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A normal form for nonlinear resonance of embedded solitons

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A normal form for nonlinear resonance of embedded solitons is derived for a coupled two-wave system that generalizes the second-harmonic-generating model. This wave system is non-Hamiltonian in general. An embedded soliton is a localized mode of the nonlinear system that coexists with the linear wave spectrum. It occurs as a result of a codimension-one bifurcation of non-local wave solutions. Nonlinearity couples the embedded soliton and the linear wave spectrum and induces a one-sided radiation-driven decay of embedded solitons. The normal form shows that the embedded soliton is semi-stable, i.e. it survives under perturbations of one sign, but is destroyed by perturbations of the opposite sign. When a perturbed embedded soliton sheds continuous wave radiation, the radiation amplitude is generally not minimal, even if the wave system is Hamiltonian. The results of the analytical theory are confirmed by numerical computations.

Keywords: embedded solitons; nonlinear resonance; second-harmonic-generating wave system; radiation-driven semi-stability; normal form; spectral analysis

1. Introduction

Progress in the analytical theory for partial differential equations (PDEs) and solitons is marked by two recent discoveries: inverse scattering and Evans-function methods. The first is a method to solve an initial-value problem for a class of integrable nonlinear PDEs. Within this method, the nonlinear spectral transform is based on a Riemann–Hilbert or a $\bar{\partial}$ formalism for linear (Lax) operators (Ablowitz & Clarkson 1991; Ablowitz & Fokas 1997). The other is a method to study the spectra of linearized operators in soliton stability problems (Evans 1972*a*–*c*, 1975; Alexander *et al*. 1990; Pego & Weinstein 1992). Both theories treat nonlinear problems through classical spectral analysis of linear equations of mathematical physics (Hislop & Sigal 1996; Pego & Weinstein 1978).

Quite often, linear spectral problems exhibit special situations when eigenvalues of discrete spectrum are embedded in the essential spectrum (Hislop & Sigal 1996). Analytical theory for embedded eigenvalues and linear resonances is well studied in quantum mechanics (Merkli & Sigal 1999; Soffer & Weinstein 1998). Embedded eigenvalues are generally destroyed and disappear under deformations of the potential. In time-evolution problems, the decay of resonant modes that correspond to embedded eigenvalues is exponential within the linear theory and is described by the Fermi golden rule (Soffer & Weinstein 1998).

Similar linear resonances also arise in the inverse scattering theory and Evans-function method. In inverse scattering, embedded eigenvalues correspond to non-generic initial data that separate propagation and non-propagation of solitons in nonlinear time-evolution problems (Pelinovsky & Sulem 1998, 2000*a*). In Evans-function applications, embedded eigenvalues correspond to Hopf bifurcations (Bridges & Derks 1999; Li & Promislow 1998; Pego *et al*. 1995; Sandstede & Scheel 1999) and edge bifurcations (Barashenkov *et al*. 1998; Kapitula & Sandstede 2002; Kivshar *et al*. 1998) in soliton stability problems. The structural instability of embedded eigenvalues is typically one sided in such problems, i.e. the way the embedded eigenvalues disappear is different depending on the sign of a perturbation term. Under one sign, the discrete spectrum of a linear problem acquires new eigenvalues emerging from the embedded eigenvalue in an appropriate spectral space. Under the opposite sign, the discrete spectrum simply loses the embedded eigenvalue.

Recent works on solitons in optical communications revealed two new classes of problems where the linear resonance concept is generalized for nonlinear problems. The first class corresponds to a problem in which an isolated eigenvalue of a linear spectral problem couples with the essential spectrum due to nonlinearity-induced generation of multiple frequencies. This mechanism leads to radiative decay of resonant modes in nonlinear time-evolution problems (Buslaev & Perel'man 1995; Pelinovsky & Yang 2000; Pelinovsky *et al*. 1998; Soffer & Weinstein 1999; Yang 1997). Although this mechanism is of the same nature as the linear quantum resonance and is described by a nonlinear variant of the Fermi golden rule, the decay rate is algebraic and it depends on how many frequencies are required for coupling the discrete and essential spectrum. The general dispersive Hamiltonian normal form for nonlinear-through-linear resonance was found in Pelinovsky *et al*. (1998) and Soffer & Weinstein (1999).

The second class of problems possesses a true nonlinear resonance. It occurs when a nonlinear PDE exhibits solitary waves that have parameters lying in the continuous spectrum of the linear wave system. Such solitary waves were recently discovered in a number of physical systems and were referred to as *embedded solitons* (Champneys *et al*. 1998, 2001; Yang *et al*. 1999, 2001). When the nonlinear PDE is linearized around embedded solitons, part of the discrete spectrum of the linearized operator lies inside the continuous spectrum of this same operator. Within the linear stability theory, single-hump embedded solitons are often neutrally stable, with no coupling between the discrete and continuous spectra. However, heuristic arguments based on energy estimates (Yang *et al*. 1999) conjecture a one-sided nonlinear instability due to nonlinear coupling of embedded solitons with the linear wave spectrum. In the one-sided instability, embedded solitons survive under perturbations of one sign and decay under perturbations of the opposite sign. These estimates were confirmed by numerical simulations of PDEs for perturbed embedded solitons (Champneys *et al*. 2001; Yang *et al*. 1999, 2001).

Analytical theory for nonlinear resonances of embedded solitons remains an open problem, except for two special situations where recent progress was made (Pelinovsky & Sulem 2001; Yang 2001). The first situation occurs for embedded solitons in integrable nonlinear PDEs solvable through a linear (Lax) operator. In particular, the modified Korteweg–de Vries (KdV) equation in one dimension (Pelinovsky & Grimshaw 1997) and the DSII equation in two dimensions (Gadyl'shin & Kiselev 1999; Pelinovsky & Sulem 2000*b*) exhibit embedded solitons where the nonlinear instability can be traced to linear resonance in the Lax operator (Pelinovsky & Sulem 2001). The second situation occurs for embedded solitons in nearly integrable PDEs, such as the perturbed integrable fifth-order KdV equation (Yang 2001). In this case, a soliton perturbation theory enables one to explain the one-sided nonlinear instability by reducing the PDE to an analytical equation for the velocity of the embedded soliton (see also Pelinovsky & Grimshaw 1996). This analytical equation describes the nonlinear coupling between embedded solitons and the linear wave spectrum that occurs due to mixing of small external perturbations with the underlying integrable PDE. Thus progress of the analytical theory for nonlinear resonance of embedded solitons was limited by the integrability of the underlying PDE.

In this paper, we derive a general normal form for nonlinear resonance of embedded solitons for a coupled wave system that is not close to an integrable system. This coupled system is even non-Hamiltonian in general. Our method is an internal perturbation technique applied to the underlying PDE (Pelinovsky & Grimshaw 1996; Pelinovsky *et al*. 1996). When nonlinearities are quadratic and cubic, the coupled wave system considered here generalizes the second-harmonic-generation (SHG) model studied in Yang *et al*. (1999). The normal form we obtained shows that the embedded soliton does suffer a one-sided nonlinear instability, as the heuristic argument in Yang *et al*. (1999) predicted. Our results also indicate that when a perturbed embedded soliton sheds continuous wave radiation (tails) the tail amplitude is generally not the minimum of all possible tail amplitudes, even for the Hamiltonian case. This contrasts with the analytical results of Yang (2001) for the perturbed integrable fifth-order KdV equation and with numerical conjectures of Boyd (1998) for the non-integrable fifth-order KdV equation. Excellent agreement is obtained between our analytical results and numerical computations.

The paper is organized as follows: § 2 formulates the problem and describes our main results. Proofs of auxiliary results for a linear stability problem associated with embedded solitons are given in § 3. Section 4 presents proofs of the main results by using nonlinear decomposition for nonlinear wave equations. Section 5 reports numerical results that confirm our analytical formulae. Section 6 summarizes the main results and briefly discusses other types of embedded solitons.

2. Formulation and main results

We consider a simple two-wave system that exhibits embedded solitons and nonlinear resonances. This system is a generalization of the SHG model with quadratic and cubic nonlinearities, where embedded solitons have been discovered (Yang *et al*. 1999). The general system can be written as

$$\mathrm{i}u_t + u_{xx} + f(u,v) = 0, \tag{2.1}$$

$$\mathrm{i}v_t + Dv_{xx} + \Delta v + g(u,v) = 0, \tag{2.2}$$

where $u, v \in \mathbb{C}$, $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, $D, \Delta \in \mathbb{R}$ and the functions $f, g : \mathbb{C}^2 \to \mathbb{C}$ satisfy the conditions

$$f(0,v) = 0, \quad \left.\frac{\partial f(u,v)}{\partial u}\right|_{u=0} = \left.\frac{\partial f(u,v)}{\partial v}\right|_{u=0} = 0, \quad v \in \mathbb{C}, \tag{2.3}$$

$$g(0,0) = 0, \quad \left.\frac{\partial g}{\partial u}\right|_{u=v=0} = \left.\frac{\partial g}{\partial v}\right|_{u=v=0} = 0. \tag{2.4}$$

The conditions (2.3), (2.4) ensure that the linear wave spectrum, defined in the limit $\|u\|_{L^2}, \|v\|_{L^2} \to 0$, becomes uncoupled,

$$v = 0, \quad u = u_0 e^{i(kx+\omega t)} : \quad \omega = \omega_1(k) = -k^2, \tag{2.5}$$

$$u = 0, \quad v = v_0 e^{i(kx+\omega t)} : \quad \omega = \omega_2(k) = \Delta - Dk^2. \tag{2.6}$$

Note that our system (2.1), (2.2) is non-Hamiltonian in general. Its continuous spectrum of the linear system is located at $\omega \in (-\infty, 0] \cup (-\infty, \Delta]$ for $D > 0$ or at $\omega \in (-\infty, 0] \cup [\Delta, \infty)$ for $D < 0$. Several assumptions are used to set the problem for embedded solitons in the coupled wave system (2.1), (2.2).

Assumption 2.1 *There exists a real number $\alpha > 0$ such that the system (2.1), (2.2) is phase invariant under the transformation*

$$u \to u e^{i\theta_0}, \qquad v \to v e^{i\alpha\theta_0} \tag{2.7}$$

for every $\theta_0 \in \mathbb{R}$. There exists a conserved quantity (power) related to the phase invariance,

$$Q = \int_{-\infty}^{\infty} dx\, (|u|^2 + \alpha|v|^2). \tag{2.8}$$

Corollary 2.2. *Nonlinear functions $f(u,v)$ and $g(u,v)$ in the system (2.1), (2.2) satisfy several constraints related to the symmetry in assumption 2.1. It follows from the phase invariance (2.7) and the system (2.1), (2.2) that*

$$\left.\begin{aligned} u\frac{\partial f}{\partial u} - \bar{u}\frac{\partial f}{\partial \bar{u}} + \alpha\left(v\frac{\partial f}{\partial v} - \bar{v}\frac{\partial f}{\partial \bar{v}}\right) &= f(u,v), \\ u\frac{\partial g}{\partial u} - \bar{u}\frac{\partial g}{\partial \bar{u}} + \alpha\left(v\frac{\partial g}{\partial v} - \bar{v}\frac{\partial g}{\partial \bar{v}}\right) &= \alpha g(u,v), \end{aligned}\right\} \tag{2.9}$$

where $\bar{u}$, $\bar{v}$ are complex conjugates of u, v. On the other hand, existence of the conserved quantity (2.8) implies additional constraints on $f(u,v)$ and $g(u,v)$,

$$\left.\begin{aligned} \bar{u}\frac{\partial f}{\partial u} - u\frac{\partial \bar{f}}{\partial u} + \alpha\left(\bar{v}\frac{\partial g}{\partial u} - v\frac{\partial \bar{g}}{\partial u}\right) &= \bar{f}(u,v), \\ \bar{u}\frac{\partial f}{\partial v} - u\frac{\partial \bar{f}}{\partial v} + \alpha\left(\bar{v}\frac{\partial g}{\partial v} - v\frac{\partial \bar{g}}{\partial v}\right) &= \alpha\bar{g}(u,v). \end{aligned}\right\} \tag{2.10}$$

Remark 2.3. The system (2.1), (2.2) may also conserve other quantities, such as momentum and the Hamiltonian. However, the semi-stability theory of nonlinear resonance of embedded solitons works for general non-Hamiltonian wave systems that conserve a single positive-definite L^2-type functional such as the power Q (2.8).

Assumption 2.4 *The system (2.1), (2.2) has embedded-soliton solutions that are bounded in $L^2(\mathbb{R})$, i.e. $Q < \infty$. The solutions are given in the form*

$$u_{\mathrm{ES}}(x,t) = \Phi_u(x)\mathrm{e}^{\mathrm{i}(\omega_{\mathrm{ES}}t+\theta_0)}, \qquad v_{\mathrm{ES}}(x,t) = \Phi_v(x)\mathrm{e}^{\mathrm{i}\alpha(\omega_{\mathrm{ES}}t+\theta_0)}, \tag{2.11}$$

where $\theta_0 \in \mathbb{R}$ is arbitrary, $\Phi_u, \Phi_v : \mathbb{R} \to \mathbb{R}$ and ω_{ES} is the propagation constant lying inside the continuous spectrum of the linear v-component system, i.e.

$$\omega_{\mathrm{ES}} \notin \omega_1(k) \quad \textit{and} \quad \alpha\omega_{\mathrm{ES}} \in \omega_2(k). \tag{2.12}$$

Here, $\omega_1(k)$ and $\omega_2(k)$ denote the continuous spectra of the u and v components, respectively, i.e.

$$\omega_1(k) = (-\infty, 0], \qquad \omega_2(k) = \begin{cases} (-\infty, \Delta], & D > 0, \\ [\Delta, \infty), & D < 0. \end{cases} \tag{2.13}$$

The free-parameter space of the embedded-soliton solutions (2.11) is defined by the parameter θ_0 associated with the symmetry (2.7). We assume that it is the only free parameter of the solutions (2.11). For convenience, we consider symmetric embedded solitons, i.e. $\Phi_{u,v}(-x) = \Phi_{u,v}(x)$. If a translational symmetry ($x \to x - x_0$) exists, another trivial parameter, x_0, may be included in (2.11), but it does not change our analysis.

Assumption 2.5 *The embedded-soliton solution (2.11) is a single isolated solution of (2.1), (2.2) under the boundary conditions $\lim_{|x|\to\infty} |u|, |v| = 0$, i.e. solution (2.11) is unique for $\omega = \omega_{\mathrm{ES}}$ and no such solutions exist for $|\omega - \omega_{\mathrm{ES}}| \leqslant \epsilon$, where ϵ is a small number.*

Assumption 2.6 *The embedded soliton (2.11) is spectrally stable, i.e. the linearized problem (see (3.3) below) has no localized eigenfunctions except for $\lambda = 0$, which is a double embedded eigenvalue into the linear continuous spectrum. Alternatively, the Evans function has no zeros at any λ except for a double zero at $\lambda = 0$.*

The latter assumption excludes stable embedded eigenvalues at $\mathrm{Re}(\lambda) = 0$ and $\mathrm{Im}(\lambda) \neq 0$, as well as unstable eigenvalues at $\mathrm{Re}(\lambda) \neq 0$. The stable embedded eigenvalues for $|\mathrm{Im}(\lambda)| \geqslant \epsilon > 0$ could be included in the analysis below and do not affect the final result. On the other hand, unstable eigenvalues make the embedded solitons linearly (exponentially) unstable. The problem of nonlinear resonance and weak (algebraic) one-sided instability of embedded solitons makes no sense if embedded solitons are linearly (exponentially) unstable.

Before presenting our main results, we give a simple example of embedded solitons in the coupled system (2.1), (2.2) and reformulate the mathematical problem for embedded solitons in terms of *codimension-one bifurcation* of non-local wave solutions.

Example 2.7. The system (2.1), (2.2) includes the SHG model (Yang *et al*. 1999), where

$$\left.\begin{aligned} f(u,v) &= \bar{u}v + \gamma_1(|u|^2 + 2|v|^2)u, \\ g(u,v) &= \tfrac{1}{2}u^2 + \gamma_2(2|u|^2 + |v|^2)v. \end{aligned}\right\} \tag{2.14}$$

Assumption 2.1 and corollary 2.2 are satisfied with these nonlinear functions, so that the parameter α in the symmetry transformation (2.7) is $\alpha = 2$. Note that the SHG model (2.14) is Hamiltonian if and only if $\gamma_1 = \gamma_2$. Embedded solitons (2.11) exist for this model in two cases (Champneys *et al.* 2001; Yang *et al.* 1999),

$$\mathrm{I}: \quad D < 0, \quad \Delta > 0, \quad \gamma_{1,2} < 0, \quad 2\omega_{\mathrm{ES}} \in [\Delta, \infty), \tag{2.15}$$

$$\mathrm{II}: \quad D > 0, \quad \Delta > 0, \quad \gamma_{1,2} > 0, \quad 2\omega_{\mathrm{ES}} \in [0, \Delta]. \tag{2.16}$$

These embedded solitons satisfy assumptions 2.4–2.6.

According to (2.12), the propagation constant ω_{ES} of the embedded soliton (2.11) lies in the linear continuous spectrum of the v component, but not the u component. This fact implies that the existence of embedded solitons is a codimension-one bifurcation of non-local waves at $\omega = \omega_{\mathrm{ES}}$. The reason will be clear when we reformulate this problem in an algebraic form below.

Lemma 2.8. *Suppose that special non-local solutions of (2.1), (2.2) are sought in the form*

$$\left.\begin{aligned} u_{\mathrm{NL}}(x,t) &= U(x;\omega,\delta)\mathrm{e}^{\mathrm{i}(\omega t+\theta_0)}, \\ v_{\mathrm{NL}}(x,t) &= V(x;\omega,\delta)\mathrm{e}^{\mathrm{i}\alpha(\omega t+\theta_0)}, \end{aligned}\right\} \tag{2.17}$$

where $U, V : \mathbb{R} \to \mathbb{R}$, $\theta_0 \in \mathbb{R}$ is arbitrary, $\delta \in [0, 2\pi)$, $\omega > 0$ and $\alpha\omega \in \omega_2(k)$. The boundary conditions for the non-local wave solutions (2.17) as $|x| \to \infty$ are

$$U(x;\omega,\delta) \to O(\mathrm{e}^{-\sqrt{\omega}|x|}), \tag{2.18}$$

$$V(x;\omega,\delta) \to r(\omega,\delta)V_{\mathrm{tail}}(x;\omega,\delta) + o(\mathrm{e}^{-\sqrt{\omega}|x|}). \tag{2.19}$$

Here, δ denotes phase and $r = r(\omega,\delta)$ is the amplitude for a periodic tail function, $V_{\mathrm{tail}}(x + x_p;\omega,\delta) = V_{\mathrm{tail}}(x;\omega,\delta)$, normalized by the condition $\max_{x\in\mathbb{R}}|V_{\mathrm{tail}}| = 1$. Besides the parameter θ_0 of the embedded soliton (2.11) and the wave frequency ω, the non-local wave solutions (2.17) have an additional parameter δ.

Proof. The functions $U(x)$ and $V(x)$ in (2.17) solve the system

$$U_{xx} - \omega U + f(U,V) = 0, \tag{2.20}$$

$$DV_{xx} + (\Delta - \alpha\omega)V + g(U,V) = 0. \tag{2.21}$$

Constructing a symmetric solution of (2.20), (2.21), we define initial conditions at $x = 0$,

$$U(0) = U_0, \quad V(0) = V_0, \quad U'(0) = 0, \quad V'(0) = 0.$$

As $|x| \to \infty$, we have to satisfy a single constraint on two parameters U_0 and V_0. The constraint is imposed to eliminate exponential growth of eigenfunctions following from (2.20). Therefore, there is a free parameter in addition to ω and θ_0. We choose the additional parameter as the phase δ of the periodic tail function $V_{\mathrm{tail}}(x)$ that solves the equation

$$DV_{\mathrm{tail},xx} + (\Delta - \alpha\omega)V_{\mathrm{tail}} + \frac{1}{r}g(0, rV_{\mathrm{tail}}) = 0.$$

Here and in (2.18), (2.19), we have used the conditions (2.3), (2.4) on the nonlinear functions. In the limit $|r(\omega,\delta)| \to 0$, the periodic tail function $V_{\mathrm{tail}}(x)$ can be easily found as

$$V_{\mathrm{tail}}(x) = \sin[k(\omega)|x| + \delta], \quad k(\omega) = \left[\frac{\Delta - \alpha\omega}{D}\right]^{1/2}. \tag{2.22}$$

Notice that the tail amplitude $r(\omega, \delta)$ is generally non-zero, since the two oscillatory terms $\sin k|x|$ and $\cos k|x|$ cannot be removed simultaneously by a single condition on U_0 and V_0. ■

The embedded soliton (2.11) corresponds to a codimension-one bifurcation, i.e. the existence of embedded solitons is defined by a zero of the scalar function $r = r(\omega, \delta)$ such that $r(\omega_{\mathrm{ES}}, \delta) = 0$. Zeros ω_{ES} of $r(\omega, \delta)$ must be uniform for any value of δ as the parameter δ disappears in the limits,

$$\lim_{\omega \to \omega_{\mathrm{ES}}} U(x; \omega, \delta) = \Phi_u(x), \qquad \lim_{\omega \to \omega_{\mathrm{ES}}} V(x; \omega, \delta) = \Phi_v(x). \tag{2.23}$$

Therefore, the form (2.11) implies that the function $r(\omega, \delta)$ has a zero at $\omega = \omega_{\mathrm{ES}}$, according to the representation

$$r(\omega, \delta) \to (\omega - \omega_{\mathrm{ES}})^n R(\delta) \quad \text{as } \omega \to \omega_{\mathrm{ES}}, \tag{2.24}$$

where n is the order of the zero. When $n = 1$, $R(\delta)$ is the slope of function $r(\omega; \delta)$ at $\omega = \omega_{\mathrm{ES}}$ and is given by (3.24) below.

Definition 2.9. The index of multiplicity of the embedded soliton (2.11) is the order of zero of the function $r(\omega, \delta)$ at $\omega = \omega_{\mathrm{ES}}$,

$$n = \mathrm{ind}_{\omega_{\mathrm{ES}}}(r) : \quad \left.\frac{\partial^n r}{\partial \omega^n}\right|_{\omega = \omega_{\mathrm{ES}}} \neq 0, \quad \left.\frac{\partial^j r}{\partial \omega^j}\right|_{\omega = \omega_{\mathrm{ES}}} = 0, \quad 0 \leqslant j < n. \tag{2.25}$$

In the nonlinear problem (2.1), (2.2), the two solutions (2.6) and (2.11) coexist, i.e. the embedded soliton is a resonant mode of the linear wave spectrum. However, there is no coupling between (2.6) and (2.11), since the amplitude $r(\omega, \delta)$ in (2.19) vanishes at $\omega = \omega_{\mathrm{ES}}$. The coupling will occur in a nonlinear time-evolution problem when the embedded soliton is perturbed. What happens then is that the deformed embedded soliton will shed radiation in the form of continuous waves (tails) in both directions of the x-axis. These tails travel at the group velocity of linear waves with wavenumber $k_{\mathrm{r}} = k(\omega_{\mathrm{ES}})$ in the v component, and their amplitudes are generally not minimal. These results are detailed below.

Proposition 2.10 (tail amplitude). *Let us suppose that the embedded soliton $(u_{\mathrm{ES}}, v_{\mathrm{ES}})(x, t)$ is perturbed by a small symmetric deformation, $\theta_0 \to \theta(t)$, such that $|\dot{\theta}(t)| < C_\theta \epsilon$, where $\epsilon \ll 1$ and C_θ is constant for $t \in [0, T]$. The perturbed embedded solitons generates radiation $(u_{\mathrm{RD}}, v_{\mathrm{RD}})(x, t)$ that propagates with the group velocity C_{g} in both directions of the x-axis, where*

$$C_{\mathrm{g}} = |\omega_2'(k_{\mathrm{r}})| = 2|D| k_{\mathrm{r}}, \qquad k_{\mathrm{r}} = k(\omega_{\mathrm{ES}}) = \left[\frac{\Delta - \alpha \omega_{\mathrm{ES}}}{D}\right]^{1/2}. \tag{2.26}$$

The radiation fronts have the tail amplitude $R = R(\delta_{\mathrm{rad}})$, where δ_{rad} is the radiation tail phase defined in (4.22). This radiation tail amplitude is not minimal, but is related to the minimum of $R(\delta)$ as

$$R(\delta_{\mathrm{rad}}) = \frac{R(\delta_{\min})}{\cos(\delta_{\min} - \delta_{\mathrm{rad}})}, \tag{2.27}$$

where $\delta_{\min}$ is the tail phase defined in (3.21) that minimizes $|R(\delta)|$. At large x and t values such that

$$|x| \gg 1, \quad t \gg 1, \quad \frac{|x|}{t} = C_x < \infty, \tag{2.28}$$

where C_x is a constant, the radiation fronts are given asymptotically as

$$\left.\begin{aligned} &u_{\mathrm{RD}}(x,t) \to 0, \\ &v_{\mathrm{RD}}(x,t) \to -\mathrm{i}\,\mathrm{sgn}(D)\dot{\theta}^n R(\delta_{\mathrm{rad}})\mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(D)(k_{\mathrm{r}}|x|+\delta_{\mathrm{rad}})} H(C_{\mathrm{g}}t - |x|), \end{aligned}\right\} \tag{2.29}$$

where n is the index of the embedded soliton (2.25) and $H(z)$ is the step-function $H(z) = 1$ at $z > 0$ and $H(z) = 0$ at $z < 0$.

We note that, in general, $\delta_{\min} \neq \delta_{\mathrm{rad}}$, even if the system (2.1), (2.2) is Hamiltonian (see § 5 for an example). Thus the radiation amplitude $R(\delta_{\mathrm{rad}}) > R(\delta_{\min})$, i.e. the tail amplitude $R(\delta_{\mathrm{rad}})$ is not minimal.

Our main result formulated in proposition 2.11 describes the nonlinear resonance and transformation of a perturbed embedded soliton (2.11) in the coupled wave system (2.1), (2.2). For convenience, we define the following quantity (see remark 2.13 below):

$$E_1(\delta) = 2\int_{-\infty}^{\infty} \left(\Phi_u(x) \frac{\partial U(x;\omega,\delta)}{\partial \omega}\bigg|_{\omega=\omega_{\mathrm{ES}}} + \alpha \Phi_v(x) \frac{\partial V(x;\omega,\delta)}{\partial \omega}\bigg|_{\omega=\omega_{\mathrm{ES}}} \right) \mathrm{d}x. \tag{2.30}$$

Proposition 2.11 (normal form for nonlinear resonance). *Consider the initial-value problem for the coupled wave system (2.1), (2.2) with initial data*

$$u_0(x) = u(x,0), \qquad v_0(x) = v(x,0). \tag{2.31}$$

Assume that u_0, v_0 are symmetric and integrable functions in L^2, $u_0, v_0 : \mathbb{R} \to \mathbb{C}$ and $u_0, v_0 \in L^2(\mathbb{R})$. Also assume that the initial data are close to the embedded soliton (2.11) in the sense

$$\left.\begin{aligned} \|u_0(x) - \Phi_u(x)\|_{L^2} \leqslant \epsilon \ll 1, \\ \|v_0(x) - \Phi_v(x)\|_{L^2} \leqslant \epsilon \ll 1, \end{aligned}\right\} \tag{2.32}$$

where $\|u(x)\|_{L^2}$ is a suitable L^2 norm in a complex function space. The leading-order time-dependent solution for (2.1), (2.2) in the asymptotic region (2.28) is

$$\begin{pmatrix} u \\ v \end{pmatrix}(x,t) = \left[\begin{pmatrix} U \\ V \end{pmatrix}(x; \omega_{\mathrm{ES}} + \dot{\theta}, \delta_{\mathrm{rad}}) + \mathrm{i}\epsilon^n \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}(x,t) + \mathrm{O}(\epsilon^{n+1}) \right] \\ \times H(C_{\mathrm{g}}t - |x|) \exp\left[\mathrm{i} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} (\omega_{\mathrm{ES}} t + \theta(t)) \right], \tag{2.33}$$

where $|\dot{\theta}(t)| < C_\theta \epsilon$ and the transverse perturbation term is $(\delta u, \delta v) = c_2 \boldsymbol{\psi}_{\mathrm{sym}(-)}(x)$, where c_2 is given by (4.21) and (4.23) and $\boldsymbol{\psi}_{\mathrm{sym}(-)}(x)$ is defined in lemma 3.2 below. Under the constraint $e_1 \equiv E_1(\delta_{\mathrm{rad}}) \neq 0$, the embedded-soliton orbit parameter $\theta(t)$ satisfies the asymptotic equation

$$e_1 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\Gamma \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} \right)^{2n}, \tag{2.34}$$

where the coefficient Γ is defined by

$$\Gamma = 4\alpha k_{\mathrm{r}} |D| |R(\delta_{\mathrm{rad}})|^2 > 0. \tag{2.35}$$

Corollary 2.12. *A small symmetric perturbation to the embedded soliton results in the one-sided instability of the embedded soliton.*

Proof. Setting $\Omega = \dot{\theta}$, we can solve the initial-value problem for (2.34),

$$\Omega(t) = \frac{\Omega_0}{[1 + (2n-1)e_1^{-1}\Gamma\Omega_0^{2n-1}t]^{1/(2n-1)}}, \tag{2.36}$$

where $\Omega_0 = \dot{\theta}(0)$. When $\Omega_0 > 0$, the dynamical soliton parameter $\omega_{\mathrm{ES}}+\Omega(t)$ is locally increased, while in the other case, when $\Omega_0 < 0$, the soliton parameter is locally decreased (the parameter ω_{ES} is positive). If $e_1 > 0$, a local increase in the soliton constant ω_{ES} ($\Omega_0 > 0$) results in decay of the perturbation, $\lim_{t\to+\infty}\Omega(t) = 0^+$. The embedded soliton is asymptotically stable in this case. On the other hand, a local decrease in the soliton constant ω_{ES} ($\Omega_0 < 0$) results in blow-up of the negative perturbation, $\lim_{t\to T_\infty}\Omega(t) = -\infty$, where

$$T_\infty = \frac{-e_1}{(2n-1)\Gamma\Omega_0^{2n-1}}.$$

This blowing-up perturbation eventually destroys the embedded soliton, typically through its radiative decay (Yang *et al.* 1999). Thus the embedded soliton is unstable only under a perturbation of a special sign, i.e. its instability is one-sided, as conjectured in Yang *et al.* (1999). If $e_1 < 0$, the one-sided instability is inverted with respect to the sign of initial perturbation Ω_0. ∎

Remark 2.13. Define a local energy of the non-local wave (2.17) as a function of ω and δ,

$$E(\omega,\delta) = \int_{-\infty}^{\infty} [U^2(x;\omega,\delta) + \alpha(V^2(x;\omega,\delta) - r^2(\omega,\delta)V_{\mathrm{tail}}^2(x;\omega,\delta))]\,\mathrm{d}x, \tag{2.37}$$

where the integral is bounded due to the boundary conditions (2.18), (2.19). Then the coefficient $E_1(\delta)$ appears in the Taylor expansion of $E(\omega,\delta)$ around $\omega = \omega_{\mathrm{ES}}$,

$$E(\omega,\delta) = E_{\mathrm{ES}} + E_1(\delta)(\omega-\omega_{\mathrm{ES}}) + O(\omega-\omega_{\mathrm{ES}})^2,$$

where E_{ES} is the energy value for the embedded soliton (2.11). Within this context, the asymptotic equation (2.34) reproduces equation (12) of Yang *et al.* (1999), found with the use of physical semi-qualitative arguments. We associate the stable scenario (decay of the perturbation) with a local increase in the wave energy $E = E(\omega_{\mathrm{ES}} + \Omega_0, \delta_{\mathrm{rad}})$ induced by the perturbation and the unstable scenario (the decay of the embedded soliton) with a local decrease in the wave energy.

Remark 2.14. The asymptotic equation (2.34) breaks down for $e_1 = E_1(\delta_{\mathrm{rad}}) = 0$ or for $\delta_{\mathrm{min}} = \delta_{\mathrm{rad}} + \frac{1}{2}\pi$. In the former case, the soliton orbit parameter $\theta(t)$ satisfies a third-order differential equation, which usually results in both-sided instabilities of solitons (Pelinovsky *et al.* 1996). Since no examples are available for such a bifurcation to occur at this time, we do not extend the derivation of the asymptotic equation (2.34) for that special case.

3. Properties of linear spectrum for embedded solitons

Here we consider the linearized problem for stability of embedded solitons and derive several general results on locations and properties of the linear spectrum. These properties are used in § 4 for the proof of propositions 2.10 and 2.11.

We use a standard linearization technique (Kaup 1990) to reduce (2.1), (2.2) to a linear eigenvalue problem. Small perturbations to the embedded soliton are written as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \left[\begin{pmatrix} \Phi_u \\ \Phi_v \end{pmatrix}(x) + \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}(x,t) \right] \exp\left[\mathrm{i} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} (\omega_{\mathrm{ES}} t + \theta_0) \right], \tag{3.1}$$

where $\|\delta u(x,t)\|_{L^2}, \|\delta v(x,t)\|_{L^2} \ll 1$. The spectrum of the embedded soliton (2.11) can be decomposed in the form

$$\begin{pmatrix} \delta u \\ \delta v \end{pmatrix}(x,t) = \sum_{\lambda} \begin{pmatrix} \delta u_\lambda \\ \delta v_\lambda \end{pmatrix}(x) \exp(\lambda t), \tag{3.2}$$

where $\sum_\lambda$ denotes a continuous and discrete sum of all eigenfunctions $(\delta u_\lambda, \delta v_\lambda)(x)$ of the linearized problem. We construct the perturbation vector

$$\boldsymbol{\psi} = [\delta u_\lambda, \delta v_\lambda, \delta \bar{u}_\lambda, \delta \bar{v}_\lambda]^{\mathrm{T}},$$

where the superscript stands for the matrix transpose. The perturbation vector satisfies the linear eigenvalue problem following from (2.1), (2.2), (3.1) and (3.2),

$$\hat{\mathcal{H}} \boldsymbol{\psi} = [\hat{\mathcal{M}} - \hat{\mathcal{W}}(x)] \boldsymbol{\psi} = \lambda \mathcal{J} \boldsymbol{\psi}. \tag{3.3}$$

Here, $\mathcal{J} = \mathrm{diag}(\mathrm{i}, \mathrm{i}, -\mathrm{i}, -\mathrm{i})$, $\hat{\mathcal{M}} = \mathrm{diag}(\mathcal{L}_u, \mathcal{L}_v, \mathcal{L}_u, \mathcal{L}_v)$ is a diagonal self-adjoint operator with entries $\mathcal{L}_u = -\partial_x^2 + \omega_{\mathrm{ES}}$, $\mathcal{L}_v = -D\partial_x^2 + \alpha\omega_{\mathrm{ES}} - \Delta$ and $\hat{\mathcal{W}}(x)$ is a real-valued matrix given by

$$\hat{\mathcal{W}}(x) = \left. \begin{bmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} & \dfrac{\partial f}{\partial \bar{u}} & \dfrac{\partial f}{\partial \bar{v}} \\ \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} & \dfrac{\partial g}{\partial \bar{u}} & \dfrac{\partial g}{\partial \bar{v}} \\ \dfrac{\partial \bar{f}}{\partial u} & \dfrac{\partial \bar{f}}{\partial v} & \dfrac{\partial \bar{f}}{\partial \bar{u}} & \dfrac{\partial \bar{f}}{\partial \bar{v}} \\ \dfrac{\partial \bar{g}}{\partial u} & \dfrac{\partial \bar{g}}{\partial v} & \dfrac{\partial \bar{g}}{\partial \bar{u}} & \dfrac{\partial \bar{g}}{\partial \bar{v}} \end{bmatrix} \right|_{\substack{u=\bar{u}=\Phi_u(x), \\ v=\bar{v}=\Phi_v(x)}} .$$

The operator $\hat{\mathcal{H}}$ is not self-adjoint, since the matrix $\hat{\mathcal{W}}(x)$ is generally not symmetric. Many properties of the linear problem (3.3) are well known, such as the location of the continuous spectrum, the null-spectrum associated with symmetries of solitons, and the discrete non-null spectrum for internal modes and for unstable modes (Kivshar *et al*. 1998; Li & Promislow 1998; Pelinovsky & Yang 2000). In application to embedded solitons, these properties can be formulated as lemmas 3.1–3.4 below. We assume here that the embedded soliton emerges from a codimension-one bifurcation (2.12), i.e. $\alpha\omega_{\mathrm{ES}} \in \omega_2(k)$, but $\omega_{\mathrm{ES}} \notin \omega_1(k)$.

Lemma 3.1. *The continuous spectrum of (3.3) is*

$$\mathrm{ess}_\lambda(\hat{\mathcal{H}}) = \mathrm{ess}_\lambda(\hat{\mathcal{M}}) = \{\lambda : \mathrm{Re}(\lambda) = 0;\\ |\,\mathrm{Im}(\lambda)| > \omega_{\mathrm{ES}} \cup \mathrm{Im}(\lambda) < |\alpha\omega_{\mathrm{ES}} - \Delta| \cup \mathrm{Im}(\lambda) > -|\alpha\omega_{\mathrm{ES}} - \Delta|\}. \tag{3.4}$$

Proof. When $|x| \to \infty$, $|\hat{\mathcal{W}}(x)| \to O(\mathrm{e}^{-\sqrt{\omega_{\mathrm{ES}}}|x|})$ (see (2.3), (2.4), (2.18) and (2.19) for $r(\omega_{\mathrm{ES}}, \delta) = 0$). Given this estimate, the continuous spectrum of (3.3) coincides with that of $\hat{\mathcal{M}}\boldsymbol{\psi} = \lambda\mathcal{J}\boldsymbol{\psi}$, i.e. with (3.1). ∎

Due to the nature of embedded solitons, the gap in the continuous spectrum at $\mathrm{Re}(\lambda) = 0$, which is typical for a linearized problem associated with the nonlinear Schrödinger (NLS) equation, closes up and disappears. As a result, the null-spectrum eigenvalue $\lambda = 0$ becomes embedded into two branches of the continuous spectrum (3.1). The eigenfunctions for $\lambda = 0$ are described below.

Lemma 3.2. *The problem $\hat{\mathcal{H}}\boldsymbol{\psi} = 0$ has six bounded solutions,*

$$\boldsymbol{\psi} = \{\boldsymbol{\psi}_{\mathrm{dis}(+)}(x), \boldsymbol{\psi}_{\mathrm{sym}(+)}(x), \boldsymbol{\psi}_{\mathrm{as}(+)}(x)\} \otimes \{\boldsymbol{\psi}_{\mathrm{dis}(-)}(x), \boldsymbol{\psi}_{\mathrm{sym}(-)}(x), \boldsymbol{\psi}_{\mathrm{as}(-)}(x)\}. \tag{3.5}$$

Here, $\boldsymbol{\psi}_{\mathrm{dis}(\pm)}(x)$ are localized eigenfunctions in the null space of $\hat{\mathcal{H}}$. They are associated with translational and rotational (2.7) symmetries of the system (2.1), (2.2),

$$\boldsymbol{\psi}_{\mathrm{dis}(+)}(x) = \begin{pmatrix} \dfrac{\partial\Phi_u}{\partial x} \\ \dfrac{\partial\Phi_v}{\partial x} \\ \dfrac{\partial\Phi_u}{\partial x} \\ \dfrac{\partial\Phi_v}{\partial x} \end{pmatrix}, \qquad \boldsymbol{\psi}_{\mathrm{dis}(-)}(x) = \begin{pmatrix} \Phi_u \\ \alpha\Phi_v \\ -\Phi_u \\ -\alpha\Phi_v \end{pmatrix}. \tag{3.6}$$

The other four eigenfunctions $\boldsymbol{\psi}_{\mathrm{sym}(\pm)}(x)$ and $\boldsymbol{\psi}_{\mathrm{as}(\pm)}(x)$ are non-local and belong to the continuous spectrum of $\hat{\mathcal{H}}$ at $\lambda = 0$. They have the form

$$\boldsymbol{\psi}_{\mathrm{sym}(\pm)}(x) = \begin{pmatrix} u_{\mathrm{sym}(\pm)} \\ v_{\mathrm{sym}(\pm)} \\ \pm u_{\mathrm{sym}(\pm)} \\ \pm v_{\mathrm{sym}(\pm)} \end{pmatrix}, \qquad \boldsymbol{\psi}_{\mathrm{as}(\pm)}(x) = \begin{pmatrix} u_{\mathrm{as}(\pm)} \\ v_{\mathrm{as}(\pm)} \\ \pm u_{\mathrm{as}(\pm)} \\ \pm v_{\mathrm{as}(\pm)} \end{pmatrix}, \tag{3.7}$$

where $u_{\mathrm{sym}(\pm)}(x)$, $v_{\mathrm{sym}(\pm)}(x)$ and $u_{\mathrm{as}(\pm)}(x)$, $u_{\mathrm{as}(\pm)}(x)$ are symmetric and antisymmetric functions, respectively. As $|x| \to \infty$, these non-local eigenfunctions can be normalized by the asymptotic behaviours

$$\boldsymbol{\psi}_{\mathrm{sym}(\pm)}(x) \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ \pm 1 \end{pmatrix} \sin(k_{\mathrm{r}}|x| + \delta_\pm) \tag{3.8}$$

and

$$\boldsymbol{\psi}_{\mathrm{as}(\pm)}(x) \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ \pm 1 \end{pmatrix} \sin(k_{\mathrm{r}}|x| + \hat{\delta}_{\pm}\,\mathrm{sgn}(x)), \tag{3.9}$$

where $k_{\mathrm{r}} = k(\omega_{\mathrm{ES}})$ *and* $\delta_{\pm}$, $\hat{\delta}_{\pm}$ *are unique numbers specified by solutions of (3.3) at* $\lambda = 0$.

Proof. The system (3.3) can be decomposed for $\delta u_{\lambda} = u_{\mathrm{r}} + \mathrm{i}u_i$ and $\delta v_{\lambda} = v_{\mathrm{r}} + \mathrm{i}v_i$,

$$\begin{pmatrix} \mathcal{L}_+ & \mathcal{O} \\ \mathcal{O} & \mathcal{L}_- \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_+ \\ \boldsymbol{u}_- \end{pmatrix} = \lambda \begin{pmatrix} \mathcal{O} & -\mathcal{I} \\ \mathcal{I} & \mathcal{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_+ \\ \boldsymbol{u}_- \end{pmatrix}, \tag{3.10}$$

where $\boldsymbol{u}_+ = (u_{\mathrm{r}}, v_{\mathrm{r}})^{\mathrm{T}}$, $\boldsymbol{u}_- = (u_i, v_i)^{\mathrm{T}}$, $\mathcal{I}$ and $\mathcal{O}$ are unit and zero matrices in $\mathbb{R}^2$ and

$$\mathcal{L}_{\pm} = \left.\begin{bmatrix} -\partial_x^2 + \omega_{\mathrm{ES}} - \left(\dfrac{\partial f}{\partial u} \pm \dfrac{\partial f}{\partial \bar{u}}\right) & -\left(\dfrac{\partial f}{\partial v} \pm \dfrac{\partial f}{\partial \bar{v}}\right) \\ -\left(\dfrac{\partial g}{\partial u} \pm \dfrac{\partial g}{\partial \bar{u}}\right) & -D\partial_x^2 + \alpha\omega_{\mathrm{ES}} - \Delta - \left(\dfrac{\partial g}{\partial v} \pm \dfrac{\partial g}{\partial \bar{v}}\right) \end{bmatrix}\right|_{\substack{u=\bar{u}=\Phi_u(x),\\ v=\bar{v}=\Phi_v(x)}}. \tag{3.11}$$

In deriving this system, we have used the fact that $\hat{\mathcal{W}}(x)$ is a real-valued matrix. When $\lambda = 0$, each equation $\mathcal{L}_{\pm}\boldsymbol{u}_{\pm} = \mathbf{0}$ has four solutions: a localized solution, symmetric and antisymmetric bounded solutions and an unbounded solution. The localized solutions generate two localized eigenfunctions (3.6), as follows from (3.11) under the constraints (2.9). The bounded solutions can be normalized and reduced to (3.7)–(3.9) by taking the inverse transformation from $\boldsymbol{u}_+$ and $\boldsymbol{u}_-$ back to δu_{λ} and δv_{λ}. ■

The operator $\hat{\mathcal{H}}$ in the problem (3.3) is not self-adjoint in general. The solutions in the null space of the adjoint operator $\hat{\mathcal{H}}^+$ is needed for inner product and decomposition properties. It is described below.

Lemma 3.3. *The problem* $\hat{\mathcal{H}}^+\boldsymbol{\phi} = 0$ *has six bounded solutions,*

$$\boldsymbol{\phi} = \{\boldsymbol{\phi}_{\mathrm{dis}(+)}(x), \boldsymbol{\phi}_{\mathrm{sym}(+)}(x), \boldsymbol{\phi}_{\mathrm{as}(+)}(x)\} \otimes \{\boldsymbol{\psi}_{\mathrm{dis}(-)}(x), \boldsymbol{\psi}_{\mathrm{sym}(-)}(x), \boldsymbol{\psi}_{\mathrm{as}(-)}(x)\}, \tag{3.12}$$

where $\boldsymbol{\phi}_{\mathrm{dis}(+)}(x)$ *is an antisymmetric localized eigenfunction,* $\boldsymbol{\phi}_{\mathrm{sym}(+)}(x)$, $\boldsymbol{\phi}_{\mathrm{as}(+)}(x)$ *are bounded symmetric and antisymmetric functions, and eigenfunctions* $\boldsymbol{\psi}_{\mathrm{dis}(-)}(x)$, $\boldsymbol{\psi}_{\mathrm{sym}(-)}(x)$ *and* $\boldsymbol{\psi}_{\mathrm{as}(-)}(x)$ *are the same as in (3.6)–(3.9).*

Proof. It follows from (2.9) and (2.10) that

$$\left.\left(\frac{\partial f}{\partial v} - \frac{\partial f}{\partial \bar{v}}\right)\right|_{\substack{u=\bar{u}=\Phi_u(x),\\ v=\bar{v}=\Phi_v(x)}} = \left.\left(\frac{\partial g}{\partial u} - \frac{\partial g}{\partial \bar{u}}\right)\right|_{\substack{u=\bar{u}=\Phi_u(x),\\ v=\bar{v}=\Phi_v(x)}}. \tag{3.13}$$

As a result, the operator $\mathcal{L}_-$ in (3.11) is self-adjoint, i.e. $\mathcal{L}_-^+ = \mathcal{L}_-$. Therefore, the spectrum of $\mathcal{L}_-^+$ is the same as that of $\mathcal{L}_-$. ■

Similar to the linearized NLS problem, the operator $\hat{\mathcal{H}}$ of (3.3) also supports a generalized eigenfunction, which is associated with the parameter $\omega = \omega_{\mathrm{ES}}$ of the embedded soliton. The generalized eigenfunction is described below.

Lemma 3.4. *If $E_1(\delta) \neq 0$, where E_1 is defined by (2.30), then the operator $\hat{\mathcal{H}}$ has a simple symmetric generalized eigenfunction at $\lambda = 0$,*

$$\boldsymbol{\psi}_{\mathrm{gen}(-)}(x;\delta) = \begin{pmatrix} \dfrac{\partial U(x;\omega,\delta)}{\partial \omega} \\ \dfrac{\partial V(x;\omega,\delta)}{\partial \omega} \\ \dfrac{\partial U(x;\omega,\delta)}{\partial \omega} \\ \dfrac{\partial V(x;\omega,\delta)}{\partial \omega} \end{pmatrix}_{\omega=\omega_{\mathrm{ES}}}, \tag{3.14}$$

which solves the non-homogeneous linear problem

$$\hat{\mathcal{H}}\boldsymbol{\psi}_{\mathrm{gen}(-)} = \mathrm{i}\mathcal{J}\boldsymbol{\psi}_{\mathrm{dis}(-)}. \tag{3.15}$$

The generalized eigenfunction (3.14) is bounded for $n = 1$ but is decaying for $n \geqslant 2$, where n is the embedded soliton index (2.25).

Proof. By taking the derivative with respect to ω in (2.20) and (2.21) and setting it at $\omega = \omega_{\mathrm{ES}}$, it is proved that solution (3.14) satisfies (3.15). When $n = 1$, the solution has the boundary condition due to (2.18), (2.19), (2.22) and (2.24),

$$\boldsymbol{\psi}_{\mathrm{gen}(-)}(x;\delta) \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} R(\delta)\sin(k_{\mathrm{r}}|x| + \delta) \quad \text{as } |x| \to \infty. \tag{3.16}$$

When $n \geqslant 2$, the solution (3.14) has a zero boundary condition at infinity.

It remains to be shown that the operator $\hat{\mathcal{H}}$ has no double generalized eigenfunction $\boldsymbol{\psi}_{\mathrm{gen}_2(-)}(x)$ at $\lambda = 0$. The double generalized eigenfunction, if it exists, solves the non-homogeneous system

$$\hat{\mathcal{H}}\boldsymbol{\psi}_{\mathrm{gen}_2(-)} = \mathrm{i}\mathcal{J}\boldsymbol{\psi}_{\mathrm{gen}(-)}. \tag{3.17}$$

Since the linear operator $\hat{\mathcal{H}}$ in (3.3) is not self-adjoint in a Hilbert space $\mathcal{H}(\mathbb{R})$, we define a skew-symmetric inner product (2-form) for eigenfunctions $\boldsymbol{f}(x)$ of the operator $\hat{\mathcal{H}}^{+}$ and eigenfunctions $\boldsymbol{g}(x)$ of $\hat{\mathcal{H}}$ as

$$\langle \boldsymbol{f} \mid \boldsymbol{g} \rangle_{\mathcal{J}} = \int_{-\infty}^{\infty} \mathrm{d}x\, (f^{(1)}g^{(1)} + f^{(2)}g^{(2)} - f^{(3)}g^{(3)} - f^{(4)}g^{(4)}), \tag{3.18}$$

where superscripts denote components of a vector. It follows from Fredholm's alternative for the non-homogeneous equation (3.17) that a bounded solution $\boldsymbol{\psi}_{\mathrm{gen}_2(-)}(x)$ exists if

$$\langle \boldsymbol{\phi}_{\mathrm{dis}(+)} \mid \boldsymbol{\psi}_{\mathrm{gen}(-)} \rangle_{\mathcal{J}} = 0 \tag{3.19}$$

and

$$\langle \boldsymbol{\psi}_{\mathrm{dis}(-)} \mid \boldsymbol{\psi}_{\mathrm{gen}(-)} \rangle_{\mathcal{J}} = 0. \tag{3.20}$$

Since $\boldsymbol{\psi}_{\mathrm{gen}(-)}(x)$ is symmetric and $\boldsymbol{\phi}_{\mathrm{dis}(+)}(x)$ antisymmetric, the first condition is automatically satisfied. We can verify from (2.30), (3.6), (3.14) and (3.18) that $E_1(\delta) = \langle \boldsymbol{\psi}_{\mathrm{dis}(-)} \mid \boldsymbol{\psi}_{\mathrm{gen}(-)} \rangle_{\mathcal{J}}$. Thus the second condition (3.20) indicates that a bounded solution to (3.17) cannot exist if $E_1(\delta) \neq 0$. ■

Remark 3.5. The operator $\hat{\mathcal{H}}$ can have another generalized eigenstate $\boldsymbol{\psi}_{\mathrm{gen}(+)}(x)$ associated with the mode $\boldsymbol{\psi}_{\mathrm{dis}(+)}(x)$ if the coupled two-wave system (2.1), (2.2) is Galilean invariant. This state is given by the derivative of the embedded soliton (U, V) with respect to its free velocity parameter. We neglect this additional state, even if it exists, since it is not relevant to the analysis below.

The embedded soliton (2.11) and the linear wave spectrum (2.6) are in nonlinear resonance, which is represented by (2.12). This nonlinear resonance results in the linear resonance in the null-space of operator $\hat{\mathcal{H}}$. Indeed, localized modes $\boldsymbol{\psi}_{\mathrm{dis}(\pm)}(x)$ coexist with the essential spectrum modes $\boldsymbol{\psi}_{\mathrm{sym}(\pm)}(x)$ and $\boldsymbol{\psi}_{\mathrm{as}(\pm)}(x)$ at $\lambda = 0$ (see (3.5)). However, all eigenfunctions are separated within the linear problem due to linear decomposition properties. It is only the nonlinearity in the basic model (2.1), (2.2) that couples eigenfunctions of discrete and essential spectrum and induces the embedded-soliton transformation.

We conclude this section by proving a technical but important result for the non-homogeneous linear problem (3.15). The δ dependence in (3.14) suggests that there is a family of simple generalized eigenfunctions $\boldsymbol{\psi}_{\mathrm{gen}(-)}(x;\delta)$ parametrized by δ for any $n \geqslant 1$. The family exists for a single eigenfunction $\boldsymbol{\psi}_{\mathrm{dis}(-)}(x)$. This is a specific feature of the nonlinear resonance of embedded solitons. The feature is explained by the homogeneous solutions of the operator $\hat{\mathcal{H}}$ at $\lambda = 0$. In fact, these homogeneous solutions with the same resonant wavenumber k_{r} are useful for computations of the minimal tail amplitude for the family of the generalized eigenfunctions $\boldsymbol{\psi}_{\mathrm{gen}(-)}(x;\delta)$.

Lemma 3.6. *Suppose $n = 1$ for embedded solitons (2.11). Then the solution $\boldsymbol{\psi}_{\mathrm{gen}(-)}(x;\delta)$ of the non-homogeneous problem (3.15) has a minimal tail amplitude at infinity for $\delta = \delta_{\min}$ and $\delta = \delta_{\min} - \pi$, where*

$$\delta_{\min} = \delta_+ + \tfrac{1}{2}\pi, \tag{3.21}$$

i.e. the value $\delta_{\min}$ minimizes $R(\delta)$ in (3.16).

Proof. A general symmetric solution of the non-homogeneous linear problem (3.15) can be written as

$$\boldsymbol{\psi}_{\mathrm{gen}(-)}(x;\delta) = \boldsymbol{\psi}_{\mathrm{gen}(-)}(x;\delta_0) + c(\delta,\delta_0)\boldsymbol{\psi}_{\mathrm{sym}(+)}(x), \tag{3.22}$$

where $c(\delta,\delta_0)$ is a constant coefficient, δ_0 is any fixed value and $\boldsymbol{\psi}_{\mathrm{sym}(-)}(x)$ is not included into (3.22) due to the symmetry conditions (3.7) and (3.14). The boundary condition for the right-hand side of (3.22) as $|x| \to \infty$ can be obtained from (3.8) and (3.16). Matching this boundary condition with the boundary condition (3.16) for $\boldsymbol{\psi}_{\mathrm{gen}(-)}(x;\delta)$, we find that

$$R(\delta) = \frac{R(\delta_0)\sin(\delta_0 - \delta_+)}{\sin(\delta - \delta_+)}, \qquad c(\delta,\delta_0) = -\frac{R(\delta_0)\sin(\delta - \delta_0)}{\sin(\delta - \delta_+)}. \tag{3.23}$$

Since δ_0 is fixed, the first formula (3.23) reduces to the form

$$R(\delta) = \frac{r_0}{\sin(\delta - \delta_+)}, \tag{3.24}$$

where r_0 is constant, $r_0 = R(\delta_0)\sin(\delta_0 - \delta_+)$. Clearly, the function $|R(\delta)|$ diverges when $\delta = \delta_+ \bmod(\pi)$ and is minimal when $\delta = \delta_{\min}$ and $\delta = \delta_{\min} - \pi$. If $\delta_0 = \delta_{\min}$, then the constant r_0 is the minimal tail amplitude, $r_0 = R(\delta_{\min})$. Obviously,

$$R(\delta_{\min} - \pi) = -R(\delta_{\min}).$$

■

Corollary 3.7. *Suppose $n = 1$ for embedded solitons (2.11). When $|\omega - \omega_{\mathrm{ES}}| \ll 1$, the tail amplitude $|r(\delta;\omega)|$ of non-local waves (2.17) is minimal when $\delta = \delta_{\min}$, where $\delta_{\min}$ is given in (3.21). In addition,*

$$r(\omega, \delta) = \frac{r(\omega, \delta_{\min})}{\sin(\delta - \delta_+)} + O(\omega - \omega_{\mathrm{ES}})^2. \tag{3.25}$$

Proof. When $|\omega - \omega_{\mathrm{ES}}| \ll 1$, the tail amplitude $r(\omega, \delta)$ of non-local waves (2.17) is given by (2.24), i.e. for $n = 1$,

$$r(\omega, \delta) = (\omega - \omega_{\mathrm{ES}})R(\delta) + O(\omega - \omega_{\mathrm{ES}})^2. \tag{3.26}$$

Then (3.25) is directly obtained from (3.24) and (3.26). ■

It is easy to see from lemmas 3.2 and 3.3 that the vector $\boldsymbol{\phi}_{\mathrm{sym}(+)}(x)$ has the same structure as $\boldsymbol{\psi}_{\mathrm{sym}(+)}(x)$ and it satisfies the following boundary condition as $|x| \to \infty$,

$$\boldsymbol{\phi}_{\mathrm{sym}(+)}(x) \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \sin(k_{\mathrm{r}}|x| + \gamma_+), \tag{3.27}$$

where γ_+ is the phase constant. The next lemma relates the phase γ_+ of the adjoint solution $\boldsymbol{\phi}_{\mathrm{sym}(+)}(x)$ to the phase δ_+ of the solution $\boldsymbol{\psi}_{\mathrm{sym}(+)}(x)$ in (3.5) and (3.8). In addition, an analytical expression for $R(\delta_{\min})$ is also obtained.

Lemma 3.8. *Suppose $n = 1$ for embedded solitons (2.11). The phase γ_+ of the adjoint solution $\boldsymbol{\phi}_{\mathrm{sym}(+)}(x)$ and the phase δ_+ of the solution $\boldsymbol{\psi}_{\mathrm{sym}(+)}(x)$ are the same, i.e. $\gamma_+ = \delta_+$. In addition,*

$$R(\delta_{\min}) = -\frac{\langle \boldsymbol{\phi}_{\mathrm{sym}(+)} \mid \boldsymbol{\psi}_{\mathrm{dis}(-)} \rangle_{\mathcal{J}}}{4Dk_{\mathrm{r}}}. \tag{3.28}$$

Proof. The general symmetric solution $\boldsymbol{\psi}_{\mathrm{gen}(-)}(x;\delta)$ of the non-homogeneous problem (3.15) has the asymptotic behaviour (3.16), where δ is an arbitrary constant. Taking the inner product between (3.15) and the null-space equation $\hat{\mathcal{H}}^+ \boldsymbol{\phi}_{\mathrm{sym}(+)} = 0$, we find that

$$\langle \boldsymbol{\phi}_{\mathrm{sym}(+)} \mid \boldsymbol{\psi}_{\mathrm{dis}(-)} \rangle_{\mathcal{J}} = 2\sum_{k=1}^{2} c_k \left(\phi^{(k)}_{\mathrm{sym}(+)} \frac{\partial \psi^{(k)}_{\mathrm{gen}(-)}}{\partial x} - \psi^{(k)}_{\mathrm{gen}(-)} \frac{\partial \phi^{(k)}_{\mathrm{sym}(+)}}{\partial x} \right) \Bigg|_{x\to-\infty}^{x\to\infty}, \tag{3.29}$$

where $c_1 = 1$, $c_2 = D$ and the superscript (k) represents the kth component of a vector function. Using boundary conditions (3.16) and (3.27), we can easily find that

$$\langle \phi_{\mathrm{sym}(+)} \mid \psi_{\mathrm{dis}(-)} \rangle_{\mathcal{J}} = -4Dk_{\mathrm{r}} R(\delta) \sin(\delta - \gamma_+). \tag{3.30}$$

The inner product on the left-hand side of (3.30) does not depend on δ. Therefore, the tail amplitude function $R(\delta)$ is

$$R(\delta) = -\frac{\langle \phi_{\mathrm{sym}(+)} \mid \psi_{\mathrm{dis}(-)} \rangle_{\mathcal{J}}}{4Dk_{\mathrm{r}} \sin(\delta - \gamma_+)}. \tag{3.31}$$

Comparing this equation with (3.24) for $r_0 = R(\delta_{\min})$, we immediately find that $\gamma_+ = \delta_+$ and the formula (3.28) for $R(\delta_{\min})$ is proved. ∎

4. Nonlinear transformation of embedded solitons

Here we prove the main results of the paper: propositions 2.10 and 2.11. We start with a general Fourier analysis of linear PDEs. The linear non-homogeneous wave equation of Schrödinger type can be written as a system of two equations,

$$\left.\begin{aligned} -W_t + \delta U_{xx} + \kappa U &= f(x), \\ U_t + \delta W_{xx} + \kappa W &= 0, \end{aligned}\right\} \tag{4.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given localized source in $L^2(\mathbb{R})$ and $\delta, \kappa \in \mathbb{R}$. First, we prove a technical result useful for further analysis.

Lemma 4.1. *Consider the initial-value problem for the system (4.1) with zero initial values, $U(x,0) = 0$ and $W(x,0) = 0$. Under the resonance condition $\mathrm{sgn}(\kappa\delta) = 1$, the time-dependent solution of (4.1) has the long-term asymptotic limit in the asymptotic region (2.28),*

$$\begin{pmatrix} U \\ W \end{pmatrix}(x,t) \to \frac{1}{2\delta\kappa_{\mathrm{r}}} |F(\kappa_{\mathrm{r}})| \begin{pmatrix} \sin(\kappa_{\mathrm{r}}|x| \pm \arg(F(\kappa_{\mathrm{r}}))) \\ -\,\mathrm{sgn}(\delta)\cos(\kappa_{\mathrm{r}}|x| \pm \arg(F(\kappa_{\mathrm{r}}))) \end{pmatrix} H(2|\delta|\kappa_{\mathrm{r}} t - |x|), \tag{4.2}$$

where the plus-minus signs stand for two separate regions, $x \gg 1$ and $x \ll -1$, respectively. In addition, $H(z)$ is the step function defined below (2.29), $F(k)$ is the Fourier transform of $f(x)$,

$$F(k) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x = \bar{F}(-k), \tag{4.3}$$

and κ_{r} is the resonant wavenumber defined by

$$\kappa_{\mathrm{r}} = \sqrt{\frac{\kappa}{\delta}} \quad \textit{(greater than zero).}$$

Proof. The time-dependent solution of the problem (4.1) with zero initial data can be found by using the Fourier transform in the form

$$\begin{pmatrix} U \\ W \end{pmatrix}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(k)\mathrm{e}^{\mathrm{i}kx}}{\kappa - \delta k^2} \begin{pmatrix} 1 - \cos(\kappa - \delta k^2)t \\ -\sin(\kappa - \delta k^2)t \end{pmatrix} \mathrm{d}k. \tag{4.4}$$

The integral in (4.4) is non-singular for finite $t \geqslant 0$ and, therefore, the poles at $k = \pm\kappa_{\mathrm{r}}$ do not result in a singular contribution for boundary values of U and W as $|x| \to \infty$. Indeed, a localized source $f(x)$ generates at $t > 0$ the outgoing waves that reach the boundary region $|x| \to \infty$ only in the limit when $t \to +\infty$. In the later limit, the integrand in (4.4) becomes singular. We use the stationary phase method (Ablowitz & Fokas 1997) to analyse the singular limit of (4.4). We decompose the denominator in (4.4) as

$$\frac{1}{\kappa - \delta k^2} = \frac{1}{2\delta\kappa_{\mathrm{r}}}\left(\frac{1}{k+\kappa_{\mathrm{r}}} - \frac{1}{k-\kappa_{\mathrm{r}}}\right). \tag{4.5}$$

Substituting $k = \mp\kappa_{\mathrm{r}} + z/t$ for the two terms separately, we rewrite (4.4) as

$$\begin{aligned}\begin{pmatrix} U \\ W \end{pmatrix}(x,t) = \frac{1}{4\pi\delta\kappa_{\mathrm{r}}}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{z}\bigg[&F(-\kappa_{\mathrm{r}} + z/t)\mathrm{e}^{-\mathrm{i}\kappa_{\mathrm{r}}x + \mathrm{i}zx/t}\begin{pmatrix} 1 - \cos(2\kappa_{\mathrm{r}}\delta z - \delta z^2/t) \\ -\sin(2\kappa_{\mathrm{r}}\delta z - \delta z^2/t)\end{pmatrix} \\ &- F(\kappa_{\mathrm{r}} + z/t)\mathrm{e}^{\mathrm{i}\kappa_{\mathrm{r}}x + \mathrm{i}zx/t}\begin{pmatrix} 1 - \cos(2\kappa_{\mathrm{r}}\delta z + \delta z^2/t) \\ \sin(2\kappa_{\mathrm{r}}\delta z + \delta z^2/t)\end{pmatrix}\bigg].\end{aligned} \tag{4.6}$$

Keeping $|x|/t = C_x$ as a constant of order $O(1)$, and using the integral

$$\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}zp}\,\mathrm{d}z}{z} = \pi\mathrm{i}\,\mathrm{sgn}(p),$$

we reduce (4.6) in the asymptotic limit $t \to \infty$ to the form

$$\begin{aligned}\begin{pmatrix} U \\ W \end{pmatrix}(x,t) \to{}& \frac{1}{4\delta\kappa_{\mathrm{r}}}|F(\kappa_{\mathrm{r}})| \\ &\times\begin{pmatrix}\sin(\kappa_{\mathrm{r}}x + \arg(F(\kappa_{\mathrm{r}})))(2\,\mathrm{sgn}(x/t) - \mathrm{sgn}(x/t + 2\delta\kappa_{\mathrm{r}}) - \mathrm{sgn}(x/t - 2\delta\kappa_{\mathrm{r}})) \\ -\cos(\kappa_{\mathrm{r}}x + \arg(F(\kappa_{\mathrm{r}})))(\mathrm{sgn}(x/t + 2\delta\kappa_{\mathrm{r}}) - \mathrm{sgn}(x/t - 2\delta\kappa_{\mathrm{r}}))\end{pmatrix}.\end{aligned} \tag{4.7}$$

This formula reduces to (4.2) in the two separate regions, $x \gg 1$ and $x \ll -1$. ∎

Remark 4.2. The system (4.1) is equivalent to a linear Schrödinger equation for complex function $\phi(x,t) = U(x,t) + \mathrm{i}W(x,t)$,

$$\mathrm{i}\phi_t + \delta\phi_{xx} + \kappa\phi = f(x). \tag{4.8}$$

The time-dependent solution of the zero initial-value problem associated with (4.8) has a long-term asymptotic limit in the asymptotic region (2.28),

$$\phi(x,t) \to -\frac{\mathrm{i}|F(\kappa_{\mathrm{r}})|}{2|\delta|\kappa_{\mathrm{r}}}\mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(\delta)(\kappa_{\mathrm{r}}|x| \pm \arg(F(\kappa_{\mathrm{r}})))}H(2|\delta|\kappa_{\mathrm{r}}t - |x|), \tag{4.9}$$

where the plus-minus signs stand for two separate regions, $x \gg 1$ and $x \ll -1$, respectively. This formula follows from (4.2). In fact, it reproduces the Sommerfeld radiation condition (see, for example, Ablowitz & Fokas 1997, ch. 4.6). According to that condition, a localized source $f(x)$ can generate at $t > 0$ only the outgoing waves that have the form $\mathrm{e}^{\mathrm{i}k(|x| - ct) + \mathrm{i}\kappa t}$ at infinity, where $c > 0$. The incoming waves that have the form $\mathrm{e}^{-\mathrm{i}k(|x| + ct) + \mathrm{i}\kappa t}$ should be eliminated if no sources are located at infinity. The incoming wave is eliminated in the limits $x \to \pm\infty$ if $k = \pm\,\mathrm{sgn}(\delta)\kappa_{\mathrm{r}}$, as in (4.9).

Now we extend lemma 4.1 for a linear non-homogeneous problem with asymptotically constant coefficients. The linear problem occurs later in analysis as the first-order perturbation reduction of the coupled NLS equations (2.1), (2.2),

$$\hat{\mathcal{H}}\boldsymbol{\phi} = [\hat{\mathcal{M}} - \hat{\mathcal{W}}(x)]\boldsymbol{\phi} = \mathcal{J}\frac{\partial \boldsymbol{\phi}}{\partial t} + \gamma \boldsymbol{f}(x), \tag{4.10}$$

where $\boldsymbol{f} = [f_u, f_v, f_u, f_v]^{\mathrm{T}}$, $f_u, f_v : \mathbb{R} \to \mathbb{R}$ is a given localized source in $L^2(\mathbb{R})$ and $\gamma \in \mathbb{R}$. We use assumption 2.6 and properties of the linearized problem described in lemmas 3.1–3.4.

Lemma 4.3. *Consider the initial-value problem for (4.10) with zero initial value, $\boldsymbol{\phi}(x,0) = 0$. The time-dependent solution of (4.10) has the long-term asymptotic limit in the asymptotic region (2.28),*

$$\boldsymbol{\phi}(x,t) \to \begin{pmatrix} 0 \\ \mathrm{i}G\mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(D)(k_{\mathrm{r}}|x|+g)} \\ 0 \\ -\mathrm{i}G\mathrm{e}^{-\mathrm{i}\,\mathrm{sgn}(D)(k_{\mathrm{r}}|x|+g)} \end{pmatrix} H(C_{\mathrm{g}}t - |x|), \tag{4.11}$$

where C_{g}, k_{r} are given by (2.26), $G \in [0, \infty)$ and $g \in [-\pi, \pi]$.

Proof. We assume here that the four branches of the continuous spectrum (3.1) and the two localized eigenfunctions (3.6) form a complete basis for the linearized problem (3.3). Solution to the non-homogeneous time-dependent problem (4.10) can be decomposed as

$$\boldsymbol{\phi}(x,t) = \gamma \boldsymbol{F}(x) + \sum_{n=1}^{4}\int_{-\infty}^{\infty} \mathrm{d}k\, c_n(k)\mathrm{e}^{\mathrm{i}\Omega_n(k)t}\boldsymbol{\psi}_n(x,k) + \gamma_+\boldsymbol{\psi}_{\mathrm{dis}(+)}(x) + \gamma_-\boldsymbol{\psi}_{\mathrm{dis}(-)}(x), \tag{4.12}$$

where $\boldsymbol{F} = [F_u, F_v, F_u, F_v]^{\mathrm{T}}$, $F_u, F_v : \mathbb{R} \to \mathbb{R}$ is a solution of the non-homogeneous time-independent problem, while $c_n(k)$ and $\gamma_\pm$ are some constants. We order the branches of the continuous spectrum by the boundary conditions

$$\boldsymbol{\psi}_n(x,k) \to \boldsymbol{e}_n\mathrm{e}^{\mathrm{i}kx} \quad \text{as } x \to +\infty.$$

Then $\Omega_3(k) = -\Omega_1(k) = \omega_{\mathrm{ES}} + k^2$ and $\Omega_2(k) = -\Omega_4(k) = \Delta - \alpha\omega_{\mathrm{ES}} - Dk^2$. The non-homogeneous solution $\boldsymbol{F}(x)$ can be decomposed through the complete set of eigenfunctions as

$$\boldsymbol{F}(x) = \sum_{n=1}^{4}\int_{-\infty}^{\infty} \mathrm{d}k\, b_n(k)\boldsymbol{\psi}_n(x,k) + \beta_+\boldsymbol{\psi}_{\mathrm{dis}(+)}(x) + \beta_-\boldsymbol{\psi}_{\mathrm{dis}(-)}(x), \tag{4.13}$$

where $b_3(k) = \bar{b}_1(-k)$ and $b_4(k) = \bar{b}_2(-k)$. The zero initial-value problem (4.10) then has the explicit solution

$$\boldsymbol{\phi}(x,t) = \gamma\sum_{n=1}^{4}\int_{-\infty}^{\infty} \mathrm{d}k\, b_n(k)(1 - \mathrm{e}^{\mathrm{i}\Omega_n(k)t})\boldsymbol{\psi}_n(x,k). \tag{4.14}$$

Suppose that $\boldsymbol{F}(x)$ has the oscillatory boundary conditions as $|x| \to \infty$,

$$\boldsymbol{F}(x) \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} R(\delta) \sin(k_{\mathrm{r}}|x| + \delta). \tag{4.15}$$

Then the spectral coefficients $b_2(k)$ and $b_4(k)$ are singular at $k = \pm k_{\mathrm{r}}$, where k_{r} is given by (2.26),

$$b_2(k) = \frac{B(k)}{k_{\mathrm{r}}^2 - k^2}, \qquad b_4(k) = \frac{\bar{B}(-k)}{k_{\mathrm{r}}^2 - k^2}, \tag{4.16}$$

where $|B(\pm k_{\mathrm{r}})| < \infty$. Since F_u, F_v are real functions, we have the symmetry relation $B(k) = \bar{B}(-k)$. In the asymptotic region (2.28) for $x \gg 1$, the second and fourth integrals in (4.14) reproduce the same solution (4.4) for $\phi_2(x,t) = U + \mathrm{i}W$, where $\delta = D$, $\kappa = \Delta - \alpha\omega_{\mathrm{ES}}$ and the Fourier transform $F(k)$ is replaced by $F(k) = 2\pi\delta B(k)$. Therefore, we can use the same method as in lemma 4.1 to analyse the singular contribution of the integrals (4.14) in the asymptotic region (2.28) such that $x \gg 1$. As a result, we derive the radiation boundary condition (4.11) with

$$G = -\gamma\frac{\pi|B(k_{\mathrm{r}})|}{k_{\mathrm{r}}}, \qquad g = \arg(DB(k_{\mathrm{r}})). \tag{4.17}$$

Since the solution $\boldsymbol{F}(x)$ and $\boldsymbol{\phi}(x,t)$ is symmetric in x, the analysis in the other region $x \ll -1$ is not required. ■

By using lemma 4.3, we now prove propositions 2.10 and 2.11 that describe solutions of the initial-value problem for the nonlinear system (2.1), (2.2).

Proof of proposition 2.10. Consider a small deformation of the embedded soliton (2.11), $\theta_0 \to \theta(t)$, such that $|\dot{\theta}| < C_\theta\epsilon$, where $\epsilon \ll 1$ and C_θ is constant for $t \in [0,T]$. Since the perturbed embedded soliton is not stationary, a small perturbation vector appears in the time-dependent initial-value problem associated with the NLS system (2.1), (2.2). The order of the perturbation vector depends on the index n of the embedded soliton (2.25).

Case 1 ($\boldsymbol{n = 1}$). At the leading order of $|\dot{\theta}| < C_\theta\epsilon$, a solution to (2.1), (2.2) is written as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \left[\begin{pmatrix} \varPhi_u \\ \varPhi_v \end{pmatrix}(x) + \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}(x,t)\right] \exp\left[\mathrm{i}\begin{pmatrix} 1 \\ \alpha \end{pmatrix}(\omega_{\mathrm{ES}}t + \theta(t))\right], \tag{4.18}$$

where the perturbation vector $\boldsymbol{\phi}(x,t) = (\delta u, \delta v, \delta\bar{u}, \delta\bar{v})^{\mathrm{T}}$ satisfies the linearized non-homogeneous problem (4.10), with $\gamma = \dot{\theta}$ and $\boldsymbol{f}(x) = \mathrm{i}\mathcal{J}\boldsymbol{\psi}_{\mathrm{dis}(-)}(x)$. The non-homogeneous solution $\boldsymbol{F}(x)$ in (4.12) is simply $\boldsymbol{F}(x) = \boldsymbol{\psi}_{\mathrm{gen}(-)}(x)$, where $\boldsymbol{\psi}_{\mathrm{gen}(-)}(x)$ is given by (3.14). The non-homogeneous solution is bounded but not decaying at infinity (cf. (3.16) and (4.15)).

If the change of $\theta(t)$ is adiabatic, then the parameter $\gamma = \dot{\theta}$ is treated as a constant. In the limit $t \to \infty$, the solution of the initial-value problem for (4.10) becomes time independent for $|x| < \infty$. The time-independent solution for (4.10) is a combination of the non-homogeneous solution $\boldsymbol{F}(x)$ and bounded homogeneous solutions (3.5) of the null-spectrum of the operator $\hat{\mathcal{H}}$. Assuming that the embedded soliton is symmetric, we construct a general time-independent symmetric solution for $\lim_{t\to\infty}\boldsymbol{\phi}(x,t)$ in the form

$$\lim_{t\to\infty}\boldsymbol{\phi}(x,t) = \dot{\theta}\boldsymbol{\psi}_{\mathrm{gen}(-)}(x;\delta_0) + c_1\boldsymbol{\psi}_{\mathrm{sym}(+)}(x) + \mathrm{i}c_2\boldsymbol{\psi}_{\mathrm{sym}(-)}(x), \tag{4.19}$$

where δ_0 is a fixed value and c_1 and c_2 are some real constants. At infinity, i.e. as $|x|\to\infty$, the time-independent solution (4.19) must match the time-independent asymptotic limit (4.11). In other words, the constants δ_0, c_1, and c_2 are to be chosen from the Sommerfeld radiation condition for the resonant waves generated by a localized source. Using (3.8), (3.16), (4.11) and (4.19), we match the limits $|x|\to\infty$ and arrive at the following system of relations:

$$\left.\begin{aligned} \dot{\theta}R(\delta_0)\sin(k_{\mathrm{r}}|x|+\delta_0) + c_1\sin(k_{\mathrm{r}}|x|+\delta_+) &= -G\,\mathrm{sgn}(D)\sin(k_{\mathrm{r}}|x|+g),\\ c_2\sin(k_{\mathrm{r}}|x|+\delta_-) &= G\cos(k_{\mathrm{r}}|x|+g). \end{aligned}\right\} \tag{4.20}$$

Solving the second equation in (4.20), we match the coefficients

$$c_2 = -G, \qquad g = \delta_{\mathrm{rad}}, \tag{4.21}$$

where the radiation phase δ_{rad} is

$$\delta_{\mathrm{rad}} = \delta_- + \tfrac{1}{2}\pi. \tag{4.22}$$

Solving the first equation in (4.20), we match the other coefficients,

$$c_1 = -\frac{\dot{\theta}R(\delta_0)\cos(\delta_0-\delta_-)}{\cos(\delta_+-\delta_-)} \tag{4.23}$$

and

$$G = -\,\mathrm{sgn}(D)\frac{\dot{\theta}R(\delta_0)\sin(\delta_0-\delta_+)}{\cos(\delta_+-\delta_-)} = -\,\mathrm{sgn}(D)\frac{\dot{\theta}R(\delta_{\min})}{\cos(\delta_{\min}-\delta_{\mathrm{rad}})}, \tag{4.24}$$

where we have used (3.24) with $r_0 = R(\delta_{\min})$. It is clear from (4.11), (4.21) and (4.24) that the small non-localized perturbation vector

$$(u_{\mathrm{RD}}, v_{\mathrm{RD}})(x,t) = \lim_{t\to\infty}(\delta u, \delta v)(x,t)$$

has, in the asymptotic region (2.28), the boundary conditions (2.29) at $n=1$.

Case 2 ($n \geqslant 2$). The non-homogeneous solution $\boldsymbol{F}(x) = \boldsymbol{\psi}_{\mathrm{gen}(-)}(x)$ is decaying exponentially in the limit $|x|\to\infty$ and so is the perturbation vector $\boldsymbol{\phi}(x,t)$ considered above. Then the constants c_1 and c_2 in the time-independent localized solution (4.19) must be zero in the case $n\geqslant 2$, $c_1=c_2=0$. By summing such localized (up to the $(n-1)$th order) solutions to the embedded soliton, we include the non-localized perturbation to the leading order of $|\dot{\theta}|^n \leqslant C_\theta^n\epsilon^n$. At the leading order, a solution to (2.1), (2.2) is written as

$$\begin{pmatrix} u\\ v\end{pmatrix} = \left[\begin{pmatrix}\Phi_u\\ \Phi_v\end{pmatrix}(x) + \sum_{k=0}^{n-1}\frac{1}{k!}\dot{\theta}^k \left.\begin{pmatrix}\dfrac{\partial^k U}{\partial\omega^k}\\[2mm] \dfrac{\partial^k V}{\partial\omega^k}\end{pmatrix}\right|_{\omega=\omega_{\mathrm{ES}}}(x) + \begin{pmatrix}\delta u\\ \delta v\end{pmatrix}(x,t)\right] \times\exp\left[\mathrm{i}\begin{pmatrix}1\\ \alpha\end{pmatrix}(\omega_{\mathrm{ES}}t+\theta(t))\right]. \tag{4.25}$$

The perturbation vector $\boldsymbol{\phi}(x,t) = (\delta u, \delta v, \delta\bar{u}, \delta\bar{v})^{\mathrm{T}}$ now satisfies the linearized non-homogeneous problem (4.10), with $\gamma = \dot{\theta}^n$ and $\boldsymbol{f} = \boldsymbol{N}_n(x)$, where the non-homogeneous vector $\boldsymbol{N}_n(x)$ is due to nonlinearity of the system (2.1), (2.2) and is computed through the lower-order terms of (4.25). Existence of non-local wave solutions of (2.20), (2.21) ensures that the non-homogeneous solution $\boldsymbol{F}(x)$ in (4.12) is simply

$$\boldsymbol{F}(x) = \frac{1}{n!}\dot{\theta}^n \boldsymbol{\psi}_{(n)}(x;\delta),$$

where

$$\boldsymbol{\psi}_{(n)}(x;\delta) = \left.\begin{pmatrix} \dfrac{\partial^n U(x;\omega,\delta)}{\partial\omega^n} \\ \dfrac{\partial^n V(x;\omega,\delta)}{\partial\omega^n} \\ \dfrac{\partial^n U(x;\omega,\delta)}{\partial\omega^n} \\ \dfrac{\partial^n V(x;\omega,\delta)}{\partial\omega^n} \end{pmatrix}\right|_{\omega=\omega_{\mathrm{ES}}}. \tag{4.26}$$

In the limit $t \to \infty$, the solution of the initial-value problem for (4.10) matches the following time-independent symmetric solution,

$$\lim_{t\to\infty} \boldsymbol{\phi}(x,t) = \frac{1}{n!}\dot{\theta}^n \boldsymbol{\psi}_{(n)}(x;\delta_0) + c_1\boldsymbol{\psi}_{\mathrm{sym}(+)}(x) + \mathrm{i}c_2\boldsymbol{\psi}_{\mathrm{sym}(-)}(x), \tag{4.27}$$

where again δ_0, c_1 and c_2 are coefficients to be found. Then the same analysis of the radiation problem in the asymptotic region (2.28) leads to the same solution (4.20)–(4.24), where $\dot{\theta}$ is replaced by $\dot{\theta}^n$. One can show that the value $\delta = \delta_{\min}$ still minimizes the tail amplitude $|R(\delta)|$ by extending the method of lemmas 3.6 and 3.8 for $n \geqslant 2$. This proof results in the same relation (3.24), with

$$r_0 = R(\delta_{\min}) = \frac{\mathrm{i}}{4Dk_{\mathrm{r}}}\langle \boldsymbol{\phi}_{\mathrm{sym}(+)} \mid \boldsymbol{N}_n\rangle_{\mathcal{J}}. \tag{4.28}$$

Note that for $n = 1$, $\boldsymbol{N}_1(x) = \mathrm{i}\boldsymbol{\psi}_{\mathrm{dis}(-)}(x)$ and (4.28) matches (3.28). Finally, the non-localized part of (4.27) leads to the boundary condition (2.29) for $n \geqslant 2$, where the radiation field is

$$(u_{\mathrm{RD}}, v_{\mathrm{RD}})(x,t) = \lim_{t\to\infty}(\delta u, \delta v)(x,t).$$

∎

Remark 4.4. The two alternative expressions for G and g given by (4.17) and (4.21)–(4.24) coincide. In order to prove it, we compute the singular contribution in the non-homogeneous solution $\boldsymbol{F}(x)$ given by (4.13) and (4.16) in the region $x \gg 1$,

$$\int_{-\infty}^{\infty} \frac{B(k)\mathrm{e}^{\mathrm{i}kx}}{k_{\mathrm{r}}^2 - k^2}\,\mathrm{d}k \to \frac{\pi\mathrm{i}}{2k_{\mathrm{r}}}(B(k_{\mathrm{r}})\mathrm{e}^{\mathrm{i}k_{\mathrm{r}}x} - \bar{B}(k_{\mathrm{r}})\mathrm{e}^{-\mathrm{i}k_{\mathrm{r}}x}). \tag{4.29}$$

Matching (4.29) with the boundary condition (4.15), we arrive at the relations

$$R(\delta) = -\operatorname{sgn}(D)\frac{\pi|B(k_{\mathrm{r}})|}{k_{\mathrm{r}}}, \qquad \delta = \arg(DB(k_{\mathrm{r}})).$$

When $\delta = \delta_{\rm rad}$, the two expressions for G and g in (4.17) and in (4.21)–(4.24) become equivalent.

Proof of proposition 2.11. Suppose that the initial data (2.31) are close to the embedded soliton according to (2.32), where ϵ is an explicit small parameter. The small initial perturbation leads to two main events: (i) deformation of the embedded soliton (2.11) according to the change of $\theta_0 \to \theta(t)$ and (ii) generation of the radiation fronts. These two events are, in fact, self-consistent, i.e. the deformation of the embedded soliton depends on the radiation fronts and vice versa. We capture this dynamics by developing formal perturbation analysis in terms of the small parameter ϵ.

Case 1 ($n = 1$). Let $T = \epsilon t$ be slow time and define a perturbation series

$$\begin{pmatrix} u \\ v \end{pmatrix} = \left[\begin{pmatrix} \Phi_u \\ \Phi_v \end{pmatrix}(x) + \epsilon \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}(x,t) + \epsilon^2 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}(x,t) + O(\epsilon^3) \right] \exp\left[\mathrm{i} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} (\omega_{\rm ES} t + \theta(T)) \right]. \tag{4.30}$$

Then the perturbation vectors $\boldsymbol{\psi}_j(x,t) = (u_j, v_j, \bar{u}_j, \bar{v}_j)^{\rm T}$ satisfy the non-homogeneous systems

$$\hat{\mathcal{H}} \boldsymbol{\psi}_1 = \mathcal{J} \frac{\partial \boldsymbol{\psi}_1}{\partial t} + \mathrm{i} \dot{\theta} \mathcal{J} \boldsymbol{\psi}_{{\rm dis}(-)}(x), \tag{4.31}$$

$$\hat{\mathcal{H}} \boldsymbol{\psi}_2 = \mathcal{J} \frac{\partial \boldsymbol{\psi}_2}{\partial t} + \dot{\theta}^2 \mathcal{J} \boldsymbol{N}_2(x) + \ddot{\theta} \mathcal{J} \hat{\boldsymbol{\psi}}_1(x), \tag{4.32}$$

where $\boldsymbol{\psi}_1(x,t) = \dot{\theta} \hat{\boldsymbol{\psi}}_1(x)$ and the term $N_2(x)$ is due to the nonlinearity of the system (2.1), (2.2). The symmetric initial data for the problem (4.31) are defined as $u_1(x,0) = (u_0(x) - \Phi_u(x))/\epsilon$ and $v_1(x,0) = (v_0(x) - \Phi_v(x))/\epsilon$. A particular solution to the problem (4.31) with zero initial data is constructed in proposition 2.10 under the adiabaticity assumption, i.e. $\dot{\theta}$ is constant in time t (which is justified by the asymptotic method, since $\theta = \theta(T = \epsilon t)$). As a result, the time-independent solutions for $\boldsymbol{\psi}_1(x,t)$ in the region $|x| < \infty$ is given by (4.19), with c_1 and c_2 defined in (4.21)–(4.23). In the limit $|x| \to \infty$, this solution is non-vanishing due to the radiation condition (4.11). The general solution of (4.31) is a superposition of the homogeneous solution, which is induced by non-zero initial data for $u_1(x,0)$ and $v_1(x,0)$, and the same non-homogeneous solution constructed in proposition 2.10. The homogeneous solutions vanish at infinity $|x| \to \infty$ and do not thus affect the radiation condition (2.29) in the asymptotic region (2.28).

The second-order linear non-homogeneous problem (4.32) may have a secularly growing solution $\boldsymbol{\psi}_2(x,t)$ in time t if the right-hand side of (4.32) is not orthogonal to the localized homogeneous solution $\boldsymbol{\psi}_{{\rm dis}(-)}(x)$ of the operator $\hat{\mathcal{H}}^+$ (Fredholm's alternative). In order to ensure the non-secular behaviour of $\boldsymbol{\psi}_2(x,t)$ in t, we define the time evolution of $\theta = \theta(T)$ from the Fredholm alternative for (4.32) as

$$\ddot{\theta} \langle \boldsymbol{\psi}_{{\rm dis}(-)} \mid \hat{\boldsymbol{\psi}}_1 \rangle_{\mathcal{J}} + \dot{\theta}^2 \langle \boldsymbol{\psi}_{{\rm dis}(-)} \mid \boldsymbol{N}_2 \rangle_{\mathcal{J}} = 0. \tag{4.33}$$

Although the computation of the second inner product in (4.33) could be quite lengthy, there is a shortcut to a final formula (2.34). The symmetry (2.7) and the conserved quantity (2.8) imply that the system (2.1), (2.2) has a balance equation,

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} (|u|^2 + \alpha |v|^2)\, \mathrm{d}x = \mathrm{i} [\bar{u} u_x - u \bar{u}_x + \alpha D (\bar{v} v_x - v \bar{v}_x)] \big|_{x \to -\infty}^{x \to \infty}. \tag{4.34}$$

In order to simplify the normal form equation, we choose δ_0 in (4.19) as $\delta_0 = \delta_{\mathrm{rad}}$. Then $c_1 = 0$ from (4.22), (4.23). By setting the expansion (4.30) into the balance equation (4.34) and equating the leading-order terms of $O(\epsilon^2)$, we match the inner products as $\langle \boldsymbol{\psi}_{\mathrm{dis}(-)} \mid \hat{\boldsymbol{\psi}}_1 \rangle_{\mathcal{J}} = E_1(\delta_{\mathrm{rad}}) = e_1$, where $E_1(\delta)$ is defined by (2.30) and $\langle \boldsymbol{\psi}_{\mathrm{dis}(-)} \mid \boldsymbol{N}_2 \rangle_{\mathcal{J}} = \Gamma$, where

$$\Gamma = -\mathrm{i}[\bar{u}_{\mathrm{RD}} u_{\mathrm{RD}x} - u_{\mathrm{RD}} \bar{u}_{\mathrm{RD}x} + \alpha D(\bar{v}_{\mathrm{RD}} v_{\mathrm{RD}x} - v_{\mathrm{RD}} \bar{v}_{\mathrm{RD}x})]\Big|_{x\to-\infty}^{x\to\infty} = 4\alpha k_{\mathrm{r}} |D| |R(\delta_{\mathrm{rad}})|^2.$$

Here we have used the boundary conditions (2.29) for $(u_{\mathrm{RD}}, v_{\mathrm{RD}})(x,t)$.

Case 2 ($\boldsymbol{n \geqslant 2}$). We now set $T = \epsilon^{2n-1} t$ as a slow time for $\theta = \theta(T)$ and define the perturbation series as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \left[\begin{pmatrix} \Phi_u \\ \Phi_v \end{pmatrix} (x) + \sum_{n=1}^{2n} \epsilon \begin{pmatrix} u_n \\ v_n \end{pmatrix} (x,t) + O(\epsilon^{2n+1}) \right] \exp\left[\mathrm{i} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \left(\omega_{\mathrm{ES}} t + \frac{1}{\epsilon^{2n-2}} \theta(T) \right) \right], \tag{4.35}$$

such that $|\dot{\theta}|$ is still of order $O(\epsilon)$. The perturbation series satisfies a set of linear non-homogeneous problems up to the order of $O(\epsilon^{2n})$, where the non-homogeneous equation has the form

$$\hat{\mathcal{H}} \boldsymbol{\psi}_{2n} = \mathcal{J} \frac{\partial \boldsymbol{\psi}_{2n}}{\partial t} + \dot{\theta}^{2n} \mathcal{J} \boldsymbol{N}_{2n}(x) + \ddot{\theta} \mathcal{J} \hat{\boldsymbol{\psi}}_1(x), \tag{4.36}$$

where $\hat{\boldsymbol{\psi}}_1(x) = \boldsymbol{\psi}_{\mathrm{gen}(-)}(x, \delta_{\mathrm{rad}})$ in the limit $t \to \infty$. Suppression of the secular growth of $\boldsymbol{\psi}_{2n}(x,t)$ in time t leads to the orthogonality condition

$$\ddot{\theta} \langle \boldsymbol{\psi}_{\mathrm{dis}(-)} \mid \boldsymbol{\psi}_{\mathrm{gen}(-)} \rangle_{\mathcal{J}} + \dot{\theta}^{2n} \langle \boldsymbol{\psi}_{\mathrm{dis}(-)} \mid \boldsymbol{N}_{2n} \rangle_{\mathcal{J}} = 0,$$

where

$$\langle \boldsymbol{\psi}_{\mathrm{dis}(-)} \mid \boldsymbol{\psi}_{\mathrm{gen}(-)} \rangle_{\mathcal{J}} = E_1(\delta_{\mathrm{rad}}) = e_1.$$

The leading-order radiation $(u_{\mathrm{RD}}, v_{\mathrm{RD}})(x,t)$ is given by (2.29). Combining the boundary condition (2.29) and the balance equation (4.34), we equate the leading-order terms of $O(\epsilon^{2n})$, which reproduce the same asymptotic equation (2.34) for any $n \geqslant 2$. ■

5. Numerical results

Here we verify numerically the main formula (2.34) for the semi-stability of embedded solitons. We consider the coupled two-wave equations (2.1) and (2.2) with the nonlinear functions (2.14). We choose parameters corresponding to the type-I embedded soliton (2.15)

$$D = -1, \qquad \Delta = 1, \qquad \gamma_1 = \gamma_2 = -0.05. \tag{5.1}$$

At $\omega_{\mathrm{ES}} = 0.8114$, the system has an embedded soliton with index $n = 1$. The profile $(\Phi_u(x), \Phi_v(x))$ of the embedded soliton is shown in figure 1*a*, while the symmetric eigenfunction $\boldsymbol{\psi}_{\mathrm{sym}(+)}(x)$ is shown in figure 1*b*. The phase δ_+ was found from this plot and formula (3.8) as $\delta_+ = 1.3270$. We have also computed the other symmetric eigenfunction $\boldsymbol{\psi}_{\mathrm{sym}(-)}(x)$, from which we found $\delta_- = 1.4007$. The phases $\delta_{\min}$ and δ_{rad}, defining the minimum amplitude tail and the radiation phase, were computed from (3.21) and (4.22) as $\delta_{\min} = 2.8978$ and $\delta_{\mathrm{rad}} = 2.9715$. As we can see, the radiation phase δ_{rad} is close to the minimum-amplitude phase $\delta_{\min}$, but is not equal

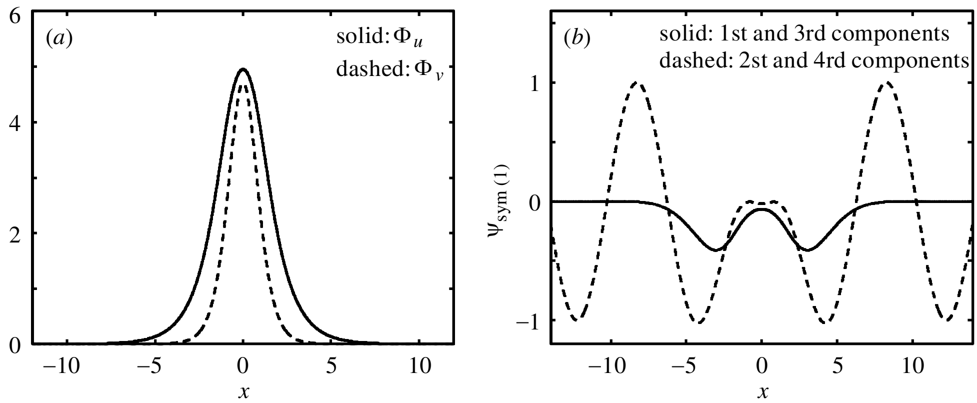

Figure 1. (a) The profile of an embedded soliton $(\Phi_u(x), \Phi_v(x))$ for the system (2.1), (2.2), with (2.14) and (5.1). The soliton exists at $\omega_{\mathrm{ES}} = 0.8114$. ($b$) The symmetric eigenfunction $\boldsymbol{\psi}_{\mathrm{sym}(1)}(x)$ of the linearized problem (3.3) defined by the boundary condition (3.8).

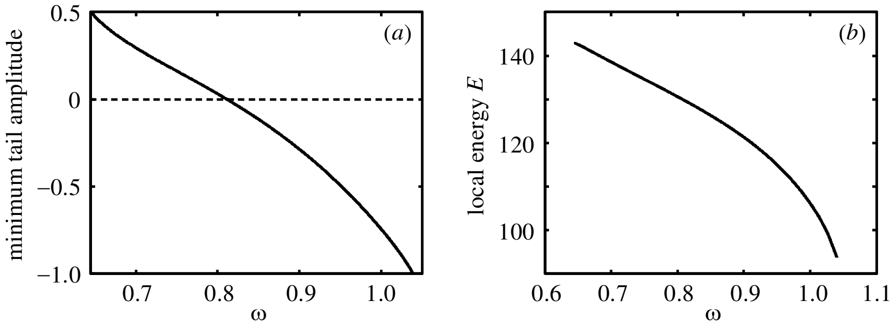

Figure 2. (a) The minimum tail amplitude $r(\omega, \delta_{\min})$ of non-local waves (2.17) for various values of ω in the system (2.1), (2.2), with (2.14) and (5.1). The phase $\delta = \delta_{\min}$ in (2.17) is chosen such that the amplitude $r(\omega, \delta_{\min})$ is minimal. (b) Local energy curve $E(\omega, \delta_{\mathrm{rad}})$, as defined by (2.37).

to it. As a result, the radiation tail amplitude $R(\delta_{\mathrm{rad}})$ is not minimal (see (2.27)). This is true even though the system (2.1), (2.2) is Hamiltonian for our choice of parameters (5.1).

In order to determine $R(\delta_{\min})$ and then $R(\delta_{\mathrm{rad}})$ via (2.27), we have numerically calculated the tail amplitude $r(\omega, \delta_{\min})$ at various values of ω and plotted them in figure 2a. This figure clearly indicates the existence of an embedded soliton at parameter $\omega_{\mathrm{ES}} = 0.8114$. The slope of this curve at ω_{ES} is $R(\delta_{\min}) = -2.806$. This value was also obtained independently from (3.28) for $\boldsymbol{\phi}_{\mathrm{sym}(+)}(x,t) = \boldsymbol{\psi}_{\mathrm{sym}(+)}(x,t)$ in the Hamiltonian case (5.1). With these values, we find the coefficient Γ from (2.35) to be $\Gamma = 49.98$.

Local energy $E(\omega, \delta_{\mathrm{rad}})$, as defined in (2.37), has also been found numerically for various values of ω. It is plotted in figure 2b. As we can see from the figure, the graph of $E(\omega, \delta_{\mathrm{rad}})$ is a decreasing function of ω. The slope of this curve at $\omega = \omega_{\mathrm{ES}} = 0.8114$ is $e_1 = E_1(\delta_{\mathrm{rad}}) = -85.3$. The same value of e_1 was also found from (2.30) by direct computation of the integral.

The asymptotic equation (2.34) has the exact solution (2.36), which can be rewrit-

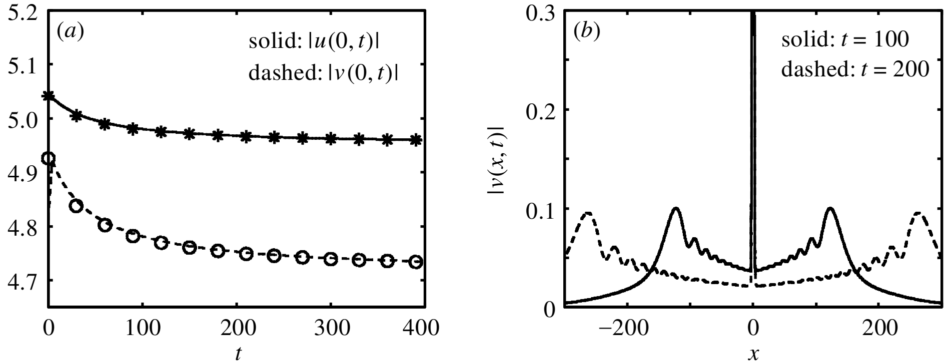

Figure 3. Numerical simulation of evolution of the embedded soliton under energy-enhancing perturbation (5.3) with $a_1 = a_2 = 0.1$. (a) Evolutions of $|u(0,t)|$ and $|v(0,t)|$. (b) Profile of $|v(x,t)|$ at two values of t. Theoretical predictions for $|u(0,t)|$ and $|v(0,t)|$ found from (5.2) are shown in (a) by stars and circles, respectively.

ten with the found constants as

$$\Omega(t) = \frac{\Omega_0}{[1 - 0.586\Omega_0 t]}, \tag{5.2}$$

where $\Omega_0 = \Omega(0)$. We compare this solution with direct numerical simulations of the two-wave system (2.1), (2.2). As initial conditions for the direct numerical simulations, we choose

$$u(x,0) = \Phi_u(x) + a_1 \operatorname{sech} 2x, \qquad v(x,0) = \Phi_v(x) + a_2 \operatorname{sech} 2x, \tag{5.3}$$

where (Φ_u, Φ_v) is the embedded soliton and a_1 and a_2 are small perturbation coefficients. First we take $a_1 = a_2 = 0.1$, where the perturbed solution has more energy than the embedded soliton. According to our analysis (see corollary 2.12), the perturbed solution will asymptotically approach the embedded soliton and shed off extra energy in the form of radiation. This is clearly confirmed by our numerical solution shown in figure 3. To compare these results quantitatively with our analytical solution (5.2), we first need to choose the initial value Ω_0. Naturally, one wants to choose Ω_0 such that the central part of the non-local solitary wave (U, V) is close to the initial condition (5.3) (note that this non-local wave has frequency $\omega_{\mathrm{ES}} + \Omega_0$ and phase δ_{rad}). Under this criterion, we found that $\Omega_0 \approx -0.0414$ when $a_1 = a_2 = 0.1$ in (5.3). With this Ω_0 value, the solution $\Omega(t)$ is then determined for all time. We then determine the non-local wave with frequency $\omega_{\mathrm{ES}} + \Omega(t)$. Its central part will be our analytical approximation for the true solution. We measured the amplitudes $|u(0,t)|$ and $|v(0,t)|$ of these analytically predicted non-local waves at $x = 0$, and plotted them in figure 3a together with the same quantities from direct numerical simulations. The agreement is excellent. We note that the true analytical solution (2.33) contains not only the non-local wave, but also a transverse perturbation term. But this transverse perturbation is one order smaller in the expansion for $|u(0,t)|$ and $|v(0,t)|$ than the non-local wave at the wave centre. Thus it is neglected when we compare the analytical solution (2.33) with the numerical solution at $x = 0$.

Next we choose energy-reducing perturbations (5.3) with $a_1 = a_2 = -0.1$. The direct numerical simulation results are shown in figure 4. Consistent with our the-

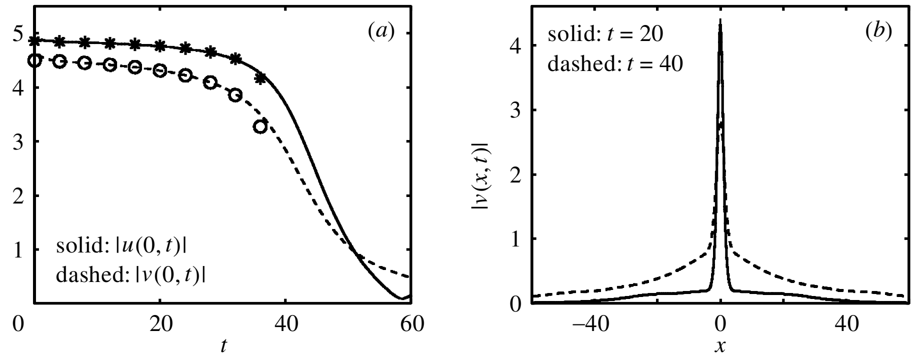

Figure 4. Numerical simulation of evolution of the embedded soliton under energy-reducing perturbation (5.3) with $a_1 = a_2 = -0.1$. Shown are (a) evolutions of $|u(0,t)|$ and $|v(0,t)|$ and (b) profile of $|v(x,t)|$ at two values of t. The theoretical prediction found from (5.2) are shown in (a) by stars and circles.

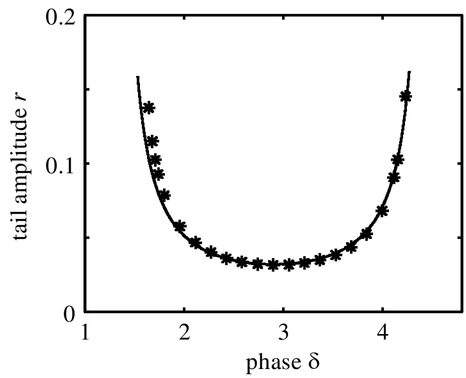

Figure 5. The tail amplitude $r(\omega,\delta)$ of non-local waves (2.17) as a function of δ for $\omega = 0.8$. The solid line is the analytical formula (5.4), and stars are numerically obtained values.

oretical predictions, the embedded soliton is destroyed in the case of the energy-reducing perturbation. To get a quantitative comparison, we first note that for this perturbation, $\Omega_0 \approx 0.0386$ in (5.2). We then determined the non-local waves with frequency $\omega_{\mathrm{ES}} + \Omega(t)$, where $\Omega(t)$ is given by (5.2). The amplitudes of these waves at $x = 0$ are shown in figure 4a at various times. Again, these analytically predicted centre amplitudes of the solutions agree well with the direct numerical simulation results. As the solution decays far away from the embedded soliton, the analytical solution (5.2) will become invalid. Thus the analytical predictions and numerical solutions will deviate apart at large times (see figure 4a).

The analytical formula (3.25) for the tail amplitude of non-local waves (U,V) has also been confirmed numerically. According to (3.25), the tail amplitude $r(\omega,\delta)$ of non-local waves with ω close to ω_{ES} is (to leading order)

$$r(\omega,\delta) \approx \frac{(\omega-\omega_{\mathrm{ES}})R(\delta_{\min})}{\sin(\delta-\delta_+)}, \tag{5.4}$$

where $R(\delta_{\min}) = -2.806$, $\delta_+ = 1.327$, $\omega_{\mathrm{ES}} = 0.8114$ (see above). In order to check this formula, we take $\omega = 0.8$ and plot $r(\omega,\delta)$ as a function of δ in figure 5 (solid line).

On the other hand, we determined numerically the dependence $r(\omega, \delta)$ for several δ values and also plotted them in figure 5 (stars) for comparison. The agreement between numerics and formula (5.4) is quite satisfactory.

6. Conclusion

We have derived the normal form for nonlinear resonance of embedded solitons in a coupled two-wave system (2.1), (2.2). The normal form is given by the asymptotic equation (2.34), and it captures the dynamics of the embedded soliton and its resonant wave radiation. This result is valid for general non-Hamiltonian wave systems under certain assumptions on the equation (assumption 2.1) and on the existence and linearized stability of embedded solitons (assumptions 2.4–2.6). Applications of the theory to the second-harmonic-generating system (2.14) shows good agreement between numerics and the theory. Some limitations of the theory are worth mentioning here.

Embedded solitons can also arise in nonlinear wave systems where assumption 2.1 is not satisfied, i.e. the conserved quantity (2.8) does not exist. In this case, the normal form should be generally different from (2.34). For instance, it happens for an extended KdV equation (Yang 2001). Two branches of embedded solitons exist there, one of which is linearly (exponentially) unstable (i.e. it does not satisfy our assumption 2.6) and the other one is linearly and nonlinearly stable (i.e. it does not compile with our proposition 2.11) (Yang 2001).

Another kind of embedded soliton occurs in non-local systems when assumption 2.5 is violated. Such embedded solitons exist for a certain interval of possible values of ω in the linear spectrum (2.6) rather than for a single value $\omega = \omega_{\mathrm{ES}}$. It is unclear how to introduce the tail amplitude $r(\omega, \delta)$ for the non-local case and what is the normal form for embedded solitons. One example was given for an integral NLS equation describing dispersion-managed optical solitons (Pelinovsky 2000). Two families of embedded solitons were identified numerically in the normal regime of the dispersion-managed fibre, one is linearly (exponentially) unstable and the other one is linearly and nonlinearly stable. Thus our main results on the normal form for embedded solitons are clearly violated in non-local wave systems.

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