.2).

We shall study here persistence of solutions of the differential advance–delay equation (1.9) near $\beta = \frac{\pi}{2}$ for finite (not necessary small) values of $\kappa > 0$. Assuming that there exists a one-parameter family of traveling solutions on the line $\beta = \frac{\pi}{2}$ for some parameter configurations of the nonlinear function (1.2), we shall find the sufficient conditions for persistence or termination of this solution family with respect to parameter continuations. The analysis that leads to the persistence result also predicts spectral stability of the solution family with respect to time evolution of the discrete NLS equation (1.1). The point $\beta = \frac{\pi}{2}$ is rather special in analysis of the differential advance–delay equation (1.6) since the center manifold of a dynamical system has a multiple zero eigenvalue and no other eigenvalues [18].

As a starting point for our analysis we shall take the one-parameter family of exact traveling solutions known analytically for the case $(\alpha_2, \alpha_3) \in \mathbb{R}^2$ with $\alpha_2 > \alpha_3$ and $\alpha_j = 0$ for $j = 1$ and $4 \leq j \leq 10$ [17]:

$$\Phi(Z) = \frac{\sinh\kappa}{\sqrt{\alpha_2 - \alpha_3}}\,\text{sech}(\kappa Z). \tag{1.12}$$

One can check by direct substitution that the function (1.12) solves the differential advance–delay equation (1.9) for $\kappa > 0$ and $\beta = \frac{\pi}{2}$. It was shown in [13] that the exact sech-solution of the discrete NLS equation (1.1) with (1.2) exists if $\alpha_1 = \alpha_8 = 0$ subject to three more relations on parameters α_j and (β, κ). We checked that none of these exact solutions exist for the models (1.3) and (1.4), except for the case $\alpha_2 > \alpha_3 \neq 0$, when the exact solution is given by the expression (1.12). Therefore, this model is used as the main example for explicit computations of the Melnikov integral.

In this paper, we shall prove that the one-parameter family (1.12) persists generally for $\alpha_3 \neq 0$ with respect to parameter continuations. In particular, the family remains on the straight line $\beta = \frac{\pi}{2}$ if $\alpha_1 = 0$, $\alpha_4 = \alpha_6$, and $\alpha_7 = 2\alpha_5$ and shifts to a local neighborhood of this line if these constraints are not met. We shall also prove that the one-parameter family (1.12) with $\alpha_3 = 0$ does not persist generally with respect to parameter continuations unless $\alpha_1 = 0$, $\alpha_4 = \alpha_6$, and $\alpha_7 = 2\alpha_5$. In particular, it does not persist in the Salerno model (1.3) for $\alpha_2 \neq 1$. These results show that the traveling solutions of the AL lattice[2] are, in this sense, less structurally stable than the traveling solutions of a non-integrable discrete NLS equation.

Our analytical results are illustrated with the numerical studies of the Salerno model (1.3) and the translationally invariant lattice (1.4). We will show that the Salerno model has no traveling solutions near $\beta = \frac{\pi}{2}$ for $\alpha_2 \neq 1$, while the translationally invariant lattice has generally a one-parameter family of traveling solutions near $\beta = \frac{\pi}{2}$ for $\alpha_3 \neq 0$.

The paper is structured as follows. Section 2 formalizes the differential advance–delay equation (1.9) as a system of two real-valued equations for real-valued functions. Section 3 describes analysis of linearized operators associated with the differential advance–delay equations. Section 4 reports analytical results on persistence of the one-parameter family of traveling solutions. Section 5 discusses spectral stability of the one-parameter family of traveling solutions. Section 6 presents relevant numerical approximations of localized solutions of the differential advance–delay equation (1.9) which confirms the theory but also uncovers several avenues for future work. Section 7 summarizes main results of our paper. Appendix A contains formal results on reductions

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[1] Moreover, when the left-hand side of the constraint (1.10) is small, the local cubic term $|\Phi|^2\Phi$ can be brought into a balance with the other two cubic terms so that the resulting ODE still admits non-trivial single-humped localized solutions [19].

[2] The one-parameter family (1.12) is a part of a two-parameter family of exact solutions in the AL lattice.

of the differential advance–delay equation (1.9) and associated linearized operators to the third-order ODE and the associated third-order derivative operators. Appendix B contains perturbation results on bifurcations of resonant poles in the linearized differential advance–delay operators.

2. Formulation of the problem

We shall start by rewriting the differential advance–delay equation (1.9) in a convenient form, which will be suitable for separation of the real and imaginary parts in the solution $\Phi(Z)$. To this end, we obtain

$$\cos \beta (\Phi_+ + \Phi_- - 2 \cosh \kappa \, \Phi) + \mathrm{i} \sin \beta \left(\Phi_+ - \Phi_- - 2 \frac{\sinh \kappa}{\kappa} \frac{\mathrm{d}\Phi}{\mathrm{d}Z} \right) + f_r + \mathrm{i} f_i = 0, \tag{2.1}$$

where

$$\begin{aligned} f_r &= \alpha_1 |\Phi|^2 \Phi + \alpha_2 \cos \beta |\Phi|^2 (\Phi_+ + \Phi_-) + \alpha_3 \cos \beta \, \Phi^2 (\bar{\Phi}_+ + \bar{\Phi}_-) + \alpha_4 (|\Phi_+|^2 + |\Phi_-|^2) \Phi \\ &\quad + \alpha_5 \cos(2\beta)(\bar{\Phi}_+ \Phi_- + \Phi_+ \bar{\Phi}_-) \Phi + \alpha_6 \cos(2\beta)(\Phi_+^2 + \Phi_-^2) \bar{\Phi} + \alpha_7 \Phi_+ \Phi_- \bar{\Phi} \\ &\quad + \alpha_8 \cos \beta (|\Phi_+|^2 \Phi_+ + |\Phi_-|^2 \Phi_-) + \alpha_9 \cos(3\beta)(\Phi_+^2 \bar{\Phi}_- + \Phi_-^2 \bar{\Phi}_+) + \alpha_{10} \cos \beta (|\Phi_+|^2 \Phi_- + |\Phi_-|^2 \Phi_+) \end{aligned}$$

and

$$\begin{aligned} f_i &= \alpha_2 \sin \beta |\Phi|^2 (\Phi_+ - \Phi_-) - \alpha_3 \sin \beta \, \Phi^2 (\bar{\Phi}_+ - \bar{\Phi}_-) - \alpha_5 \sin(2\beta)(\bar{\Phi}_+ \Phi_- - \Phi_+ \bar{\Phi}_-) \Phi \\ &\quad + \alpha_6 \sin(2\beta)(\Phi_+^2 - \Phi_-^2) \bar{\Phi} + \alpha_8 \sin \beta (|\Phi_+|^2 \Phi_+ - |\Phi_-|^2 \Phi_-) + \alpha_9 \sin(3\beta)(\Phi_+^2 \bar{\Phi}_- - \Phi_-^2 \bar{\Phi}_+) \\ &\quad - \alpha_{10} \sin \beta (|\Phi_+|^2 \Phi_- - |\Phi_-|^2 \Phi_+). \end{aligned}$$

The complex-valued differential advance–delay equation (2.1) at $\beta = \frac{\pi}{2}$ can be reduced to the scalar equation for real-valued functions $\Phi \in \mathbb{R}$ provided that

$$\alpha_1 = 0, \qquad \alpha_4 = \alpha_6, \qquad \alpha_7 = 2\alpha_5. \tag{2.2}$$

In this case, the system (2.1) is replaced by the scalar equation

$$2 \frac{\sinh \kappa}{\kappa} \frac{\mathrm{d}\Phi}{\mathrm{d}Z} = [1 + (\alpha_2 - \alpha_3) \Phi^2 + \alpha_8 (\Phi_+^2 + \Phi_+ \Phi_- + \Phi_-^2) - (\alpha_9 + \alpha_{10}) \Phi_+ \Phi_-](\Phi_+ - \Phi_-). \tag{2.3}$$

We shall add an assumption about existence of non-trivial solutions in the scalar equation (2.3), which allows us to pose two main questions on persistence of these solutions.

Assumption 2.1. There exists a parameter configuration in $(\alpha_2, \alpha_3, \alpha_8, \alpha_9, \alpha_{10})$ such that the differential advance–delay equation (2.3) has a single-humped solution $\Phi_0(Z)$ for any $\kappa > 0$ with the property

$$\Phi_0 \in H^1(\mathbb{R}): \quad \Phi_0(-Z) = \Phi_0(Z), \qquad \lim_{|Z| \to \infty} \mathrm{e}^{\kappa |Z|} \Phi_0(Z) = c_0 \tag{2.4}$$

for some $0 < c_0 < \infty$.

Remark 2.2. Due to translational invariance of the differential advance–delay equation, the family of even solutions $\Phi_0(Z)$ can be extended into a one-parameter family $\Phi_0(Z - s)$, $\forall s \in \mathbb{R}$. It is however convenient for the persistence analysis to set $s = 0$ in the rest of the article.

Example 2.3. When $\alpha_8 = \alpha_9 = \alpha_{10} = 0$, the single-humped localized solution $\Phi_0(Z)$ of the scalar equation (2.3) is known in the analytic form (1.12) for any $\kappa > 0$ and $\alpha_2 > \alpha_3$.

Question 2.4. Is the solution $\Phi_0(Z)$ structurally stable in the scalar equation (2.3) with respect to parameter continuations in $(\alpha_2, \alpha_3, \alpha_8, \alpha_9, \alpha_{10})$?

Question 2.5. Is the one-parameter family of solutions $\Phi_0(Z)$ structurally stable in the system (2.1) near $\beta = \frac{\pi}{2}$ with respect to parameter continuations in $(\alpha_1, \ldots, \alpha_{10})$?

We shall answer these questions by using the Implicit Function Theorem for the differential advance–delay equations (2.1) and (2.3) in a local neighborhood of the point Φ_0 in function space $H^1(\mathbb{R})$. To do so, we define the Frechet derivative of the system (2.1) at $\beta = \frac{\pi}{2}$ and $\Phi = \Phi_0$ for any $\kappa > 0$. When $\alpha_j = 0$ for $j = 1$ and $4 \leq j \leq 7$, the Frechet derivative diagonalizes into two

linearized operators for real and imaginary parts of the perturbation to the solution $\Phi_0(Z)$:

$$
L_+ = -2\frac{\sinh\kappa}{\kappa}\frac{d}{dZ} + [1 + (\alpha_2 - \alpha_3)\Phi^2 - 2(\alpha_9 + \alpha_{10})\Phi_+\Phi_-](\delta_+ - \delta_-)
$$
$$
+ 2(\alpha_2 - \alpha_3)\Phi(\Phi_+ - \Phi_-) + 3\alpha_8(\Phi_+^2\delta_+ - \Phi_-^2\delta_-) - (\alpha_9 + \alpha_{10})(\Phi_+^2\delta_- - \Phi_-^2\delta_+) \tag{2.5}
$$

and

$$
L_- = -2\frac{\sinh\kappa}{\kappa}\frac{d}{dZ} + [1 + (\alpha_2 + \alpha_3)\Phi^2 - 2\alpha_9\Phi_+\Phi_-](\delta_+ - \delta_-) - 2\alpha_3\Phi(\Phi_+ - \Phi_-)
$$
$$
+ \alpha_8(\Phi_+^2\delta_+ - \Phi_-^2\delta_-) + (\alpha_9 - \alpha_{10})(\Phi_+^2\delta_- - \Phi_-^2\delta_+), \tag{2.6}
$$

where operators $\delta_\pm$ act on a continuous function $U(Z)$ of $Z \in \mathbb{R}$ such that $\delta_\pm U = U(Z \pm 1)$ and the subscript of $\Phi_0(Z)$ is dropped for simplicity of notation. Assumptions on the spectrum of linearized operators $L_\pm$ and the relevant analysis are described in Section 3. Formal reductions of $L_\pm$ to third-order derivative operators (A.2)–(A.5) are reported in Appendix A.

One can ask why the line $\beta = \frac{\pi}{2}$ is so special in the existence of solutions of the differential advance–delay equation (2.1) and if there exists any other curves on the parameter plane (κ, β) which can be analyzed by a similar method. For instance, when $\beta = 0$, the system (2.1) for any set of parameters α's admits a reduction to the scalar advance–delay equation for real-valued solutions $\Phi(Z)$:

$$
[1 + (\alpha_2 + \alpha_3)\Phi^2 + \alpha_8(\Phi_+^2 - \Phi_+\Phi_- + \Phi_-^2) + (\alpha_9 + \alpha_{10})\Phi_+\Phi_-](\Phi_+ + \Phi_-)
$$
$$
+ [-2\cosh\kappa + \alpha_1\Phi^2 + (\alpha_4 + \alpha_6)(\Phi_+^2 + \Phi_-^2) + (\alpha_7 + 2\alpha_5)\Phi_+\Phi_-]\Phi = 0. \tag{2.7}
$$

This reduction corresponds to the stationary solutions (1.5) with $c = 0$ and was studied in [17] in detail. To explain why the reduction (2.7) is useless for analysis of traveling solutions with $c \neq 0$ ($\beta \neq 0$), consider the unperturbed homogeneous linear equation

$$
\cos\beta(\Phi_+ + \Phi_- - 2\cosh\kappa\,\Phi) + i\sin\beta\left(\Phi_+ - \Phi_- - 2\frac{\sinh\kappa}{\kappa}\frac{d\Phi}{dZ}\right) = 0. \tag{2.8}
$$

Applying the Laplace transform to the linear equation (2.8), we obtain the dispersion relation

$$
D(p; \kappa, \beta) \equiv \cos\beta(\cosh p - \cosh\kappa) + i\sin\beta\left(\sinh p - \frac{\sinh\kappa}{\kappa}p\right) = 0, \tag{2.9}
$$

where p is the parameter of the Laplace transform. Roots with $\operatorname{Re}(p) > 0$ and $\operatorname{Re}(p) < 0$ correspond to the unstable and stable manifolds, respectively, resulting in the spatial decay of the solution $\Phi(Z)$, while roots with $\operatorname{Re}(p) = 0$ correspond to the center manifold resulting in the oscillatory non-decaying behavior of the solution $\Phi(Z)$. For any β, the dispersion relation (2.9) always possesses a pair of real roots $p = \pm\kappa$, which provides a localization of the single-humped solution $\Phi(Z)$. However, for any $\beta \neq 0$, the dispersion relation (2.9) also has roots with $\operatorname{Re}(p) = 0$ which destroy localization of $\Phi(Z)$.

If $\beta = \frac{\pi}{2}$, the only root on the imaginary axis is at $p = 0$. When $\beta_0 < \beta < \frac{\pi}{2}$ with $\beta_0 \approx \frac{\pi}{13}$, there exists a single positive imaginary root $p = ik$ with $k > 0$. When $0 < \beta < \beta_0$, the number of imaginary roots increases dramatically as is illustrated in Fig. 1 corresponding to decreasing wave speed c. The limit $\beta \to 0^+$ is singular: no roots exist for $\beta = 0$, but finitely many roots with large values of $p = ik$ exist for any fixed small value of $\beta \neq 0$. Therefore, persistence of solutions of the scalar advance–delay equation (2.7) for $\beta \neq 0$ is a delicate, likely unsolvable problem of analysis.

On the other hand, persistence of solutions for $\beta = \frac{\pi}{2}$ is a relatively simple problem because the root of $D(p; \kappa, \frac{\pi}{2})$ is located at the origin $p = 0$, the system (2.1) can be reduced to the scalar equation (2.3) under the constraints (2.2), and there are cases when solutions are known in the analytic form, e.g. the exact solution (1.12). Therefore, we restrict our analysis to the particular case $\beta = \frac{\pi}{2}$. However, we anticipate that the analysis can be extended to the domain $\beta_0 < \beta < \frac{\pi}{2}$, where $D(p; \kappa, \beta)$ has only one purely imaginary root $p = ik$ with $k > 0$ (the white region in the right-hand panel of Fig. 1). Traveling solutions in this domain are likely to occur as a result of the bifurcation of codimension one, i.e. they exist generally as one-parameter families on the plane (κ, β) for fixed values of parameters α's.

3. Analysis of linearized differential advance–delay equations

Both operators L_+ and L_- in (2.5) and (2.6) can be written in the general form

$$
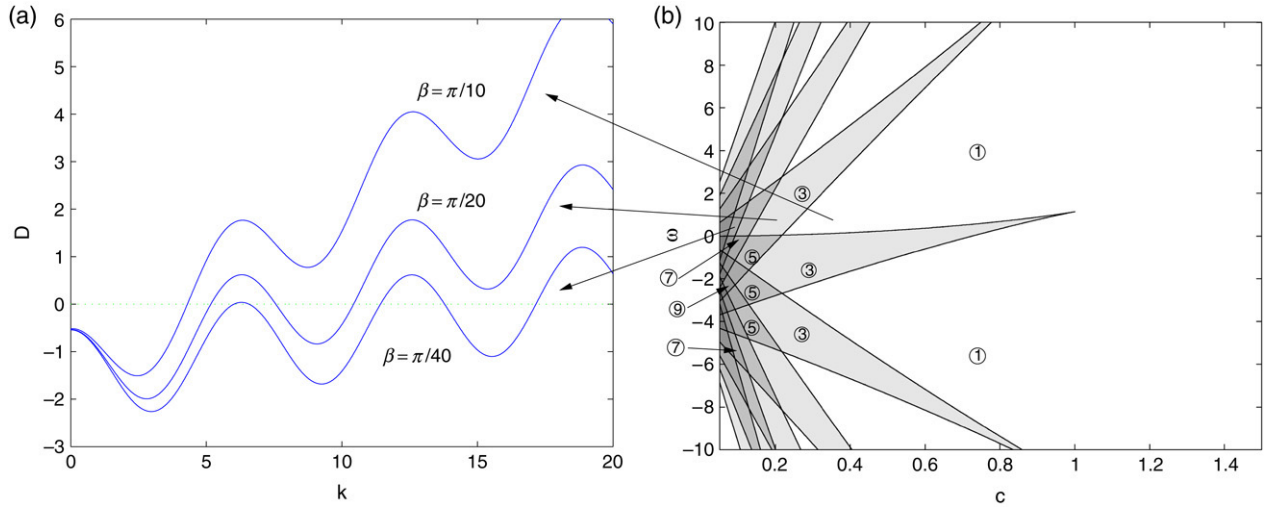L = -2\frac{\sinh\kappa}{\kappa}\frac{d}{dZ} + [1 + V_+(Z)]\delta_+ - [1 + V_-(Z)]\delta_- + V_0(Z), \tag{3.1}
$$

Fig. 1. Left: the graph of $D(ik; \kappa, \beta)$ versus k for $\kappa = 1$ and different β. Right: parameter plane (ω, c) divided into domains. Arrows show correspondence between the domains of (ω, c) with the same number of real-valued roots k of $D(ik; \kappa, \beta)$. This number of roots is indicated by the depth of shading and also the number indicated in the small circles. As c approaches zero (which is equivalent to $\beta \to 0$) the number of real roots k increases.

where $V_0(Z)$ and $V_\pm(Z)$ are bounded exponentially decaying potentials. The operator L maps continuously $H^1(\mathbb{R})$ to $L^2(\mathbb{R})$ equipped with the standard inner product

$$\forall f, g \in L^2(\mathbb{R}): \quad (f, g) = \int_\mathbb{R} \bar{f}(Z) g(Z) \mathrm{d}Z. \tag{3.2}$$

Related to the inner product (3.2), the adjoint operator L^* satisfies $(W, LU) = (L^*W, U)$ for any functions $U(Z)$ and $W(Z)$ in $H^1(\mathbb{R})$. The adjoint operator is written in the general form

$$L^* = 2\frac{\sinh \kappa}{\kappa} \frac{\mathrm{d}}{\mathrm{d}Z} + [1 + V_+(Z - 1)]\delta_- - [1 + V_-(Z + 1)]\delta_+ + V_0(Z), \tag{3.3}$$

where $V_+(Z - 1)$ may involve $\Phi(Z - 2)$ and $V_-(Z + 1)$ may involve $\Phi(Z + 2)$.

According to Assumption 2.1, the function $\Phi(Z)$ is even, such that $\Phi_+ - \Phi_-$ is odd and $\Phi_+ \Phi_-$ is even on $Z \in \mathbb{R}$. It is clear from explicit expressions of the operators $L_\pm$ in (2.5) and (2.6) that both operators $L_\pm$ change the symmetry of the eigenfunction $U(Z)$, such that

$$L_\pm : H^1_{\mathrm{ev}}(\mathbb{R}) \mapsto L^2_{\mathrm{odd}}(\mathbb{R}), \qquad L_\pm : H^1_{\mathrm{odd}}(\mathbb{R}) \mapsto L^2_{\mathrm{ev}}(\mathbb{R}), \tag{3.4}$$

where the even and odd extensions of $H^1(\mathbb{R})$ are defined by

$$H^1_{\mathrm{ev}}(\mathbb{R}) = \{U \in H^1(\mathbb{R}) : U(-Z) = U(Z)\},$$
$$H^1_{\mathrm{odd}}(\mathbb{R}) = \{U \in H^1(\mathbb{R}) : U(-Z) = -U(Z)\}$$

and similar for $L^2(\mathbb{R})$. Eigenvalues of the operator L in $H^1(\mathbb{R})$ hold some symmetry properties, which are standard for linearized Hamiltonian systems.

Lemma 3.1. *Let λ_0 be an eigenvalue of operator L with the eigenvector $U \in H^1(\mathbb{R})$. Then, $-\lambda_0$, $\bar{\lambda}_0$ and $-\bar{\lambda}_0$ are also eigenvalues of operator L with eigenvectors $U(-Z)$, $\bar{U}(Z)$, and $\bar{U}(-Z)$.*

Proof. Eigenvalue $-\lambda_0$ exists due to the symmetry property (3.4). Eigenvalue $\bar{\lambda}_0$ exists due to the fact that L has real-valued coefficients. Eigenvalue $-\bar{\lambda}_0$ exists as a consequence of the above two symmetries. $\blacksquare$

Example 3.2. When $\alpha_8 = \alpha_9 = \alpha_{10} = 0$ and $\Phi(Z)$ is given by the exact solution (1.12), the operators $L_\pm$ are written in explicit form

$$L_+ = -2\frac{\sinh \kappa}{\kappa} \frac{\mathrm{d}}{\mathrm{d}Z} + [1 + \sinh^2 \kappa \, \mathrm{sech}^2(\kappa Z)](\delta_+ - \delta_-) - 4\sinh^3 \kappa \, \sinh(\kappa Z) \, \mathrm{sech}(\kappa Z) \, \mathrm{sech}(\kappa Z + \kappa) \, \mathrm{sech}(\kappa Z - \kappa),$$

$$L_- = -2\frac{\sinh \kappa}{\kappa} \frac{\mathrm{d}}{\mathrm{d}Z} + [1 + \sinh^2 \kappa \, \mathrm{sech}^2(\kappa Z)](\delta_+ - \delta_-) + 2\nu \sinh^2 \kappa \, \mathrm{sech}^2(\kappa Z)(\delta_+ - \delta_-)$$
$$+ 4\nu \sinh^3 \kappa \, \sinh(\kappa Z) \, \mathrm{sech}(\kappa Z) \, \mathrm{sech}(\kappa Z + \kappa) \, \mathrm{sech}(\kappa Z - \kappa),$$

where $\nu = \alpha_3/(\alpha_2 - \alpha_3)$. When $\alpha_3 = 0$, the linearized operators $L_\pm$ are associated with the integrable AL lattice and the spectrum of these operators is constructed in the explicit form in Example 3.8.

3.1. Absolutely continuous part of the spectrum of $L_\pm$

We characterize all parts of the spectrum of the unbounded differential advance–delay operator L in (3.1), where L stands for either L_+ or L_-. Let us denote the potential-free operator by L_0, such that

$$L_0 = -2\frac{\sinh\kappa}{\kappa}\frac{\mathrm{d}}{\mathrm{d}Z} + \delta_+ - \delta_-. \tag{3.5}$$

The operator $L - L_0$ contains potentials $V_0(Z)$ and $V_\pm(Z)$, which are bounded, exponentially decaying functions of $Z \in \mathbb{R}$. Therefore, $L - L_0$ is a relatively compact perturbation to the unbounded operator L_0. Results from perturbation theory [11] imply that the continuous spectra of L and L_0 coincide, the residual spectrum of L is empty, and the point spectrum of L contains a finite number of isolated or embedded eigenvalues. The entire spectrum of L_0 in $H^1(\mathbb{R})$ is absolutely continuous. Its location can be found with the Fourier transform at $\lambda = \lambda_0(k)$ for $k \in \mathbb{R}$, where

$$\lambda_0(k) = 2\mathrm{i}\left(\sin k - k\frac{\sinh\kappa}{\kappa}\right), \tag{3.6}$$

is a one-to-one map from $k \in \mathbb{R}$ to $\lambda \in \mathrm{i}\mathbb{R}$. The properties of the spectrum of L are summarized as follows:

Proposition 3.3. *The spectrum of L in $H^1(\mathbb{R})$ consists of the point spectrum $\sigma_p(L)$ and a single-branched continuous spectrum $\sigma_c(L) = \{\mathrm{i}\mathbb{R}\}$.*

Some of the eigenvalues of $\sigma_p(L)$ are embedded into $\sigma_c(L)$, for instance, $0 \in \sigma_p(L)$ due to the exact relations (3.16) below. In order to separate eigenvalues of $\sigma_p(L)$ and the continuous spectrum $\sigma_c(L)$, we use the technique of exponential weighted spaces pioneered in [16]. Let

$$H_\mu^1(\mathbb{R}) = \{U \in H_{\mathrm{loc}}^1(\mathbb{R}) : \mathrm{e}^{\mu Z}U(Z) \in H^1(\mathbb{R})\}, \tag{3.7}$$

for $0 < \mu < \mu_0$ with some $\mu_0 > 0$. The adjoint space is $H_{-\mu}^1(\mathbb{R})$.

Lemma 3.4. *Let $0 < \mu < \mu_0$ with $\mu_0 = \min\{\kappa, \cosh^{-1}(\sinh\kappa/\kappa)\}$. Then, under Assumption 2.1, the continuous spectrum of L in $H_\mu^1(\mathbb{R})$ and L^* in $H_{-\mu}^1(\mathbb{R})$ is located along a curve contained within the strip $\mathrm{Re}(\lambda) \in [\lambda_-, \lambda_+]$ for some $0 < \lambda_- < \lambda_+$.*

Proof. If $|\mu| < \kappa$, the potential terms in the operator $L_\mu = \mathrm{e}^{\mu Z}L\mathrm{e}^{-\mu Z}$ decay exponentially as $|Z| \to \infty$. By the perturbation theory for linear unbounded operators [11], the continuous spectrum $\sigma_c(L_\mu)$ in $H^1(\mathbb{R})$ coincides with that of $\mathrm{e}^{\mu Z}L_0\mathrm{e}^{-\mu Z}$ in $H^1(\mathbb{R})$, i.e. it is located at $\sigma_c(L_\mu) = \{\lambda \in \mathbb{C} : \lambda = \lambda_\mu(k), k \in \mathbb{R}\}$, where

$$\lambda_\mu(k) = \lambda_0(k + \mathrm{i}\mu) = 2\left[\mu\frac{\sinh\kappa}{\kappa} - \sinh\mu\cos k\right] + 2\mathrm{i}\left[\cosh\mu\sin k - k\frac{\sinh\kappa}{\kappa}\right]. \tag{3.8}$$

In particular, $\mathrm{Re}\,\lambda_\mu(k) > 0$ for $0 < \mu < \kappa$ and $\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{Im}\,\lambda_\mu(k) < 0$ if $|\mu| < \cosh^{-1}(\sinh\kappa/\kappa)$. Therefore, if $0 < \mu < \mu_0$ and $\mu_0 = \min\{\kappa, \cosh^{-1}(\sinh\kappa/\kappa)\}$, the branch of the continuous spectrum is a one-to-one map from $k \in \mathbb{R}$ to $\lambda \in \mathbb{C}$, where λ oscillates in the strip $\mathrm{Re}\,\lambda \in [\lambda_-, \lambda_+]$ with

$$\lambda_\pm = 2\mu\left[\frac{\sinh\kappa}{\kappa} \pm \frac{\sinh\mu}{\mu}\right] > 0.$$

The location of $\sigma_c(L_\mu)$ is illustrated on Fig. 2. The continuous spectrum $\sigma_c(L_{-\mu}^*)$ of the adjoint operator $L_{-\mu}^* = \mathrm{e}^{-\mu Z}L^*\mathrm{e}^{\mu Z}$ in $H^1(\mathbb{R})$ coincides with that of $\mathrm{e}^{-\mu Z}L_0^*\mathrm{e}^{\mu Z}$ in $H^1(\mathbb{R})$, i.e. it is located along the curve $\lambda = -\lambda_0(k - \mathrm{i}\mu) = \bar{\lambda}_\mu(k) = \lambda_\mu(-k)$ on $k \in \mathbb{R}$. This curve on the λ-plane is the same as $\lambda = \lambda_\mu(k)$ but it is traversed in the reverse direction as k increases. ∎

Definition 3.5. *Let $U_\mu(Z; k)$ and $W_\mu(Z; k)$ denote eigenfunctions of the continuous spectrum of L in $H_\mu^1(\mathbb{R})$ and L^* in $H_{-\mu}^1(\mathbb{R})$ with $0 < \mu < \mu_0$, according to the equations*

$$LU_\mu(Z; k) = \lambda_\mu(k)U_\mu(Z; k), \qquad L^*W_\mu(Z; k) = \bar{\lambda}_\mu(k)W_\mu(Z; k), \tag{3.9}$$

such that $\lambda_\mu(k)$ is given by (3.8) and $U_\mu(Z; k)$, $W_\mu(Z; k)$ are normalized by the asymptotic behavior

$$\lim_{Z\to\infty}\mathrm{e}^{-\mathrm{i}kZ+\mu Z}U_\mu(Z; k) = 1, \qquad \lim_{Z\to\infty}\mathrm{e}^{-\mathrm{i}kZ-\mu Z}W_\mu(Z; k) = 1. \tag{3.10}$$
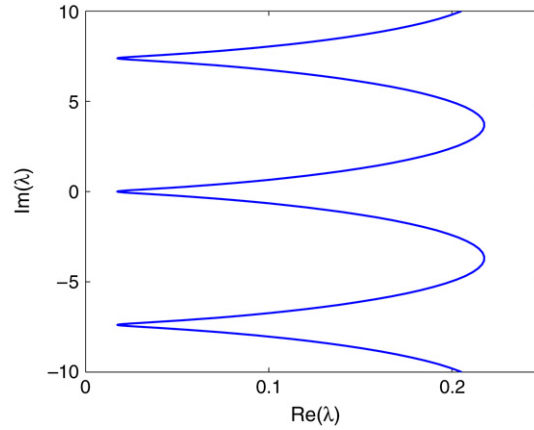
Fig. 2. Schematic depiction of the location of $\sigma_c(L_\mu)$ for $0 < \mu < \mu_0$ in the complex λ-plane.

Let the scattering data $\{a_\mu(k), b_\mu(k)\}$ be defined by the asymptotic behavior

$$\lim_{Z \to -\infty} \mathrm{e}^{-\mathrm{i}kZ + \mu Z} U_\mu(Z; k) = a_\mu(k), \qquad \lim_{Z \to -\infty} \mathrm{e}^{-\mathrm{i}kZ - \mu Z} W_\mu(Z; k) = b_\mu(k), \tag{3.11}$$

assuming that the limits exist.

Assumption 3.6. There exist constants $\Lambda_- < 0 < \Lambda_+$, such that the point spectrum $\sigma_p(L)$ in $H^1_\mu(\mathbb{R})$ for any $0 \le \mu < \mu_0$ does not include eigenvalues in the strip $\Lambda_- < \lambda < \Lambda_+$, except for the zero eigenvalue $\lambda = 0$.

Lemma 3.7. *Under Assumptions 2.1 and 3.6, the scattering data $\{a_\mu(k), b_\mu(k)\}$ in Definition 3.5 are bounded and non-zero for any $k \in \mathbb{R}$ and $0 < \mu < \mu_1$, where μ_1 is sufficiently small that $0 < \lambda_+ < \Lambda_+$.*

Proof. Existence of bounded eigenfunctions $U_\mu(Z; k)$ for operators L in $H^1_\mu(\mathbb{R})$ with $0 < \mu < \mu_0$ follows from the wave function formalism [11], because the bounded potential terms $V_0(Z)$ and $V_\pm(Z)$ decay exponentially quickly. Since the continuous spectrum $\sigma_c(L)$ in $H^1_\mu(\mathbb{R})$ has a single branch along $\lambda = \lambda_\mu(k)$ which is uniquely parameterized by $k \in \mathbb{R}$, the bounded eigenfunctions $\mathrm{e}^{\mu Z} U_\mu(Z; k)$ have the limiting behavior $\mathrm{e}^{\mathrm{i}kZ}$ as $|Z| \to \infty$. If the normalization (3.10) is introduced and the scattering data $a_\mu(k)$ is defined by (3.11), the scattering data may vanish or be unbounded for a value $k = k_0$, for which the eigenfunction $\mathrm{e}^{\mu Z} U_\mu(Z; k_0)$ decays at one of the infinities. However, such marginal eigenfunction would become the eigenfunction of the point spectrum of L in $H^1_\mu(\mathbb{R})$ for either smaller or larger value of μ. By Assumption 3.6, no such eigenfunctions may exist at least in $0 < \mu < \mu_1$, and therefore, the scattering data $a_\mu(k)$ is always bounded and non-zero on $k \in \mathbb{R}$. The same proof applies to the operator L^* in $H^1_{-\mu}(\mathbb{R})$ with $0 < \mu < \mu_1$ for eigenfunctions $W_\mu(Z; k)$ and scattering data $b_\mu(k)$. ∎

Example 3.8. We continue Example 3.2 and consider operators L_+ and L_- associated with the integrable AL lattice, when $\alpha_2 = 1$ and $\alpha_j = 0$ for all other j's (i.e. $\nu = 0$). Due to the integrability of the AL lattice [4], one can expect that the complete set of eigenfunctions of these operators is available in analytic form. Indeed, direct substitution with MATHEMATICA shows that the eigenfunctions $U_0(Z; k)$ of the continuous spectrum of L_+ for $\lambda = \lambda_0(k)$ are given by

$$U_0(Z; k) = \mathrm{e}^{\mathrm{i}kZ} \frac{1 - \cos k \cosh \kappa + \mathrm{i} \sin k \sinh \kappa \tanh(\kappa Z) + \sinh^2 \kappa \, \mathrm{sech}^2(\kappa Z)}{1 - \cos k \cosh \kappa + \mathrm{i} \sin k \sinh \kappa}, \tag{3.12}$$

while the eigenfunctions $U_0(Z; k)$ of the continuous spectrum of L_- for $\lambda = \lambda_0(k)$ are given by

$$U_0(Z; k) = \mathrm{e}^{\mathrm{i}kZ} \frac{1 - \cos k \cosh \kappa + \mathrm{i} \sin k \sinh \kappa \tanh(\kappa Z)}{1 - \cos k \cosh \kappa + \mathrm{i} \sin k \sinh \kappa}. \tag{3.13}$$

For both operators, the same spectral data $a_0(k)$ is given by

$$a_0(k) = \frac{1 - \cos k \cosh \kappa - \mathrm{i} \sin k \sinh \kappa}{1 - \cos k \cosh \kappa + \mathrm{i} \sin k \sinh \kappa} = \frac{1 - \cosh(\kappa + \mathrm{i}k)}{1 - \cosh(\kappa - \mathrm{i}k)}. \tag{3.14}$$

All eigenfunctions can be analytically extended in the strip $-\kappa < \mathrm{Im}(k) < \kappa$, such that the eigenfunctions in the weighted space $H^1_\mu(\mathbb{R})$ are given by $U_\mu(Z; k) = U_0(Z; k + \mathrm{i}\mu)$ with $0 < \mu < \mu_0$. In particular, the eigenfunctions and the spectral data are bounded and non-zero for any $k \in \mathbb{R}$ and $-\kappa < \mu < \kappa$. This property implies that no point spectrum exists in this strip, i.e. Assumption 3.6 is satisfied for the operators $L_\pm$ associated with the integrable AL lattice.

Lemma 3.9. *Under Assumptions 2.1 and 3.6, the set of eigenfunctions $U_\mu(Z; k)$ and $W_\mu(Z; k)$ in Definition 3.5 satisfies the orthogonality relation*

$$\int_{\mathbb{R}} \bar{W}_\mu(Z; k') U_\mu(Z; k) dZ = \frac{4\pi \sinh \kappa}{\sinh \kappa - \kappa \cosh \mu \cos k} \delta(k' - k), \tag{3.15}$$

where $\delta(k)$ is the Dirac delta function. In addition, $\bar{b}_\mu(k) = 1/a_\mu(k)$ on $k \in \mathbb{R}$.

Proof. Let us consider the homogeneous equations (3.9) for $U_\mu(Z; k)$ and $\bar{W}_\mu(Z; k')$, where $k, k' \in \mathbb{R}$. By integrating $\bar{W}_\mu(Z; k') L U_\mu(Z; k) - U_\mu(Z; k) L^* \bar{W}_\mu(Z; k')$ on $Z \in [-L, L]$ and extending the limit $L \to \infty$, we obtain that

$$\int_{\mathbb{R}} \bar{W}_\mu(Z; k') U_\mu(Z; k) dZ = -2 \frac{\sinh \kappa}{\kappa} \left(\lim_{L \to \infty} \frac{\bar{W}_{k'}(L) U_k(L)}{\lambda_\mu(k) - \lambda_\mu(k')} - \lim_{L \to -\infty} \frac{\bar{W}_{k'}(L) U_k(L)}{\lambda_\mu(k) - \lambda_\mu(k')} \right).$$

By using the asymptotic representations (3.10) and (3.11) of the eigenfunctions $U_\mu(Z; k)$ and $\bar{W}_\mu(Z; k')$ as $|Z| \to \infty$ and the property of the generalized functions

$$\lim_{L \to \pm\infty} \frac{e^{i(k-k')L}}{i(k - k')} = \pm \pi \delta(k - k'),$$

we obtain the orthogonality relation

$$\int_{\mathbb{R}} \bar{W}_\mu(Z; k') U_\mu(Z; k) dZ = \frac{2\pi \sinh \kappa [1 + a_\mu(k) \bar{b}_\mu(k)]}{\sinh \kappa - \kappa \cosh \mu \cos k}.$$

Let us now consider the homogeneous equations (3.9) for $U_\mu(Z; k)$ and $\bar{W}_\mu(Z; k)$ for any $k \in \mathbb{R}$. By using the same integration on $Z \in [-L, L]$ and extending the limit $L \to \infty$, we obtain that

$$\frac{2 \sinh \kappa}{\kappa} [a_\mu(k) \bar{b}_\mu(k) - 1] = 0.$$

Therefore, $\bar{b}_\mu(k) = 1/a_\mu(k)$ and the orthogonality relation takes the form (3.15).   ∎

3.2. Kernels of $L_\pm$

One can check directly from the scalar equation (2.3) and the linearized operators (2.5) and (2.6) that both L_+ and L_- have a non-empty geometric kernel in $H^1(\mathbb{R})$ with eigenfunctions Φ' and Φ respectively. That is

$$L_+ \Phi'(Z) = 0, \qquad L_- \Phi(Z) = 0. \tag{3.16}$$

In addition, L_+ has a non-empty generalized kernel in $H^1(\mathbb{R})$ with the eigenfunction

$$L_+ \frac{\partial \Phi}{\partial \kappa} = \frac{2(\kappa \cosh \kappa - \sinh \kappa)}{\kappa^2} \Phi'(Z). \tag{3.17}$$

In the case of the integrable AL lattice, when $\alpha_2 = 1$ and $\alpha_j = 0$ for all other j's (see Example 3.2), L_- has a non-empty generalized kernel in $H^1(\mathbb{R})$ with the eigenfunction

$$L_- Z \Phi(Z) = 2 \left(\cosh \kappa - \frac{\sinh \kappa}{\kappa} \right) \Phi(Z). \tag{3.18}$$

By Assumption 2.1, all eigenfunctions in (3.16)–(3.18) decay exponentially with decay rate κ as $|Z| \to \infty$. Therefore, these eigenfunctions remain in $H^1_\mu(\mathbb{R})$ for $0 \le \mu < \mu_0$. Let us denote eigenfunctions of the geometric kernel of L in $H^1(\mathbb{R})$ by $u_0(Z)$ and eigenfunctions of the generalized kernel of L in $H^1(\mathbb{R})$ by $u_1(Z)$, such that

$$L u_0 = 0, \qquad L u_1 = u_0.$$

By explicit construction, we can see that $u_0(Z)$ is odd for L_+ and even for L_- on $Z \in \mathbb{R}$, while $u_1(Z)$ is even for L_+. When $u_1(Z)$ exists for L_-, it is odd.

Lemma 3.10. *Assume that $\mathrm{DimKer}(L) = 1$ with $u_0 \in H^1(\mathbb{R})$. The operator L^* has a geometric kernel in $H^1_{-\mu}(\mathbb{R})$ with $0 < \mu < \mu_0$. If the kernel of L^* persists in $H^1(\mathbb{R})$ for $\mu = 0$, then the operators L and L^* have a non-empty generalized kernel in $H^1_\mu(\mathbb{R})$ and $H^1_{-\mu}(\mathbb{R})$, respectively, for $0 < \mu < \mu_0$.*

Proof. Since the unbounded differential operator $L_\mu = e^{\mu Z} L e^{-\mu Z}$ is Fredholm of zero index for $0 < \mu < \mu_0$, the adjoint operator $L^*_{-\mu} = e^{-\mu Z} L^* e^{\mu Z}$ has a one-dimensional geometric kernel for the same value of $0 < \mu < \mu_0$. If the kernel of L^* persists in $H^1(\mathbb{R})$, then the problem $L^* w_0 = 0$ has an eigenfunction $w_0 \in H^1(\mathbb{R})$. According to (A.6) and (A.7) of Appendix A, in the limit

$\kappa \to 0$, the eigenfunction $w_0(Z)$ is $\text{sech}(\kappa Z)$ for L_+ and $\text{sech}(\kappa Z)\tanh(\kappa Z)$ for L_-, i.e. it has the opposite symmetry than the eigenfunction $u_0 \in H^1(\mathbb{R})$. Due to the symmetry property (3.4), the symmetry of eigenfunction $w_0(Z)$ persists for any value of $\kappa \in \mathbb{R}$.

Consider the linear inhomogeneous problems $Lu_1 = u_0$ in $H^1_\mu(\mathbb{R})$ and $L^* w_1 = w_0$ in $H^1_{-\mu}(\mathbb{R})$ with $0 < \mu < \mu_0$. Since the continuous spectrum is bounded away from the origin for $0 < \mu < \mu_0$, the Fredholm Alternative Theorem guarantees existence of the generalized kernel if and only if

$$(e^{\mu Z} w_0, e^{-\mu Z} u_0) = (w_0, u_0) = 0. \tag{3.19}$$

Since $(w_0, u_0) = 0$ due to different spatial symmetries of $w_0(Z)$ and $u_0(Z)$ on $Z \in \mathbb{R}$, there is a non-empty generalized kernel of operators L and L^* in $H^1_\mu(\mathbb{R})$ and $H^1_{-\mu}(\mathbb{R})$, respectively. ■

Assumption 3.11. Assume that operators L_+ and L_- satisfy one of the following two properties:

(i) The zero eigenvalue of L and L^* in $H^1_\mu(\mathbb{R})$ and $H^1_{-\mu}(\mathbb{R})$ with $0 \le \mu < \mu_0$ is double with the generalized eigenfunctions $\{u_0(Z), u_1(Z)\} \in H^1(\mathbb{R})$ and $\{w_0(Z), w_1(Z)\} \in H^1(\mathbb{R})$, respectively.
(ii) The zero eigenvalue of L and L^* in $H^1_\mu(\mathbb{R})$ and $H^1_{-\mu}(\mathbb{R})$ with $0 < \mu < \mu_0$ is simple and $w_0 \notin H^1(\mathbb{R})$.

Remark 3.12. Due to the Fredholm Alternative Theorem, Assumption 3.11(i) implies that $(w_1, u_0) = (w_0, u_1) \neq 0$. This assumption is generally satisfied since $w_0(Z)$ and $u_1(Z)$ have the same symmetry on $Z \in \mathbb{R}$. It follows from the exact solutions (3.16) and (3.17) that the generalized kernel of L_+ has a subspace $\{u_0, u_1\} \in H^1(\mathbb{R})$. Although it does not necessarily imply that $\{w_0, w_1\} \in H^1(\mathbb{R})$, we will assume in the rest of our paper that all parts of Assumption 3.11(i) are satisfied for operator L_+.

Assumption 3.11(ii) is equivalent to the assumption that $(w_0, u_0) \neq 0$, which is only possible if $w_0(Z)$ has a component of the same symmetry as $u_0(Z)$. The operator L_- can satisfy the assertion that $w_0 \notin H^1(\mathbb{R})$ only if the solution of $L_- w_0 = 0$ has two bounded non-decaying functions (even and odd), a linear combination of which would generate a function $w_0 \in H^1_{-\mu}(\mathbb{R})$ for $0 < \mu < \mu_0$.

Example 3.13. According to Example 3.8, Assumptions 3.6 and 3.11(i) are satisfied for both linearized operators $L_\pm$ associated with the integrable AL lattice. According to Appendix B, the linearized operator L_- in Example 3.2 satisfy Assumptions 3.6 and 3.11(ii) for small $\nu \neq 0$, where $\nu = \alpha_3/(\alpha_2 - \alpha_3)$. Numerical approximations of the spectrum of L_- for $\nu = 0$ (top) and $\nu = 0.2$ (bottom) are shown on Fig. 3 for $\kappa = 1$. The numerical method is based on the sixth-order finite-difference approximation of the derivative operator and truncation of the computational domain on $Z \in [-L, L]$ with $L = 10$ and step size $h = 0.1$. The number of grid points is odd so that the number of eigenvalues in the truncated matrix problem is also odd. For $\nu = 0$ (top figures), the smallest eigenvalue with $|\lambda| = 2.9201 \times e^{-15}$ corresponds to the bounded eigenfunction (blue dots), while the next two eigenvalues with $|\lambda| = 8.5718 \times e^{-5}$ corresponds to the decaying eigenfunctions (magenta dots). This picture corresponds to Assumption 3.11(i). For $\nu = 0.2$ (bottom figures), the smallest eigenvalue with $|\lambda| = 1.6511 \times e^{-13}$ corresponds to the decaying eigenfunction (blue dots), while the next two eigenvalues with $|\lambda| = 0.0390$ corresponds to the bounded oscillatory complex-valued eigenfunctions (magenta dots). This picture corresponds to Assumption 3.11(ii).

Lemma 3.14. *Under Assumptions 2.1, 3.6 and 3.11(i), eigenfunctions $\{U_\mu(Z; k), W_\mu(Z; k)\}$ and spectral data $\{a_\mu(k), b_\mu(k)\}$ in Definition 3.5 are uniformly bounded on $k \in \mathbb{R}$ in the limit $\mu \to 0^+$. Moreover, the eigenfunctions $U_0(Z; 0)$ and $W_0(Z; 0)$ are even on $Z \in \mathbb{R}$, such that $a_0(0) = b_0(0) = 1$.*

Proof. By Assumption 3.11(i), the kernel of L persists in the space $H^1_{-\mu}(\mathbb{R})$ with $0 \le \mu < \mu_0$. No other eigenvalues exist on the imaginary axis by Assumption 3.6. Therefore, the eigenfunction $U_\mu(Z; k)$ and the scattering data $a_\mu(k)$ are bounded and $a_\mu(k) \neq 0$ for any $k \in \mathbb{R}$ for any $0 \le \mu < \mu_0$. By uniqueness of the eigenfunctions in Definition 3.5, the limit $\mu \to 0^+$ is thus uniform in $k \in \mathbb{R}$. The same proof applies to the eigenfunction $W_\mu(Z; k)$. According to (A.6) and (A.7) of Appendix A, the eigenfunctions $U_0(Z; 0)$ and $W_0(Z; 0)$ as $\kappa \to 0$ converge to the even eigenfunctions $1 - 2\text{sech}^2(\kappa Z)$ and 1. By the symmetry property (3.4) and uniqueness of eigenfunctions in Definition 3.5, the eigenfunctions remain even for any $\kappa \in \mathbb{R}$. ■

Lemma 3.15. *Under Assumptions 2.1, 3.6 and 3.11(ii), the eigenfunction $W_\mu(Z; k)$ and spectral data $b_\mu(k)$ in Definition 3.5 are uniformly bounded on $k \in \mathbb{R}$ in the limit $\mu \to 0^+$, such that $W_0(Z; 0) \in H^1_\mu(\mathbb{R})$ for $0 < \mu < \mu_0$ and $b_0(0) = 0$. The only singularity of the eigenfunction $U_\mu(Z; k)$ and spectral data $a_\mu(k)$ in the limit $\mu \to 0^+$ is a simple pole at $k = 0$ on $k \in \mathbb{R}$, such that*

$$\lim_{k \to 0} k U_0(Z; k) = C_1 u_0(Z) \in H^1(\mathbb{R}), \qquad \lim_{k \to 0} k a_0(k) = C_2$$

for some constants C_1 and C_2.

Proof. By Assumption 3.6, eigenfunctions $\{U_\mu(Z; k), W_\mu(Z; k)\}$ and spectral data $\{a_\mu(k), b_\mu(k)\}$ must be bounded for any $k \neq 0$ in the limit $\mu \to 0^+$. By Assumption 3.11(ii), the eigenfunction of $L^* w_0 = 0$ is in $H^1_\mu(\mathbb{R})$ for $0 < \mu < \mu_0$ but not in $H^1(\mathbb{R})$. Since
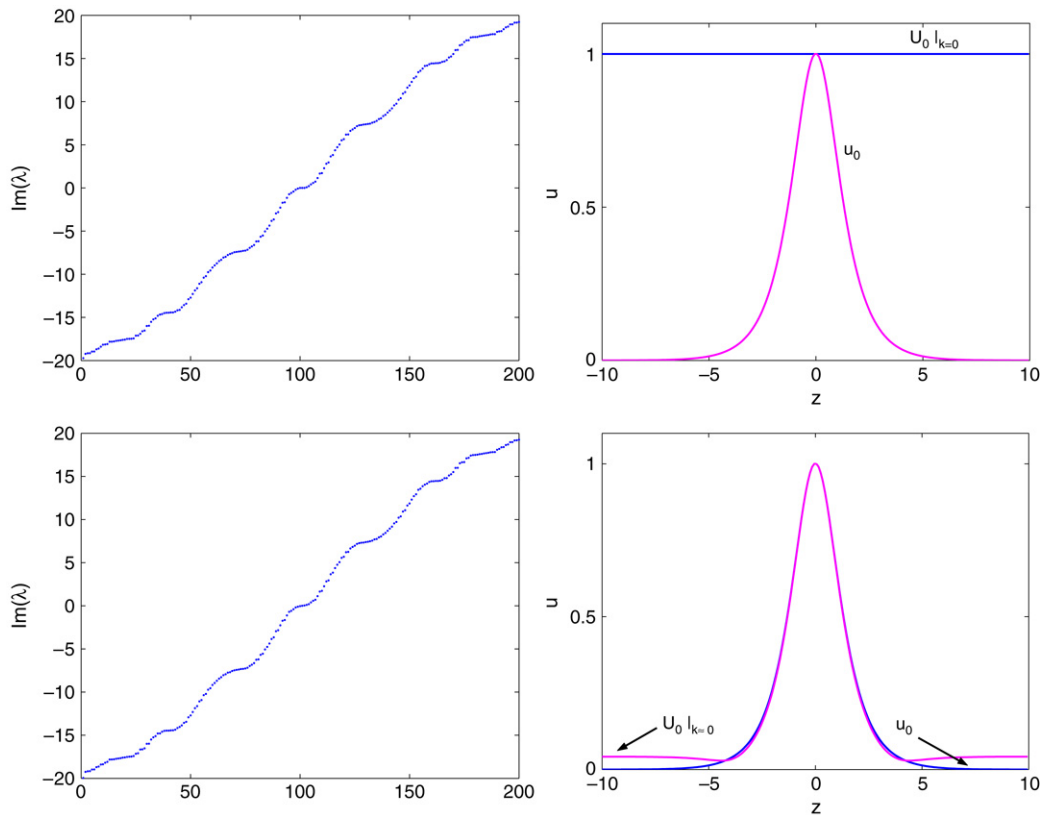
Fig. 3. Top: eigenvalues of the operator L_- for $\nu = 0$ and $\kappa = 1$ (left) and the two normalized eigenvectors for the two smallest eigenvalues (right). Bottom: the same for operator L_- for $\nu = 0.2$ and $\kappa = 1$. See text for details. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$W_0(Z; 0)$ solves the same equation, then the eigenfunction w_0 belongs to the branch of the continuous spectrum, which implies that $b_0(k)$ has a simple zero at $k = 0$. By Lemma 3.9, $a_0(k)$ has a simple pole at $k = 0$, where the only bounded eigenfunction of $Lu_0 = 0$ is the decaying eigenfunction $u_0 \in H^1(\mathbb{R})$. ■

Remark 3.16. It is important that other eigenvalues of L in $H^1_\mu(\mathbb{R})$ are bounded away from the imaginary axis. The limit $\mu \to 0^+$ pushes the continuous spectrum back to the imaginary axis, and the imaginary eigenvalues of L in $H^1_\mu(\mathbb{R})$, if they would exist, could become resonant poles of L in $H^1(\mathbb{R})$ leading to additional zeros or poles of $a_\mu(k)$ on $k \in \mathbb{R}$ in the limit $\mu \to 0$. If the simple kernel of Assumption 3.11(ii) arises as a result of the splitting of the double kernel of Assumption 3.11(i), the value of $\mu_1 < \mu_0$ in Assumption 3.6 must be chosen in such way that the continuous spectrum of L passes in between the zero eigenvalue and the non-zero resonant pole which bifurcates from the zero point. The splitting of the double kernel is described in Appendix B for operator L_- for small $\nu \neq 0$, where $\nu = \alpha_3/(\alpha_2 - \alpha_3)$.

3.3. Solutions of the linear inhomogeneous equations related to $L_\pm$

We use the previous results to find the conditions under which the linear inhomogeneous equation associated to the differential advance–delay operator L can be solved in the space $H^1(\mathbb{R})$. This is the main result of this section and it is used in Section 4 for an application of the Implicit Function Theorem.

Lemma 3.17. *Under Assumptions 2.1, 3.6 and 3.11, there exists a solution $U \in H^1_\mu(\mathbb{R})$ with $0 < \mu < \mu_1$ of the linear inhomogeneous equation $LU = F(Z)$ for $F \in L^2_\mu(\mathbb{R})$ with $-\mu_1 < \mu < \mu_1$ if and only if*

$$\int_{\mathbb{R}} w_0(Z) F(Z) \, \mathrm{d}Z = 0. \tag{3.20}$$

Moreover, $U \in H^1(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} W_0(Z; 0) F(Z) \, \mathrm{d}Z = 0, \tag{3.21}$$

in addition to (3.20).

Proof. The first assertion follows by the Fredholm Alternative, since the zero eigenvalue is isolated in $H^1_\mu(\mathbb{R})$ for $0 < \mu < \mu_0$ (even if $w_0 \notin H^1(\mathbb{R})$, the integral (3.20) makes sense since $F \in L^2_\mu(\mathbb{R})$ for $-\mu_1 < \mu < \mu_1$). By Proposition 3.3 and Assumption 3.6, the spectrum of L in $H^1_\mu(\mathbb{R})$ with $0 < \mu < \mu_1$ is given by the absolutely continuous part, the zero eigenvalue, and the point spectrum outside the strip $\Lambda_- < \lambda < \Lambda_+$. Assume that the condition (3.20) is satisfied. By Assumption 2.1, Lemma 3.9 and the wave function formalism [11], we represent the solution $U(Z)$ in $H^1_\mu(\mathbb{R})$ for $0 < \mu < \mu_1$ with the generalized Fourier transform

$$U(Z) = \int_{\mathbb{R}} \frac{\hat{F}_\mu(k) U_\mu(Z; k) \mathrm{d}k}{\lambda_\mu(k)} + \hat{F}_1 u_1(Z) + \sum_{\lambda_j \in \sigma_d(L_\mu) \backslash \{0\}} \frac{\hat{F}_j}{\lambda_j} u_j(Z), \tag{3.22}$$

where

$$\hat{F}_\mu(k) = \frac{\sinh \kappa - \kappa \cosh \mu \cos k}{4\pi \sinh \kappa} (W_0(\cdot; k), F), \qquad \hat{F}_1 = \frac{(w_1, F)}{(w_1, u_0)},$$

and $\hat{F}_j$ for $\lambda_j \in \sigma_d(L_\mu) \backslash \{0\}$ are projections to the eigenfunctions of the non-zero point spectrum of L_μ outside the strip $\Lambda_- < \lambda < \Lambda_+$. We note that the location of these eigenvalues λ_j are not affected by the weight parameter μ due to strong exponential decay of eigenfunctions. We also note that the second term $\hat{F}_1 u_1(Z)$ in (3.22) is absent if the kernel is simple by Assumption 3.11(ii).

Under Assumption 3.11(i) and Lemma 3.14, the functions $\hat{F}_\mu(k)$ and $U_\mu(Z; k)$ in the representation (3.22) are uniformly bounded in $k \in \mathbb{R}$ as $\mu \to 0^+$. The integrand of (3.22) has only one singularity at $k = 0$ of the simple pole type in the limit $\mu \to 0^+$ since $\lambda_0(0) = 0$. The integral can be split into two parts:

$$\lim_{\mu \to 0^+} \int_{\mathbb{R}} \frac{\hat{F}_\mu(k) U_\mu(Z; k) \mathrm{d}k}{\lambda_\mu(k)} = \pi i \mathrm{Res}\left[\frac{\hat{F}_0(k) U_0(Z; k)}{\lambda_0(k)}, k = 0 \right] + \lim_{\epsilon \to 0^+} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{\hat{F}_0(k) U_0(Z; k) \mathrm{d}k}{\lambda_0(k)}$$

$$= \frac{\pi \kappa \hat{F}_0(0)}{\kappa - \sinh \kappa} U_0(Z; 0) + \mathrm{p.v.} \int_{\mathbb{R}} \frac{\hat{F}_0(k) U_0(Z; k) \mathrm{d}k}{\lambda_0(k)}.$$

By Lemma 3.14 and thanks to the linear growth of $\lambda_0(k)$ in k as $|k| \to \infty$, the second term is in $H^1(\mathbb{R})$ if $F \in L^2(\mathbb{R})$. Since the first term is bounded but non-decaying and all other eigenfunctions $u_1(Z)$ and $u_j(Z)$ are in $H^1(\mathbb{R})$, we obtain that $U \in H^1(\mathbb{R})$ if and only if $\hat{F}_0(0) = 0$, i.e. under the condition (3.21).

Under Assumption 3.11(ii) and Lemma 3.15, the integral in the representation (3.22) has a double pole at $k = 0$ as $\mu \to 0^+$ if $\hat{F}_0(0) \neq 0$. By Lemma 3.15, $W_0(Z; 0)$ is proportional to $w_0(Z)$, such that the condition (3.21) follows from the condition (3.20). If $\hat{F}_0(0) = 0$, the integral can be split into two parts as above and the residue term produces now the function $\lim_{k \to 0} k U_0(Z; k)$, which is proportional to $u_0(Z) \in H^1(\mathbb{R})$ by Lemma 3.15. Therefore, we verify again that $U \in H^1(\mathbb{R})$. ∎

Corollary 3.18. *Under assumptions of Lemma 3.17,*

(i) *There exists a unique solution $U \in H^1_{\mathrm{ev}}(\mathbb{R})$ of the linear inhomogeneous equation $L_+ U = F(Z)$ if $F \in L^2_{\mathrm{odd}}(\mathbb{R})$.*
(ii) *There exists a unique solution $U \in H^1_{\mathrm{odd}}(\mathbb{R})$ of the linear inhomogeneous equation $L_- U = F(Z)$ if $F \in L^2_{\mathrm{ev}}(\mathbb{R})$ and $(W_0, F) = 0$, where $W_0 = W_0(Z; 0)$.*

Proof. (i) The operator L_+ satisfies Assumption 3.11(i) (see Remark 3.12). The eigenfunctions $w_0(Z)$ and $W_0(Z; 0)$ are even and $u_0(Z)$ is odd. As a result, $(w_0, F) = (W_0, F) = 0$, i.e. the conditions of Lemma 3.17 are satisfied. Uniqueness follows from the fact that the homogeneous solution $u_0(Z)$ is not in $H^1_{\mathrm{ev}}(\mathbb{R})$.

(ii) Under Assumption 3.11(i), the eigenfunctions $w_0(Z)$ is odd and $u_0(Z)$ is even for L_-, so that $(w_0, F) = 0$ and the homogeneous solution is not in $H^1_{\mathrm{odd}}(\mathbb{R})$. The only condition (3.21) results in $(W_0, F) = 0$. Under Assumption 3.11(ii), both conditions (3.20) and (3.21) are equivalent as $W(Z; 0)$ is proportional to $w_0(z)$, which includes both odd and even components on $Z \in \mathbb{R}$. ∎

4. Melnikov integrals for persistence of one-parameter family of solutions

According to Assumption 2.1, the scalar differential advance–delay equation (2.3) admits a solution in $H^1_{\mathrm{ev}}(\mathbb{R})$ for some parameter configurations. This solution satisfies the vector system (2.1) under the constraints (2.2). We shall now answer Questions 2.4 and 2.5 on persistence of this solution. Our technique relies on the Melnikov integral, which originates from the conditions (3.20) and (3.21) of Lemma 3.17.

In order to answer Question 2.4 about the scalar equation (2.3), we represent parameters of the equation by $\alpha_j = \alpha_j^{(0)} + \epsilon a_j$ for $j = 2, 3, 8, 9, 10$, where ϵ is small, $\alpha_j^{(0)}$ is an unperturbed value of α_j for which Assumption 2.1 holds, and ϵa_j is a perturbation to $\alpha_j^{(0)}$, for which persistence of solutions is needed to be established. We also represent a solution to the scalar differential

advance–delay (2.3) by $\Phi(Z) = \Phi_0(Z) + U(Z)$, where $U(Z)$ solves the scalar equation in the operator form

$$L_+ U = N(U) + \epsilon F(\Phi_0 + U). \tag{4.1}$$

Here L_+ is the unperturbed differential advance–delay operator given by (2.5) for $\alpha_j = \alpha_j^{(0)}$ and $\Phi = \Phi_0(Z)$, $N(U)$ is the unperturbed nonlinear vector field with the quadratic[3] and cubic terms in U, and $\epsilon F(\Phi)$ contains cubic terms in $\Phi = \Phi_0 + U$ due to the perturbations ϵa_j of the parameters α_j of the scalar equation (2.3), e.g. explicitly

$$F = (a_2 - a_3)\Phi^2(\Phi_+ - \Phi_-) + a_8(\Phi_+^3 - \Phi_-^3) - (a_9 + a_{10})\Phi_+ \Phi_-(\Phi_+ - \Phi_-).$$

It follows from the algebra property in $H^1(\mathbb{R})$ under the assumption that $\Phi_0 \in H^1(\mathbb{R})$ that there exist constants $C_1, C_2, C_3 > 0$, such that

$$\|N(U)\|_{H^1} \le C_1\|U\|_{H^1}^2 + C_2\|U\|_{H^1}^3, \qquad \|F(\Phi_0 + U)\|_{H^1} \le C_3\|\Phi_0 + U\|_{H^1}^3. \tag{4.2}$$

Therefore, the nonlinear vector field of the system (4.1) is closed in $H^1(\mathbb{R})$. By using the algebra property (4.2), the assumption that $\Phi_0 \in H^1_{ev}(\mathbb{R})$, and the symmetry of the scalar equation (2.3), we can see that

$$N, F : H^1_{ev}(\mathbb{R}) \mapsto H^1_{odd}(\mathbb{R}). \tag{4.3}$$

By Corollary 3.18(i), the operator $L_+ : H^1_{ev}(\mathbb{R}) \mapsto L^2_{odd}(\mathbb{R})$ is invertible. By Lemma 3.7, the construction of eigenfunctions of the continuous spectrum is structurally stable with respect to small perturbations of the potentials $V_0(Z)$ and $V_\pm(Z)$. Therefore, properties of Lemma 3.14 remain valid in a local neighborhood of $\Phi = \Phi_0 \in H^1_{ev}(\mathbb{R})$ and $\epsilon = 0$ and the kernel of L_+ is empty in $H^1_{ev}(\mathbb{R})$. As a result, the Frechet derivative operator of the problem (4.1) is continuously invertible in a local neighborhood of the point $U = 0 \in H^1_{ev}(\mathbb{R})$ and $\epsilon = 0 \in \mathbb{R}$. By the Implicit Function Theorem, we assert the following theorem.

Theorem 4.1. *Under Assumptions 2.1, 3.6 and 3.11(i), there exists a unique solution $U(Z) = U_\epsilon(Z) \in H^1_{ev}(\mathbb{R})$ of the scalar problem (4.1) for sufficiently small ϵ, such that $\|U_\epsilon\|_{H^1} \le C\epsilon$ for some $C > 0$.*

Corollary 4.2. *Under the assumptions of Theorem 4.1, there exists a unique continuation of the solution $\Phi_0(Z)$ of the scalar equation (2.3) with respect to the perturbed parameters $(\alpha_2, \alpha_3, \alpha_8, \alpha_9, \alpha_{10})$.*

Remark 4.3. A similar application of the Implicit Function Theorem is reported in [10] (Theorem 3) for persistence of heteroclinic orbits with small oscillatory tails. However, there are several important differences between our results and the work [10]. The ODE approach is used in [10] to guarantee that bounded continuous solutions of the truncated normal form persists in the original differential advance–delay equation. Therefore, the space $C_b^0(\mathbb{R})$ was used in [10], which does not distinguish between true heteroclinic solutions and solutions with oscillatory tails. On the other hand, we use here the spectral approach to guarantee that localized solutions of a differential advance–delay equation persist with respect to parameter continuations. The space $H^1(\mathbb{R})$ is more suitable for localized solutions of differential advance–delay equations, and, by the Sobolev Embedding Theorem, the space $H^1(\mathbb{R})$ is continuously embedded in the space $C_b^0(\mathbb{R})$.

Remark 4.4. Persistence of stationary solutions to the advance–delay equation (2.7) with respect to parameter continuation can be proven with a similar application of the Implicit Function Theorem. However, there is a very important difference in the case of stationary solutions with $\beta = 0$ compared to the case of traveling solutions with $\beta \ne 0$. The linearization operator of the advance–delay equation (2.7) at any solution $\Phi(Z) \in L^2(\mathbb{R})$ defines a map from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. If the solution $\Phi(Z)$ is a bounded continuous function for some parameter configuration (e.g. for the translationally invariant discrete NLS equation with the nonlinearity (1.4)), the solution in $L^2(\mathbb{R})$ may not be continuous, but only piecewise-continuous for perturbed parameter configurations. Therefore, the delicate property of translational invariance is not structurally stable. As shown in [17], this property arises if the second-order difference equation admits an integrable invariant, which is expressed by the first-order difference equation.

In order to answer Question 2.5 about the system (2.1), we represent parameters of the system by $\alpha_j = \alpha_j^{(0)} + \epsilon a_j$ for all j's and $\cot \beta = \varepsilon$, where ϵ and ε are small parameters. The role of the perturbations ϵa_j is the same as above with the only addition that the unperturbed parameters $(\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_7)$ must satisfy the constraints (2.2). The new parameter ε measures the distance between β and $\frac{\pi}{2}$. We also represent a solution to the system (2.1) by $\Phi(Z) = \Phi_0(Z) + U(Z) + iV(Z)$, where (U, V) are real-valued solutions of the system in the operator form:

$$L_+ U + \tilde{L}_+ V = N_+(U, V) + \varepsilon M_+(\Phi_0 + U, V) + F_+(\Phi_0 + U, V; \epsilon, \varepsilon), \tag{4.4}$$

$$L_- V = N_-(U, V) + \varepsilon M_-(\Phi_0 + U, V) + F_-(\Phi_0 + U, V; \epsilon, \varepsilon). \tag{4.5}$$

---

[3] Quadratic terms in $N(U)$ depend on $\Phi_0(Z)$.

Here L_+ and L_- are given by (2.5) and (2.6) for $\alpha_j = \alpha_j^{(0)}$ and $\Phi = \Phi_0(Z)$, while

$$\tilde{L}_+ = 2\alpha_4(\Phi_+^2 + \Phi_-^2) - 2\alpha_7 \Phi_+ \Phi_- + \alpha_7 \Phi(\Phi_+\delta_- + \Phi_-\delta_+),$$

where the constraints (2.2) have been used. Furthermore, $N_\pm(U, V)$ are the unperturbed nonlinear vector fields with the quadratic and cubic terms in (U, V), while $\varepsilon M_\pm(\Phi, V)$ and $F_\pm(\Phi, V; \epsilon, \varepsilon)$ contain, respectively, linear and cubic terms in $\Phi = \Phi_0 + U$ and V related to the perturbations ϵa_j and ε of the system (2.1). For instance, the linear terms are written explicitly as

$$M_+ = 2\cosh\kappa V - V_+ - V_-, \qquad M_- = \Phi_+ + \Phi_- - 2\cosh\kappa \Phi.$$

The same Banach algebra property (4.2) holds in $H^1(\mathbb{R}, \mathbb{C}^2)$ for $N_\pm(U, V)$ and $F_\pm(\Phi_0 + U, V; \epsilon, \varepsilon)$. Therefore, the nonlinear vector field of the system (4.4) and (4.5) is closed on $(U, V) \in H^1(\mathbb{R}, \mathbb{C}^2)$. By using the algebra property (4.2), the assumption that $\Phi_0 \in H^1_{\text{ev}}(\mathbb{R})$, and the symmetry of the original vector field in (2.1), we can see that

$$\begin{aligned}
&\tilde{L}_+, N_+, M_+, F_+ : H^1_{\text{ev}}(\mathbb{R}) \times H^1_{\text{odd}}(\mathbb{R}) \mapsto H^1_{\text{odd}}(\mathbb{R}), \\
&N_-, M_-, F_- : H^1_{\text{ev}}(\mathbb{R}) \times H^1_{\text{odd}}(\mathbb{R}) \mapsto H^1_{\text{ev}}(\mathbb{R}).
\end{aligned} \tag{4.6}$$

Moreover, $F_\pm(\Phi, V; \epsilon, \varepsilon)$ is a linear function of ϵ and analytic function of ε, such that $F_\pm(\Phi, V; 0, 0) = 0$. We give explicitly the first terms in ε and ϵ for $F_-(\Phi, 0; \epsilon, \varepsilon)$, which are used in Examples 4.7 and 4.8:

$$\begin{aligned}
F_-(\Phi, 0; \epsilon, \varepsilon) = {} & \varepsilon[(\alpha_2 + \alpha_3)\Phi^2(\Phi_+ + \Phi_-) - \alpha_8(\Phi_+^3 + \Phi_-^3) + (\alpha_9 + \alpha_{10})\Phi_+ \Phi_-(\Phi_+ + \Phi_-)] \\
& \epsilon[a_1\Phi^3 + (a_4 - a_6)\Phi(\Phi_+^2 + \Phi_-^2) + (a_7 - 2a_5)\Phi\Phi_+\Phi_-] + O(\varepsilon^2, \epsilon\varepsilon).
\end{aligned}$$

By using the same proof as in Theorem 4.1, we find that the Frechet derivative operator of Eq. (4.4) is continuously invertible in a local neighborhood of the point $U = 0 \in H^1_{\text{ev}}(\mathbb{R})$, $\epsilon = 0 \in \mathbb{R}$, and $\varepsilon = 0 \in \mathbb{R}$. Therefore, there exists a map $U_{\epsilon,\varepsilon}(V) : H^1_{\text{odd}}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \mapsto H^1_{\text{ev}}(\mathbb{R})$ for sufficiently small ϵ and ε, such that $\|U_{\epsilon,\varepsilon}(V)\|_{H^1(\mathbb{R})} \le C(|\epsilon| + |\varepsilon| + \|V\|_{H^1(\mathbb{R})})$. By Corollary 3.18(ii), the operator $L_- : H^1_{\text{odd}}(\mathbb{R}) \mapsto L^2_{\text{ev}}(\mathbb{R})$ is invertible if the scalar Fredholm condition is satisfied, which leads to the Melnikov integral

$$\Delta_{\epsilon,\varepsilon}(U, V) = \int_{\mathbb{R}} W_0(Z; 0)[N_-(U, V) + \varepsilon M_-(\Phi_0 + U, V) + F_-(\Phi_0 + U, V; \epsilon, \varepsilon)]\mathrm{d}Z, \tag{4.7}$$

where $U = U_{\epsilon,\varepsilon}(V)$ is constructed above. By repeating the same argument on the structural stability of the spectrum of L_- in $V \in H^1_{\text{odd}}(\mathbb{R})$, we conclude that the Frechet derivative operator of the system (4.4) and (4.5) is continuously invertible in a local neighborhood of the point $(U, V) = (0, 0) \in H^1_{\text{ev}}(\mathbb{R}) \times H^1_{\text{odd}}(\mathbb{R})$, $\epsilon = 0 \in \mathbb{R}$, and $\varepsilon = 0 \in \mathbb{R}$ provided that $\Delta_{\epsilon,\varepsilon}(U, V) = 0$. By the Implicit Function Theorem, we assert the following theorem.

Theorem 4.5. *Under Assumptions 2.1, 3.6 and 3.11, there exists a unique solution $U(Z) = U_{\epsilon,\varepsilon}(Z) \in H^1_{\text{ev}}(\mathbb{R})$ and $V(Z) = V_{\epsilon,\varepsilon}(Z) \in H^1_{\text{odd}}(\mathbb{R})$ of the system (4.4) and (4.5) for sufficiently small ϵ and ε, such that $\|U_{\epsilon,\varepsilon}\|_{H^1} \le C_U(|\epsilon| + |\varepsilon|)$ and $\|V_{\epsilon,\varepsilon}\|_{H^1} \le C_V(|\epsilon| + |\varepsilon|)$ for some $C_U, C_V > 0$, provided that $\Delta(\epsilon, \varepsilon) = 0$, where $\Delta(\epsilon, \varepsilon) = \Delta_{\epsilon,\varepsilon}(U_{\epsilon,\varepsilon}, V_{\epsilon,\varepsilon})$ is given by (4.7), such that $\Delta(0, 0) = 0$.*

Corollary 4.6. *Under the assumptions of Theorem 4.5,*
 (i) *There exists a unique continuation of the solution $\Phi_0(Z)$ of the system (2.1) for $\beta = \frac{\pi}{2}$ under the constraints (2.2) with respect to perturbed parameters α_j and β if $\partial_\varepsilon \Delta(0, 0) \ne 0$ for any $\kappa \in \mathbb{R}$.*
 (ii) *There does not exist any continuation of the solution $\Phi_0(Z)$ of the system (2.1) for $\beta = \frac{\pi}{2}$ under the constraint (2.2) with respect to perturbed parameters α_j and β if $\Delta(0, \varepsilon) = 0$ and $\Delta(\epsilon, 0) \ne 0$ for any $\kappa \in \mathbb{R}$ in a local neighborhood of $\epsilon = \varepsilon = 0$.*

Example 4.7. According to Examples 3.8 and 3.13, the linearized operators $L_\pm$ of the AL lattice satisfy the assumptions of our analysis. The system (2.1) can be rewritten for the Salerno model (1.3) in the compact form

$$(1 + |\Phi|^2)(\Phi_+ - \Phi_-) - 2\frac{\sinh\kappa}{\kappa}\Phi' = \mathrm{i}\varepsilon[(1 + |\Phi|^2)(\Phi_+ + \Phi_-) - 2\cosh\kappa\Phi] + \mathrm{i}\epsilon|\Phi|^2\Phi, \tag{4.8}$$

where $\epsilon = \frac{2(1-\alpha_2)}{\alpha_2 \sin\beta}$, $\varepsilon = \cot\beta$, and the amplitude of $\Phi(Z)$ is rescaled by the factor $\sqrt{\alpha_2}$ for $\alpha_2 \ne 0$. Since $\Phi(Z) = \sinh\kappa \operatorname{sech}(\kappa Z)$ is a solution of

$$(1 + |\Phi|^2)(\Phi_+ + \Phi_-) - 2\cosh\kappa\Phi = 0,$$

it is clear that $\Delta(0, \varepsilon) = 0$ in Theorem 4.5. According to (A.6) and (A.7) of Appendix A, $W_0(Z; 0) \to 1 - 2\operatorname{sech}^2(\kappa Z)$ and $\Phi(Z) \to \kappa \operatorname{sech}(\kappa Z)$ as $\kappa \to 0$, such that we have

$$\partial_\epsilon \Delta(0, 0) = \int_{\mathbb{R}} W_0(Z; 0)\Phi^3(Z)\mathrm{d}Z \to \kappa^2 \int_{\mathbb{R}}(1 - 2\operatorname{sech}^2\zeta)\operatorname{sech}^3\zeta\,\mathrm{d}\zeta = -\frac{\kappa^2}{2}\int_{\mathbb{R}}\frac{\mathrm{d}\zeta}{\cosh^3\zeta} < 0,$$

in the limit of small κ. By Corollary 4.6(ii), the solution $\Phi(Z)$ can not be continued for any $\epsilon \neq 0$ at least for small $\kappa > 0$. Therefore the two-parameter family of solutions[4] terminates near $\beta = \frac{\pi}{2}$ in the Salerno model (1.3) with $\alpha_2 \neq 1$.

Example 4.8. According to Example 3.13 and Appendix B, linearized operators $L_\pm$ of the discrete NLS equations (1.1) and (1.2) with $\alpha_2 > \alpha_3$, and $\alpha_j = 0$ for all other j's satisfy assumptions of our analysis at least for small $\nu \neq 0$, where $\nu = \alpha_3/(\alpha_2 - \alpha_3)$. By using the exact solution (1.12) and the explicit computation,

$$\partial_\varepsilon \Delta(0, 0) = \int_{\mathbb{R}} W_0(Z; 0)[(1 + (\alpha_2 + \alpha_3)\Phi^2)(\Phi_+ + \Phi_-) - 2 \cosh \kappa \, \Phi] dZ,$$

we obtain for $\alpha_3 \neq 0$ that

$$\partial_\varepsilon \Delta(0, 0) = 2\alpha_3 \int_{\mathbb{R}} W_0(Z; 0) \Phi^2(\Phi_+ + \Phi_-) dZ \rightarrow \frac{4\kappa^2 \alpha_3}{(\alpha_2 - \alpha_3)^{3/2}} \int_{\mathbb{R}} (1 - 2 \operatorname{sech}^2 \zeta) \operatorname{sech}^3 \zeta \, d\zeta \neq 0,$$

where the last approximation is valid for small κ and small α_3. By Corollary 4.6(i), there exists a unique continuation of the solution (1.12) with $\alpha_3 \neq 0$ and $\alpha_2 > \alpha_3$ with respect to perturbed parameters α_j and β. Details of the continuation depends on the perturbations a_j to the parameters α_j, due to the explicit computation

$$\partial_\epsilon \Delta(0, 0) = \int_{\mathbb{R}} W_0(Z; 0)[a_1 \Phi^3 + (a_4 - a_6)\Phi(\Phi_+^2 + \Phi_-^2) + (a_7 - 2a_5)\Phi\Phi_+\Phi_-] dZ$$

$$\rightarrow (a_1 + 2a_4 - 2a_6 + a_7 - 2a_5) \int_{\mathbb{R}} (1 - 2 \operatorname{sech}^2 \zeta) \operatorname{sech}^3 \zeta \, d\zeta$$

for small κ. For the translationally invariant lattice with parameters (1.4), the exact solution (1.12) exists for $\alpha_4 = \alpha_6 = \alpha_8 = \alpha_9 = 0$ and $\alpha_2 > \alpha_3$. When $\alpha_4 = \alpha_6 = 0$ is preserved for $\epsilon \neq 0$, the family persists on the line $\kappa > 0$ and $\beta = \frac{\pi}{2}$ by Corollary 4.2. When either $\alpha_4 \neq 0$ or $\alpha_6 \neq 0$ for $\epsilon \neq 0$, the curve on the plane (κ, β) is located near the line $\kappa > 0$ and $\beta = \frac{\pi}{2}$. If in addition, $\alpha_4 = \alpha_6$, the curve on the plane (κ, β) intersects the point $(\kappa, \beta) = (0, \frac{\pi}{2})$ at least to first order of the perturbation theory.

Remark 4.9. The Melnikov integral (4.7) can be useful for the cases when $\partial_\varepsilon \Delta(0, 0)$ vanishes at a particular point $\kappa = \kappa_0$. In this case, the one-parameter family of solutions on the plane (κ, β) may display interesting behavior such as a branch point near the point $\kappa = \kappa_0$ and $\beta = \frac{\pi}{2}$. Similarly, interesting behavior is expected if $\Delta(0, \varepsilon) = 0$ for all $\kappa \in \mathbb{R}$ and $\Delta(\epsilon, 0) = 0$ for a point $\kappa = \kappa_0$. At the present time, we do not have any numerical examples, which would motivate studies of these bifurcations.

5. Spectral stability of the traveling solutions

By using the transformation

$$u_n(t) = \frac{1}{h} \Phi(Z, T) e^{i\beta Z + i\omega h^2 T}, \quad Z = \frac{hn - 2ct}{h}, \quad T = \frac{t}{h^2},$$

and the parametrization (1.7), the discrete NLS equation (1.1) converts to the time-dependent version of the differential advance–delay equation (1.9):

$$i\Phi_T + \cos\beta[\Phi_+ + \Phi_- - 2 \cosh \kappa \, \Phi] + i \sin\beta \left[\Phi_+ - \Phi_- - 2\frac{\sinh \kappa}{\kappa} \Phi_Z\right] + f(\Phi_- e^{-i\beta}, \Phi, \Phi_+ e^{i\beta}) = 0,$$

where the subscript denotes partial derivatives and $\Phi_\pm = \Phi(Z \pm 1, T)$. Suppose Assumption 2.1 is satisfied, $\beta = \frac{\pi}{2}$, and $\alpha_j = 0$ for $j = 1$ and $4 \leq j \leq 7$. The standard linearization of the solution $\Phi(Z, T)$ by the substitution $\Phi(Z, T) = \Phi_0(Z) + [U(Z, T) + iV(Z, T)] + O(\|U\|^2, \|V\|^2)$ results in the uncoupled linearized problem

$$U_T + L_+ U = 0, \qquad V_T + L_- V = 0, \tag{5.1}$$

where operators $L_\pm$ are given by (2.5) and (2.6). Therefore, the spectrum of operators $L_\pm$ investigated in Section 3 for analysis of persistence of solution $\Phi_0(Z)$ is also important for predictions of spectral stability of the solution $\Phi_0(Z)$ with respect to time evolution. If Assumption 3.6 is replaced with a stronger spectral assumption, one can immediately formulate results on neutral stability of the solution family $\Phi_0(Z)$ in the linearized time evolution problem (5.1).

Assumption 5.1. The point spectrum $\sigma_p(L)$ in $H^1(\mathbb{R})$ does not include any eigenvalues on $\lambda \in \mathbb{C}$, except for the zero eigenvalue $\lambda = 0$.

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[4] The function $\Phi(Z) = \sinh \kappa \operatorname{sech}(\kappa Z)$ is obviously a solution of the Eq. (4.8) for any $\varepsilon \in \mathbb{R}$ and $\epsilon = 0$, i.e. it is actually the two-parameter family of solutions of the AL lattice.

Proposition 5.2. *Let Assumptions 2.1, 3.11 and 5.1 be satisfied. Then, the solution $\Phi_0(Z)$ is neutrally stable with respect to the time evolution of the linearized problem (5.1), such that*

$$\sup_{t \geq 0}(\|U(\cdot, T)\|_{H^1(\mathbb{R})} + \|V(\cdot, T)\|_{H^1(\mathbb{R})}) \leq C < \infty,$$

for some $C > 0$.

Remark 5.3. In a similar context, time evolution of embedded solitons in nonlinear partial differential equations was studied in [19]. It is suggested that the solution $\Phi_0(Z)$ is spectrally and nonlinearly stable in the time evolution if Assumption 3.11(ii) is satisfied and the resonance pole of L_- shifts to the right half-plane of λ, in accordance with Appendix B. This analogy suggests nonlinear stability of the time evolution of the one-parameter family of traveling solutions (1.12) in the discrete NLS equations (1.1) and (1.2) with $\alpha_2 > \alpha_3 \neq 0$ and $\alpha_j = 0$ for $j = 1$ and $4 \leq j \leq 10$. We note that the nonlinear stability of the two-parameter family of traveling solutions of the AL lattice (with $\alpha_3 = 0$) follows from the integrability of the AL lattice [4].

6. Numerical approximations of solution families

Numerical approximations of solutions of the differential advance–delay equation (1.9) are based on the pseudo-spectral method, similarly to the work [1]. In this method, the solution $\Phi(Z)$ is extended into a periodic function over a large but finite period L and the periodic function is approximated with Fourier series

$$\Phi(Z) = \sum_{j=1}^{N} a_j \cos\left(\frac{2\pi j}{L} Z\right) + \mathrm{i} b_j \sin\left(\frac{2\pi j}{L} Z\right). \tag{6.1}$$

If the solution satisfies the reversibility constraint $\Phi(-Z) = \bar{\Phi}(Z)$, the Fourier coefficients $\{a_j, b_j\}_{j=1}^{N}$ are real-valued. When the Fourier series (6.1) is substituted into the differential advance–delay equation (1.9), the equation transforms into a large system of coupled algebraic equations at the collocation points $Z_i = \frac{Li}{2(N+1)}$, $i = -(N+1), -N, \ldots, N, (N+1)$ for the unknown coefficients $\{a_j, b_j\}_{j=1}^{N}$. The system of algebraic equations is solved for some initial values of $\{a_j, b_j\}_{j=1}^{N}$ by using the Powell hybrid method [20]. The numerical solution has generally a non-zero amplitude radiation tail near the end points $Z = \pm \frac{1}{2}L$. We measure the radiation tail by the signed amplitude $\Delta = \mathrm{Im}\, \Phi\left(\frac{L}{2}\right)$. If a zero of the radiation tail is detected for some parameter configurations, the zero of Δ is continued with respect to perturbed parameters by using AUTO [3]. The solution $\Phi(Z)$ with zero radiation tail corresponds to a localized traveling wave.

6.1. Translationally invariant discrete NLS lattice

We consider the discrete NLS equation (1.1) with the nonlinearity (1.2) and (1.4). To simplify our work, we set $\alpha_4 = \alpha_6, \alpha_9 = 0$, and add the normalization constraint

$$\alpha_2 + \alpha_3 + 4\alpha_6 + 2\alpha_8 = 1. \tag{6.2}$$

Then, the solution $\Phi(Z, T)$ has three independent parameters $(\alpha_3, \alpha_6, \alpha_8)$ in addition to two internal parameters κ and β. Fig. 4 show zeros of the tail amplitude for $\kappa = 1$, three values of $\beta = \{0.51\pi, 0.55\pi, 0.6\pi\}$ and three parameter configurations $(\alpha_6, \alpha_8) = (-1, 1)$ (left), $(\alpha_3, \alpha_8) = (-1, 1)$ (center), and $(\alpha_3, \alpha_6) = (-1, -1)$ (right). The solution profiles along the curve $\beta = 0.6\pi$ for $(\alpha_3, \alpha_6) = (-1, -1)$ are shown in Fig. 5. We can clearly see that the condition $\Delta = 0$ for $\mathrm{Im}\, \Phi\left(\frac{L}{2}\right)$ corresponds to the zero tail amplitude for $\mathrm{Re}\, \Phi\left(\frac{L}{2}\right)$ as well. We can also see that a simple zero of the tail amplitude persists for general parameter configurations but may be non-unique as it happens on the left panel of Fig. 4.[5]

If $\beta = \frac{\pi}{2}$ and $\alpha_6 = 0$, the differential advance–delay equation (1.9) reduces to the scalar equation (2.3). Corollary 4.2 states that the family of localized traveling solutions persists on the line $\beta = \frac{\pi}{2}$. This fact is confirmed numerically in Fig. 6, where different single-humped solutions are found on the line $\beta = \frac{\pi}{2}$ for parameter continuations in α_3 (left) and α_8 (right). We can see that the wave amplitude of the single-humped solutions grows as α_3 increases and α_8 decreases.

When $\alpha_6 \neq 0$, the differential advance–delay equation (1.9) is equivalent to the full complex system (2.1). Corollary 4.6(i) and Example 4.8 state that the family of solutions persists along a curve in the (κ, β)-plane near the line $\kappa > 0$ and $\beta = \frac{\pi}{2}$ for small

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[5] It would be interesting to investigate the tail amplitude as a function of β, in this case we would expect to see regular curves (as a function of β) of 'U', 'n' and 'S' shapes, see [14], where 'U' and 'n' branches are bounded away from $\Delta = 0$ containing only solutions with non-zero radiation tails and 'S' branches which contain a single solution with zero radiation tail ($\Delta = 0$). However since, by definition, β is proportional to the wave number of the radiation tail and the wave number is zero for $\beta = \frac{\pi}{2}$, then a small change in β can give rise to a large variation in the radiation tail behavior. Therefore, the behavior of Δ as a function of β near to the point $\beta = \frac{\pi}{2}$ is a very computationally expensive task.
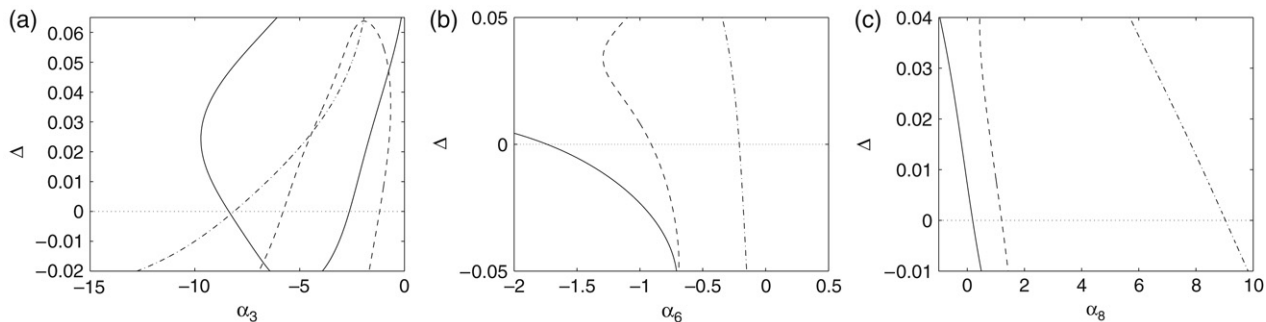
Fig. 4. The tail amplitude Δ versus α_3 for $(\alpha_6, \alpha_8) = (-1, 1)$ (left); α_6 for $(\alpha_3, \alpha_8) = (-1, 1)$ (center), and α_8 for $(\alpha_3, \alpha_6) = (-1, -1)$ (right). The other values are $\kappa = 1$ and $\beta = 0.6\pi$ (solid), $\beta = 0.55\pi$ (dashed), and $\beta = 0.51\pi$ (dash–dotted).
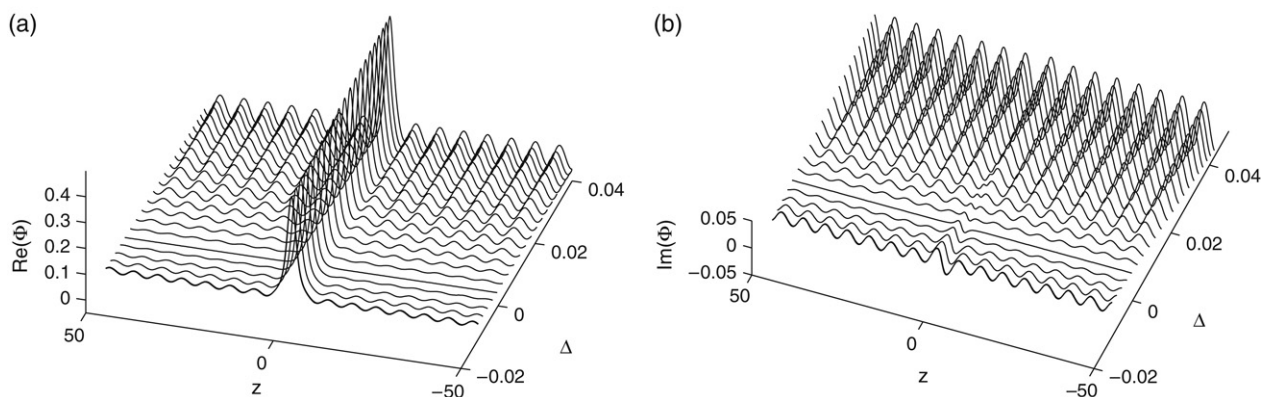

Fig. 5. Profiles of solution (a) Re $\Phi(Z)$ and (b) Im $\Phi(Z)$ versus tail amplitude Δ when the solution is continued for $\kappa = 1$ and $\beta = 0.6\pi$ versus α_8 for $(\alpha_3, \alpha_6) = (-1, -1)$. The value $\Delta = 0$ corresponds to the localized solution.
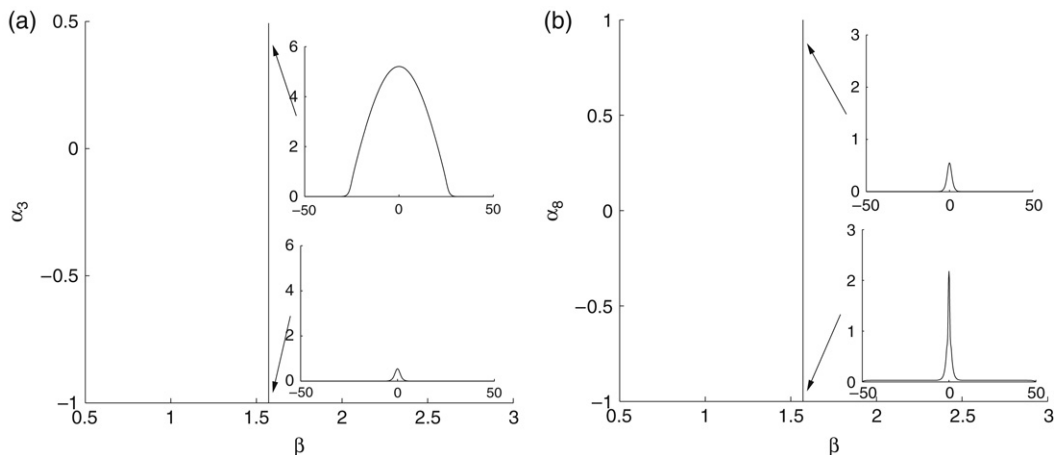
Fig. 6. Persistence of the localized solution for $\kappa = 1$ and $\beta = \frac{\pi}{2}$ versus α_3 for $(\alpha_6, \alpha_8) = (0, 1)$ (left) and α_8 for $(\alpha_3, \alpha_6) = (-1, 0)$ (right). The insets show the single-humped profiles of the localized solution.

$\alpha_6 \neq 0$. Moreover, the curve only intersects the line $\beta = \frac{\pi}{2}$ at the point $\kappa = 0$. Fig. 7(a) illustrates this fact for fixed values $(\alpha_3, \alpha_8) = (-1, 1)$ and different values of $\alpha_6 = (0.5, 0.25, 0, -0.5, -1, -1.5, -2)$. In Fig. 7(b) we project the solution curves onto the (c, ω)-plane, where the values of c and ω are given by the parametrization (1.7). All the solution curves can be seen to intersect the point $(c, \omega) = (1, \pi - 2)$ which corresponds for $h = 1$ to the point $(\kappa, \beta) = (0, \frac{\pi}{2})$. As α_6 decreases the solution curves in the (c, ω)-plane move toward the lower half-plane.

Fig. 8 shows that the localized traveling waves undergo a fold bifurcation for positive values of α_6 as κ is increased. At the fold bifurcation, two solution branches merge, one corresponding to a single-humped solution and the other one becoming double-humped as we move away from the fold point. The insets to the figure show the solution profiles along the solution curves for $\alpha_6 = 0.5$. The amplitudes of both single-humped and doubled-humped solutions grow with increasing values of κ up to the
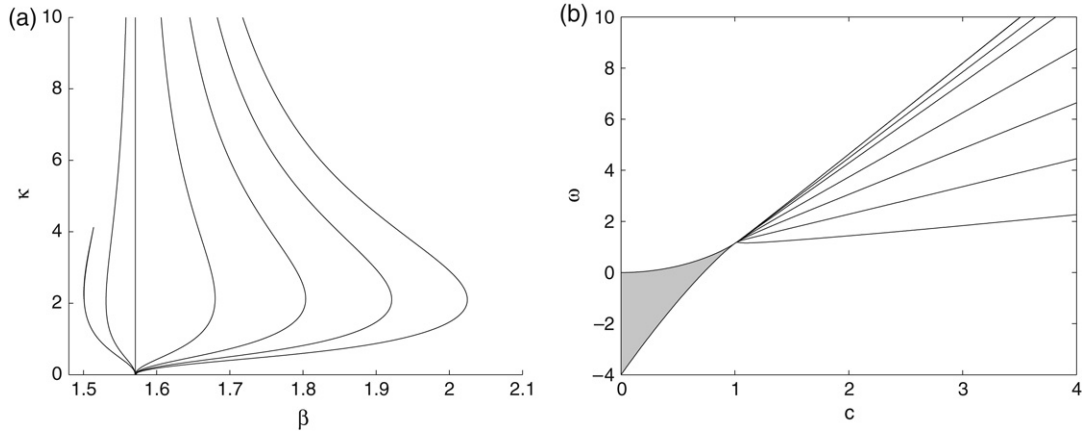
Fig. 7. Persistence of solutions for $(\alpha_3, \alpha_8) = (-1, 1)$ as κ is varied on the (β, κ)-plane (left) and on the (c, ω)-plane (right). Different curves correspond to different values of $\alpha_6 = 0.5, 0.25, 0, -0.5, -1, -1.5, -2$ from left to right on the left panel and from top to bottom on the right panel. The shaded area in the right panel indicates the boundary of the existence domain at $\kappa = 0$ and $\beta \in [0, \pi]$.
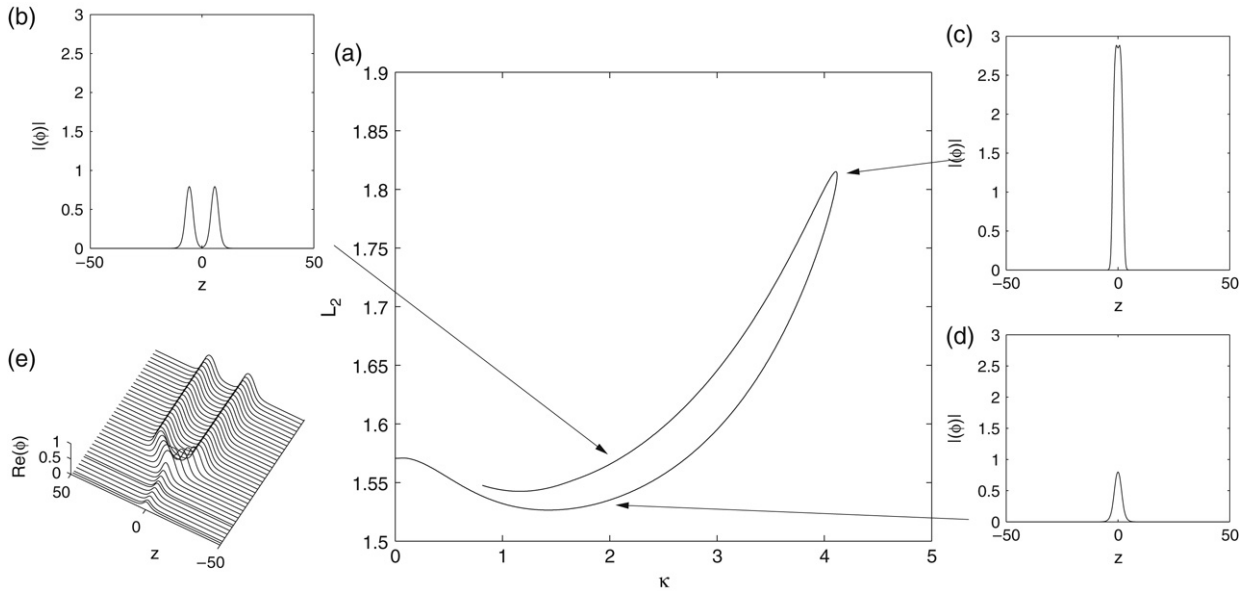

Fig. 8. (a) Fold bifurcation at which single-humped and doubled-humped solutions coalesce for $\alpha_6 = 0.5$ and $(\alpha_3, \alpha_8) = (-1, 1)$. Solution profiles are shown in panels (b)–(d). Panel (e) shows the fold bifurcation for fixed $\kappa = 1$ as α_6 varies. The fold bifurcation only occurs when α_6 is positive.

maximum amplitude at $\kappa \approx 4$. The branch containing the double-humped solutions in Fig. 8 is expected to continue to the value $\kappa = 0$ but computations in this limit are extremely difficult as the distance between the two humps becomes unbounded as $\kappa \to 0$.

Numerical simulations of the temporal dynamics of the initial-value problem were performed for the discrete NLS equation (1.1) using a variable-order, variable-timestep Runge–Kutta method, subject to periodic boundary conditions in n. The results illustrate spectral stability of the localized traveling waves of the translationally invariant NLS lattice, in accordance with Proposition 5.2. We have computed numerical approximations for $\alpha_6 = \alpha_8 = 0$ using as initial data the exact solution (1.12) with the amplitude modulation:

$$u_n(0) = \mu \, \Phi(n) e^{i\beta n}, \tag{6.3}$$

where $\Phi(Z)$ is given by (1.12) and μ is an amplitude parameter near $\mu = 1$. Fig. 9 presents the time evolution of the perturbed localized traveling wave for $\alpha_2 = 1$ and $\alpha_3 = 0.5$, when $\mu = 0.9$ (top) and $\mu = 1.1$ (bottom). The qualitative behavior for both values of the amplitude perturbation parameter are the same, that is the initial perturbation is shed as radiation for a short time frame, after which the radiation is separated from the solution core and moves across the lattice at a smaller wave speed than the solitary core of the solution.[6] After the radiation has been shed, the remaining solitary wave now travels across the lattice as a wave of permanent form, with no further shedding of radiation, in exactly the same manner as an unperturbed solution.

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[6] Note that waves for later times appear at the right end of the computational domain, this is just an artifact of the periodic boundary conditions we have used.
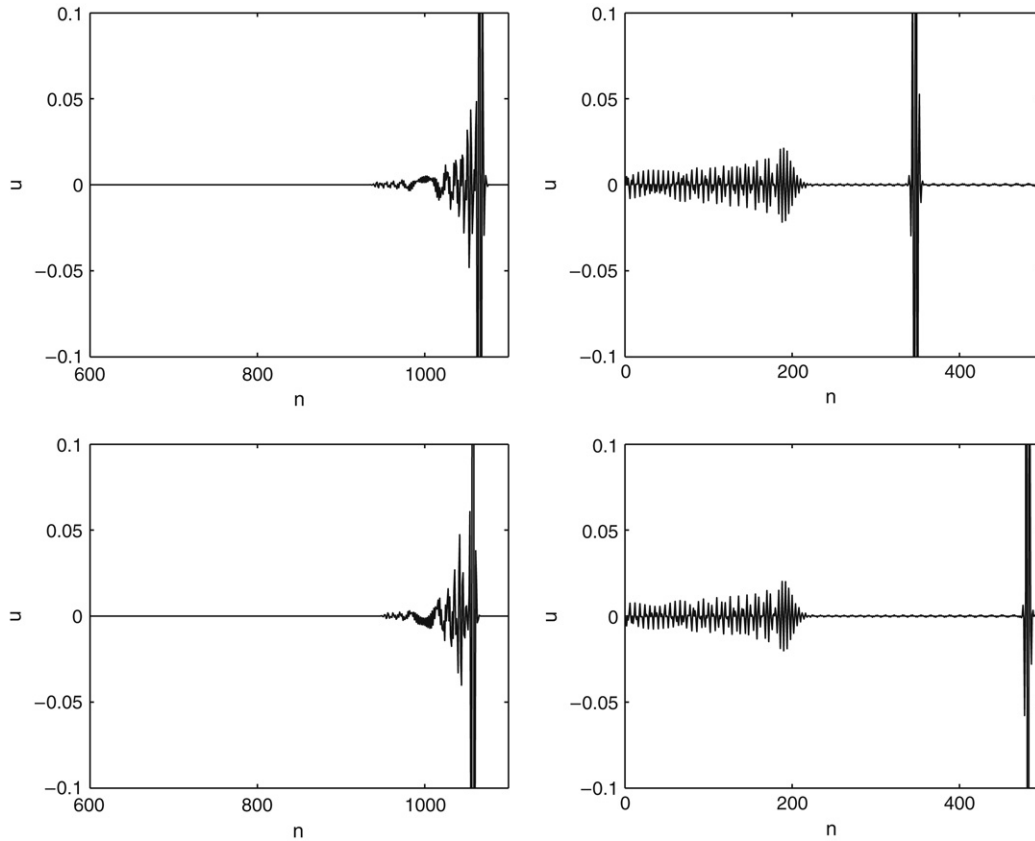
Fig. 9. Evolution of the initial data (6.3) for $\alpha_2 = 1$, $\alpha_3 = 0.5$, $\kappa = 1$, where $\mu = 0.9$ (top) with $t = 29$ (left) and $t = 600$ (right) and $\mu = 1.1$ (bottom) with $t = 23$ (left) and $t = 600$ (right).

We have checked that the same scenario holds for the values of $\alpha_6 \neq 0$ and $\alpha_8 \neq 0$, when the exact solution (1.12) is replaced by a numerical approximation of the localized traveling solution. The only difference is that in some numerical simulations (not shown), we have observed that the radiation moves across the lattice with a greater wavespeed than that of the solitary wave core.

6.2. Salerno model

We consider the Salerno model (1.1) and (1.3) parameterized by the only parameter α_2. Corollary 4.6(ii) and Example 4.7 state that the localized traveling solutions of the Ablowitz–Ladik model with $\alpha_2 = 1$ and $\beta = \frac{\pi}{2}$ do not persist for $\alpha_2 \neq 1$. Fig. 10(a) shows the tail amplitude Δ versus κ for different values of α_2 and β. The tail amplitude remains non-zero in a neighborhood of the point $\alpha_2 = 1$ and $\beta = \frac{\pi}{2}$. However, Fig. 10(b) shows that non-trivial zeros of the tail amplitude can appear far from the point of our studies: two zero tail solutions are found for values of κ near $\kappa = 1$. This shows that in addition to the known solutions of the AL lattice at $\alpha_2 = 1$, other non-trivial localized solutions can exist away from the limit $\beta = \frac{\pi}{2}$ and $\alpha_2 = 1$. The persistence of these solutions and how far they can be continued toward the pure cubic discrete NLS equation at $\alpha_2 = 0$ is still an open question.

7. Conclusion

We have proved two main results in this article. Firstly, we have found that traveling localized solutions are structurally stable in the discrete NLS equation at the particular point $\beta = \frac{\pi}{2}$, provided that the system of differential advance–delay equations can be reduced to a scalar equation. Secondly, we have proved that the existence of traveling solutions in the general case is a bifurcation of codimension one, such that a one-parameter family of solutions on the parameter plane (β, κ) (or equivalently, (ω, c)) is structurally stable with respect to parameter continuations near to the line $\beta = \frac{\pi}{2}$ and $\kappa > 0$, provided that a Melnikov function admits non-trivial zeros. We have illustrated these two main results with numerical approximations of traveling solutions in the two models, namely the translationally invariant discrete NLS equation and the Salerno model. The first model has a structurally stable family of traveling solutions near $\beta = \frac{\pi}{2}$, while the Salerno model has no traveling solutions near this point except for in the limit $\alpha_2 = 1$, which corresponds to the integrable AL lattice. A full treatment of the stability of the traveling wave solutions remains an avenue of future research, particularly the stability of single-humped and double-humped solutions around the fold bifurcation. The existence of traveling wave solutions away from the point $\beta = \frac{\pi}{2}$ is still an open question. Since the amplitude of the tail radiation becomes exponentially small as a function of κ for $\beta \neq \frac{\pi}{2}$, the existence of traveling wave solutions becomes a beyond all orders problem
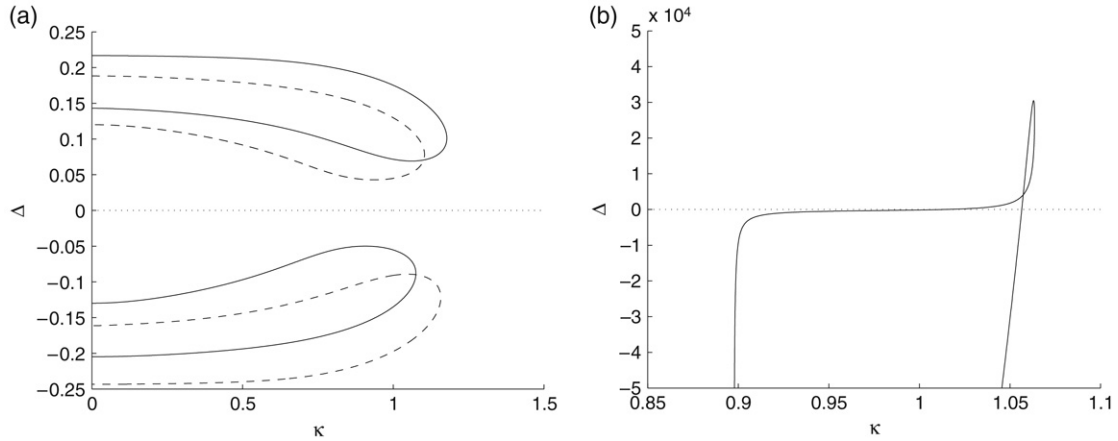
Fig. 10. (a) Tail amplitude of the Salerno model as a function of κ for $\beta = 0.65\pi$ (solid) and $\beta = 0.35\pi$ (dashed). The curves with $\Delta > 0$ correspond to $\alpha_2 = 1.1$ and the curves with $\Delta < 0$ correspond to $\alpha_2 = 0.9$. (b) Existence of non-trivial zero tail amplitude solutions for $\alpha_2 = 0.7$ and $\beta = 0.875\pi$.

which can be solved by computing zeros of the so-called Stokes constants. The computation of the Stokes constants is on-going work and will be reported elsewhere.

Acknowledgements

Appendix A. Formal reductions to the third-order ODE and associated operators

Let $\beta = \frac{\pi}{2}$ and $\Phi(Z) = \kappa\varphi(\zeta)$ with $\zeta = \kappa Z$ be a smooth solution of the differential advance–delay equation (1.9). The linear terms are expanded in powers of κ by

$$i\left(\Phi_+ - \Phi_- - 2\frac{\sinh\kappa}{\kappa}\Phi'\right) = i\kappa(\varphi(\zeta + \kappa) - \varphi(\zeta - \kappa) - 2\sinh\kappa\,\varphi'(\zeta)) = \frac{i}{3}\kappa^4(\varphi''' - \varphi') + O(\kappa^6),$$

while the expansion of the nonlinear terms is given by

$$f(-i\Phi_-, \Phi, i\Phi_+) = (\alpha_1 + 2\alpha_4 - 2\alpha_5 - 2\alpha_6 + \alpha_7)\kappa^3|\varphi|^2\varphi$$
$$+ 2i\kappa^4[(\alpha_2 + 2\alpha_8 - 2\alpha_9)|\varphi|^2\varphi' + (-\alpha_3 + \alpha_8 + \alpha_9 - \alpha_{10})\varphi^2\bar{\varphi}'] + O(\kappa^5).$$

When the constraint (1.10) is satisfied, the leading-order balance occurs at the third-order ODE, which admits a reduction to the real-valued function $\varphi(\zeta)$

$$\varphi''' - \varphi' + 6\gamma\varphi^2\varphi' = 0, \quad \varphi : \mathbb{R} \mapsto \mathbb{R}, \tag{A.1}$$

where $\gamma = \alpha_2 - \alpha_3 + 3\alpha_8 - \alpha_9 - \alpha_{10}$. When $\gamma > 0$, there exists an exact single-humped localized solution of the third-order ODE (A.1) in the form $\varphi = \frac{1}{\sqrt{\gamma}}\operatorname{sech}\zeta$.

A similar reduction can be performed for the differential advance–delay operators $L_\pm$ given by (2.5) and (2.6). When $U(Z) = u(\zeta)$ and $W(Z) = w(\zeta)$ are smooth functions of $\zeta = \kappa Z$, the operators $L_\pm$ are reduced at the leading order $O(\kappa^3)$ to the form of the third-order derivative operators:

$$L_+ = \frac{1}{3}\left(\frac{d^3}{d\zeta^3} - \frac{d}{d\zeta}\right) + 2\operatorname{sech}^2\zeta\frac{d}{d\zeta} - 4\operatorname{sech}^3\zeta\sinh\zeta \tag{A.2}$$

$$L_- = \frac{1}{3}\left(\frac{d^3}{d\zeta^3} - \frac{d}{d\zeta}\right) + 2\operatorname{sech}^2\zeta\frac{d}{d\zeta} + 4\nu\left(\operatorname{sech}^2\zeta\frac{d}{d\zeta} + \operatorname{sech}^3\zeta\sinh\zeta\right), \tag{A.3}$$

where

$$\nu = \frac{\alpha_3 - \alpha_8 - \alpha_9 + \alpha_{10}}{\alpha_2 - \alpha_3 + 3\alpha_8 - \alpha_9 - \alpha_{10}}.$$

The adjoint operators are

$$L_+^* = -\frac{1}{3}\left(\frac{d^3}{d\zeta^3} - \frac{d}{d\zeta}\right) - 2\,\text{sech}^2\,\zeta\,\frac{d}{d\zeta} \tag{A.4}$$

$$L_-^* = -\frac{1}{3}\left(\frac{d^3}{d\zeta^3} - \frac{d}{d\zeta}\right) - 2\,\text{sech}^2\,\zeta\,\frac{d}{d\zeta} + 4\,\text{sech}^3\,\zeta\,\sinh\zeta - 4\nu\left(\text{sech}^2\,\zeta\,\frac{d}{d\zeta} - 3\,\text{sech}^3\,\zeta\,\sinh\zeta\right). \tag{A.5}$$

It follows from explicit computations that

$$L_+\frac{\sinh\zeta}{\cosh^2\zeta} = 0, \qquad L_+\left(\text{sech}\,\zeta - \frac{\zeta\sinh\zeta}{\cosh^2\zeta}\right) = -\frac{2\sinh\zeta}{3\cosh^2\zeta}, \qquad L_+(1 - 2\,\text{sech}^2\,\zeta) = 0, \tag{A.6}$$

and

$$L_+^*\,\text{sech}\,\zeta = 0, \qquad L_+^*\,\zeta\,\text{sech}\,\zeta = -\frac{2}{3}\,\text{sech}\,\zeta, \qquad L_+^*1 = 0. \tag{A.7}$$

When $\nu = 0$, all eigenfunctions of L_- and L_-^* follow from the expressions above since $L_- = -L_+^*$ and $L_-^* = -L_+$. When $\nu \neq 0$, the eigenfunctions of the operators L_- and L_-^* were computed numerically in [19]. In the general case (which excludes two special integrable cases with $\nu = 0$ and $\nu = -\frac{1}{4}$), the operator L_- has one decaying eigenfunction $\text{sech}\,\zeta$ and no bounded non-decaying eigenfunctions, while the operator L_-^* has no decaying eigenfunctions and two bounded non-decaying eigenfunctions (even and odd in ζ) (see Fig. 2(c), (d) in [19]). It was also shown that the generalized eigenfunction of $L_-u_1 = \text{sech}\,\zeta$ is a bounded non-decaying function (odd in ζ) and therefore, it does not exist in the exponentially weighted space $H_\mu^1(\mathbb{R})$ for $\mu \neq 0$. It was shown in [19] by the perturbation theory and numerically that the double zero eigenvalue in $H_\mu^1(\mathbb{R})$ for $\nu = 0$ and $0 < \mu < \mu_0$ splits into a simple zero eigenvalue and a small positive eigenvalue for small $\nu \neq 0$. The positive eigenvalue in $H_\mu^1(\mathbb{R})$ with $0 < \mu < \mu_0$ corresponds to the resonance pole of the linearized operator L_- in $H^1(\mathbb{R})$.

Appendix B. Perturbation theory for eigenfunctions of the zero eigenvalue

Let us rewrite the differential advance–delay operator L_- of Example 3.2 in the form $L_- = L_0 + 2\nu L_1$, where

$$L_0 = -2\frac{\sinh\kappa}{\kappa}\frac{d}{dZ} + [1 + \sinh^2\kappa\,\text{sech}^2(\kappa Z)](\delta_+ - \delta_-)$$

$$L_1 = \sinh^2\kappa\,\text{sech}^2(\kappa Z)(\delta_+ - \delta_-) + 2\sinh^3\kappa\,\sinh(\kappa Z)\,\text{sech}(\kappa Z)\,\text{sech}(\kappa Z + \kappa)\,\text{sech}(\kappa Z - \kappa).$$

According to the exact solutions (3.16) and (3.18), we have

$$L_0\Phi = 0, \qquad L_1\Phi = 0, \qquad L_0 Z\Phi = C_0\Phi,$$

where $\Phi(Z) = \text{sech}(\kappa Z)$ and $C_0 = 2(\cosh\kappa - \frac{\sinh\kappa}{\kappa}) \neq 0$. Following the perturbation theory in [19], we develop perturbation expansions for the eigenvalue λ and the eigenfunction $U(Z)$ associated to the double zero eigenvalue in $H_\mu^1(\mathbb{R})$ with $0 < \mu < \mu_0$ for $\nu = 0$:

$$U = C_0\Phi + (2\nu)^2\lambda_2 Z\Phi + (2\nu)^3(\lambda_3 Z\Phi + \lambda_2 U_3) + (2\nu)^4(\lambda_4 Z\Phi + U_4) + O(\nu^5),$$
$$\lambda = (2\nu)^2\lambda_2 + (2\nu)^3\lambda_3 + (2\nu)^4\lambda_4 + O(\nu^5),$$

where the first-order corrections are zero due to the fact that $L_1\Phi = 0$. The first non-trivial equations for U_3 and U_4 are read as follows:

$$L_0 U_3 + L_1 Z\Phi = 0, \qquad L_0 U_4 + \lambda_2 L_1 U_3 = \lambda_2^2 Z\Phi.$$

By Fredholm Alternative in the weighted space $H_\mu^1(\mathbb{R})$ with $0 < \mu < \mu_0$, the solutions U_3 and U_4 exist in $H_\mu^1(\mathbb{R})$ if and only if

$$(w_0, L_1 Z\Phi) = 0, \qquad \lambda_2[(w_0, L_1 U_3) - \lambda_2(w_0, Z\Phi)] = 0,$$

where w_0 is the eigenfunction of $L_0^* w_0 = 0$. Because L_0 satisfies Assumption 3.11(i) (see Example 3.8), w_0 is in fact in $H^1(\mathbb{R})$ and, by Lemma 3.10, $w_0(Z)$ is odd on $Z \in \mathbb{R}$. Therefore, $(w_0, L_1 Z\Phi) = 0$ is satisfied. Since 0 is a double zero eigenvalue of L_0 by the same Assumption 3.11(i), we have $(w_0, Z\Phi) \neq 0$, such that the splitting in $H_\mu^1(\mathbb{R})$ occurs if $(w_0, L_1 U_3) \neq 0$ with a new simple eigenvalue

$$\lambda = (2\nu)^2\frac{(w_0, L_1 U_3)}{(w_0, Z\Phi)} + O(\nu^3).$$

In the limit of small κ, $w_0 \to \tanh(\kappa Z)\mathrm{sech}(\kappa Z)$ and $\Phi \to \mathrm{sech}(\kappa Z)$, such that $(w_0, Z\Phi) < 0$. It was shown analytically and numerically in [19] that $(w_0, L_1 U_3) < 0$ in the same limit. Therefore, $\lambda > 0$ for small κ.

By a similar method, one can show that the bounded eigenfunction $U_0(Z; 0)$ of the operator L_- does not exist for small $\nu \neq 0$. According to the exact solution (3.13), we have $L_0 1 = 0$. The perturbation expansion for $U_0(Z; 0)$ at $\lambda = 0$ is given by $U_0(Z; 0) = 1 + 2\nu U_1(Z) + O(\nu^2)$, where $U_1(Z)$ is a bounded function of the inhomogeneous problem $L_0 U_1 + L_1 1 = 0$. No bounded solutions exist unless $(w_0, L_1 1) = 0$, which is generally violated as w_0 and $L_1 1$ are both odd on $Z \in \mathbb{R}$.

Finally, one can show by the same technique for small $\nu \neq 0$ that the eigenfunction w_0 of $L_-^* w_0 = 0$ exists in $H^1_\mu(\mathbb{R})$ for $0 < \mu < \mu_0$ but does not exist in $H^1(\mathbb{R})$.

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