Purely Nonlinear Instability of Standing Waves with Minimal Energy

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Abstract

We consider Hamiltonian systems with $U(1)$ symmetry. We prove that in the generic situation the standing wave that has the minimal energy among all other standing waves is unstable, in spite of the absence of linear instability. Essentially, the instability is caused by higher algebraic degeneracy of the zero eigenvalue in the spectrum of the linearized system. We apply our theory to the nonlinear Schrödinger equation. © 2003 Wiley Periodicals, Inc.

1 Introduction

Let us consider the nonlinear Schrödinger equation with $U(1)$ symmetry:

$$iu_t = -\Delta u + g(|u|^2)u\,, \tag{1.1}$$

where $u = u(t, x)$ is complex valued, $t \geq 0$, $x \in \mathbb{R}^n$. As shown in [20] and later generalized in [1], for a large class of nonlinearities, equation (1.1) admits quasi-stationary solitary waves of finite energy, dubbed as standing waves: $u(t, x) = e^{-i\omega t}\phi_\omega(x)$, where ω is from a certain interval $\Omega \subset \mathbb{R}$, and the wave profile $\phi_\omega(x)$ decays as $|x| \to \infty$. We are interested in the stability properties of standing waves with respect to perturbations of the initial data.

More generally, following [8], we consider an abstract $U(1)$-invariant Hamiltonian system of the form

$$\frac{d\boldsymbol{u}}{dt} = JE'(\boldsymbol{u}(t))\,, \tag{1.2}$$

where E is the energy functional and J is a skew-symmetric linear operator. System (1.2) is assumed to be invariant under a representation $T(\,\cdot\,)$ of the group $U(1)$ on X. We assume that, for $\omega \in (\omega_1, \omega_2) \subset \mathbb{R}$, system (1.2) admits standing-wave solutions $\boldsymbol{u}(t) = T(\omega t)\boldsymbol{\phi}_\omega$ and that the map $\omega \mapsto \boldsymbol{\phi}_\omega$ is sufficiently smooth. We also assume that the linearized Hamiltonian $H_\omega = E''(\boldsymbol{\phi}_\omega) - \omega Q''(\boldsymbol{\phi}_\omega)$ has at most one negative eigenvalue; here $Q(\boldsymbol{u})$ is the charge functional that is the Noether integral of motion conserved due to the $U(1)$ invariance of the system.

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Standing-wave solutions with different ω can have different stability properties. As proven in [8] (see also the review in [22]), the standing wave $\boldsymbol{u}(t) = T(\omega t)\boldsymbol{\phi}_\omega$ is orbitally stable if $d''(\omega) > 0$, where

$$d(\omega) = E(\boldsymbol{\phi}_\omega) - \omega Q(\boldsymbol{\phi}_\omega)\,.$$

Using the relation

$$\frac{d}{d\omega} E(\boldsymbol{\phi}_\omega) = \omega \frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega)\,, \tag{1.3}$$

which follows from the Lagrangian equation $E'(\boldsymbol{\phi}_\omega) = \omega Q'(\boldsymbol{\phi}_\omega)$, we can rewrite the condition for the standing wave $\boldsymbol{u}(t) = T(\omega t)\boldsymbol{\phi}_\omega$ to be orbitally stable as Vakhitov-Kolokolov's stability criterion [24]:

$$\frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega) < 0\,. \tag{1.4}$$

If $d''(\omega) < 0$, the standing-wave solution $\boldsymbol{u}(t) = T(\omega t)\boldsymbol{\phi}_\omega$ is linearly unstable. The resulting (nonlinear) instability was proven in [8] (see also [7, appendix]).

Remark 1.1. The instability in the systems with linear instability is usually taken for granted.[1] Under general assumptions, the (nonlinear) instability of zero solution to the general evolution equations of the form $\frac{du}{dt} = Lu + F(u)$ in the case when the zero solution is linearly unstable (the spectrum of L meets the right half-plane $\{\operatorname{Re}\lambda > 0\}$) was recently proven by Shatah and Strauss [17].

Let us consider quasi-stationary solutions of minimal energy, which are of significance in the physical sciences. If the energy $E(\boldsymbol{\phi}_\omega)$ as a function of ω has a minimum (local or global) at a certain point $\omega_* \in (\omega_1, \omega_2)$, $\omega_* \neq 0$, then one immediately concludes from (1.3) that

$$d''(\omega_*) = -\frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega)\bigg|_{\omega_*} = -\frac{1}{\omega_*}\frac{d}{d\omega} E(\boldsymbol{\phi}_\omega)\bigg|_{\omega_*}$$

vanishes, and the stability criterion (1.4) breaks down. At the same time, there is no (exponential) instability in the linearized system at $d''(\omega_*) = 0$, and one needs a more careful stability analysis of (1.2).

The instability of the standing waves with minimal energy, which are at the border of the stable and unstable standing waves, has not been proven even in simpler systems. The continuity-based argument in [8, corollary 5.1] has a fatal flaw: The subset of stable stationary states does not have to be open inside the set of all stationary states (see Figure 1.1). Essentially, in this paper we will prove that the subset of stable stationary states is open in the case of *Hamiltonian systems* with $U(1)$ symmetry.

As shown in [16] for the nonlinear Schrödinger equation and Klein-Gordon equation, in dimensions two and higher, the corresponding standing wave does not

[1] Moreover, the term *instability* is sometimes used to refer to the instability of the linearized system. The type of instability we are interested in is often referred to as "nonlinear instability."

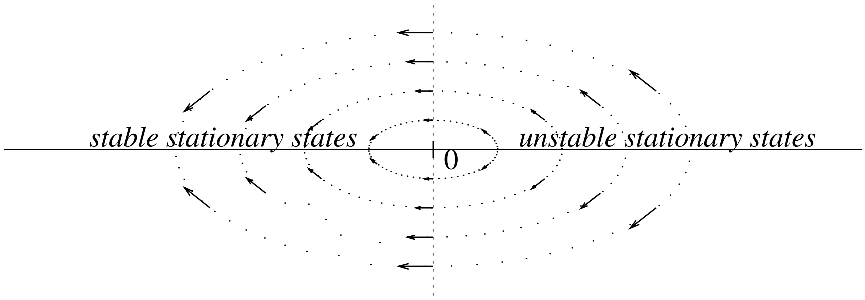

FIGURE 1.1. Planar flow. The set of stationary states is $\mathbb{R}$; the subset of stable states is $S = (-\infty, 0]$.

minimize the energy under the charge constraint. In [5], this result was generalized to a general Hamiltonian system with $U(1)$ symmetry.[2] As a result, one expects the minimal-energy standing waves to be unstable. The qualitative analysis of this instability by formal asymptotic methods was developed in [11] (see also [3]) in the context of the nonlinear Schrödinger equation in one dimension.

The main result of our paper is the proof of the (nonlinear) instability of minimal-energy standing-wave solutions to abstract Hamiltonian systems (1.2) with $U(1)$ symmetry. We also consider how our theory applies to the nonlinear Schrödinger equation (1.1) and justify some of the asymptotical results in heuristic papers [11, 13]. As mentioned above, in the case of minimal-energy standing waves there is no linear instability, in the sense that the spectrum of the linearized system does not meet the right half-plane. However, the spectrum contains a zero eigenvalue of higher algebraic multiplicity, and this eventually leads to instability. It was reckoned in [12] that the degenerate zero eigenvalue leads to the growth of particular perturbations and that this growth is only algebraic in time; we will justify this result.

Let us also mention that the asymptotic stability of the standing waves was proven in certain cases in [4, 6, 18, 19, 25]. Currently, there is an increasing interest in this subject.

The paper is structured as follows: In Section 2, we formulate our assumptions and the main results. Section 3 discusses the details of the spectral decomposition near the degenerate zero eigenvalue. We derive the equations that govern the evolution of the perturbation in Section 4. In Section 5, we derive the bounds on the growth of the part of the perturbation that corresponds to the continuous spectrum of JH_ω. The asymptotics on the part of the solution that corresponds to the discrete

[2] It was also shown in [5] that if the standing waves exist for $\omega \in (\omega_1, \omega_2]$, the minimum of the map $\omega \mapsto E(\boldsymbol{\phi}_\omega)$ is achieved at ω_2, and the stability criterion (1.4) is satisfied uniformly near ω_2, so that $\lim_{\omega\to\omega_2-0} \frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega) < 0$, then the standing wave $T(\omega_2 t)\boldsymbol{\phi}_{\omega_2}$ is orbitally stable as long as one considers the initial data $\boldsymbol{u}(0)$ of the same charge as the standing wave itself: $Q(\boldsymbol{u}(0)) = Q(\boldsymbol{\phi}_{\omega_2})$.

spectrum of JH_ω are obtained in Section 6. In that section we also estimate the errors and complete the proof the main theorem. Details relevant to the nonlinear Schrödinger equation (1.1) are considered in Section 7. The operator JH_ω is unbounded on its domain; in Appendix A, we define abstract Sobolev spaces where JH_ω acts continuously. In Appendix B we give estimates on the remainder in the Taylor series in the function spaces; these estimates are needed for applications of the theory to the nonlinear Schrödinger equation.

2 Main Results

We first give the background from papers [8, 9] and from [21]. Let X be a real Hilbert space with the inner product $(\cdot\,,\cdot)$. Let I be the natural isomorphism $I : X \to X^*$,

$$\langle I\boldsymbol{u}, \boldsymbol{v}\rangle = (\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in X\,,$$

where $\langle\cdot\,,\cdot\rangle$ is the canonical pairing $X^* \times X \to \mathbb{R}$. Let $E : X \to \mathbb{R}$ be the energy functional, which we assume to be C^2 on all of X. We write its derivative as $\langle E'(\boldsymbol{u}), \boldsymbol{v}\rangle$, where $E' : X \to X^*$.

We consider an abstract Hamiltonian system of the form

$$\frac{d\boldsymbol{u}}{dt} = JE'(\boldsymbol{u}(t))\,, \quad \boldsymbol{u} : \mathbb{R} \to X\,, \tag{2.1}$$

where J is a closed linear operator from X^* to X with dense domain $D(J) \subset X^*$, which is skew symmetric:

$$\langle \boldsymbol{\xi}, J\boldsymbol{\eta}\rangle = -\langle \boldsymbol{\eta}, J\boldsymbol{\xi}\rangle \quad \forall \boldsymbol{\xi}, \boldsymbol{\eta} \in D(J) \subset X^*\,.$$

We assume that J is one-to-one and onto.

Let $T : U(1) \to \mathrm{Aut}(X)$ be a unitary representation of $U(1)$ on X so that $\|T(s)\boldsymbol{u}\|_X = \|\boldsymbol{u}\|_X$ for each $s \in U(1)$ and each $\boldsymbol{u} \in X$. We identify $U(1)$ with $\mathbb{R} \bmod 2\pi$ under addition. We assume that the functional E is invariant under the action of $U(1)$:

$$E(T(s)\boldsymbol{u}) = E(\boldsymbol{u}) \quad \forall s \in U(1),\ \forall \boldsymbol{u} \in X\,. \tag{2.2}$$

We assume that J "commutes" with T in the sense that

$$T(s)J = JT^*(-s)\,.$$

We assume that there is a bounded linear operator $B : X \to X^*$, self-adjoint with respect to the pairing $\langle\cdot\,,\cdot\rangle$ and such that JB is an extension of $T'(0)$. (After having described the setup from [8], we will assume that $T'(0)$ is defined on the entire space X so that $JB = T'(0)$.) We define the "charge" functional $Q : X \to \mathbb{R}$ by

$$Q(\boldsymbol{u}) = \frac{1}{2}\langle B\boldsymbol{u}, \boldsymbol{u}\rangle\,. \tag{2.3}$$

The charge functional $Q(\boldsymbol{u})$ is invariant under the action of T:

$$\frac{d}{ds}Q(T(s)\boldsymbol{u}) = \langle Q'(T(s)\boldsymbol{u}), T'(0)T(s)\boldsymbol{u}\rangle = \langle BT(s)\boldsymbol{u}, JBT(s)\boldsymbol{u}\rangle = 0$$

so that $Q(T(s)\boldsymbol{u}) = Q(\boldsymbol{u})$ for all $s \in U(1)$, $\boldsymbol{u} \in D(T'(0))$. For $\boldsymbol{u} \notin D(T'(0))$, we prove that $Q(T(s)\boldsymbol{u}) = Q(\boldsymbol{u})$ approximating $\boldsymbol{u}$ by the elements from $D(T'(0))$.

Note that the values of the energy and charge functionals are (formally) conserved under the flow of (2.1):

$$\frac{d}{dt}E(\boldsymbol{u}) = \langle E'(\boldsymbol{u}), \dot{\boldsymbol{u}}\rangle = \langle E'(\boldsymbol{u}), JE'(\boldsymbol{u})\rangle = 0\,,$$

$$\begin{aligned}\frac{d}{dt}Q(\boldsymbol{u}) &= \langle Q'(\boldsymbol{u}), \dot{\boldsymbol{u}}\rangle = \langle Q'(\boldsymbol{u}), JE'(\boldsymbol{u})\rangle = -\langle E'(\boldsymbol{u}), JQ'(\boldsymbol{u})\rangle \\ &= -\langle E'(\boldsymbol{u}), JB\boldsymbol{u}\rangle = -\langle E'(\boldsymbol{u}), T'(0)\boldsymbol{u}\rangle = \frac{d}{ds}E(T(s)\boldsymbol{u})\bigg|_{s=0} = 0\,.\end{aligned}$$

The equation will be considered in the weak sense. We review the main definitions from [8] and the three assumptions needed for the proofs of stability/instability theorems.

DEFINITION 2.1 By "solution of (2.1) in a time interval $\mathscr{I}$," we mean a function

$$\boldsymbol{u} \in C(\mathscr{I}, X)$$

such that

$$\frac{d}{dt}\langle \boldsymbol{\xi}, \boldsymbol{u}(t)\rangle = \langle E'(\boldsymbol{u}(t)), -J\boldsymbol{\xi}\rangle \quad \text{in } \mathscr{D}'(\mathscr{I}) \text{ for all } \boldsymbol{\xi} \in D(J) \subset X^*.$$

DEFINITION 2.2 The *standing-wave solution* (or the *bound state*) is a solution of evolution equation (2.1) of the special form

$$\boldsymbol{u}(t) = T(\omega t)\boldsymbol{\phi}_\omega\,,$$

where $\omega \in \mathbb{R}$ and $\boldsymbol{\phi}_\omega \in X$. The function $\boldsymbol{\phi}_\omega$ satisfies the "stationary" equation

$$E'(\boldsymbol{\phi}_\omega) = \omega Q'(\boldsymbol{\phi}_\omega)\,.$$

DEFINITION 2.3 The $\boldsymbol{\phi}_\omega$ orbit $\{T(\omega t)\boldsymbol{\phi}_\omega : t \in \mathbb{R}\}$ is *stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If $\|\boldsymbol{u}_0 - \boldsymbol{\phi}_\omega\| < \delta$ and $\boldsymbol{u}(t)$ is a solution of (2.1) in some interval $[0, t_1)$, then $\boldsymbol{u}(t)$ can be continued to a solution in $0 \le t < \infty$ and

$$\sup_{0<t<\infty}\ \inf_{s\in\mathbb{R}} \|\boldsymbol{u}(t) - T(s)\boldsymbol{\phi}_\omega\| < \varepsilon\,.$$

Otherwise the $\boldsymbol{\phi}_\omega$ orbit is called *unstable*.

ASSUMPTION 2.4 (Existence of Solutions) For each $\boldsymbol{u}_0 \in X$ there exists $t_1 > 0$ depending only on $\|\boldsymbol{u}_0\|$, and there exists a solution of (2.1) in the interval $[0, t_1)$ such that

(a) $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and

(b) $E(\boldsymbol{u}(t)) = E(\boldsymbol{u}_0)$ and $Q(\boldsymbol{u}(t)) = Q(\boldsymbol{u}_0)$ for $t \in [0, t_1)$.

ASSUMPTION 2.5 (Existence of Bound States) There exist an open interval

$$(\omega_1, \omega_2) \subset \mathbb{R}, \quad \omega_1 < \omega_2 ,$$

and a mapping $\omega \mapsto \boldsymbol{\phi}_\omega$ from (ω_1, ω_2) into X such that everywhere on (ω_1, ω_2)

(a) the mapping $\omega \mapsto \boldsymbol{\phi}_\omega$ is C^2,
(b) $E'(\boldsymbol{\phi}_\omega) = \omega Q'(\boldsymbol{\phi}_\omega)$,
(c) $\boldsymbol{\phi}_\omega \in D(T'(0)^3) \cap D(JIT'(0)^2)$, and
(d) $T'(0)\boldsymbol{\phi}_\omega \neq 0$.

Define the operator $H_\omega : X \to X^*$ by

$$H_\omega = E''(\boldsymbol{\phi}_\omega) - \omega Q''(\boldsymbol{\phi}_\omega) = E''(\boldsymbol{\phi}_\omega) - \omega B , \tag{2.4}$$

where $E''(\boldsymbol{\phi}_\omega)$ and $Q''(\boldsymbol{\phi}_\omega)$ are considered as linear maps from X to X^*.

ASSUMPTION 2.6 For each $\omega \in (\omega_1, \omega_2)$, H_ω has exactly one simple negative eigenvalue $-\Lambda_\omega^2$, the kernel of H_ω is spanned by $T'(0)\boldsymbol{\phi}_\omega$, and the rest of its spectrum is positive and bounded away from zero.

In order to check that $T'(0)\boldsymbol{\phi}_\omega$ belongs to $\ker H_\omega$, we differentiate the relation $E'(T(\omega t)\boldsymbol{\phi}_\omega) = \omega Q'(T(\omega t)\boldsymbol{\phi}_\omega)$ with respect to t at $t = 0$, obtaining

$$E''(\boldsymbol{\phi}_\omega)T'(0)\boldsymbol{\phi}_\omega = \omega B T'(0)\boldsymbol{\phi}_\omega , \tag{2.5}$$

and hence $H_\omega T'(0)\boldsymbol{\phi}_\omega = 0$.

According to [8], the standing wave $\boldsymbol{u}(t) = T(\omega t)\boldsymbol{\phi}_\omega$ is orbitally stable if $d''(\omega) > 0$ and unstable if $d''(\omega) < 0$. We will analyze the case $d''(\omega) = 0$.

Besides assumptions from [8], we impose additional (very mild) assumptions needed for the proof of instability in the absence of linear instability.

ASSUMPTION 2.7 $T'(0)$ is defined on the entire space X.

ASSUMPTION 2.8 There is a symmetry transformation $\mathrm{C} : X \to X$, $\mathrm{C}^2 = 1$, that is self-adjoint with respect to $(\cdot, \cdot)$ and so that the following conditions are satisfied:

(a) $E(\mathrm{C}\boldsymbol{u}) = E(\boldsymbol{u})$ for all $\boldsymbol{u} \in X$.
(b) For any $\omega \in (\omega_1, \omega_2)$, one has $\mathrm{C}\boldsymbol{\phi}_\omega = \boldsymbol{\phi}_\omega$ and $\mathrm{C}\boldsymbol{\chi}_\omega = \boldsymbol{\chi}_\omega$, where $\boldsymbol{\chi}_\omega$ is the eigenvector that corresponds to the simple negative eigenvalue $-\Lambda_\omega^2$ of H_ω.
(c) C is compatible with both the representation $T : U(1) \to \mathrm{Aut}(X)$ and the operator $J : X^* \to X$ in the following sense:

$$\mathrm{C}T(s) = T(-s)\mathrm{C} \quad \forall s \in \mathbb{R} , \qquad \mathrm{C}J = -J\mathrm{C}^* ,$$

where $\mathrm{C}^* : X^* \to X^*$ is the adjoint of C with respect to $\langle \cdot, \cdot \rangle$.

The operator C can be thought of as "complex conjugation" on X. We assume that the operator C is extended onto the complexification $X_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} X$ by antilinearity:

$$\mathrm{C}(z\boldsymbol{u}) = \bar{z}\mathrm{C}(\boldsymbol{u}) \quad \forall z \in \mathbb{C}, \ \forall \boldsymbol{u} \in X_\mathbb{C} . \tag{2.6}$$

Differentiating the relation $\mathrm{C}T(s) = T(-s)\mathrm{C}$ with respect to s at $s = 0$, we see that C and $T'(0)$ anticommute:

$$\{\mathrm{C}, T'(0)\} = 0\,. \tag{2.7}$$

It follows that $\{\mathrm{C}, JB\} = 0$, so that $J\mathrm{C}^*B = -\mathrm{C}JB = JB\mathrm{C}$. Since J is invertible and Range $B \subset D(J)$ ($JB = T'(0)$ is defined on the entire space X), we conclude that $\mathrm{C}^*B = B\mathrm{C}$, and thus

$$Q(\mathrm{C}\boldsymbol{u}) = \frac{1}{2}\langle B\mathrm{C}\boldsymbol{u}, \mathrm{C}\boldsymbol{u}\rangle = \frac{1}{2}\langle B\boldsymbol{u}, \boldsymbol{u}\rangle = Q(\boldsymbol{u}) \quad \forall \boldsymbol{u} \in X\,.$$

The operator JH_ω is unbounded on its domain $D(JH_\omega) \subset X$ (let us also mention that since $H_\omega : X \to X^*$ is continuous and $D(J)$ is dense in X^*, $D(JH_\omega)$ is dense in X). Following the definition of standard Sobolev spaces, we define the Hilbert spaces $\mathcal{H}^k_\omega$ associated to JH_ω.

DEFINITION 2.9 Let $\zeta \in \mathbb{C}$ be any point from the resolvent set $\rho(JH_\omega)$ of JH_ω. For a nonnegative integer k, we define the space $\mathcal{H}^k_{\omega,\zeta} \subset X$ as the closure of $D((JH_\omega)^k) \subset X$ with respect to the norm

$$\|\boldsymbol{u}\|_{\mathcal{H}^k_{\omega,\zeta}} = \|(JH_\omega - \zeta)^k\boldsymbol{u}\|_X\,.$$

We also define $\mathcal{H}^\infty_{\omega,\zeta} = \bigcap_{k=0}^\infty \mathcal{H}^k_{\omega,\zeta}$.

The norms $\|\boldsymbol{u}\|_{\mathcal{H}^k_{\omega,\zeta}}$ and $\|\boldsymbol{u}\|_{\mathcal{H}^k_{\omega,\zeta'}}$ that correspond to different $\zeta, \zeta' \in \rho(JH_\omega)$ are equivalent. For simplicity, we assume that $\zeta \in \rho(JH_\omega)$ is the same for all values of ω between ω_1 and ω_2, and in the future will write $\mathcal{H}^k_\omega$ without the subscript ζ. Clearly, $\mathcal{H}^0_\omega \equiv X \supset \mathcal{H}^1_\omega \supset \mathcal{H}^2_\omega \supset \cdots$, and JH_ω is continuous from $\mathcal{H}^{k+1}_\omega$ to $\mathcal{H}^k_\omega$ for any nonnegative integer k. See Appendix A for details.

The following is the key assumption that we need to tackle the nonlinearity.

ASSUMPTION 2.10 There exists a nonnegative integer l such that for any $\omega \in (\omega_1, \omega_2)$ the following conditions are satisfied:

(a) The following maps are continuous:

$$\partial_\omega H_\omega : X \to X^*\,, \quad \partial_\omega(JH_\omega) : \mathcal{H}^{k+1}_\omega \to \mathcal{H}^k_\omega\,, \quad 0 \le k \le l.$$

(b) There is a constant $c < \infty$ so that for any $\boldsymbol{\rho} \in \mathcal{H}^l_\omega$ with $\|\boldsymbol{\rho}\|_{\mathcal{H}^l_\omega} \le 1$

$$\|J(E'(\boldsymbol{\phi}_\omega + \boldsymbol{\rho}) - E'(\boldsymbol{\phi}_\omega) - E''(\boldsymbol{\phi}_\omega)\boldsymbol{\rho})\|_{\mathcal{H}^l_\omega} \le c\|\boldsymbol{\rho}\|^2_{\mathcal{H}^l_\omega}\,.$$

(c) The map $JE'''(\boldsymbol{\phi}_\omega) : \mathcal{H}^l_\omega \times \mathcal{H}^l_\omega \to X$ is continuous, and there is a constant $c' < \infty$ so that for any $\boldsymbol{\rho} \in \mathcal{H}^l_\omega$ with $\|\boldsymbol{\rho}\|_{\mathcal{H}^l_\omega} \le 1$

$$\left\| J\left(E'(\boldsymbol{\phi}_\omega + \boldsymbol{\rho}) - E'(\boldsymbol{\phi}_\omega) - E''(\boldsymbol{\phi}_\omega)\boldsymbol{\rho} - \frac{E'''(\boldsymbol{\phi}_\omega)(\boldsymbol{\rho}, \boldsymbol{\rho})}{2} \right) \right\|_{\mathcal{H}^l_\omega} \le c'\|\boldsymbol{\rho}\|^3_{\mathcal{H}^l_\omega}.$$

$E'''(\boldsymbol{\phi}_\omega)$ is considered a bilinear map from $X \times X$ to X^*.

Remark 2.11. In the case of a nonlinear Schrödinger equation in $\mathbb{R}^n$, we take $X = H^1_r(\mathbb{R}^n)$ (spherically symmetric Sobolev functions of degree 1), and then $\mathcal{H}^k_\omega = H^{2k+1}_r(\mathbb{R}^n)$ (see Section 7 for details). For Assumption 2.10(b) and (c) to be satisfied, l should be taken so that $2l+1 > \frac{n}{2}$ (see Appendix B).

Our main result is the instability of the standing-wave solution $\boldsymbol{\phi}_\omega$ with minimal energy.

THEOREM 2.12 (Instability of Standing Waves with Minimal Energy) *If the map*

$$\omega \mapsto E(\boldsymbol{\phi}_\omega)$$

assumes a local minimum at $\omega_* \in (\omega_1, \omega_2)$, *then the standing wave* $\boldsymbol{u}(t) = T(\omega_* t)\boldsymbol{\phi}_{\omega_*}$ *is unstable in the sense of Definition* 2.3.

Remark 2.13. If we require that $\frac{d^2}{d\omega^2} Q(\boldsymbol{\phi}_\omega) > 0$ at ω_*, then the proofs become shorter. We also do not need Assumption 2.10(c).

All the cases when $E(\boldsymbol{\phi}_\omega)$ has a strict local minimum (or nonstrict local minimum) at a point ω_* are characterized by $d''(\omega) = -\frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega)$ being strictly negative (nonpositive, respectively) for ω from a one-sided open neighborhood of ω_*. Indeed, if $\omega_* > 0$, then $d''(\omega) \le 0$ for $\omega \ge \omega_*$. If $\omega_* < 0$, then $d''(\omega) \le 0$ for $\omega \le \omega_*$. If $\omega_* = 0$, $d''(\omega)$ is strictly negative (nonpositive, respectively) for ω from a punctured open neighborhood of $\omega_* = 0$. If, moreover, $\frac{d^2}{d\omega^2} E(\boldsymbol{\phi}_\omega)\big|_{\omega=0} > 0$, then

$$d''(0) = -\frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega)\bigg|_{\omega=0} = \lim_{\omega\to 0}\left(-\frac{1}{\omega}\frac{d}{d\omega} E(\boldsymbol{\phi}_\omega)\right) = -\frac{d^2}{d\omega^2} E(\boldsymbol{\phi}_\omega)\bigg|_{\omega=0} < 0\,,$$

so that the standing wave $T(\omega_* t)\boldsymbol{\phi}_{\omega_*}$, where $\omega_* = 0$, is linearly unstable, and its (nonlinear) instability follows from [8].

We will formulate the main theorem in terms of the behavior of $d(\omega)$ near the inflection point. Formally, we need to prove that there is $\varepsilon > 0$ such that for any $\delta > 0$ there exists a finite $t_1 = t_1(\delta, \varepsilon)$ and a solution $\boldsymbol{u}(t)$, $0 \le t \le t_1$, such that $\|\boldsymbol{u}(0) - \boldsymbol{\phi}_{\omega_*}\| < \delta$ and

$$\sup_{0<t\le t_1} \inf_{s\in\mathbb{R}} \|\boldsymbol{u}(t) - T(s)\boldsymbol{\phi}_{\omega_*}\| > \varepsilon\,.$$

Our approach is based on the observation that the solution $\boldsymbol{u}(t)$ with some particular initial data near $\boldsymbol{\phi}_{\omega_0}$, where ω_0 is δ-close to ω_*, at later moments t will be passing near the orbits spanned by $\boldsymbol{\phi}_\omega$ with $\omega = \omega(t)$, with $\omega(t)$ leaving the ε-neighborhood of ω_* in some finite time.

THEOREM 2.14 (Main Theorem) *Let* $T(\omega t)\boldsymbol{\phi}_\omega$ *be the standing-wave solutions of* (2.1). *Assume that* $\omega_* \in (\omega_1, \omega_2)$ *is the inflection point of* $d(\omega)$ *so that* $d''(\omega_*) = 0$ *and* $d''(\omega) \le 0$ *in a one-sided open neighborhood of* ω_*. *Then there is* $\varepsilon > 0$ *such that for any* $\delta > 0$ *there exist* $t_1 = t_1(\delta, \varepsilon) < \infty$ *and a pair of functions*

$$(\omega, \boldsymbol{\rho}) \in C^1([0, t_1], (\omega_1, \omega_2)) \times C^1([0, t_1], X)$$

such that

$$\boldsymbol{u}(t) = T\left(\int_0^t \omega(t')dt'\right)(\boldsymbol{\phi}_{\omega(t)} + \boldsymbol{\rho}(t))$$

is a solution to (2.1) *and such that*

$$|\omega(0) - \omega_*| < \delta\,, \quad \|\boldsymbol{\rho}(t)\|_X \le \beta(\omega(t) - \omega_*)\,, \quad \textit{and} \quad |\omega(t_1) - \omega_*| > \varepsilon\,,$$

where $\beta \in C(\mathbb{R})$ *is such that* $\beta(s) \ge 0$, $\beta(s) = o(s)$.

For definiteness, we prove this theorem assuming that $d''(\omega) \le 0$ for $\omega \ge \omega_*$. The other case ($d''(\omega) \le 0$ for $\omega \le \omega_*$) is covered if we consider the standing wave $T(\omega t)\phi_\omega$ as the standing wave that corresponds to $-\omega$ (formally, this amounts to substituting the representation $T(s)$ by $T(-s)$).

Since

$$\inf_{s\in\mathbb{R}} \|\boldsymbol{u}(t) - T(s)\boldsymbol{\phi}_{\omega_*}\|_X = \inf_{s\in\mathbb{R}} \|\boldsymbol{\phi}_{\omega(t)} - T(s)\boldsymbol{\phi}_{\omega_*}\|_X + \beta(\omega(t) - \omega_*)\,,$$

the theorem implies that

$$\inf_{s\in\mathbb{R}} \|\boldsymbol{u}(t) - T(s)\boldsymbol{\phi}_{\omega_*}\|_X = O(\omega(t) - \omega_*) \tag{2.8}$$

as long as $t \le t_1$. Therefore, at the moment $t_1 = t_1(\delta, \varepsilon)$ the solution $\boldsymbol{u}(t)$ that was initially $O(\delta)$-close to the orbit spanned by $\boldsymbol{\phi}_{\omega_*}$ leaves the $O(\varepsilon)$-neighborhood of this orbit. This proves the instability of the standing wave $\boldsymbol{u}(t) = T(\omega_* t)\boldsymbol{\phi}_{\omega_*}$. Figure 2.1 gives a schematic picture of the evolution of $\boldsymbol{u}(t)$ that leads to instability, plotted on the graph of the dependence of $Q(\boldsymbol{\phi}_\omega)$ versus ω, in the notation of Theorem 2.14.

3 Spectral Decomposition near Degenerate Zero Eigenvalue

In what follows, we restrict our attention to an open neighborhood $\mathcal{O}(\omega_*)$ of ω_* that is small enough so that its closure belongs to the range of the frequencies of the standing waves,

$$\overline{\mathcal{O}(\omega_*)} \subset (\omega_1, \omega_2)\,.$$

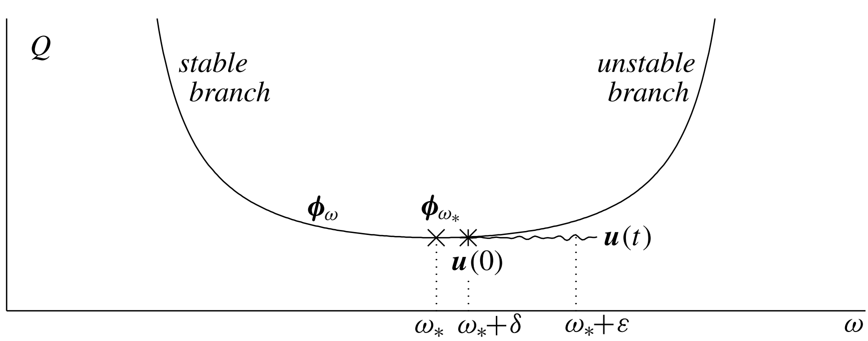

FIGURE 2.1. Drift of $\boldsymbol{u}(t)$ near the orbits spanned by unstable standing waves.

The spectrum of JH_ω is a disjoint union of the discrete and continuous spectra:

$$\sigma(JH_\omega) = \sigma_d(JH_\omega) \cup \sigma_c(JH_\omega)\,.$$

Without loss of generality, we assume that

$$\sigma_d(JH_{\omega_*}) = \{0\}\,. \tag{3.1}$$

Remark 3.1. If this is not so, then in our proof we substitute $\sigma_c(JH_\omega)$ by $\sigma_c(JH_\omega)\cup \sigma_d^*(JH_\omega)$, where $\sigma_d^*(JH_\omega)$ consists of eigenvalues that are continuations of eigenvalues from $\sigma_d(JH_{\omega_*}) \setminus \{0\}$ (note that the points of the discrete spectrum are continuous functions of ω). We need to choose an open neighborhood $\mathcal{O}(\omega_*)$ small enough so that for $\omega \in \mathcal{O}(\omega_*)$ the set $\sigma_c(JH_\omega) \cup \sigma_d^*(JH_\omega)$ is uniformly separated away from 0.

We denote the projection onto the discrete spectral subspace of JH_ω (spectral subspace that corresponds to $\sigma_d(JH_\omega)$) by

$$P_d(\omega) : X \to X_{d,\omega} \subset X\,,$$

and the projection onto the continuous spectral subspace of JH_ω by

$$P_c(\omega) = 1 - P_d(\omega) : X \to X_{c,\omega} \subset X\,,$$

where $X_{d,\omega}$ and $X_{c,\omega}$ are ranges of the mapping $P_d(\omega)$ and $P_c(\omega)$ at a point $\omega \in \mathcal{O}(\omega_*)$ (discrete and continuous spectral subspaces). There is the ω-dependent decomposition of X into the direct sum

$$X = X_{d,\omega} \oplus X_{c,\omega}\,.$$

LEMMA 3.2 *For any $\omega \in \mathcal{O}(\omega_*)$ and any nonnegative integer k the projections $P_d(\omega)$ and $P_c(\omega) = 1 - P_d(\omega)$ define the continuous mappings*

$$P_d(\omega) : \mathcal{H}_\omega^k \to \mathcal{H}_\omega^{k+1}\,, \quad P_c(\omega) : \mathcal{H}_\omega^k \to \mathcal{H}_\omega^k\,.$$

For any $\omega \in \mathcal{O}(\omega_*)$ and any nonnegative integer $k \le l$, where l is as in Assumption 2.10*, the derivatives $P_d'(\omega) \equiv \partial_\omega P_d(\omega)$ and $P_c'(\omega) \equiv \partial_\omega P_c(\omega)$ define the continuous mappings*

$$P_d'(\omega) : \mathcal{H}_\omega^k \to \mathcal{H}_\omega^{k+1}\,, \quad P_c'(\omega) : \mathcal{H}_\omega^k \to \mathcal{H}_\omega^{k+1}\,.$$

PROOF: According to Assumption 2.5, the operator JH_ω is continuous in ω. As follows from [10, chap. 11, sec. 5] the eigenvalues of JH_ω depend continuously on ω. Hence, due to (3.1), we can decrease the size of the neighborhood $\mathcal{O}(\omega_*)$ to ensure that there is $r > 0$ so that for $\omega \in \mathcal{O}(\omega_*)$

$$\sigma_d(JH_\omega) \subset \mathcal{D}_r(0)\,, \quad \sigma_c(JH_\omega) \subset \mathbb{C} \setminus \overline{\mathcal{D}_r(0)}\,,$$

where $\mathcal{D}_r(0) \subset \mathbb{C}$ is the open disc of radius r around $\lambda = 0$. At a particular ω, the projector onto the subspace $X_{d,\omega}$ is given by the integral

$$P_d(\omega) = -\frac{1}{2\pi i} \oint_{|\zeta|=r} R(\zeta,\omega)\,d\zeta\,, \tag{3.2}$$

where $R(\zeta,\omega) = (JH_\omega - \zeta)^{-1}$. For more details, see [10, 15]. Since the values of ζ in (3.2) belong to the resolvent set of JH_ω, the statements about continuity of $P_d(\omega)$ and $P_c(\omega)$ follow from the definition of the spaces $\mathcal{H}^k_\omega$ (Definition 2.9).

Let us consider continuity of the derivatives of the spectral projectors. We have

$$P_d'(\omega) = \frac{1}{2\pi i}\oint R(\zeta,\omega)(\partial_\omega J H_\omega)R(\zeta,\omega)d\zeta\ ,$$

where $R(\zeta,\omega)$ is continuous from $\mathcal{H}^k_\omega$ to $\mathcal{H}^{k+1}_\omega$, while $\partial_\omega(JH_\omega)$ is continuous from $\mathcal{H}^{k+1}_\omega$ to $\mathcal{H}^k_\omega$ for any nonnegative integer $k \le l$, according to Assumption 2.10(a). We conclude that $P_d'(\omega)$ is continuous from $\mathcal{H}^k_\omega$ to $\mathcal{H}^{k+1}_\omega$ for any nonnegative integer $k \le l$. The same is true for $P_c'(\omega) = -P_d'(\omega)$. □

LEMMA 3.3 $\mathrm{C}^* H_\omega \mathrm{C} = H_\omega$, *where* $H_\omega = E''(\boldsymbol{\phi}_\omega) - \omega Q''(\boldsymbol{\phi}_\omega)$.

PROOF: For any $\boldsymbol{u},\boldsymbol{v} \in X$ and $s,t \in \mathbb{R}$, we can express

$$\partial_s\partial_t\big|_{s=0,t=0} E(\boldsymbol{\phi}_\omega + s\boldsymbol{u} + t\boldsymbol{v}) = \langle E''(\boldsymbol{\phi}_\omega)\boldsymbol{u},\boldsymbol{v}\rangle\ .$$

Using the relation $E(\boldsymbol{\phi}_\omega+\boldsymbol{\rho}) = E(\mathrm{C}(\boldsymbol{\phi}_\omega+\boldsymbol{\rho})) = E(\boldsymbol{\phi}_\omega+\mathrm{C}\boldsymbol{\rho})$, valid for any $\boldsymbol{\rho} \in X$ (we took into account that $\mathrm{C}\boldsymbol{\phi}_\omega = \boldsymbol{\phi}_\omega$), we conclude that

$$\langle E''(\boldsymbol{\phi}_\omega)\boldsymbol{u},\boldsymbol{v}\rangle = \langle E''(\boldsymbol{\phi}_\omega)\mathrm{C}\boldsymbol{u},\mathrm{C}\boldsymbol{v}\rangle = \langle \mathrm{C}^* E''(\boldsymbol{\phi}_\omega)\mathrm{C}\boldsymbol{u},\boldsymbol{v}\rangle\ . \tag{3.3}$$

We conclude that $\mathrm{C}^*E''(\boldsymbol{\phi}_\omega)\mathrm{C} = E''(\boldsymbol{\phi}_\omega)$. Applying the same argument to the charge functional, we deduce that $\mathrm{C}^*Q''(\boldsymbol{\phi}_\omega)\mathrm{C} = Q''(\boldsymbol{\phi}_\omega)$. □

Using the relation $\mathrm{C}J = -J\mathrm{C}^*$, we derive

$$\mathrm{C}JH_\omega = -J\mathrm{C}^*H_\omega = -JH_\omega\mathrm{C}\,.$$

COROLLARY 3.4 $\{\mathrm{C}, JH_\omega\} = 0$.

We use the projectors

$$\Pi^+ = \frac{1}{2}(1+\mathrm{C}) \quad \text{and} \quad \Pi^- = \frac{1}{2}(1-\mathrm{C}) \tag{3.4}$$

to decompose the space X into the direct sum of "real part" and "imaginary part":

$$X = X^+ \oplus X^-\ . \tag{3.5}$$

Since C is self-adjoint with respect to $(\,\cdot\,,\cdot\,)$, this decomposition is orthogonal.

LEMMA 3.5 *The operator* C *commutes with the projection onto* $X_{d,\omega}$:

$$[\mathrm{C}, P_d(\omega)] = 0\,.$$

PROOF: Let us give an explicit computation, trying to keep track of the signs. Using the anticommutivity relation $\{\mathrm{C}, JH_\omega\} = 0$ and the antilinearity of the extension of C onto the complexification $X_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} X$, we compute, for any $\boldsymbol{u} \in X$,

$$\begin{aligned}
\mathrm{C}P_d(\omega)\boldsymbol{u} &= \mathrm{C}\Big(-\frac{1}{2\pi i} \oint\limits_{|\zeta|=r} (JH_\omega - \zeta)^{-1}\boldsymbol{u}\, d\zeta \Big) \\
&= \frac{1}{2\pi i} \oint\limits_{|\zeta|=r} (-JH_\omega - \bar{\zeta})^{-1}(\mathrm{C}\boldsymbol{u}) d\bar{\zeta} \\
&= \frac{1}{2\pi i} \oint\limits_{|y|=r} (-JH_\omega - y)^{-1}(\mathrm{C}\boldsymbol{u}) dy \\
&= -\frac{1}{2\pi i} \oint\limits_{|z|=r} (-JH_\omega + z)^{-1}(\mathrm{C}\boldsymbol{u}) d(-z) = P_d(\omega)\mathrm{C}\boldsymbol{u}\,.
\end{aligned}$$

In the second line, we substituted $\bar{\zeta}$ by y, so that the contour of integration became oriented clockwise. In the last line, we reversed the contour of integration (gaining a negative sign) and substituted $-z$ for y.

We conclude that $[\mathrm{C}, P_d(\omega)] = 0$ for any $\omega \in \mathcal{O}(\omega_*)$. $\square$

COROLLARY 3.6 *There are the following commutation relations*:

$$[\Pi^\pm, P_d(\omega)] = [\Pi^\pm, P_d'(\omega)] = 0 \quad \textit{and} \quad [\Pi^\pm, P_c(\omega)] = [\Pi^\pm, P_c'(\omega)] = 0\,.$$

LEMMA 3.7 *Let us introduce the vectors*

$$\boldsymbol{e}_1(\omega) = T'(0)\boldsymbol{\phi}_\omega\,, \qquad \boldsymbol{e}_2(\omega) = \boldsymbol{\phi}_\omega' \equiv \frac{d\boldsymbol{\phi}_\omega}{d\omega}\,.$$

Then $\boldsymbol{e}_1(\omega) \in X^-$ *and* $\boldsymbol{e}_2(\omega) \in X^+$, *and*

$$JH_\omega \boldsymbol{e}_1(\omega) = 0\,, \qquad JH_\omega \boldsymbol{e}_2(\omega) = \boldsymbol{e}_1(\omega)\,.$$

PROOF: According to Assumption 2.8(b), $\boldsymbol{\phi}_\omega \in X^+$ for $\omega \in \mathcal{O}(\omega_*)$; therefore, $\boldsymbol{e}_2(\omega) = \boldsymbol{\phi}_\omega' \in X^+$. Using (2.7), we see that

$$\mathrm{C}T'(0)\boldsymbol{\phi}_\omega = -T'(0)\mathrm{C}\boldsymbol{\phi}_\omega = -T'(0)\boldsymbol{\phi}_\omega\,;$$

hence $\boldsymbol{e}_1(\omega) = T'(0)\boldsymbol{\phi}_\omega \in X^-$.

According to (2.5), $\boldsymbol{e}_1(\omega) = T'(0)\boldsymbol{\phi}_\omega \in \ker H_\omega$. Differentiating the relation $E'(\boldsymbol{\phi}_\omega) = \omega Q'(\boldsymbol{\phi}_\omega)$ with respect to ω, we get

$$E''(\boldsymbol{\phi}_\omega)\boldsymbol{\phi}_\omega' = \omega B \boldsymbol{\phi}_\omega' + Q'(\boldsymbol{\phi}_\omega)\,,$$

hence $H_\omega \boldsymbol{\phi}_\omega' = Q'(\boldsymbol{\phi}_\omega) = B\boldsymbol{\phi}_\omega$, or $JH_\omega\boldsymbol{\phi}_\omega' = JB\boldsymbol{\phi}_\omega = T'(0)\boldsymbol{\phi}_\omega$, so that $\boldsymbol{e}_2(\omega) \equiv \boldsymbol{\phi}_\omega'$. $\square$

LEMMA 3.8 *Assume that* $\frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega)$ *vanishes at some point* ω_*. *Then there exist vectors* $\boldsymbol{e}_3 \in X^-$ *and* $\boldsymbol{e}_4 \in X^+$ *such that*

$$JH_{\omega_*}\boldsymbol{e}_3 = \boldsymbol{e}_2(\omega_*)\,, \tag{3.6}$$

$$JH_{\omega_*}\boldsymbol{e}_4 = \boldsymbol{e}_3\,. \tag{3.7}$$

There is no $\boldsymbol{e}_5 \in X$ *such that* $JH_{\omega_*}\boldsymbol{e}_5 = \boldsymbol{e}_4$; *hence* $\dim \mathcal{N}_g(JH_{\omega_*}) = 4$, *where the generalized null space* $\mathcal{N}_g$ *is defined by*

$$\mathcal{N}_g(JH_{\omega_*}) = \{\boldsymbol{v} : (JH_{\omega_*})^k \boldsymbol{v} = 0 \textit{ for some } k \in \mathbb{N}\}\,.$$

PROOF: Since

$$\frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega) = \langle B\boldsymbol{\phi}_\omega, \boldsymbol{\phi}'_\omega\rangle = \langle B\boldsymbol{\phi}_\omega, \boldsymbol{e}_2(\omega)\rangle \tag{3.8}$$

vanishes at $\omega = \omega_*$, we conclude that

$$\boldsymbol{e}_2(\omega_*) \in (\ker(JH_{\omega_*})^*)^\perp = (\operatorname{span}\langle B\boldsymbol{\phi}_{\omega_*}\rangle)^\perp\,,$$

and hence there exists $\boldsymbol{e}_3$ such that $JH_{\omega_*}\boldsymbol{e}_3 = \boldsymbol{e}_2(\omega_*)$. Applying Π^+ to this relation and taking into account that $\boldsymbol{e}_2 \in X^+$ and $\Pi^+ JH_\omega = JH_\omega \Pi^-$ (as follows from Corollary 3.4), we obtain

$$\boldsymbol{e}_2(\omega_*) = \Pi^+ JH_{\omega_*}\boldsymbol{e}_3 = JH_{\omega_*}\Pi^-\boldsymbol{e}_3\,.$$

Hence, we may substitute $\boldsymbol{e}_3$ by $\Pi^-\boldsymbol{e}_3$, thus complying with the condition $\boldsymbol{e}_3 \in X^-$.

Furthermore, since

$$\langle B\boldsymbol{\phi}_{\omega_*}, \boldsymbol{e}_3\rangle = \langle H_{\omega_*}\boldsymbol{e}_2(\omega_*), \boldsymbol{e}_3\rangle = \langle H_{\omega_*}\boldsymbol{e}_3, \boldsymbol{e}_2(\omega_*)\rangle = \langle H_{\omega_*}\boldsymbol{e}_3, JH_{\omega_*}\boldsymbol{e}_3\rangle = 0\,,$$

we conclude that $\boldsymbol{e}_3 \in (\ker(JH_{\omega_*})^*)^\perp$; hence there exists $\boldsymbol{e}_4$ such that $JH_{\omega_*}\boldsymbol{e}_4 = \boldsymbol{e}_3$. Since $\boldsymbol{e}_3 = \Pi^-\boldsymbol{e}_3 = JH_{\omega_*}\Pi^+\boldsymbol{e}_4$, we substitute $\boldsymbol{e}_4$ by $\Pi^+\boldsymbol{e}_4 \in X^+$ to satisfy the condition $\boldsymbol{e}_4 \in X^+$.

To prove the last statement of the lemma that there is no $\boldsymbol{e}_5 \in X$ such that $JH_{\omega_*}\boldsymbol{e}_5 = \boldsymbol{e}_4$, we need to show that $\boldsymbol{e}_4 \notin (\ker(JH_{\omega_*})^*)^\perp$. This amounts to showing that

$$\langle B\boldsymbol{\phi}_{\omega_*}, \boldsymbol{e}_4\rangle$$

does not vanish. We compute

$$\begin{aligned}\langle B\boldsymbol{\phi}_{\omega_*}, \boldsymbol{e}_4\rangle = \langle H_{\omega_*}\boldsymbol{e}_2(\omega_*), \boldsymbol{e}_4\rangle &= \langle H_{\omega_*}\boldsymbol{e}_4, \boldsymbol{e}_2(\omega_*)\rangle \\ &= \langle H_{\omega_*}\boldsymbol{e}_4, JH_{\omega_*}\boldsymbol{e}_3\rangle = -\langle H_{\omega_*}\boldsymbol{e}_3, \boldsymbol{e}_3\rangle\,.\end{aligned} \tag{3.9}$$

According to Assumption 2.6, there is the following spectral decomposition of X that corresponds to H_ω:

$$\operatorname{span}\langle \boldsymbol{\chi}_\omega\rangle \oplus \operatorname{span}\langle \boldsymbol{e}_1(\omega)\rangle \oplus \mathscr{P}_\omega = X\,, \tag{3.10}$$

where $\mathscr{P}_\omega \subset X$ is the positive spectral subspace of H_ω. Since $I^{-1}H_\omega$ is self-adjoint with respect to $(\,\cdot\,,\cdot\,)$, decomposition (3.10) is orthogonal. We use spectral decomposition (3.10) at ω_* to express $\boldsymbol{e}_3$ as

$$\boldsymbol{e}_3 = a\boldsymbol{\chi}_{\omega_*} + b\boldsymbol{e}_1(\omega_*) + \boldsymbol{p}\,, \quad a, b \in \mathbb{R},\ \boldsymbol{p} \in \mathscr{P}_{\omega_*} \subset X\,.$$

Note that $\boldsymbol{\chi}_{\omega_*} \in X^+$ is orthogonal to both $\boldsymbol{e}_1, \boldsymbol{e}_3 \in X^-$ (due to mutual orthogonality of X^+ and X^-) and to $\boldsymbol{p}$ (due to orthogonality of spectral decomposition (3.10)). Therefore, $a = 0$. From $\boldsymbol{e}_3 = b\boldsymbol{e}_1 + \boldsymbol{p}$ we know that $\boldsymbol{p} \neq 0$, so that $\langle H_{\omega_*}\boldsymbol{e}_3, \boldsymbol{e}_3\rangle = \langle H_{\omega_*}\boldsymbol{p}, \boldsymbol{p}\rangle$ is strictly positive. Using (3.9), we see that

$$\langle B\boldsymbol{\phi}_{\omega_*}, \boldsymbol{e}_4\rangle = -\langle H_{\omega_*}\boldsymbol{e}_3, \boldsymbol{e}_3\rangle = -\langle H_{\omega_*}\boldsymbol{p}, \boldsymbol{p}\rangle < 0 \,. \tag{3.11}$$

□

LEMMA 3.9 *If $\mathcal{O}(\omega_*)$ is sufficiently small, then one can continue $\boldsymbol{e}_3 \in X^-_{d,\omega_*}$ and $\boldsymbol{e}_4 \in X^+_{d,\omega_*}$, defined in Lemma 3.8, to C^1 functions from $\mathcal{O}(\omega_*)$ to X, which we denote by $\boldsymbol{e}_3(\omega)$ and $\boldsymbol{e}_4(\omega)$, so that*

$$\boldsymbol{e}_3(\omega) \in X^-_{d,\omega}\,, \qquad \boldsymbol{e}_4(\omega) \in X^+_{d,\omega}\,.$$

Thus in the frame $\{\boldsymbol{e}_j(\omega) \in X_{d,\omega} : 1 \leq j \leq 4\}$ the restriction of the operator JH_ω onto $X_{d,\omega}$ has the form

$$JH_\omega\big|_{X_{d,\omega}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a(\omega) & 0 \end{bmatrix}, \tag{3.12}$$

where $a(\omega)$ is a differentiable function of ω that satisfies $a(\omega_*) = 0$.

For $1 \leq j \leq 4$, $\boldsymbol{e}_j(\omega) \in \mathcal{H}^\infty_\omega$.

PROOF: Let $\tilde{\boldsymbol{e}}_4(\omega) = P_d(\omega)\boldsymbol{e}_4 \in X^+_{d,\omega} \equiv X_{d,\omega} \cap X^+$ so that $\tilde{\boldsymbol{e}}_4(\omega)$ is defined for $\omega \in \mathcal{O}(\omega_*)$ and with the values in $X^+_{d,\omega}$ (recall that $\boldsymbol{e}_4 \in X^+_{d,\omega_*} \subset X$ does not depend on ω). Apparently, $\tilde{\boldsymbol{e}}_4(\omega_*) = \boldsymbol{e}_4$. Note that $\tilde{\boldsymbol{e}}_4'(\omega) = P_d'(\omega)\boldsymbol{e}_4 \in \mathcal{H}^1_\omega$ so that $\tilde{\boldsymbol{e}}_4(\omega)$ is C^1 for $\omega \in \mathcal{O}(\omega_*)$. Applying the bootstrapping argument to the relation $\tilde{\boldsymbol{e}}_4(\omega) = P_d(\omega)\tilde{\boldsymbol{e}}_4(\omega)$, where $P_d(\omega) : \mathcal{H}^k_\omega \to \mathcal{H}^{k+1}_\omega$ for any $k \geq 0$ (see Lemma 3.2), we conclude that $\tilde{\boldsymbol{e}}_4(\omega) \in \mathcal{H}^\infty_\omega$ so that $(JH_\omega)^2\tilde{\boldsymbol{e}}_4(\omega)$ is well-defined as an element of $X^+_{d,\omega}$. Since $\{\boldsymbol{e}_2(\omega), \tilde{\boldsymbol{e}}_4(\omega)\}$ is a frame in $X^+_{d,\omega}$, we can write

$$(JH_\omega)^2\tilde{\boldsymbol{e}}_4(\omega) = b(\omega)\boldsymbol{e}_2(\omega) + a(\omega)\tilde{\boldsymbol{e}}_4(\omega)\,, \tag{3.13}$$

where, according to Assumption 2.5, $a(\omega)$ and $b(\omega)$ are differentiable functions of ω (we only know they are C^1, because so is $\boldsymbol{e}_2(\omega) = \boldsymbol{\phi}'_\omega$). Evaluating (3.13) at ω_* and using the relation $JH_{\omega_*}\boldsymbol{e}_3 = \boldsymbol{e}_2(\omega_*)$ (see Lemma 3.8), we see that $a(\omega_*) = 0$ and $b(\omega_*) = 1$. We assume that $\mathcal{O}(\omega_*)$ is small enough so that $b(\omega) \neq 0$ on the closure of $\mathcal{O}(\omega_*)$. We then define

$$\boldsymbol{e}_4(\omega) = \frac{1}{b(\omega)}\tilde{\boldsymbol{e}}_4(\omega) \in X^+_{d,\omega}\,, \tag{3.14}$$

$$\boldsymbol{e}_3(\omega) = JH_\omega\boldsymbol{e}_4(\omega) \in X^-_{d,\omega}\,. \tag{3.15}$$

Apparently, $\boldsymbol{e}_3(\omega)$ and $\boldsymbol{e}_4(\omega)$ are C^1. According to (3.13) and (3.14), there is a relation $JH_\omega\boldsymbol{e}_3(\omega) = \boldsymbol{e}_2(\omega) + a(\omega)\boldsymbol{e}_4(\omega)$.

The statement of the lemma that $\boldsymbol{e}_j(\omega) \in \mathcal{H}^\infty_\omega$ for $1 \leq j \leq 4$ is proven by applying the bootstrapping argument to $\boldsymbol{e}_j(\omega) = P_d(\omega)\boldsymbol{e}_j(\omega)$. □

According to (3.11), the open neighborhood $\mathcal{O}(\omega_*)$ can be shrunk so that the inequality

$$\langle B\boldsymbol{\phi}_\omega, \boldsymbol{e}_4(\omega)\rangle < 0 \tag{3.16}$$

is satisfied everywhere in $\mathcal{O}(\omega_*)$ and holds uniformly (the left-hand side is separated away from 0 uniformly in ω).

LEMMA 3.10 *For $\omega \in \mathcal{O}(\omega_*)$, the following relation is true*:

$$a(\omega) = \frac{1}{m(\omega)} \frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega)\,, \tag{3.17}$$

where $m(\omega) = -\langle B\boldsymbol{\phi}_\omega, \boldsymbol{e}_4(\omega)\rangle > 0$.

PROOF: According to Lemma 3.9, there is a relation

$$JH_\omega \boldsymbol{e}_3(\omega) = \boldsymbol{e}_2(\omega) + a(\omega)\boldsymbol{e}_4(\omega)\,.$$

We take the pairing of this relation with $B\boldsymbol{\phi}_\omega$. Since

$$\langle B\boldsymbol{\phi}_\omega, JH_\omega \boldsymbol{e}_3(\omega)\rangle = -\langle H_\omega \boldsymbol{e}_1(\omega), \boldsymbol{e}_3(\omega)\rangle = 0\,,$$

while $\langle B\boldsymbol{\phi}_\omega, \boldsymbol{e}_2(\omega)\rangle = \frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega)$, we obtain the desired relation

$$\frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega) + a(\omega)\langle B\boldsymbol{\phi}_\omega, \boldsymbol{e}_4(\omega)\rangle = 0\,.$$

The inequality $m(\omega) \equiv -\langle B\boldsymbol{\phi}_\omega, \boldsymbol{e}_4(\omega)\rangle > 0$ follows from (3.16). □

Remark 3.11. If $\frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega) = 0$ in an interval $\mathscr{I} \subset \mathcal{O}(\omega_*)$, then there exists $\boldsymbol{e}_3(\omega)$ and $\boldsymbol{e}_4(\omega)$ that satisfy (3.6) and (3.7) in $\omega \in \mathscr{I} \subset \mathcal{O}(\omega_*)$. Due to the differentiability of the projector $P_d(\omega)$ onto the space $X_{d,\omega} = \operatorname{span}\langle \boldsymbol{e}_j(\omega) : 1 \le j \le 4\rangle$, as well as differentiability of $\boldsymbol{e}_1(\omega)$ and $\boldsymbol{e}_2(\omega)$, the eigenvectors $\boldsymbol{e}_3(\omega)$ and $\boldsymbol{e}_4(\omega)$ can be chosen differentiable in ω.

Remark 3.12. According to (3.12), the characteristic equation on $JH_\omega\big|_{X_{d,\omega}}$ is given by

$$\lambda^2(\lambda^2 - a(\omega)) = 0\,.$$

If $a(\omega) \neq 0$ at a given point $\omega \in \mathcal{O}(\omega_*)$, then $\dim \mathcal{N}_g(JH_\omega) = 2$, and JH_ω has two simple eigenvalues $\pm\sqrt{a(\omega)}$.

As follows from Lemma 3.10, $a(\omega)$ is of the same sign as $\frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega)$. Therefore, if $\frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega) > 0$, then the operator JH_ω has a real positive eigenvalue that results in linear instability.

If $\frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega) < 0$, both eigenvalues are imaginary; the orbital stability in the latter case was proven in [8]. This agrees with Vakhitov-Kolokolov's stability criterion (1.4).

If $\frac{d}{d\omega} Q(\boldsymbol{\phi}_\omega) = 0$, then the zero eigenvalue has a higher algebraic multiplicity: $\dim \mathcal{N}_g(JH_\omega) = 4$. According to Theorem 2.14, there is a (nonlinear) instability in this case.

We depict all three cases in Figure 3.1.

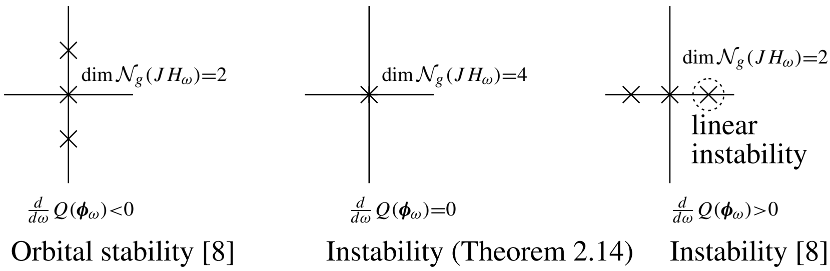

FIGURE 3.1. Discrete spectrum of JH_ω near $\lambda = 0$.

4 Evolution of the Perturbed Standing Wave

In this section we derive the equations that describe the time evolution of the spectral components of the perturbation $\boldsymbol{\rho}$. We consider the evolution equation (2.1) with the initial data $\boldsymbol{u}(0)$ near $\boldsymbol{\phi}_{\omega_*}$. We write the solution $\boldsymbol{u}(t)$ in the form

$$\boldsymbol{u}(t) = T\left(\int_0^t \omega(t')dt'\right)(\boldsymbol{\phi}_{\omega(t)} + \boldsymbol{\rho}(t)), \tag{4.1}$$

where $\omega(t)$ is some function of t. The function $\boldsymbol{\rho}(t)$ satisfies the equation

$$\dot{\omega}(t)\boldsymbol{\phi}'_{\omega(t)} + \dot{\boldsymbol{\rho}} = JE'(\boldsymbol{\phi}_{\omega(t)} + \boldsymbol{\rho}) - \omega(t)T'(0)(\boldsymbol{\phi}_{\omega(t)} + \boldsymbol{\rho}). \tag{4.2}$$

We rewrite this equation as

$$\dot{\boldsymbol{\rho}}(t) = JH_{\omega(t)}\boldsymbol{\rho}(t) - \dot{\omega}(t)\boldsymbol{e}_2(\omega(t)) + \boldsymbol{N}(\omega(t), \boldsymbol{\rho}(t)), \tag{4.3}$$

where

$$JH_\omega = J(E''(\boldsymbol{\phi}_\omega) - \omega Q''(\boldsymbol{\phi}_\omega)) = JE''(\boldsymbol{\phi}_\omega) - \omega T'(0) \tag{4.4}$$

is the linearization of system (2.1) near the soliton and

$$\boldsymbol{N}(\omega, \boldsymbol{\rho}) = JE'(\boldsymbol{\phi}_\omega + \boldsymbol{\rho}) - JE'(\boldsymbol{\phi}_\omega) - JE''(\boldsymbol{\phi}_\omega)\boldsymbol{\rho} \tag{4.5}$$

is the nonlinearity. The norm of $\boldsymbol{N}(\omega, \boldsymbol{\rho})$ in the space $\mathcal{H}^l_\omega$ is bounded due to Assumption 2.10(b) and (c).

At a moment t, we split $\boldsymbol{\rho}(t)$ into two components that lie in the discrete and continuous spectral subspaces of X that correspond to $JH_{\omega(t)}$:

$$\boldsymbol{\rho}(t) = \boldsymbol{\rho}_d(t) + \boldsymbol{\rho}_c(t) \in X_{d,\omega(t)} \oplus X_{c,\omega(t)} = X,$$
$$\boldsymbol{\rho}_d(t) = P_d(\omega(t))\boldsymbol{\rho}(t), \quad \boldsymbol{\rho}_c(t) = P_c(\omega(t))\boldsymbol{\rho}(t).$$

Let $\rho_j(t)$, $1 \le j \le 4$, be the components of the function $\boldsymbol{\rho}_d(t) \in X_{d,\omega(t)}$ in the frame $\{\boldsymbol{e}_j(\omega) : 1 \le j \le 4\}$:

$$\boldsymbol{\rho}_d(t) = \sum_{j=1}^4 \boldsymbol{e}_j(\omega(t))\rho_j(t). \tag{4.6}$$

Let us derive the evolution equations on the parts of $\boldsymbol{\rho}(t)$ that live in discrete and continuous spectral subspaces of JH_ω. For this, we project equation (4.3) onto $X_{d,\omega(t)}$ and $X_{c,\omega(t)}$. First, let us find the projection of $\dot{\boldsymbol{\rho}}$ onto $X_{d,\omega(t)}$:

$$
\begin{aligned}
P_d(\omega)\dot{\boldsymbol{\rho}}(t) &= P_d(\omega)(\boldsymbol{e}_j\dot{\rho}_j(t) + \dot{\omega}(t)\boldsymbol{e}_j'(\omega)\rho_j(t) + \dot{\boldsymbol{\rho}}_c(t)) \\
&= \boldsymbol{e}_j\dot{\rho}_j(t) + \dot{\omega}(t)P_d(\omega)\boldsymbol{e}_j'(\omega)\rho_j(t) + P_d(\omega)\dot{\boldsymbol{\rho}}_c(t)\,,
\end{aligned}
\tag{4.7}
$$

where the summation in j is assumed.

LEMMA 4.1 *Let $\Gamma_{ij}(\omega)$, $1 \le i, j \le 4$ be the "Christoffel symbols" so that*

$$
P_d(\omega)\boldsymbol{e}_j'(\omega) = \sum_{i=1}^{4} \boldsymbol{e}_i(\omega)\Gamma_{ij}(\omega)\,.
$$

The symbols $\Gamma_{ij}(\omega)$ are continuous functions of ω, and $\Gamma_{ij}(\omega) \equiv 0$ when $i + j$ is odd.

PROOF: This lemma is here to introduce $\Gamma_{ij}(\omega)$. The proof of the first statement is trivial ($\Gamma_{ij}(\omega)$ are known to be continuous only because the map $\omega \mapsto \boldsymbol{\phi}_\omega$ was assumed to be C^2, so that $\boldsymbol{e}_2'(\omega) = \partial_\omega^2\boldsymbol{\phi}_\omega$ is known to be continuous only in ω). The statement about vanishing of $\Gamma_{ij}(\omega) \equiv 0$ when $i+j$ is odd follows from the inclusions $\boldsymbol{e}_1(\omega), \boldsymbol{e}_3(\omega) \in X^-$ and $\boldsymbol{e}_2(\omega), \boldsymbol{e}_4(\omega) \in X^+$, valid for all $\omega \in \mathcal{O}(\omega_*)$. $\square$

Using the relation

$$
P_d(\omega)\dot{\boldsymbol{\rho}}_c(t) = P_d(\omega)\frac{d}{dt}(P_c(\omega)\boldsymbol{\rho}(t)) = \dot{\omega}P_d(\omega)P_c'(\omega)\boldsymbol{\rho} = -\dot{\omega}P_d(\omega)P_d'(\omega)\boldsymbol{\rho}
$$

and squinting at Lemma 4.1, we simplify expression (4.7) for $P_d(\omega(t))\dot{\boldsymbol{\rho}}(t)$ to the form

$$
P_d(\omega)\dot{\boldsymbol{\rho}}(t) = \boldsymbol{e}_j(\omega)\dot{\rho}_j(t) + \dot{\omega}\boldsymbol{e}_i(\omega)\Gamma_{ij}(\omega)\rho_j(t) - \dot{\omega}P_d(\omega)P_d'(\omega)\boldsymbol{\rho}\,.
\tag{4.8}
$$

Using (4.8), we project (4.3) onto $X_{d,\omega}$ at $\omega = \omega(t)$:

$$
\begin{aligned}
\boldsymbol{e}_j(\omega)\dot{\rho}_j &= \boldsymbol{e}_1(\omega)\rho_2 + \boldsymbol{e}_2(\omega)\rho_3 + \boldsymbol{e}_3(\omega)\rho_4 + a(\omega)\boldsymbol{e}_4(\omega)\rho_3 \\
&\quad - \dot{\omega}\boldsymbol{e}_2(\omega) + P_d(\omega)\boldsymbol{N}(\omega, \boldsymbol{\rho}) \\
&\quad + \dot{\omega}P_d(\omega)P_d'(\omega)\boldsymbol{\rho} - \dot{\omega}\boldsymbol{e}_i(\omega)\Gamma_{ij}(\omega)\rho_j\,.
\end{aligned}
\tag{4.9}
$$

Let $\{\boldsymbol{\xi}_i(\omega) \in X^* : 1 \le i \le 4, \omega \in \mathcal{O}(\omega_*)\}$ be the frame that is dual to $\{\boldsymbol{e}_i(\omega) : 1 \le i \le 4, \omega \in \mathcal{O}(\omega_*)\}$:

$$
\begin{aligned}
\langle \boldsymbol{\xi}_j(\omega), \boldsymbol{e}_i(\omega)\rangle &= \delta_{ij} \quad \forall \omega \in \mathcal{O}(\omega_*),\ 1 \le i, j \le 4, \\
\langle \boldsymbol{\xi}_i(\omega), \boldsymbol{u}\rangle &= 0 \qquad \forall \boldsymbol{u} \in X_{c,\omega}\,.
\end{aligned}
\tag{4.10}
$$

We use the frame $\{\boldsymbol{\xi}_j(\omega) : 1 \le j \le 4\}$ to rewrite equation (4.9) in the form of a system:

$$\begin{cases} \dot\rho_1 = \rho_2 + \langle \boldsymbol{\xi}_1(\omega), \boldsymbol{N}(\omega,\boldsymbol{\rho})\rangle + \dot\omega(\langle \boldsymbol{\xi}_1(\omega), P_d'(\omega)\boldsymbol{\rho}\rangle - \Gamma_{1j}(\omega)\rho_j) \\ \dot\rho_2 + \dot\omega = \rho_3 + \langle \boldsymbol{\xi}_2(\omega), \boldsymbol{N}(\omega,\boldsymbol{\rho})\rangle + \dot\omega(\langle \boldsymbol{\xi}_2(\omega), P_d'(\omega)\boldsymbol{\rho}\rangle - \Gamma_{2j}(\omega)\rho_j) \\ \dot\rho_3 = \rho_4 + \langle \boldsymbol{\xi}_3(\omega), \boldsymbol{N}(\omega,\boldsymbol{\rho})\rangle + \dot\omega(\langle \boldsymbol{\xi}_3(\omega), P_d'(\omega)\boldsymbol{\rho}\rangle - \Gamma_{3j}(\omega)\rho_j) \\ \dot\rho_4 = a(\omega)\rho_3 + \langle \boldsymbol{\xi}_4(\omega), \boldsymbol{N}(\omega,\boldsymbol{\rho})\rangle + \dot\omega(\langle \boldsymbol{\xi}_4(\omega), P_d'(\omega)\boldsymbol{\rho}\rangle - \Gamma_{4j}(\omega)\rho_j). \end{cases} \tag{4.11}$$

The summation in the repeated indices is assumed.

Using the relation $P_c(\omega)\dot{\boldsymbol{\rho}} = \dot{\boldsymbol{\rho}}_c - \dot\omega P_c'(\omega)\boldsymbol{\rho}$, we project (4.3) onto $X_{c,\omega}$ at $\omega = \omega(t)$:

$$\dot{\boldsymbol{\rho}}_c = JH_\omega \boldsymbol{\rho}_c + P_c\boldsymbol{N}(\omega,\boldsymbol{\rho}) + \dot\omega P_c'(\omega)\boldsymbol{\rho}\,.$$

Applying the projectors $\Pi^+ = \frac{1}{2}(1 + \mathbb{C})$ and $\Pi^- = \frac{1}{2}(1 - \mathbb{C})$, we get

$$\begin{cases} \dot{\boldsymbol{\rho}}_c^+ = JH_\omega\boldsymbol{\rho}_c^- + \Pi^+ P_c\boldsymbol{N}(\omega,\boldsymbol{\rho}) + \dot\omega P_c'(\omega)\boldsymbol{\rho}^+ \\ \dot{\boldsymbol{\rho}}_c^- = JH_\omega\boldsymbol{\rho}_c^+ + \Pi^- P_c\boldsymbol{N}(\omega,\boldsymbol{\rho}) + \dot\omega P_c'(\omega)\boldsymbol{\rho}^-, \end{cases} \tag{4.12}$$

where $\boldsymbol{\rho}^\pm \equiv \Pi^\pm\boldsymbol{\rho}$ and $\boldsymbol{\rho}_c^\pm \equiv \Pi^\pm\boldsymbol{\rho}_c$. We used the relations $\Pi^\pm JH_\omega = JH_\omega\Pi^\mp$ that follow from Corollary 3.4. Let us note that according to Corollary 3.6 the projections $\Pi^\pm$ commute with $P_c'(\omega)$.

5 Control of the Continuous Part of Perturbation

To proceed to the control of $\boldsymbol{\rho}_c$, we need the following important result.

PROPOSITION 5.1 *For $\omega \in \mathcal{O}(\omega_*)$, the quadratic form defined by H_ω is positive definite on $X_{c,\omega}$. Namely, there exists $C > 0$ so that*

$$\langle H_\omega\boldsymbol{\psi}, \boldsymbol{\psi}\rangle \ge C\|\boldsymbol{\psi}\|_X^2 \quad \forall \boldsymbol{\psi} \in X_{c,\omega}\,. \tag{5.1}$$

In the context of the nonlinear Schrödinger equation with particular nonlinearities, the analogous result is proven in [25, theorem 2.5]. In the general context, a similar idea appears in [8, theorem 5.2].

PROOF: Both H_ω and $X_{c,\omega}$ (the continuous subspace of JH_ω) depend continuously on ω. Therefore, since we may shrink $\mathcal{O}(\omega_*)$, it suffices to check that (5.1) holds at $\omega = \omega_*$.

There is a decomposition $X = \mathcal{N}_g(JH_\omega) \oplus \mathcal{N}_g^\perp(H_\omega J)$; here "$\perp$" denotes orthogonality with respect to the pairing $\langle\,\cdot\,,\cdot\,\rangle$. Hence, $X_{c,\omega} \subset \mathcal{N}_g^\perp(H_\omega J)$. At ω_*, where $X_{d,\omega_*} = \mathcal{N}_g(JH_{\omega_*})$, one has $X_{c,\omega_*} = \mathcal{N}_g^\perp(H_{\omega_*}J)$:

$$\langle \boldsymbol{\eta}, \boldsymbol{\psi}\rangle = (-1)^k\big\langle (H_{\omega_*}J)^k\boldsymbol{\eta}, (JH_{\omega_*}|_{X_{c,\omega_*}})^{-k}\boldsymbol{\psi}\big\rangle = 0 \tag{5.2}$$

for any $\boldsymbol{\psi} \in X_{c,\omega_*}$ and $\boldsymbol{\eta} \in \mathcal{N}_g(H_{\omega_*}J)$.

Since H_ω is self-adjoint, the corresponding spectral decomposition

$$X = \operatorname{span}\langle \boldsymbol{\chi}_\omega\rangle \oplus \operatorname{span}\langle \boldsymbol{e}_1(\omega)\rangle \oplus \mathscr{P}_\omega \tag{5.3}$$

is orthogonal with respect to $(\cdot, \cdot)$. Recall that $\boldsymbol{\chi}_\omega$ is the eigenvector that corresponds to the simple negative eigenvalue $-\Lambda_\omega^2$ of H_ω, and that $\mathscr{P}_\omega$ is the positive subspace of H_ω. We assume that $\|\boldsymbol{\chi}_\omega\|_X = 1$.

Let $\boldsymbol{e}_2(\omega_*) = a_0\boldsymbol{\chi}_{\omega_*} + b_0\boldsymbol{e}_1(\omega_*) + \boldsymbol{p}_0$, where $\boldsymbol{p}_0 \in \mathscr{P}_{\omega_*}$. Noting that $\boldsymbol{e}_1(\omega_*)$ is orthogonal to $\boldsymbol{e}_2(\omega_*)$ (due to the orthogonality of X^- and X^+ with respect to $(\cdot, \cdot)$) and to both $\boldsymbol{\chi}_{\omega_*}$ and $\boldsymbol{p}_0$ (due to orthogonality of decomposition (5.3)), we conclude that $b_0 = 0$. Since

$$
\begin{aligned}
0 = \left.\frac{dQ(\boldsymbol{\phi}_\omega)}{d\omega}\right|_{\omega_*} &= \langle B\boldsymbol{\phi}_\omega, \partial_\omega\boldsymbol{\phi}_\omega\rangle\big|_{\omega_*} = \langle J^{-1}\boldsymbol{e}_1(\omega_*), \boldsymbol{e}_2(\omega_*)\rangle \\
&= \langle H_{\omega_*}\boldsymbol{e}_2(\omega_*), \boldsymbol{e}_2(\omega_*)\rangle = -a_0^2\Lambda_{\omega_*}^2 + \langle H_{\omega_*}\boldsymbol{p}_0, \boldsymbol{p}_0\rangle \,,
\end{aligned}
$$

we learn that a_0 is different from zero, so that $(\boldsymbol{\chi}_{\omega_*}, \boldsymbol{e}_2(\omega_*)) = a_0 \neq 0$. Hence we may define the projector

$$
\pi(\boldsymbol{\psi}) = \boldsymbol{\psi} - \frac{(\boldsymbol{e}_1(\omega_*), \boldsymbol{\psi})}{\|\boldsymbol{e}_1(\omega_*)\|^2}\boldsymbol{e}_1(\omega_*) - \frac{(\boldsymbol{\chi}_{\omega_*}, \boldsymbol{\psi})}{(\boldsymbol{\chi}_{\omega_*}, \boldsymbol{e}_2(\omega_*))}\boldsymbol{e}_2(\omega_*) \,, \quad \boldsymbol{\psi} \in X \,,
$$

so that

$$
\ker\pi = \operatorname{span}\langle\boldsymbol{e}_1(\omega_*), \boldsymbol{e}_2(\omega_*)\rangle, \quad \operatorname{Range}(\pi) \subset \mathscr{P}_{\omega_*} \,.
$$

The last inclusion follows from (5.3) and from the identities

$$
(\boldsymbol{\chi}_{\omega_*}, \pi(\boldsymbol{\psi})) = 0 = (\boldsymbol{e}_1(\omega_*), \pi(\boldsymbol{\psi})) \,.
$$

We use spectral decomposition (5.3) corresponding to H_{ω_*} to express $\boldsymbol{\psi}$ as

$$
\boldsymbol{\psi} = a\boldsymbol{\chi}_{\omega_*} + b\boldsymbol{e}_1(\omega_*) + \boldsymbol{p} \,, \quad a, b \in \mathbb{R},\ \boldsymbol{p} \in \mathscr{P}_{\omega_*} \subset X \,. \tag{5.4}
$$

We have

$$
\langle H_{\omega_*}\boldsymbol{\psi}, \boldsymbol{\psi}\rangle = -a^2\Lambda_{\omega_*}^2 + \langle H_{\omega_*}\boldsymbol{p}, \boldsymbol{p}\rangle \geq -a^2\Lambda_{\omega_*}^2 + C\|\boldsymbol{p}\|_X^2 \tag{5.5}
$$

for some constant $C > 0$.

From now on, let $\boldsymbol{\psi} \in X_{c,\omega_*} = P_c(\omega_*)X$. Using the relations $H_{\omega_*}\boldsymbol{e}_1(\omega_*) = 0$ and $\langle H_{\omega_*}\boldsymbol{e}_2(\omega_*), \boldsymbol{\psi}\rangle = 0$ (valid due to (5.2) since $H_{\omega_*}\boldsymbol{e}_2(\omega_*) \in \mathscr{N}_g(H_{\omega_*}J)$ and $\boldsymbol{\psi} \in X_{c,\omega_*}$), and

$$
\langle H_{\omega_*}\boldsymbol{e}_2(\omega_*), \boldsymbol{e}_2(\omega_*)\rangle = \left.\frac{dQ(\boldsymbol{\phi}_\omega)}{d\omega}\right|_{\omega_*} = 0 \,,
$$

one concludes that $\langle H_{\omega_*}\boldsymbol{\psi}, \boldsymbol{\psi}\rangle = \langle H_{\omega_*}\pi(\boldsymbol{\psi}), \pi(\boldsymbol{\psi})\rangle$, and, because of the inclusion $\operatorname{Range}(\pi) \subset \mathscr{P}_{\omega_*}$,

$$
\langle H_{\omega_*}\boldsymbol{\psi}, \boldsymbol{\psi}\rangle = \langle H_{\omega_*}\pi(\boldsymbol{\psi}), \pi(\boldsymbol{\psi})\rangle \geq C\|\pi(\boldsymbol{\psi})\|_X^2 \tag{5.6}
$$

for the same constant $C > 0$ as in (5.5).

In the rest of the proof of Proposition 5.1, the vectors $\boldsymbol{e}_j(\omega)$ are considered at the point ω_*. For brevity, we will often write $\boldsymbol{e}_j$ instead of $\boldsymbol{e}_j(\omega_*)$. We take the pairing of $\pi(\boldsymbol{\psi})$ with $H_{\omega_*}\boldsymbol{e}_4 \in \mathscr{N}_g(H_{\omega_*}J)$:

$$
\langle H_{\omega_*}\boldsymbol{e}_4, \pi(\boldsymbol{\psi})\rangle = \langle H_{\omega_*}\boldsymbol{e}_4, \boldsymbol{\psi}\rangle - \frac{(\boldsymbol{e}_1, \boldsymbol{\psi})}{\|\boldsymbol{e}_1\|^2}\langle H_{\omega_*}\boldsymbol{e}_4, \boldsymbol{e}_1\rangle - \frac{(\boldsymbol{\chi}_{\omega_*}, \boldsymbol{\psi})}{(\boldsymbol{\chi}_{\omega_*}, \boldsymbol{e}_2)}\langle H_{\omega_*}\boldsymbol{e}_4, \boldsymbol{e}_2\rangle \,. \tag{5.7}
$$

The first term in the right-hand side vanishes due to (5.2) with $j = 3$. The second term also vanishes:

$$\langle H_{\omega_*}\boldsymbol{e}_4, \boldsymbol{e}_1\rangle = \langle H_{\omega_*}\boldsymbol{e}_4, JH_{\omega_*}\boldsymbol{e}_2\rangle = -\langle H_{\omega_*}\boldsymbol{e}_2, JH_{\omega_*}\boldsymbol{e}_4\rangle = -\langle H_{\omega_*}\boldsymbol{e}_2, \boldsymbol{e}_3\rangle$$
$$= -\langle H_{\omega_*}\boldsymbol{e}_3, \boldsymbol{e}_2\rangle = -\langle H_{\omega_*}\boldsymbol{e}_3, JH_{\omega_*}\boldsymbol{e}_3\rangle = 0\,.$$

Since

$$\langle H_{\omega_*}\boldsymbol{e}_4, \boldsymbol{e}_2\rangle = \langle H_{\omega_*}\boldsymbol{e}_2, \boldsymbol{e}_4\rangle = \langle B\boldsymbol{\phi}_{\omega_*}, \boldsymbol{e}_4\rangle < 0\,,$$

as follows from (3.11), the last term in the right-hand side of (5.7) provides the bound $\|\pi(\boldsymbol{\psi})\|_X \geq C|a|$ for some $C > 0$. From (5.6) we conclude that

$$\langle H_{\omega_*}\boldsymbol{\psi}, \boldsymbol{\psi}\rangle = \langle H_{\omega_*}\pi(\boldsymbol{\psi}), \pi(\boldsymbol{\psi})\rangle \geq Ca^2 \tag{5.8}$$

for some $C > 0$.

It remains to take into account the coefficient b in decomposition (5.4). For this, we evaluate the pairing of $\pi(\boldsymbol{\psi})$ with $J^{-1}\boldsymbol{e}_4 \in \mathcal{N}_g(H_{\omega_*}J)$:

$$\langle J^{-1}\boldsymbol{e}_4, \pi(\boldsymbol{\psi})\rangle = \langle J^{-1}\boldsymbol{e}_4, \boldsymbol{\psi}\rangle - \frac{(\boldsymbol{e}_1, \boldsymbol{\psi})}{\|\boldsymbol{e}_1\|^2}\langle J^{-1}\boldsymbol{e}_4, \boldsymbol{e}_1\rangle - \frac{(\boldsymbol{\chi}, \boldsymbol{\psi})}{(\boldsymbol{\chi}, \boldsymbol{e}_2)}\langle J^{-1}\boldsymbol{e}_4, \boldsymbol{e}_2\rangle\,. \tag{5.9}$$

The first term in the right-hand side vanishes due to (5.2) with $j = 4$. The last term in the right-hand side of (5.9) vanishes since

$$\langle J^{-1}\boldsymbol{e}_4, \boldsymbol{e}_2\rangle = \langle J^{-1}\boldsymbol{e}_4, (JH_{\omega_*})^2\boldsymbol{e}_4\rangle = -\langle H_{\omega_*}\boldsymbol{e}_4, JH_{\omega_*}\boldsymbol{e}_4\rangle = 0\,.$$

Due to (3.11), there is the inequality

$$\langle J^{-1}\boldsymbol{e}_4, \boldsymbol{e}_1\rangle = \langle J^{-1}\boldsymbol{e}_4, (JH_{\omega_*})^2\boldsymbol{e}_3\rangle = \langle H_{\omega_*}\boldsymbol{e}_3, \boldsymbol{e}_3\rangle < 0\,;$$

hence the second term in the right-hand side of (5.9) leads to the bound $\|\pi(\boldsymbol{\psi})\|_X \geq C|(\boldsymbol{e}_1, \boldsymbol{\psi})| = C\|\boldsymbol{e}_1\|_X^2|b|$ for some $C > 0$, and using (5.6) we obtain

$$\langle H_{\omega_*}\boldsymbol{\psi}, \boldsymbol{\psi}\rangle \geq Cb^2 \tag{5.10}$$

for some constant $C > 0$. The weighted arithmetic mean of (5.5), (5.8), and (5.10) results in the desired bound

$$\langle H_{\omega_*}\boldsymbol{\psi}, \boldsymbol{\psi}\rangle \geq C\|\boldsymbol{\psi}\|_X^2$$

for some $C > 0$. □

DEFINITION 5.2 Let l be a nonnegative integer from Assumption 2.10. We define

$$M_l(\omega) = (-1)^l(H_\omega J)^{2l}H_\omega\,.$$

We claim that on $X_{c,\omega}$ the quadratic form $\langle M_l(\omega)\,\cdot\,, \,\cdot\,\rangle$ defines a metric that is equivalent to $\|\cdot\|_{\mathcal{H}^l_\omega}$.

LEMMA 5.3 *For any nonnegative integer l there exist positive constants C_1 and C_2 so that for any $\omega \in \mathcal{O}(\omega_*)$*

$$|\langle M_l(\omega)\boldsymbol{\psi}, \boldsymbol{\psi}\rangle| \leq C_2\|\boldsymbol{\psi}\|^2_{\mathcal{H}^l_\omega} \quad \text{for any } \boldsymbol{\psi} \in \mathcal{H}^l_\omega\,, \tag{5.11}$$

$$\langle M_l(\omega)\boldsymbol{\psi}, \boldsymbol{\psi}\rangle \geq C_1\|\boldsymbol{\psi}\|^2_{\mathcal{H}^l_\omega} \quad \text{for any } \boldsymbol{\psi} \in \mathcal{H}^l_\omega \cap X_{c,\omega}\,, \tag{5.12}$$

where $X_{c,\omega}$ is the continuous spectral subspace that corresponds to JH_ω.

PROOF: Since we assumed that both $E''(\boldsymbol{\phi}_\omega)$ and $Q''(\boldsymbol{\phi}_\omega) = B$ are continuous from X to X^*, the quadratic form defined by $H_\omega = E''(\boldsymbol{\phi}_\omega) - \omega Q''(\boldsymbol{\phi}_\omega)$ is bounded everywhere on X so that $|\langle H_\omega \boldsymbol{\psi}, \boldsymbol{\psi}\rangle| \le C\|\boldsymbol{\psi}\|_X^2$, and we obtain the following bound:

$$\begin{aligned}|\langle M_l(\omega)\boldsymbol{\psi}, \boldsymbol{\psi}\rangle| &= |\langle H_\omega (JH_\omega)^l \boldsymbol{\psi}, (JH_\omega)^l \boldsymbol{\psi}\rangle| \\ &\le C\|(JH_\omega)^l \boldsymbol{\psi}\|_X^2 \le C\|\boldsymbol{\psi}\|_{\mathcal{H}_\omega^l}^2 \quad \forall \boldsymbol{\psi} \in X\,.\end{aligned}$$

We used the continuity of the map $(JH_\omega)^l : \mathcal{H}_\omega^l \to X$. This proves (5.11).

Let us now prove (5.12). Since $(JH_\omega)^l \boldsymbol{\psi}\rangle \in X_{c,\omega}$, Proposition 5.1 provides the following lower bound:

$$\langle M_l(\omega)\boldsymbol{\psi}, \boldsymbol{\psi}\rangle = \langle H_\omega (JH_\omega)^l \boldsymbol{\psi}, (JH_\omega)^l \boldsymbol{\psi}\rangle \ge C\|(JH_\omega)^l \boldsymbol{\psi}\|_X^2\,, \quad C > 0\,.$$

Since $0 \notin \sigma(JH_\omega\big|_{X_{c,\omega}})$, for $\boldsymbol{\psi} \in X_{c,\omega}$ the norm $\|(JH_\omega - \zeta)^l \boldsymbol{\psi}\|_X$ with $\zeta = 0$ is equivalent to $\|\boldsymbol{\psi}\|_{\mathcal{H}_\omega^l}$ (see Lemma A.1). This finishes the proof of (5.12). $\square$

We will use the notation $\|\boldsymbol{u}\|_{M_l}^2 = \langle M_l(\omega)\boldsymbol{u}, \boldsymbol{u}\rangle$, where

$$M_l(\omega) = (-1)^l (H_\omega J)^{2l} H_\omega\,.$$

PROPOSITION 5.4 *Let l be a nonnegative integer from Assumption* 2.10. *The following bounds are satisfied:*

$$\left|\frac{d}{dt}\|\boldsymbol{\rho}_c^+\|_{M_l}\right| \le C\big[\|\boldsymbol{\rho}^+\|_{\mathcal{H}_\omega^l}\|\boldsymbol{\rho}^-\|_{\mathcal{H}_\omega^l} + \|\boldsymbol{\rho}\|_{\mathcal{H}_\omega^l}^3 + \dot\omega\|\boldsymbol{\rho}^+\|_{\mathcal{H}_\omega^l}\big]\,, \tag{5.13}$$

$$\left|\frac{d}{dt}\|\boldsymbol{\rho}_c^-\|_{M_l}\right| \le C\big[\|\boldsymbol{\rho}\|_{\mathcal{H}_\omega^l}^2 + \dot\omega\|\boldsymbol{\rho}^-\|_{\mathcal{H}_\omega^l}\big]\,. \tag{5.14}$$

PROOF: Let us first deal with the growth of $\|\boldsymbol{\rho}_c^+\|$. We have

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{\rho}_c^+\|_{M_l}^2 &= \frac{d}{dt}\langle M_l(\omega)\boldsymbol{\rho}_c^+, \boldsymbol{\rho}_c^+\rangle \\ &= \dot\omega\langle M_l'(\omega)\boldsymbol{\rho}_c^+, \boldsymbol{\rho}_c^+\rangle + \langle M_l(\omega)\dot{\boldsymbol{\rho}}_c^+, \boldsymbol{\rho}_c^+\rangle + \langle M_l(\omega)\boldsymbol{\rho}_c^+, \dot{\boldsymbol{\rho}}_c^+\rangle\,.\end{aligned}$$

We use the first equation from system (4.12) to express $\dot{\boldsymbol{\rho}}_c^+$. The principal terms

$$\langle M_l(\omega) JH_\omega \boldsymbol{\rho}^+, \boldsymbol{\rho}^+\rangle + \langle M_l(\omega)\boldsymbol{\rho}^+, JH_\omega\boldsymbol{\rho}^+\rangle$$

add up to zero since

$$\langle M_l(\omega)\boldsymbol{\rho}^+, JH_\omega\boldsymbol{\rho}^+\rangle = -\langle H_\omega\boldsymbol{\rho}^+, JM_l(\omega)\boldsymbol{\rho}^+\rangle = -\langle H_\omega JM_l(\omega)\boldsymbol{\rho}^+, \boldsymbol{\rho}^+\rangle\,.$$

The first identity is due to skew symmetry of J, $\langle \boldsymbol{\xi}, J\boldsymbol{\eta}\rangle = -\langle \boldsymbol{\eta}, J\boldsymbol{\xi}\rangle$ for any $\boldsymbol{\xi}, \boldsymbol{\eta} \in X^*$, while the second identity is due to self-adjointness of H_ω, $\langle H_\omega \boldsymbol{u}, \boldsymbol{v}\rangle =$

$\langle H_\omega \boldsymbol{v}, \boldsymbol{u}\rangle$ for any $\boldsymbol{u},\ \boldsymbol{v} \in X$. Recall that the conjugation is understood with respect to the pairing $\langle\,\cdot\,,\cdot\,\rangle : X^* \times X \to \mathbb{R}$. We then arrive at the following equation:

$$\|\boldsymbol{\rho}_c^+\|_{M_l}\frac{d}{dt}\|\boldsymbol{\rho}_c^+\|_{M_l} = \frac{\dot\omega}{2}\langle M_l'(\omega)\boldsymbol{\rho}_c^+,\,\boldsymbol{\rho}_c^+\rangle + \langle M_l(\omega)\boldsymbol{\rho}_c^+,\,\Pi^+ P_c \boldsymbol{N}(\omega,\boldsymbol{\rho})\rangle + \dot\omega\langle M_l(\omega)\boldsymbol{\rho}_c^+,\,P_c'(\omega)\boldsymbol{\rho}^+\rangle\,, \tag{5.15}$$

where we used the self-adjointness of M_l (with respect to $\langle\,\cdot\,,\cdot\,\rangle$).

The term $\langle M_l'(\omega)\boldsymbol{\rho}_c^+,\,\boldsymbol{\rho}_c^+\rangle$ can be written as

$$\begin{aligned}\langle M_l'(\omega)\boldsymbol{\rho}_c^+,\,\boldsymbol{\rho}_c^+\rangle &= \langle(\partial_\omega H_\omega)(JH_\omega)^l\boldsymbol{\rho}_c^+,\,(JH_\omega)^l\boldsymbol{\rho}_c^+\rangle\\ &\quad+\langle H_\omega\partial_\omega(JH_\omega)^l\boldsymbol{\rho}_c^+,\,(JH_\omega)^l\boldsymbol{\rho}_c^+\rangle\\ &\quad+\langle H_\omega(JH_\omega)^l\boldsymbol{\rho}_c^+,\,\partial_\omega(JH_\omega)^l\boldsymbol{\rho}_c^+\rangle\,.\end{aligned}$$

Let us find bounds for the terms in the right-hand side. We know that the Hamiltonian operator $H_\omega = E''(\boldsymbol{\phi}_\omega) - \omega Q''(\boldsymbol{\phi}_\omega)$ defines a continuous map $X \to X^*$. The map $\partial_\omega H_\omega : X \to X^*$ is continuous due to Assumption 2.10(a). Therefore, the bilinear forms $\langle H_\omega\,\cdot\,,\,\cdot\,\rangle$ and $\langle\partial_\omega H_\omega\,\cdot\,,\,\cdot\,\rangle$ are continuous (bounded by $C\|\cdot\|_X\,\|\cdot\|_X$) everywhere on $X\times X$. As follows from Definition 2.9, the operator $(JH_\omega)^l$ is continuous from $\mathcal{H}^l_\omega$ to X, and, according to Assumption 2.10(a), so is $\partial_\omega(JH_\omega)^l$. We conclude that the first term in the right-hand side of (5.15) is bounded by

$$C|\dot\omega|\,\|\boldsymbol{\rho}_c^+\|^2_{\mathcal{H}^l_\omega}\,. \tag{5.16}$$

Since the quadratic form $\langle M_l\,\cdot\,,\,\cdot\,\rangle$ is positive definite on $X_{c,\omega}$, the second term in the right-hand side of (5.15) is bounded by

$$\left|\langle M_l(\omega)\boldsymbol{\rho}_c^+,\,\Pi^+P_c\boldsymbol{N}(\omega,\boldsymbol{\rho})\rangle\right| \le C\|\boldsymbol{\rho}_c^+\|_{M_l}\|\Pi^+P_c\boldsymbol{N}(\omega,\boldsymbol{\rho})\|_{M_l}\,. \tag{5.17}$$

From Lemma 3.2, the last term in the right-hand side of (5.15) is bounded by

$$\begin{aligned}\left|\dot\omega\langle M_l(\omega)\boldsymbol{\rho}_c^+,\,P_c'(\omega)\boldsymbol{\rho}^+\rangle\right| &\le C|\dot\omega|\,\|\boldsymbol{\rho}_c^+\|_{M_l}\|P_c'(\omega)\boldsymbol{\rho}^+\|_{M_l}\\ &\le C|\dot\omega|\,\|\boldsymbol{\rho}_c^+\|_{\mathcal{H}^l_\omega}\|\boldsymbol{\rho}^+\|_{\mathcal{H}^l_\omega}\,.\end{aligned} \tag{5.18}$$

Collecting bounds (5.16), (5.17), and (5.18), we rewrite (5.15) as

$$\left|\frac{d}{dt}\|\boldsymbol{\rho}_c^+\|_{M_l}\right| \le \|\Pi^+P_c\boldsymbol{N}(\omega,\boldsymbol{\rho})\|_{\mathcal{H}^l_\omega} + C\dot\omega\|\boldsymbol{\rho}^+\|_{\mathcal{H}^l_\omega}\,. \tag{5.19}$$

To cancel the factor of $\|\boldsymbol{\rho}_c^+\|_{M_l}$, we used the bound $\|\boldsymbol{\rho}_c^+\|_{\mathcal{H}^l_\omega} \le C_1^{-1}\|\boldsymbol{\rho}_c^+\|_{M_l}$ that follows from (5.12).

According to Assumption 2.10(b), $\|\boldsymbol{N}(\omega,\boldsymbol{\rho})\|_{\mathcal{H}^l_\omega} \le C\|\boldsymbol{\rho}\|^2_{\mathcal{H}^l_\omega}$, so that

$$\left|\frac{d}{dt}\|\boldsymbol{\rho}_c^+\|_{M_l}\right| \le C\big(\|\boldsymbol{\rho}\|^2_{\mathcal{H}^l_\omega} + \dot\omega\|\boldsymbol{\rho}^+\|_{\mathcal{H}^l_\omega}\big)\,. \tag{5.20}$$

Similarly, one derives the bound on the derivative of $\|\boldsymbol{\rho}_c^-\|_{M_l}$:

$$\left|\frac{d}{dt}\|\boldsymbol{\rho}_c^-\|_{M_l}\right| \le C\big(\|\boldsymbol{\rho}\|^2_{\mathcal{H}^l_\omega} + \dot\omega\|\boldsymbol{\rho}^-\|_{\mathcal{H}^l_\omega}\big)\,. \tag{5.21}$$

Bound (5.21) agrees with (5.14). We need to show that there is an improvement of bound (5.20), which is a consequence of the symmetry of the energy functional under the action of C.

LEMMA 5.5 *For any $\omega \in \mathcal{O}(\omega_*)$, the restriction of the map*

$$\Pi^+ J E'''(\boldsymbol{\phi}_\omega) : \mathcal{H}^l_\omega \times \mathcal{H}^l_\omega \to X$$

onto $(\mathcal{H}^l_\omega \cap X^+) \times (\mathcal{H}^l_\omega \cap X^+)$ or $(\mathcal{H}^l_\omega \cap X^-) \times (\mathcal{H}^l_\omega \cap X^-)$ is identically zero.

PROOF: Similarly to how we derived (3.3), we deduce

$$\langle E'''(\boldsymbol{\phi}_\omega)(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}\rangle = \langle \mathrm{C}^* E'''(\boldsymbol{\phi}_\omega)(\mathrm{C}\boldsymbol{u}, \mathrm{C}\boldsymbol{v}), \boldsymbol{w}\rangle \,.$$

Note that due to Assumption 2.10(c), both sides make sense for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}^l_\omega$ and $\boldsymbol{w} \in X$. Therefore,

$$(J E'''(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}) = (J \mathrm{C}^* E'''(\mathrm{C}\boldsymbol{u}, \mathrm{C}\boldsymbol{v}), \boldsymbol{w}) = -(J E'''(\mathrm{C}\boldsymbol{u}, \mathrm{C}\boldsymbol{v}), \mathrm{C}\boldsymbol{w})\,,$$

where E''' is evaluated at the point $\boldsymbol{\phi}_\omega$. This identity shows, in particular, that for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}^l_\omega \cap X^+$ (or, similarly, $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}^l_\omega \cap X^-$) and $\boldsymbol{w} \in X^+$,

$$(J E'''(\boldsymbol{\phi}_\omega)(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}) = 0\,.$$

Since $\Pi^+ \boldsymbol{w} = \boldsymbol{w}$ and the projector Π^+ is self-adjoint, we conclude that

$$\Pi^+ J E'''(\boldsymbol{\phi}_\omega)(\boldsymbol{u}, \boldsymbol{v}) = 0$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}^l_\omega \cap X^+$ (or $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}^l_\omega \cap X^-$). □

The bound stated in Assumption 2.10(b) can be refined:

LEMMA 5.6 *For any $\omega \in \mathcal{O}(\omega_*)$ and any $\boldsymbol{\rho} \in \mathcal{H}^l_\omega$ with $\|\boldsymbol{\rho}\|_{\mathcal{H}^l_\omega} \le 1$,*

$$\|\Pi^+ \boldsymbol{N}(\omega, \boldsymbol{\rho})\|_{\mathcal{H}^l_\omega} \le 2c \|\boldsymbol{\rho}^+\|_{\mathcal{H}^l_\omega} \|\boldsymbol{\rho}^-\|_{\mathcal{H}^l_\omega} + c' \|\boldsymbol{\rho}\|^3_{\mathcal{H}^l_\omega}\,,$$

where $\boldsymbol{N}(\omega, \boldsymbol{\rho}) = J(E'(\boldsymbol{\phi}_\omega + \boldsymbol{\rho}) - E'(\boldsymbol{\phi}_\omega) - E''(\boldsymbol{\phi}_\omega)\boldsymbol{\rho})$ and the constants c and c' are the same as in Assumption 2.10.

PROOF: We bound $\|\Pi^+ \boldsymbol{N}(\omega, \boldsymbol{\rho})\|_{\mathcal{H}^l_\omega}$ by

$$\left\|\Pi^+ \left(\boldsymbol{N}(\omega, \boldsymbol{\rho}) - \frac{1}{2} J E'''(\boldsymbol{\phi}_\omega)(\boldsymbol{\rho}, \boldsymbol{\rho})\right)\right\|_{\mathcal{H}^l_\omega} + \frac{1}{2} \|\Pi^+ J E'''(\boldsymbol{\phi}_\omega)(\boldsymbol{\rho}, \boldsymbol{\rho})\|_{\mathcal{H}^l_\omega}\,.$$

As follows from Assumption 2.10(c), the first term is bounded by $c' \|\boldsymbol{\rho}\|^3_{\mathcal{H}^l_\omega}$. Due to Lemma 5.5, the second term is bounded by

$$\frac{1}{2} \|\Pi^+ J E'''(\boldsymbol{\phi}_\omega)(\boldsymbol{\rho}, \boldsymbol{\rho})\|_{\mathcal{H}^l_\omega} \le \|J E'''(\boldsymbol{\phi}_\omega)(\boldsymbol{\rho}^+, \boldsymbol{\rho}^-)\|_{\mathcal{H}^l_\omega} \le 2c \|\boldsymbol{\rho}^+\|_{\mathcal{H}^l_\omega} \|\boldsymbol{\rho}^-\|_{\mathcal{H}^l_\omega}\,.$$

In the last inequality we used the bound

$$\|J E'''(\boldsymbol{\phi}_\omega)(\boldsymbol{u}, \boldsymbol{v})\|_{\mathcal{H}^l_\omega} \le 2c \|\boldsymbol{u}\|_{\mathcal{H}^l_\omega} \|\boldsymbol{v}\|_{\mathcal{H}^l_\omega}\,,$$

valid for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}^l_\omega$, which follows from Assumption 2.10(b) and (c). This finishes the proof of Lemma 5.6. □

Lemma 5.6, together with (5.19), proves inequality (5.13). This finishes the proof of Proposition 5.4. □

6 Growth of Perturbation

The system on $\rho_j(t)$, $1 \le j \le 4$, $\omega(t)$, $\boldsymbol{\rho}_c^+$, and $\boldsymbol{\rho}_c^-$, which is a union of equations from (4.11) and (4.12), is underdetermined. We claim that one can find a solution such that $\rho_2(t) \equiv 0$. We rewrite system (4.11), considering ρ_j as functions of ω and substituting $\rho_2 \equiv 0$:

$$(6.1)\qquad \begin{cases} \dfrac{d\rho_1}{d\omega} = \dfrac{1}{\dot\omega}\langle \boldsymbol{\xi}_1, \boldsymbol{N}\rangle + \langle \boldsymbol{\xi}_1, P_d'(\omega)\boldsymbol{\rho}\rangle - \displaystyle\sum_{j=1,3} \Gamma_{1j}\rho_j \\ \dfrac{d\rho_3}{d\omega} = \dfrac{1}{\dot\omega}(\rho_4 + \langle \boldsymbol{\xi}_3, \boldsymbol{N}\rangle) + \langle \boldsymbol{\xi}_3, P_d'(\omega)\boldsymbol{\rho}\rangle - \displaystyle\sum_{j=1,3} \Gamma_{3j}\rho_j \\ \dfrac{d\rho_4}{d\omega} = \dfrac{1}{\dot\omega}(a(\omega)\rho_3 + \langle \boldsymbol{\xi}_4, \boldsymbol{N}\rangle) + \langle \boldsymbol{\xi}_4, P_d'(\omega)\boldsymbol{\rho}\rangle - \Gamma_{44}\rho_4, \end{cases}$$

where we remembered that the Christoffel symbols $\Gamma_{ij}(\omega)$ vanish when $i+j$ is odd (see Lemma 4.1). Let us remember that $\boldsymbol{\xi}_j = \boldsymbol{\xi}_j(\omega)$ is the frame dual to $\boldsymbol{e}_j(\omega)$ (see (4.10)), and $\boldsymbol{N} = \boldsymbol{N}(\omega, \boldsymbol{\rho})$ is the nonlinear term defined by (4.5). The expression for $\dot\omega$ comes from the second equation in (4.11):

$$(6.2)\qquad \dot\omega = \frac{\rho_3 + \langle \boldsymbol{\xi}_2(\omega), \boldsymbol{N}(\omega, \boldsymbol{\rho})\rangle}{1 - \langle \boldsymbol{\xi}_2(\omega), P_d'(\omega)\boldsymbol{\rho}\rangle + \Gamma_{24}(\omega)\rho_4}.$$

Remark 6.1. Since ρ_2 is a component of $\boldsymbol{\rho}_d$ in the direction of $\boldsymbol{e}_2(\omega) = \boldsymbol{\phi}_\omega'$, the condition that $\rho_2(t)$ be equal to 0 means that we are adjusting $\omega(t)$ so that at the moment t the solution $\boldsymbol{u}(t) = T(\int_0^t \omega(t')dt')(\boldsymbol{\phi}_{\omega(t)} + \boldsymbol{\rho}(t))$ passes, in a certain sense, near the orbit spanned by $\boldsymbol{\phi}_{\omega(t)}$.

Let us write down the principal part of the equations from (6.1), neglecting all nonlinear terms and terms with Γ_{ij} and $P_d'(\omega)$ (the latter group of terms appeared in (4.9) due to the dependence of ω on time, which is expected to be a higher-order effect):

$$(6.3)\qquad \begin{cases} \dfrac{d\rho_1}{d\omega} = 0 \\ \dfrac{d\rho_3}{d\omega} = \dfrac{1}{\dot\omega}\rho_4 \\ \dfrac{d\rho_4}{d\omega} = \dfrac{1}{\dot\omega}a(\omega)\rho_3. \end{cases}$$

In our approximation, (6.2) becomes $\dot\omega = \rho_3$. We illustrate the behavior of solutions to (6.3) in Section 6.1. In Section 6.2, we derive bounds on particular solutions to (6.3). We analyze the error terms in Section 6.3, proving that the solution to the complete system (4.11) and (4.12) with conveniently chosen initial data remains close to the solution to the reduced system (6.3) as long as ω remains in a small open neighborhood of ω_*. We use this result for the proof of Theorem 2.14 in Section 6.4.

6.1 Analysis of the "Normal Form" Equation

System (6.3) is equivalent to the following scalar equation:

$$\dddot{\omega} = a(\omega)\dot{\omega} \,. \tag{6.4}$$

We call this scalar equation the "normal form" for instability of standing waves of minimal energy by analogy with the central manifold reduction theorem. Simple analysis of (6.4) shows that the time evolution of the system leads to unbounded growth of $\omega(t)$ in $\mathcal{O}(\omega_*)$.

LEMMA 6.2 *Assume that the differentiable function $a(\omega)$ is nonnegative for $\omega \geq \omega_*$ in $\omega \in \mathcal{O}(\omega_*)$ with $a(\omega_*) = 0$. There exists $\varepsilon > 0$ such that for any $\delta > 0$ and any solution $\omega(t)$ such that*

$$0 < \omega(0) - \omega_* < \delta \,, \quad 0 < \dot{\omega}(0) \leq o(\delta) \,, \quad 0 \leq \ddot{\omega}(0) \leq o(\delta) \,, \tag{6.5}$$

there exists $t_1 = t_1(\delta, \varepsilon) > 0$ so that $\omega(t_1) - \omega_* > \varepsilon$.

PROOF: The proof follows immediately from the two integrations of the normal form equation (6.4):

$$\ddot{\omega}(t) = \int_{\omega(0)}^{\omega(t)} a(\omega')d\omega' + \ddot{\omega}(0) \,, \tag{6.6}$$

$$\frac{\dot{\omega}^2(t)}{2} = \int_{\omega(0)}^{\omega(t)} d\omega' \int_{\omega(0)}^{\omega'} a(\omega'')d\omega'' + \ddot{\omega}(0)(\omega(t) - \omega(0)) + \frac{\dot{\omega}^2(0)}{2} \,, \tag{6.7}$$

where the constants of integration are related to the initial data. Under the condition that $a(\omega)$ is nonnegative for $\omega \geq \omega_*$, the function $\omega(t)$ increases in time t for the initial data (6.5). □

The phase portrait for system (6.4) in the plane $(\omega, \dot{\omega})$ is shown in Figure 6.1.

COROLLARY 6.3 *If $a(\omega) = 0$ for $\omega \in \mathscr{I} \subset \mathcal{O}(\omega_*)$, the function $\omega(t)$ grows parabolically as*

$$\omega(t) = \omega(0) + \dot{\omega}(0)t + \frac{1}{2}\ddot{\omega}(0)t^2 \,. \tag{6.8}$$

Remark 6.4. Relation (3.17) between $a(\omega)$ and $\frac{d}{d\omega}Q(\boldsymbol{\phi}_\omega)$ shows that $a(\omega)$ is positive for $\omega > \omega_*$ if

$$\frac{d}{d\omega}Q(\boldsymbol{\phi}_\omega) = -d''(\omega) \,.$$

also is. Therefore, the statement of Lemma 6.2 is in agreement with Theorem 2.14. The instability evolves as the gradient descent towards the family of unstable standing waves with $\omega > \omega_*$.

6.2 Asymptotics for the Discrete Part of Perturbation

We rewrite the nontrivial part of system (6.3) with $\dot{\omega} = \rho_3$:

$$\frac{d\rho_3}{d\omega} = \frac{\rho_4}{\rho_3}, \qquad \frac{d\rho_4}{d\omega} = a(\omega). \tag{6.9}$$

LEMMA 6.5 *Let ω_* be such that $a(\omega_*) = 0$. Assume that $a(\omega) \in C^1(\mathbb{R})$ is strictly positive and nondecreasing for $\omega > \omega_*$, $\omega \in \mathcal{O}(\omega_*)$. Fix $\omega_0 > \omega_*$, $\omega_0 \in \mathcal{O}(\omega_*)$. Let $(R_3(\omega), R_4(\omega))$ be a solution to* (6.9) *with the initial data at ω_0 such that $R_3(\omega_0) > 0$ and $R_4(\omega_0) > 0$ satisfy*

$$R_3^2(\omega_0) \le 2(\omega_0 - \omega_*) R_4(\omega_0), \tag{6.10}$$

$$R_4^2(\omega_0) \le a(\omega_0) R_3^2(\omega_0). \tag{6.11}$$

Then for $\omega \ge \omega_0$, $\omega \in \mathcal{O}(\omega_*)$, the functions $R_3(\omega)$ and $R_4(\omega)$ are strictly positive, strictly increasing, differentiable, and satisfy

$$R_3^2(\omega) \le 2(\omega - \omega_*) R_4(\omega), \tag{6.12}$$

$$R_4^2(\omega) \le a(\omega) R_3^2(\omega), \tag{6.13}$$

for $\omega \ge \omega_0$, $\omega \in \mathcal{O}(\omega_*)$.

PROOF: Since we assumed that $a(\omega)$ is strictly positive for $\omega > \omega_*$, $\omega \in \mathcal{O}(\omega_*)$, $R_3(\omega_0) > 0$, and $R_4(\omega_0) \ge 0$, the statement about differentiability, monotonicity, and positiveness of R_3 and R_4 for $\omega \ge \omega_0$ follows from equations (6.9).

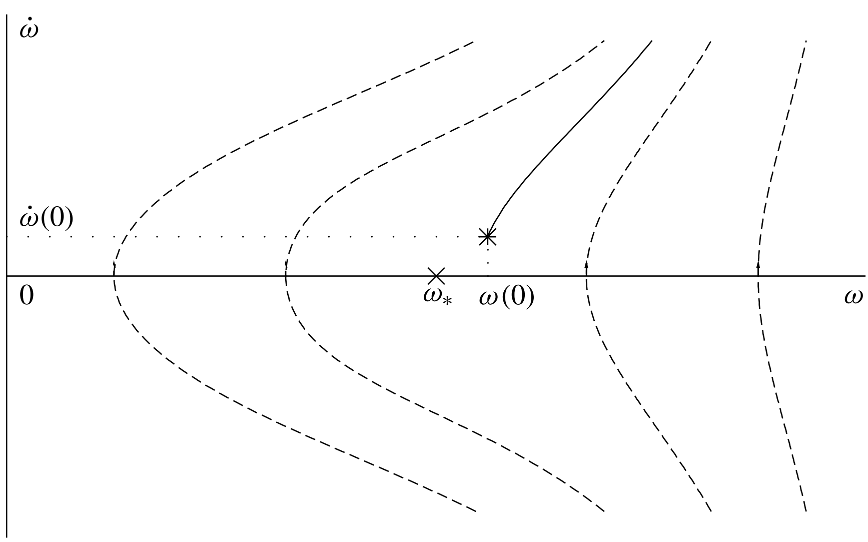

FIGURE 6.1. Phase plane $(\omega, \dot{\omega})$ for system (6.7) in the case of nonnegative $a(\omega)$ for $\omega \ge \omega_*$.

Inequality (6.12) is valid for $\omega > \omega_0$ since it is satisfied at $\omega = \omega_0$ and it only strengthens with the growth of ω:

$$\frac{d}{d\omega}\left(2(\omega-\omega_*)R_4(\omega) - R_3^2(\omega)\right) = 2(\omega-\omega_*)a(\omega) > 0\,,$$

where we used equations (6.9). Inequality (6.13) also strengthens for $\omega > \omega_0$:

$$\frac{d}{d\omega}\left(a(\omega)R_3^2(\omega) - R_4^2(\omega)\right) = a'(\omega)R_3^2(\omega) \geq 0\,.$$

The right-hand side is nonnegative since $a(\omega)$ was assumed to be nondecreasing for $\omega > \omega_*$, $\omega \in \mathcal{O}(\omega_*)$. □

Remark 6.6. A trivial manipulation with (6.12) and (6.13) yields the following bounds for $\omega \geq \omega_0$ and $\omega \in \mathcal{O}(\omega_*)$:

$$a^{-\frac{1}{2}}(\omega)R_4(\omega) \leq R_3(\omega) \leq 2(\omega-\omega_*)a^{\frac{1}{2}}(\omega)\,. \tag{6.14}$$

$$R_4(\omega) \leq 2(\omega-\omega_*)a(\omega)\,. \tag{6.15}$$

Let us also mention that choosing initial conditions that satisfy (6.12) and (6.13) at ω_0 is always possible. We may first choose $R_4(\omega_0)$ complying with (6.15) at ω_0; according to (6.15), we will be able to choose $R_3(\omega_0)$ complying with both inequalities in (6.14).

For the reader's convenience, we give the asymptotics of a particular solution that satisfies conditions (6.10) and (6.11) of Lemma 6.5 for the case when all the derivatives of $Q(\boldsymbol{\phi}_\omega)$ of order up to $k \geq 1$ vanish at ω_* and

$$c_k \equiv \frac{1}{(k+1)!}\frac{d^{k+1}}{d\omega^{k+1}}Q(\omega)\bigg|_{\omega_*} > 0\,.$$

In this case, there is a solution

$$R_3(\omega) \sim \left(\frac{2c_k}{(k+2)m}\right)^{\frac{1}{2}}(\omega-\omega_*)^{\frac{k}{2}+1}\,, \qquad R_4 \sim \frac{c_k}{m}(\omega-\omega_*)^{k+1}\,,$$

where $m \equiv -\langle B\boldsymbol{\phi}_{\omega_*}, \boldsymbol{e}_4(\omega_*)\rangle > 0$ according to Lemma 3.10.

Using $\dot{\omega}(t) \approx \rho_3(t) \approx R_3(\omega(t))$, as follows from (6.2), we also obtain

$$\omega(t) - \omega_* \sim \left((\omega_0-\omega_*)^{-\frac{k}{2}} - \frac{kt}{2}\left(\frac{2c_k}{(k+2)m}\right)^{\frac{1}{2}}\right)^{-\frac{2}{k}}\,,$$

where $\omega_0 = \omega(0)$ could be chosen arbitrarily close to ω_*. The asymptotics are valid as long as $\omega(t) - \omega_* \ll 1$.

6.3 Control of the Error

We are ready for the culmination: We are going to prove that $\rho_j(t)$, $j = 1, 3, 4$, $\|\boldsymbol{\rho}_c^+\|_{\mathcal{H}^l_\omega}$, and $\|\boldsymbol{\rho}_c^-\|_{\mathcal{H}^l_\omega}$ grow slower than $\omega(t) - \omega_*$, with $\rho_3(t)$ and $\rho_4(t)$ essentially equal to $R_3(\omega(t))$ and $R_4(\omega(t))$ (according to (6.14) and (6.15), both $R_3(\omega)$ and $R_4(\omega)$ are $o(\omega - \omega_*)$).

We will use the following induction argument:

LEMMA 6.7 *Assume that $a(\omega)$ is strictly positive and nondecreasing for $\omega > \omega_*$ in $\mathcal{O}(\omega_*)$ and $a(\omega_*) = 0$. Let κ_1, κ_3, κ_4, and $\kappa_c^\pm$ be constants such that*

$$\kappa_1 > 0\,, \quad \kappa_3 \in \left(0, \frac{2}{3}\right), \quad \kappa_4 \in \left(0, 2\kappa_3 - 3\kappa_3^2\right), \quad \kappa_c^\pm > 0\,,$$

and assume that $\mathcal{O}(\omega_*)$ is sufficiently small. Let

$$\rho_1 = r_1\,, \quad \rho_3 = R_3(\omega) + r_3\,, \quad \rho_3 = R_3(\omega) + r_3\,,$$

where $R_3(\omega)$ and $R_4(\omega)$ are solutions to (6.9) *that satisfy conditions* (6.10) *and* (6.11) *in Lemma* 6.5. *If at some point ω in $\mathcal{O}(\omega_*)$ one has inequalities*

$$|r_1| \le \kappa_1 R_3(\omega)\,, \tag{6.16}$$

$$|r_3| \le \kappa_3 R_3(\omega)\,, \qquad |r_4| \le \kappa_4 R_4(\omega)\,, \tag{6.17}$$

$$\|\Pi^+ \boldsymbol{\rho}_c\|_{M_l} \le \kappa_c^+ R_4(\omega)\,, \quad \|\Pi^- \boldsymbol{\rho}_c\|_{M_l} \le \kappa_c^- R_3(\omega)\,, \tag{6.18}$$

then the following bounds are satisfied:

$$\left|\frac{dr_1}{d\omega}\right| < \kappa_1 R_3'(\omega) = \kappa_1 \frac{R_4(\omega)}{R_3(\omega)}\,, \tag{6.19}$$

$$\left|\frac{d}{d\omega}(r_3 R_3(\omega))\right| < \kappa_3 \left(R_3^2(\omega)\right)' = 2\kappa_3 R_4(\omega)\,, \tag{6.20}$$

$$\left|\frac{dr_4}{d\omega}\right| < \kappa_4 R_4'(\omega) = \kappa_4 a(\omega)\,, \tag{6.21}$$

$$\left|\frac{d}{d\omega}\|\Pi^+ \boldsymbol{\rho}_c\|_{M_l}\right| < \kappa_c^+ R_4'(\omega) = \kappa_c^+ a(\omega)\,, \tag{6.22}$$

$$\left|\frac{d}{d\omega}\|\Pi^- \boldsymbol{\rho}_c\|_{M_l}\right| < \kappa_c^- R_3'(\omega) = \kappa_c^- \frac{R_4(\omega)}{R_3(\omega)}\,. \tag{6.23}$$

According to Lemma 6.7, if the bounds (6.16), (6.17), and (6.18) are valid at ω_0, then they are also valid for all $\omega > \omega_0$, $\omega \in \mathcal{O}(\omega_*)$.

Remark 6.8. Lemma 6.7 proves the existence of particular solutions to (4.11) and (4.12): We first find the approximate solution $(R_3(\omega), R_4(\omega))$, then use (4.11), (4.12), and (6.2) to express r_1, r_3, r_4, and $\boldsymbol{\rho}_c^\pm$ in terms of ω (bounds stated in Lemma 6.7 apply as long as $\omega \in \mathcal{O}(\omega_*)$), and then use (6.2) to express ω in terms of time.

PROOF: We rewrite (6.1) in terms of known functions $R_3(\omega)$, $R_4(\omega)$, and unknowns r_j, considered as functions of ω. We retain the notation ρ_j in marginal terms.

$$(6.24)\quad \begin{cases} \dfrac{dr_1}{d\omega} = \dfrac{1-\langle \boldsymbol{\xi}_2, P_d'\boldsymbol{\rho}\rangle + \Gamma_{24}\rho_4}{R_3(\omega)+r_3+\langle \boldsymbol{\xi}_2, \boldsymbol{N}\rangle}\langle \boldsymbol{\xi}_1, \boldsymbol{N}\rangle + \langle \boldsymbol{\xi}_1, P_d'(\omega)\boldsymbol{\rho}\rangle - \displaystyle\sum_{j=1,3}\Gamma_{1j}\rho_j \\ \dfrac{dr_3}{d\omega} = \dfrac{1-\langle \boldsymbol{\xi}_2, P_d'(\omega)\boldsymbol{\rho}\rangle + \Gamma_{24}\rho_4}{R_3(\omega)+r_3+\langle \boldsymbol{\xi}_2, \boldsymbol{N}\rangle}(R_4(\omega)+r_4+\langle \boldsymbol{\xi}_3, \boldsymbol{N}\rangle) \\ \qquad + \langle \boldsymbol{\xi}_3, P_d'(\omega)\boldsymbol{\rho}\rangle - \displaystyle\sum_{j=1,3}\Gamma_{3j}\rho_j - \dfrac{R_4(\omega)}{R_3(\omega)} \\ \dfrac{dr_4}{d\omega} = \dfrac{1-\langle \boldsymbol{\xi}_2, P_d'\boldsymbol{\rho}\rangle + \Gamma_{24}\rho_4}{R_3(\omega)+r_3+\langle \boldsymbol{\xi}_2, \boldsymbol{N}\rangle}(a(\omega)(R_3(\omega)+r_3)+\langle \boldsymbol{\xi}_4, \boldsymbol{N}\rangle) \\ \qquad + \langle \boldsymbol{\xi}_4, P_d'(\omega)\boldsymbol{\rho}\rangle - \Gamma_{44}\rho_4 - a(\omega)\,. \end{cases}$$

In this system, $\boldsymbol{\xi}_j = \boldsymbol{\xi}_j(\omega)$, $\Gamma_{ij} = \Gamma_{ij}(\omega)$, $\boldsymbol{N} = \boldsymbol{N}(\omega, \boldsymbol{\rho})$,

$$\boldsymbol{\rho} = \sum_{j=1,3,4} \boldsymbol{e}_j(\omega)\rho_j + \boldsymbol{\rho}_c\,,$$

$\rho_1 = r_1$, $\rho_3 = R_3(\omega) + r_3$, and $\rho_4 = R_4(\omega) + r_4$.

Since $\boldsymbol{\rho}^+ = \boldsymbol{e}_4(\omega)\rho_4 + \boldsymbol{\rho}_c^+$ and $\boldsymbol{\rho}^- = \boldsymbol{e}_1(\omega)\rho_1 + \boldsymbol{e}_3(\omega)\rho_3 + \boldsymbol{\rho}_c^-$, there are the bounds

$$\|\boldsymbol{\rho}^+\|_{\mathcal{H}_\omega^l} \le C(|\rho_4| + \|\boldsymbol{\rho}_c^+\|_{M_l})\,,$$
$$\|\boldsymbol{\rho}^-\|_{\mathcal{H}_\omega^l} \le C(|\rho_1| + |\rho_3| + \|\boldsymbol{\rho}_c^-\|_{M_l})\,.$$

We used the bounds $\|\boldsymbol{\rho}_c^+\|_{\mathcal{H}_\omega^l} \le C\|\boldsymbol{\rho}_c^+\|_{M_l}$ and $\|\boldsymbol{\rho}_c^-\|_{\mathcal{H}_\omega^l} \le C\|\boldsymbol{\rho}_c^-\|_{M_l}$ that follow from (5.12), as well as the bounds $\|\boldsymbol{e}_j(\omega)\|_{\mathcal{H}_\omega^l} < \infty$, $1 \le j \le 4$, that follow from Lemma 3.9. According to the assumptions of Lemma 6.7,

$$\|\boldsymbol{\rho}^-\|_{\mathcal{H}_\omega^l} \le CR_3(\omega)\,, \quad \|\boldsymbol{\rho}^+\|_{\mathcal{H}_\omega^l} \le CR_4(\omega)\,.$$

Taking into account inequality (6.13), we conclude that

$$\|\boldsymbol{\rho}\|_{\mathcal{H}_\omega^l} \le CR_3(\omega)\,.$$

According to Assumption 2.10(b), there are the bounds

$$\|\boldsymbol{N}(\omega, \boldsymbol{\rho})\|_{\mathcal{H}_\omega^l} \le C\|\boldsymbol{\rho}\|^2_{\mathcal{H}_\omega^l} \le CR_3^2(\omega)$$

so that

$$(6.25)\qquad |\langle \boldsymbol{\xi}_j(\omega), \boldsymbol{N}(\omega, \boldsymbol{\rho})\rangle| \le CR_3^2(\omega)\,, \quad 1 \le j \le 4\,.$$

There is a refinement of this bound for $j = 2, 4$. Indeed, for $j = 2, 4$, $\boldsymbol{e}_j(\omega) \in X^+$, so that $\langle \boldsymbol{\xi}_j(\omega), \boldsymbol{N}(\omega, \boldsymbol{\rho})\rangle = \langle \boldsymbol{\xi}_j(\omega), \Pi^+\boldsymbol{N}(\omega, \boldsymbol{\rho})\rangle$, and Lemma 5.6 yields

$$\left|\langle \boldsymbol{\xi}_j(\omega), \boldsymbol{N}(\omega, \boldsymbol{\rho})\rangle\right| \le C\left(\|\boldsymbol{\rho}^+\|_{\mathcal{H}_\omega^l}\|\boldsymbol{\rho}^-\|_{\mathcal{H}_\omega^l} + \|\boldsymbol{\rho}\|^3_{\mathcal{H}_\omega^l}\right), \quad j = 2, 4.$$

Taking into account inequality (6.12), we may write

$$\left|\langle \boldsymbol{\xi}_j(\omega), \boldsymbol{N}(\omega, \boldsymbol{\rho})\rangle\right| \le C R_3(\omega) R_4(\omega)\,, \qquad j = 2, 4. \tag{6.26}$$

First, let us note that due to the assumption that $r_3(\omega) \le \kappa_3 R_3(\omega)$, where $\kappa_3 < \frac{2}{3}$, the term

$$R_3(\omega) + r_3 + \langle \boldsymbol{\xi}_2, \boldsymbol{N}\rangle\,,$$

which is in the denominator in all three equations in (6.24), is not smaller than $R_3(\omega)/3$ (as long as $\mathcal{O}(\omega_*)$ is sufficiently small).

Bound (6.19) is straightforward, since all the terms in the right-hand side of the first equation in (6.24) are bounded by $C R_3(\omega)$. According to (6.12), this is smaller than $\kappa_1 R_4(\omega)/R_3(\omega)$ if $\mathcal{O}(\omega_*)$ is sufficiently small.

To prove (6.20), we multiply the second equation from system (6.24) by $R_3(\omega)$ and then add $r_3(\omega) R_3'(\omega) = r_3(\omega) R_4(\omega)/R_3(\omega)$ to both sides of the equation. This gives

$$\begin{aligned}(r_3 R_3)' = {} & R_3 \frac{1 - \langle \boldsymbol{\xi}_2, P_d' \boldsymbol{\rho}\rangle + \Gamma_{24}\rho_4}{R_3 + r_3 + \langle \boldsymbol{\xi}_2, \boldsymbol{N}\rangle} (R_4 + r_4 + \langle \boldsymbol{\xi}_3, \boldsymbol{N}\rangle) \\ & + R_3 \Bigg(\langle \boldsymbol{\xi}_3, P_d' \boldsymbol{\rho}\rangle - \sum_{j=1,3} \Gamma_{3j}\rho_j \Bigg) - R_4 + \frac{r_3 R_4}{R_3}\,. \end{aligned} \tag{6.27}$$

If we neglect the terms $\Gamma_{ij}\rho_j$, $\langle \boldsymbol{\xi}_j, P_d' \boldsymbol{\rho}\rangle$, and $\langle \boldsymbol{\xi}_j, \boldsymbol{N}\rangle$, then the remaining terms in the right-hand side add up to

$$\frac{R_3(R_4 + r_4)}{R_3 + r_3} - R_4 + \frac{r_3 R_4}{R_3} = \frac{R_3^2 r_4 + r_3^2 R_4}{R_3(R_3 + r_3)}\,.$$

Due to (6.17), this expression is bounded by

$$\frac{\kappa_3^2 + \kappa_4}{1 - \kappa_3} R_4(\omega)\,. \tag{6.28}$$

Let us return to the terms in (6.27) that we neglected. Bounding $\Gamma_{ij}\rho_j$ and $\langle \boldsymbol{\xi}_j, P_d' \boldsymbol{\rho}\rangle$ by $C R_3$ and $\langle \boldsymbol{\xi}_j, \boldsymbol{N}\rangle$ by $C R_3^2$, we see that the contribution of all such terms to the right-hand side of (6.27) is bounded by $C R_3^2(\omega)$. According to (6.12), this quantity is negligible, in the sense that, taking a smaller neighborhood $\mathcal{O}(\omega_*)$ if needed, we can ensure that $C R_3^2(\omega)$ is smaller than

$$\left(2\kappa_3 - \frac{\kappa_3^2 + \kappa_4}{1 - \kappa_3} \right) R_4(\omega)\,, \tag{6.29}$$

where

$$2\kappa_3 - \frac{\kappa_3^2 + \kappa_4}{1 - \kappa_3} = \frac{2\kappa_3 - 3\kappa_3^2 - \kappa_4}{1 - \kappa_3} > 0$$

according to the choice of constants in Lemma 6.7. Taking the sum of (6.28) and (6.29), we bound the right-hand side of (6.27) by $2\kappa_3 R_4(\omega)$, arriving at (6.20).

To prove (6.21), we note that if we neglect the terms containing $\Gamma_{ij}\rho_j$, $\langle \boldsymbol{\xi}_2, P_d' \boldsymbol{\rho}\rangle$, and $\langle \boldsymbol{\xi}_j, \boldsymbol{N}\rangle$ in the right-hand side of the last equation in (6.24), then the remaining terms in the right-hand side add up to 0. Due to Corollary 3.6, Π^+ commutes with $P_d'(\omega)$; therefore, for $j = 2, 4$,

$$\langle \boldsymbol{\xi}_j, P_d' \boldsymbol{\rho} \rangle = \langle \boldsymbol{\xi}_j, \Pi^+ P_d' \boldsymbol{\rho} \rangle = \langle \boldsymbol{\xi}_j, P_d' \boldsymbol{\rho}^+ \rangle .$$

This is bounded by $C\|\boldsymbol{\rho}^+\|_{\mathcal{H}^l_\omega} \le CR_4$. Bounding $\langle \boldsymbol{\xi}_j, \boldsymbol{N} \rangle$, $j = 2, 4$, by CR_3R_4, we see that all the terms in the right-hand side of the last equation in system (6.24) are bounded by $CR_4(\omega)$. As follows from (6.15), this is smaller than $\kappa_4 a(\omega)$ if $\mathcal{O}(\omega_*)$ is sufficiently small. This proves bound (6.21).

According to Proposition 5.4,

$$\left| \frac{d}{d\omega} \|\boldsymbol{\rho}_c^+\|_{M_l} \right| \le C \left[\frac{1}{\dot{\omega}} \|\boldsymbol{\rho}^+\|_{\mathcal{H}^l_\omega} \|\boldsymbol{\rho}^-\|_{\mathcal{H}^l_\omega} + \|\boldsymbol{\rho}\|^3_{\mathcal{H}^l_\omega} + \|\boldsymbol{\rho}^+\|_{\mathcal{H}^l_\omega} \right], \tag{6.30}$$

$$\left| \frac{d}{dt} \|\boldsymbol{\rho}_c^-\|_{M_l} \right| \le C \left[\frac{1}{\dot{\omega}} \|\boldsymbol{\rho}\|^2_{\mathcal{H}^l_\omega} + \|\boldsymbol{\rho}^-\|_{\mathcal{H}^l_\omega} \right]. \tag{6.31}$$

It follows that $\left|\frac{d}{d\omega}\|\boldsymbol{\rho}_c^+\|_{M_l}\right| \le CR_4(\omega)$ and $\left|\frac{d}{d\omega}\|\boldsymbol{\rho}_c^-\|_{M_l}\right| \le CR_3(\omega)$, leading to bounds (6.22) and (6.23) (if $\mathcal{O}(\omega_*)$ is sufficiently small).

This finishes the proof of Lemma 6.7. □

6.4 Proof of the Main Theorem

We will consider separately the cases when $a(\omega) > 0$ for $\omega > \omega_*$ (nondegenerate case) and when $a(\omega) \equiv 0$ for $\omega \ge 0$ (degenerate case).

PROOF OF THEOREM 2.14, NONDEGENERATE CASE: In this case we assume that $a(\omega) > 0$ for $\omega > \omega_*$, $\omega \in \mathcal{O}(\omega_*)$. For any $\delta > 0$, we are going to construct a solution to (2.1) of the form

$$\boldsymbol{u}(t) = T\left(\int_0^t \omega(t')dt'\right)(\boldsymbol{\phi}_{\omega(t)} + \boldsymbol{\rho}(t)) \tag{6.32}$$

so that

$$|\omega(0) - \omega_*| < \delta\,, \quad \|\boldsymbol{\rho}\|_{\mathcal{H}^l_\omega} \le \beta(\omega(t) - \omega_*)\,, \quad |\omega(t_1) - \omega_*| > \varepsilon\,,$$

where $\varepsilon > 0$ does not depend on δ, $t_1 = t_1(\delta, \varepsilon) < \infty$, and $\beta \in C^1(\mathbb{R})$ is such that $\beta(s) = o(s)$.

Remark 6.9. To prove Theorem 2.14, we only need $\|\boldsymbol{\rho}\|_X \le \beta(\omega - \omega_*)$, while we are going to prove a stronger bound $\|\boldsymbol{\rho}\|_{\mathcal{H}^l_\omega} \le \beta(\omega - \omega_*)$. In the case of nonlinear Schrödinger or Klein-Gordon equations, when $\mathcal{H}^l_\omega \subset H^{2l+1}(\mathbb{R}^n)$ and l is such that $2l + 1 > \frac{n}{2}$ (for details, see Section 7), this means that (6.32) is a *strong* solution to (2.1).

We use Lemma 6.7 to construct $\boldsymbol{\rho}$ as a function of ω. We pick the values for the constants κ_j, $j = 1, 3, 4$, and $\kappa_c^\pm$ that satisfy the conditions of Lemma 6.7 (for example, set them all equal to $\frac{1}{4}$). We choose $\mathcal{O}(\omega_*) \subset (\omega_1, \omega_2)$ small enough so that Lemma 6.7 becomes applicable. Pick any ε such that $\omega_* + \epsilon \in \mathcal{O}(\omega_*)$. For any $\delta \in (0, \varepsilon)$, take $\omega_0 = \omega_* + \delta$. Let $(R_3(\omega), R_4(\omega))$ be a solution to (6.3) with the initial data at ω_0 satisfying conditions stated in Lemma 6.5, and let ρ_j, $j = 1, 3, 4$, and $\boldsymbol{\rho}_c$ satisfy the initial conditions stated in Lemma 6.7 (one can take $\rho_1|_{t=0} = 0$, $\rho_3|_{t=0} = R_3(\omega_0)$, $\rho_4|_{t=0} = R_4(\omega_0)$, and $\boldsymbol{\rho}_c|_{t=0} = 0$). As follows from Lemma 6.7,

$$\boldsymbol{\rho} = \sum_{j=1,3,4} \boldsymbol{e}_j(\omega)\rho_j + \boldsymbol{\rho}_c\,,$$

considered as a function of ω, will satisfy

$$\|\boldsymbol{\rho}\|_{\mathcal{H}^l_\omega} \le C R_3(\omega)$$

as long as ω does not exit $\mathcal{O}(\omega_*)$. We integrate (6.2) to obtain $\omega(t)$; this allows us to express $\boldsymbol{\rho}$ in terms of time. According to (6.14), there is the inequality $R_3(\omega) \le 2(\omega - \omega_*)a^{1/2}(\omega) = o(\omega - \omega_*)$. We conclude that

$$\|\boldsymbol{\rho}(t)\|_{\mathcal{H}^l_\omega} \le \beta(\omega(t) - \omega_*) \quad \text{where } \beta(s) = o(s).$$

Let us prove the statement of Theorem 2.14 that $\omega(t)$ leaves the ε-neighborhood of ω_* in finite time. As is clear from (6.2), $\dot\omega \ge C R_3(\omega) \ge C R_3(\omega_0) > 0$; therefore, ω will exceed $\omega_0 + \varepsilon$ at the moment

$$t_1(\delta, \varepsilon) \le \frac{C(\varepsilon - \delta)}{R_3(\omega_0)} < \infty\,.$$

This finishes the proof of Theorem 2.14 in the nondegenerate case. □

PROOF OF THEOREM 2.14, DEGENERATE CASE: We now prove the main theorem in the case when $a(\omega) \equiv 0$ for $\omega \in \mathscr{I}$, where $\mathscr{I}$ is an interval of nonzero measure. Without loss of generality, we may assume that $a(\omega) \equiv 0$ for $\omega \ge \omega_*$, $\omega \in \mathcal{O}(\omega_*)$; for these values of ω, $Q(\omega) = Q(\omega_*)$ and $E(\omega) = E(\omega_*)$. We will show that there exist solutions $\boldsymbol{u}(t)$ that crawl adiabatically near the orbits spanned by the standing waves that correspond to $\omega \in \mathcal{O}(\omega_*)$. These solutions have the form $\boldsymbol{u}(t) \approx T(\theta(t))\boldsymbol{\phi}_{\omega(t)}$ with $\omega(t) \approx \omega_* + \epsilon t$, where ϵ is an arbitrarily small positive number (and $\theta(t)$ is some function we are not interested in). Our approximation works as long as $t \le C\epsilon^{-1}$, with the error at $t \approx C\epsilon^{-1}$ being of magnitude ϵ. This resembles the indifferent equilibrium in classical mechanics, when a body can stay at rest or move with a constant speed.

LEMMA 6.10 *Assume that $a(\omega) \equiv 0$ for $\omega \ge \omega_*$ in $\mathcal{O}(\omega_*)$. Let $0 < \epsilon \le 1$, and let $R_3(\omega) \equiv \epsilon$ and $R_4(\omega) \equiv 0$ be a particular solution of system (6.9). If $\mathcal{O}(\omega_*)$ is sufficiently small, then there exists a constant $K > 0$, independent of ϵ, such that if at some point $\omega_* \le \omega \le \omega_* + K^{-1}$, $\omega \in \mathcal{O}(\omega_*)$, one has inequalities*

$$|r_1(\omega)| \le K\epsilon(\omega - \omega_*)\,, \quad |r_3(\omega)| \le K\epsilon(\omega - \omega_*)\,, \quad |r_4(\omega)| \le K\epsilon^2(\omega - \omega_*)\,,$$

and

$$\|\boldsymbol{\rho}_c^-\|_{M_l} \le K\epsilon(\omega - \omega_*)\,, \quad \|\boldsymbol{\rho}_c^+\|_{M_l} \le K\epsilon^2(\omega - \omega_*)\,,$$

then the following bounds are satisfied:

$$|r_1'(\omega)| < K\epsilon\,, \quad |r_3'(\omega)| < K\epsilon\,, \quad |r_4'(\omega)| < K\epsilon^2\,, \tag{6.33}$$

$$\frac{d}{d\omega}\|\boldsymbol{\rho}_c^-\|_{M_l} < K\epsilon\,, \quad \frac{d}{d\omega}\|\boldsymbol{\rho}_c^+\|_{M_l} < K\epsilon^2\,. \tag{6.34}$$

PROOF: Since for $\omega \le \omega_* + K^{-1}$ (whatever value we are to assign to K) we have

$$|r_1(\omega)| \le \epsilon\,, \quad |r_3(\omega)| \le \epsilon\,, \quad |r_4(\omega)| \le \epsilon^2\,, \quad \|\boldsymbol{\rho}_c^-\|_{M_l} \le \epsilon\,, \quad \|\boldsymbol{\rho}_c^+\|_{M_l} \le \epsilon^2\,,$$

we know that $\|\boldsymbol{\rho}^-\|_{M_l} \le C\epsilon$, $\|\boldsymbol{\rho}^+\|_{M_l} \le C\epsilon^2$, and Lemma 5.6 together with (5.11) and (5.12) gives

$$\|\Pi^+ \boldsymbol{N}(\omega, \boldsymbol{\rho})\|_{\mathcal{H}^l_\omega} \le C\epsilon^3\,.$$

Above, C stands for positive constants that do not depend on ϵ. According to (5.13), (5.14), and (6.1), we can take K independent of ϵ so that inequalities (6.33) and (6.34) hold. □

Taking a larger value for K if necessary, we may assume that $\omega_* + K^{-1} \in \mathcal{O}(\omega_*)$. Let $(\boldsymbol{\rho}(t), \omega(t))$ be a solution to (4.11)–(4.12) with $\rho_2 \equiv 0$ and the initial data

$$\rho_1\big|_{t=0} = \rho_4\big|_{t=0} = 0\,, \qquad \rho_3\big|_{t=0} = \epsilon\,, \qquad \boldsymbol{\rho}_c\big|_{t=0} = 0\,, \qquad \omega\big|_{t=0} = \omega_*\,.$$

Then $\boldsymbol{u}(t) = \boldsymbol{\phi}_{\omega(t)} + \boldsymbol{\rho}(t)$ is $O(\epsilon)$-close to ϕ_{ω_*} at $t = 0$:

$$\inf_{s\in\mathbb{R}} \|\boldsymbol{u}(0) - T(s)\boldsymbol{\phi}_{\omega_*}\| \le \|\boldsymbol{\rho}\big|_{t=0}\| = \epsilon\|\boldsymbol{e}_3(\omega_*)\|\,.$$

Take $R_3(\omega) \equiv \epsilon$ and $R_4(\omega) \equiv 0$. Then the inequalities

$$|r_1(\omega)| \le K\epsilon(\omega - \omega_*)\,, \quad |r_3(\omega)| \le K\epsilon(\omega - \omega_*)\,, \quad |r_4(\omega)| \le K\epsilon^2(\omega - \omega_*)\,,$$

and

$$\|\boldsymbol{\rho}_c^-\|_{M_l} \le K\epsilon(\omega - \omega_*)\,, \quad \|\boldsymbol{\rho}_c^+\|_{M_l} \le K\epsilon^2(\omega - \omega_*)$$

are trivially satisfied at $t = 0$, $\omega = \omega_*$. According to Lemma 6.10, these inequalities also hold for all $\omega \in [\omega_*, \omega_* + K^{-1}]$. Substituting $\rho_3(\omega) = \epsilon + r_3(\omega)$ into (6.2) and using the bound $|r_3(\omega)| \le K\epsilon(\omega - \omega_*)$, we see that

$$\dot\omega \ge \rho_3(\omega) - O(\|\boldsymbol{\rho}^+\| + \|\boldsymbol{N}(\omega, \boldsymbol{\rho})\|) \ge \epsilon - K\epsilon(\omega - \omega_*) - o(\epsilon)\,,$$

which is not smaller than $\frac{\epsilon}{3}$ as long as $\omega \le \omega_1 \equiv \omega_* + \frac{1}{2K}$ and ϵ is sufficiently small. Then there exists $t_1 \le 3(\omega_1 - \omega_*)/\epsilon$ such that $\omega(t_1) = \omega_1$, with $\omega(t)$ monotonically increasing for $t \le t_1$. Hence, for ϵ sufficiently small,

$$\begin{aligned}\inf_{s\in\mathbb{R}} \|\boldsymbol{u}(t_1) - T(s)\boldsymbol{\phi}_{\omega_*}\| &\ge \inf_{s\in\mathbb{R}} \|\boldsymbol{\phi}_{\omega_1} - T(s)\boldsymbol{\phi}_{\omega_*}\| - \|\boldsymbol{\rho}(t_1)\| \\ &\ge O(\omega_1 - \omega_*) - O(\epsilon)\,.\end{aligned}$$

Since we could take ϵ to be arbitrarily small, we have proved that the standing wave $\boldsymbol{\phi}_{\omega_*}$ is unstable in the sense of Definition 2.3. This finishes the proof of Theorem 2.14 in the degenerate case, when $a(\omega) \equiv 0$. □

7 Example: The Nonlinear Schrödinger Equation

Consider the nonlinear Schrödinger equation with $U(1)$ symmetry

$$iu_t = -\Delta u + g(|u|^2)u\,, \tag{7.1}$$

where $u = u(t,x)$ is complex valued, $t \geq 0$, $x \in \mathbb{R}^n$. The energy functional corresponding to equation (7.1) is given by

$$E(u) = \int_{-\infty}^{\infty} \left(\frac{1}{2}|\nabla u|^2 + F(u)\right) dx\,, \tag{7.2}$$

where

$$F(u) = \int_0^{|u|} g(v^2)v\,dv\,.$$

The charge associated with $U(1)$ symmetry is given by

$$Q(u) = \frac{1}{2}\int_{-\infty}^{\infty} |u|^2\,dx\,. \tag{7.3}$$

The functions $E(u)$ and $Q(u)$ are conserved due to the Noether theorem for system (7.1).

The standing-wave solutions have the form $u(t,x) = e^{-i\omega t}\phi_\omega(x)$, where the profiles $\phi_\omega(x)$ satisfy the stationary equation

$$\omega\phi_\omega = -\Delta\phi_\omega + g(\phi_\omega^2)\phi_\omega\,. \tag{7.4}$$

We assume that $g \in C^\infty$. The existence of standing-wave solutions $u(t,x) = e^{-i\omega t}\phi_\omega(x)$ of (7.1) in $\mathbb{R}^n$ for a large class of nonlinearities $g(s)$ was shown in [20]. Local existence (Assumption 2.4) and existence of bound states (Assumption 2.5) was proven for a large class of nonlinear functions in [16]. Existence of bound states in the most general case was considered in [1]. For a large class of nonlinearities, there exist standing waves of minimal energy (see, for example, [11]).

The nonlinear Schrödinger equation (7.1) can be formulated in the abstract form (2.1) for two-component vector functions after separating the real and imaginary parts of $u(t,x)$. The real Hilbert space for (7.1) is then $X = H^1(\mathbb{R}^n,\mathbb{R}^2) \cong H^1(\mathbb{R}^n,\mathbb{C})$, with the natural equivalence of $\mathbb{C}$ and $\mathbb{R}^2$. $(\,\cdot\,,\,\cdot\,)$ is the scalar product in $H^1(\mathbb{R}^n,\mathbb{R}^2)$, so that for any $\boldsymbol{\psi}, \boldsymbol{\vartheta} \in H^1(\mathbb{R}^n,\mathbb{R}^2)$

$$(\boldsymbol{\psi},\boldsymbol{\vartheta})_{H^1(\mathbb{R}^n,\mathbb{R}^2)} = \int\limits_{\mathbb{R}^n} \big(\boldsymbol{\psi}(x)\cdot\boldsymbol{\vartheta}(x) + \boldsymbol{\psi}'(x)\cdot\boldsymbol{\vartheta}'(x)\big)\,d^nx\,,$$

where $\boldsymbol{\psi}\cdot\boldsymbol{\vartheta} = \psi_1\vartheta_1 + \psi_1\vartheta_2$. The dual space is $X^* = H^{-1}(\mathbb{R}^n,\mathbb{R}^2)$, and the pairing $\langle\,\cdot\,,\,\cdot\,\rangle : X^*\times X \to \mathbb{R}$ is

$$\langle\boldsymbol{\xi},\boldsymbol{\psi}\rangle = \int\limits_{\mathbb{R}^n} \boldsymbol{\xi}(x)\cdot\boldsymbol{\psi}(x)d^nx\,, \quad \boldsymbol{\xi} \in H^{-1}(\mathbb{R}^n,\mathbb{R}^2)\,, \quad \boldsymbol{\psi} \in H^1(\mathbb{R}^n,\mathbb{R}^2)\,.$$

The operator $I : X \to X^*$ is represented by $(1-\Delta) : H^1(\mathbb{R}^n, \mathbb{R}^2) \to H^{-1}(\mathbb{R}^n, \mathbb{R}^2)$. The skew-symmetric operator $J : X^* \to X$ is induced in $\mathbb{R}^2$ by multiplying by $-i$ in $\mathbb{C}$:

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The representation $T : U(1) \to \mathrm{Aut}(X)$ is given by $T(s) = e^{Js}$ with $T'(0) = J$. B is the inclusion $X \hookrightarrow X^*$, and the charge functional is given by

$$Q(\boldsymbol{u}) = \frac{1}{2}\langle B\boldsymbol{u}, \boldsymbol{u}\rangle = \frac{1}{2}\|\boldsymbol{u}\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^2)} .$$

We define the distance between initial data factoring out the action of the $U(1)$ group:

$$\mathrm{dist}(u, v) = \inf_{s\in\mathbb{R}} \|u - e^{is} v\|_X . \tag{7.5}$$

Let

$$u(t, x) = e^{-i\int_0^t \omega(t')dt'}(\phi_{\omega(t)}(x) + v(t, x) + iw(t, x))$$

be a solution of (7.1), where $\phi_{\omega(t)}$, v, and w are real valued. We will write the solution $u(t, x)$ as

$$\boldsymbol{u}(t, x) = e^{-i\int_0^t \omega(t')dt'}(\boldsymbol{\phi}_{\omega(t)}(x) + \boldsymbol{\rho}(t, x)) ,$$

where

$$\boldsymbol{\phi}_\omega(x) = \begin{bmatrix} \phi_\omega(x) \\ 0 \end{bmatrix}, \qquad \boldsymbol{\rho}(t, x) = \begin{bmatrix} v(t, x) \\ w(t, x) \end{bmatrix}.$$

The functions $v(t, x)$ and $w(t, x)$ satisfy the following nonlinear system:

$$\begin{cases} v_t = -\Delta w + g(\phi_\omega^2)w - \omega w - \dot\omega\phi'_\omega + N_1 \\ w_t = -(-\Delta v + g(\phi_\omega^2)v + 2g'(\phi_\omega^2)\phi_\omega^2 v - \omega v) + N_2, \end{cases} \tag{7.6}$$

where the nonlinear terms are

$$N_1 = \big(g((\phi_\omega + v)^2 + w^2) - g(\phi_\omega^2)\big)\, w , \tag{7.7}$$

$$N_2 = 2g'(\phi_\omega^2)\phi_\omega^2 v - \big(g((\phi_\omega + v)^2 + w^2) - g(\phi_\omega^2)\big)\,(\phi_\omega + v) . \tag{7.8}$$

Following [25], we define the operators

$$L_-(\omega) = -\Delta + g\big(\phi_\omega^2\big) - \omega , \quad L_+(\omega) = -\Delta + g\big(\phi_\omega^2\big) + 2g'\big(\phi_\omega^2\big)\phi_\omega^2 - \omega .$$

We will usually omit the dependence of $L_\pm$ on ω. System (7.6) can be rewritten as the following equation in $\boldsymbol{\rho}$:

$$\dot{\boldsymbol{\rho}} = JH_\omega\boldsymbol{\rho} - \dot\omega \begin{bmatrix} \phi'_\omega \\ 0 \end{bmatrix} + \boldsymbol{N}(\omega, \boldsymbol{\rho}) , \tag{7.9}$$

where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad H_\omega = \begin{bmatrix} L_+(\omega) & 0 \\ 0 & L_-(\omega) \end{bmatrix}, \quad \boldsymbol{N}(\omega, \boldsymbol{\rho}) = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix},$$

with N_1 and N_2 from (7.7) and (7.8).

The positive spectra of operators L_- and L_+ consist of the continuous spectrum and possibly a certain number of isolated positive eigenvalues. The kernels of L_+ and L_- are generated by

$$
\begin{aligned}
\ker L_+(\omega) &= \operatorname{span}\langle \partial_{x^j}\phi_\omega : 1 \le j \le n \rangle\,, \\
\ker L_-(\omega) &= \operatorname{span}\langle \phi_\omega \rangle\,.
\end{aligned}
\tag{7.10}
$$

Since $\partial_{x^j}\phi_\omega$ has one node at $x^j = 0$, there is an eigenfunction of L_+, denoted $\chi_\omega(x)$, that has no zeros (we can pick χ_ω to be real and strictly positive) and corresponds to the negative eigenvalue $-\Lambda_\omega^2$. We assume that χ_ω is symmetric and that $\|\chi_\omega\|_{L^2} = 1$. The negative eigenvalue $-\Lambda_\omega^2$ is unique (see [21] for the details). It follows that $-\Lambda_\omega^2$ is the negative eigenvalue of H_ω that corresponds to the eigenvector

$$
\boldsymbol{\chi}_\omega(x) = \begin{bmatrix} \chi_\omega(x) \\ 0 \end{bmatrix}.
$$

Since $\partial_{x^j}\phi_\omega \in \ker L_+$ is orthogonal to $\ker L_- = \operatorname{span}\langle\phi_\omega\rangle$, there exists $\varsigma_{\omega,j}$, antisymmetric in x^j, such that $\partial_{x^j}\phi_\omega = L_-\varsigma_{\omega,j}$. The linearized operator

$$
JH_\omega = \begin{bmatrix} 0 & L_-(\omega) \\ -L_+(\omega) & 0 \end{bmatrix}
$$

satisfy Assumption 2.6 with an important modification: According to (7.10), the dimension of the kernel of JH_ω is equal to $n+1$. However, if we take $X = H_r^1(\mathbb{R}^n)$, which is a subspace of spherically symmetric functions in $H^1(\mathbb{R}^n)$, then the kernel of JH_ω becomes one-dimensional and Assumption 2.6 is satisfied. In the restricted space, the generalized null space of JH_ω is spanned by the eigenvectors

$$
\boldsymbol{e}_1(\omega) = \begin{bmatrix} 0 \\ -\phi_\omega \end{bmatrix}, \quad \boldsymbol{e}_2(\omega) = \begin{bmatrix} \phi'_\omega \\ 0 \end{bmatrix},
\tag{7.11}
$$

so that

$$
JH_\omega \boldsymbol{e}_1(\omega) = 0\,, \quad JH_\omega \boldsymbol{e}_2(\omega) = \boldsymbol{e}_1(\omega)\,.
\tag{7.12}
$$

At the point $\omega = \omega_*$, we have

$$
\frac{d}{d\omega} Q(\phi_\omega)\Big|_{\omega=\omega_*} = (\phi_{\omega_*}, \phi'_{\omega_*}) = 0\,.
$$

The function ϕ_{ω_*} is the only spherically symmetric function from the kernel of L_-; thus, ϕ'_{ω_*} is orthogonal to $\ker L_-$ at ω_*, so that there exists the function $\alpha(x)$ (which we can choose to be spherically symmetric) that satisfies $L_-\alpha = \phi'_{\omega_*}$. The operator L_+ does not have spherically symmetric functions in its kernel, allowing us to conclude that there exists a spherically symmetric function $\beta(x)$ that satisfies $L_+\beta = \alpha$. The vectors

$$
\boldsymbol{e}_3 = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \quad \boldsymbol{e}_4 = \begin{bmatrix} -\beta \\ 0 \end{bmatrix},
\tag{7.13}
$$

are the generalized null vectors of JH_{ω_*}:

$$JH_{\omega_*}\boldsymbol{e}_3 = \boldsymbol{e}_2(\omega_*)\,, \quad JH_{\omega_*}\boldsymbol{e}_4 = \boldsymbol{e}_3\,,$$
$$(JH_{\omega_*})^4\boldsymbol{e}_4 = (JH_{\omega_*})^3\boldsymbol{e}_3 = (JH_{\omega_*})^2\boldsymbol{e}_2(\omega_*) = (JH_{\omega_*})\boldsymbol{e}_1(\omega_*) = 0\,. \tag{7.14}$$

This sequence cannot be continued to the left: There is no $\boldsymbol{e}_5$ such that $JH_{\omega_*}\boldsymbol{e}_5 = \boldsymbol{e}_4$. Indeed, we would need to have $\boldsymbol{e}_5 = [\begin{smallmatrix}0\\-\gamma\end{smallmatrix}]$, with the function $\gamma(x)$ satisfying $L_-\gamma = \beta$, but this is impossible since β is not orthogonal to $\ker L_- = \operatorname{span}\langle\phi_{\omega_*}\rangle$:

$$m(\omega_*) = (\beta, \phi_\omega) = (\beta, L_+\phi'_\omega) = (L_+\beta, \phi'_\omega) = (\alpha, \phi'_\omega) = (\alpha, L_-\alpha) > 0\,.$$

The last expression is strictly positive since L_- is self-adjoint and semi–positive definite, while $\alpha \notin \ker L_-$.

In Assumption 2.8, the operator C on $\mathbb{R}^2$ is induced by the complex conjugation on $\mathbb{C}$:

$$\mathrm{C} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},$$

so that the projections onto $X^\pm$ are given by

$$\Pi^+ = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \quad \text{and} \quad \Pi^- = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

Note that, according to (7.7), the quadratic term in the Taylor series expansion of

$$\Pi^+\boldsymbol{N}(\omega, \boldsymbol{\rho}) = \Pi^+J(E'(\boldsymbol{\phi}_\omega + \boldsymbol{\rho}) - E'(\boldsymbol{\phi}_\omega) - E''(\boldsymbol{\phi}_\omega)\boldsymbol{\rho}) = \begin{bmatrix}N_1\\ 0\end{bmatrix}$$

contains the product of v and w, in agreement with Lemma 5.6.

The spaces $\mathcal{H}^k_\omega$, $k \geq 0$, coincide with the Sobolev spaces of spherically symmetric functions $H^{2k+1}_r(\mathbb{R}^n, \mathbb{R}^2)$. Part (a) of Assumption 2.10 trivially follows from $\partial_\omega H_\omega$ being an operator of multiplication by a smooth function. Let l be a nonnegative integer such that $2l + 1 > \frac{n}{2}$; then parts (b) and (c) of Assumption 2.10 on the nonlinear terms are satisfied according to Proposition B.1, which we prove in Appendix B.

The explicit form of $M_l(\omega)$ is

$$M_l = (-1)^l(H_\omega J)^{2l}H_\omega = \begin{bmatrix}L_+(L_-L_+)^{2l} & 0\\ 0 & L_-(L_+L_-)^{2l}\end{bmatrix}.$$

Let $\boldsymbol{\rho} = [\begin{smallmatrix}v\\w\end{smallmatrix}]$; then

$$\begin{aligned}\langle M_l(\omega)\boldsymbol{\rho}, \boldsymbol{\rho}\rangle &= \int_{\mathbb{R}^n} \big(vL_+(L_-L_+)^{2l}v + wL_-(L_+L_-)^{2l}w\big)dx\\ &= \int_{\mathbb{R}^n} (VL_+V + WL_-W)dx\,,\end{aligned}$$

where $V = (L_-L_+)^l v$ and $W = (L_+L_-)^l w$.

We have verified all the assumptions needed for the proof of Theorem 2.14. This proves instability of standing waves of minimal energy in the case of the nonlinear Schrödinger equation in $\mathbb{R}^n$. Standing waves in the Klein-Gordon equation are treated in the same way.

The bound estimates of Theorem 2.14 justify the use of the normal form equations for analysis of instability of the standing waves of minimal energy in [11, 12, 13]. The previous heuristic analysis was based on the equation of motion of a particle in a potential field (see [11, 12] for details of derivations):

$$E_0 = \frac{1}{2} m(\omega)\dot{\omega}^2(t) + U(\omega)\,, \tag{7.15}$$

where

$$U(\omega) = E(\phi_{\omega(t)}) + \omega(Q(\phi_{\omega(t)}) - Q(\phi_{\omega(0)}))\,,$$

$m(\omega) = (\alpha, L_{-}\alpha) > 0$, and E_0 is constant. Equation (7.15) can be recovered from the normal form equation (6.4) in the case when $m(\omega)$ is equal to $m(\omega_*)$ for $\omega \in \mathcal{O}(\omega_*)$. Indeed, integrating (6.6) with $a(\omega) = \frac{1}{m(\omega_*)} \frac{d}{d\omega} Q(\phi_\omega)$, we have

$$Q(\phi_{\omega(t)}) - m(\omega_*)\ddot{\omega}(t) = Q_0 = \text{const}\ . \tag{7.16}$$

Then, integrating (6.7) with $\frac{d}{d\omega} E(\phi_\omega) = \omega \frac{d}{d\omega} Q(\phi_\omega)$, we have

$$E(\phi_{\omega(t)}) + \omega(Q(\phi_{\omega(t)}) - Q(\phi_{\omega(0)})) + \frac{1}{2} m(\omega_*)\dot{\omega}^2(t) = E_0\,, \tag{7.17}$$

which is (7.15) for $m(\omega_*) = m(\omega)$. Although we justify the use of (7.17) for the proof of instability of standing waves of minimal energy, we do not prove convergence of the nonlinear Schrödinger equation (7.1) to the normal form equation (6.4) for all solutions near the standing wave. Our analysis was developed only for solutions that correspond to perturbations satisfying inequalities (6.16), (6.17), and (6.18) (where R_3 and R_4 satisfy inequalities (6.12) and (6.13) at the initial value of ω). Also, the analysis becomes invalid in the case $a(\omega) = 0$ in $\omega \in \mathscr{I} \subset \mathcal{O}(\omega_*)$, when $m(\omega)$ is essentially far from $m(\omega_*)$. Modified bound estimates for this critical case are obtained by Perelman in the context of the nonlinear Schrödinger equation in one dimension [14].

Remark 7.1. Our theory does not apply to traveling waves in nonlinear long-wave systems such as the Boussinesq equation and the Korteweg–de Vries equation [12]. The group of translations for such systems acts as

$$T(s) f(t,x) = T(t, x - s)\,,$$

$T'(0) = -\frac{\partial}{\partial x}$, and the symmetry operator is $\mathrm{C} f(t,x) = f(t,-x)$. For the generalized Korteweg–de Vries equation,

$$u_t + u_{xxx} - f(u)_x = 0\,, \quad u = u(t,x)\,, \quad x \in \mathbb{R},\ t \geq 0, \tag{7.18}$$

the energy functional is given by

$$E(u) = \int \left(\frac{u_x^2}{2} + F(u) \right) dx\,, \quad F'(u) = f(u)\,.$$

The Hamiltonian form of (7.18) is $\dot{u} = JE'(u)$, with $J = \frac{\partial}{\partial x}$. The charge functional is given by

$$Q(u) = -\frac{1}{2}\int_{-\infty}^{\infty} u^2(x)dx\,,$$

$B = Q''(u) = -1$, and

$$H_\omega = E''(\phi_\omega) - \omega Q''(\phi_\omega) = -\partial_x^2 + f'(\phi_\omega) + \omega\,.$$

The traveling-wave solutions have the form

$$u(t,x) = \phi_\omega(x - \omega t)\,, \quad \omega > 0\,,\ \phi_\omega(\pm\infty) = 0\,,$$

where ϕ_ω satisfies the equation $\omega Q'(\phi) = E'(\phi)$ or

$$-\omega\phi = -\phi_{xx} + f(\phi)\,.$$

Our assumptions on $T'(0)$ and J are no longer satisfied. In particular, $T'(0) = -\frac{\partial}{\partial x}$ is no longer continuous on $X = H^1(\mathbb{R})$, while $J = \frac{\partial}{\partial x} : X^* \to X$ is no longer onto. We refer to [2] for more details.

Appendix A: Abstract Sobolev Spaces

Let X be a real Hilbert space with the scalar product $(\,\cdot\,,\cdot\,)$ and the norm $\|\cdot\|_X = \sqrt{(\,\cdot\,,\cdot\,)_X}$. Let A be a linear operator from $D(A) \subset X$ to X, unbounded on its domain. In analogy to the standard Sobolev spaces, we define the spaces $\mathcal{H}^s(A)$ associated to the operator A. The role of the smoothing operator $(1-\Delta)^{-1}$ is played by the bounded operator $(A-\zeta)^{-1}$, with ζ from the resolvent set of A.

LEMMA A.1 *Let $\zeta \in \mathbb{C}$ be any point from the resolvent set $\rho(A)$ of A. For a nonnegative integer k, we define the space $\mathcal{H}_\zeta^k(A) \subset X$ as the closure of $D(A^k) \subset X$ with respect to the norm*

$$\|\boldsymbol{u}\|_{\mathcal{H}_\zeta^k} = \|(A-\zeta)^k \boldsymbol{u}\|_X\,.$$

Then we have the following:

- *The space $\mathcal{H}_\zeta^k(A)$ is a Hilbert space with the scalar product*

$$(\boldsymbol{u},\boldsymbol{v})_{\mathcal{H}_\zeta^k} = ((A-\zeta)^k\boldsymbol{u}, (A-\zeta)^k\boldsymbol{v})\,.$$

- *The embeddings $\mathcal{H}_\zeta^{k+1}(A) \hookrightarrow \mathcal{H}_\zeta^k(A)$ are continuous and so are the maps $A : \mathcal{H}_\zeta^{k+1}(A) \to \mathcal{H}_\zeta^k(A)$.*
- *The metrics $\|\cdot\|_{\mathcal{H}_\zeta^k}$ with different $\zeta \in \rho(A)$ are equivalent.*

PROOF: Let us prove that $\mathcal{H}_\zeta^k(A)$ is a Hilbert space. Apparently, $(\boldsymbol{u},\boldsymbol{u})_{\mathcal{H}_\zeta^k} = \|\boldsymbol{u}\|^2_{\mathcal{H}_\zeta^k}$. The norm $\|\cdot\|_{\mathcal{H}_\zeta^k}$ is subadditive on $D(A^k)$ because so is $\|\cdot\|_X$:

$$\begin{aligned}\|\boldsymbol{u}+\boldsymbol{v}\|_{\mathcal{H}_\zeta^k} &= \|(A-\zeta)^k(\boldsymbol{u}+\boldsymbol{v})\|_X\\ &\le \|(A-\zeta)^k\boldsymbol{u}\|_X + \|(A-\zeta)^k\boldsymbol{v}\|_X = \|\boldsymbol{u}\|_{\mathcal{H}_\zeta^k} + \|\boldsymbol{v}\|_{\mathcal{H}_\zeta^k}\,.\end{aligned}$$

The completeness of the spaces $\mathcal{H}^k_\zeta$ follows from their definition.

The continuity of the embedding $\mathcal{H}^{k+1}_\zeta(A) \hookrightarrow \mathcal{H}^k_\zeta(A)$ follows from the continuity of $(A-\zeta)^{-1}$ on X:

$$\|\boldsymbol{u}\|_{\mathcal{H}^k_\zeta} = \|(A-\zeta)^{-1}((A-\zeta)^{k+1}\boldsymbol{u})\|_X \le c\|(A-\zeta)^{k+1}\boldsymbol{u}\|_X = c\|\boldsymbol{u}\|_{\mathcal{H}^{k+1}_\zeta},$$

where $c = \|(A-\zeta)^{-1}\|_{X\to X} < \infty$. The continuity of the mapping $A : \mathcal{H}^{k+1}_\zeta(A) \to \mathcal{H}^k_\zeta(A)$ is immediate:

$$\|A\boldsymbol{u}\|_{\mathcal{H}^k_\zeta} \le \|(A-\zeta)\boldsymbol{u}\|_{\mathcal{H}^k_\zeta} + |\zeta|\|\boldsymbol{u}\|_{\mathcal{H}^k_\zeta} \le \|\boldsymbol{u}\|_{\mathcal{H}^{k+1}_\zeta} + c|\zeta|\|\boldsymbol{u}\|_{\mathcal{H}^{k+1}_\zeta}.$$

To show that the norms for different $\zeta \in \rho(A)$ are equivalent, we find the bound on the difference:

$$\begin{aligned}\Big|\|\boldsymbol{u}\|_{\mathcal{H}^k_\zeta} - \|\boldsymbol{u}\|_{\mathcal{H}^k_{\zeta'}}\Big| &= \Big|\|(A-\zeta)^k\boldsymbol{u}\|_X - \|(A-\zeta')^k\boldsymbol{u}\|_X\Big| \\ &\le \|((A-\zeta)^k - (A-\zeta')^k)\boldsymbol{u}\|_X .\end{aligned}$$

The right-hand side is bounded by

$$\sum_{k'=0}^{k-1} c_{k'}(\zeta,\zeta')\|(A-\zeta)^{k'}\boldsymbol{u}\|_X \le C(\zeta,\zeta')\|\boldsymbol{u}\|_{\mathcal{H}^k_{\zeta'}}$$

so that $\|\boldsymbol{u}\|_{\mathcal{H}^k_\zeta} \le (1 + C(\zeta,\zeta'))\|\boldsymbol{u}\|_{\mathcal{H}^k_{\zeta'}}$.

Similarly, $\|\boldsymbol{u}\|_{\mathcal{H}^k_{\zeta'}} \le (1 + C(\zeta',\zeta))\|\boldsymbol{u}\|_{\mathcal{H}^k_\zeta}$. □

Appendix B: Taylor Series Remainders in Sobolev Spaces

PROPOSITION B.1 *Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Let $f(x,y)$ be a function from $C^{m+s+1}(\mathbb{R}^n \times \mathbb{R})$ with uniformly bounded derivatives. Assume that $s \in \mathbb{N}$, $s > \frac{n}{2}$ and $\rho \in H^s(\mathbb{R}^n)$ with $\|\rho\|_{H^s(\mathbb{R}^n)} \le 1$. Then*

$$\left\| f(\,\cdot\,,\rho) - f(\,\cdot\,,0) - \rho f'_y(\,\cdot\,,0) - \cdots - \frac{\rho^m}{m!} f^{(m)}_y(\,\cdot\,,0)\right\|_{H^s(\mathbb{R}^n)} \le c\|\rho\|^{m+1}_{H^s(\mathbb{R}^n)},$$

where c depends on bounds on f and its derivatives, but not on ρ.

We will prove the proposition if we show that the ∂_x^α derivative of the expression under the norm in the left-hand side is an L^2 function for any $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \le s$. As usual, $\mathbb{Z}_+$ stands for the set of nonnegative integers, and we use the common notation

$$|\alpha| = \sum_{j=1}^n \alpha_j, \quad \alpha \in \mathbb{Z}^n_+ .$$

LEMMA B.2 *Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Let $f(x,y) \in C^{m+s+1}(\mathbb{R}^n \times \mathbb{R})$, and let $\|\rho\|_{H^s(\mathbb{R}^n)} \le 1$. For $1 \le p \le \infty$, the L^2 norm of the derivatives*

$$\partial_x^\alpha\Big(f(x,\rho) - f(x,0) - \rho f_y(x,0) - \cdots - \frac{\rho^m}{m!} f^{(m)}_y(x,0)\Big), \tag{B.1}$$

where $\alpha \in \mathbb{Z}_+^n$, $0 \le |\alpha| \le s$, can be bounded by the finite sum of L^2 norms of monomials

$$\partial_x^{\beta_1}\rho \cdots \partial_x^{\beta_k}\rho\,, \quad \beta_j \in \mathbb{Z}_+^n\,, \quad \sum_{j=1}^{k}|\beta_j| \le |\alpha|\,, \quad m+1 \le k \le m+s+1\,,$$

times the constant $c \equiv \sup_{\substack{|\alpha|\le s\\ j\le m+s+1\\ x\in\mathbb{R},\,|y|\le 1}} \left|\partial_x^\alpha \partial_y^j f(x,y)\right| < \infty$.

PROOF: We use the integral form for the remainder term in the Taylor series:

$$f(x,\rho) - \sum_{l=0}^{m} \frac{\rho^l}{l!} f_y^{(l)}(x,0) = \int_0^1 dt\,(1-t)^m \frac{\rho^{m+1}}{m!} f_y^{(m+1)}(x,t\rho)\,.$$

Since $\rho \in H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, all terms in the above expression are continuous functions of x, so that the expressions can be considered pointwise. The rest follows from the Minkovski inequality:

$$f(x,\rho) - \sum_{l=0}^{m} \frac{\rho^l}{l!} f_y^{(l)}(x,0) \le \int_0^1 dt(1-t)^m \left\|\partial_x^\alpha\left(\rho^{m+1} f_y^{(m+1)}(x,t\rho)\right)\right\|_{L^2}\,.$$

□

LEMMA B.3 (Generalized Hölder Inequality) *If $\rho_j \in L^{p_j}$ for $1 \le j \le k$, $1 \le p_j \le \infty$, then $\rho_1 \cdots \rho_k \in L^P$, where $\frac{1}{P} = \frac{1}{p_1} + \ldots + \frac{1}{p_k}$, and*

$$\|\rho_1 \cdots \rho_k\|_{L^P} \le \|\rho_1\|_{L^{p_1}} \cdots \|\rho_k\|_{L^{p_k}}\,.$$

This is an elementary exercise on proof by induction.

LEMMA B.4 *Let $\rho \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2}$. Then $\partial_x^{\alpha_1}\rho \cdots \partial_x^{\alpha_k}\rho$, where $\alpha_j \in \mathbb{Z}_+^n$ and $\sum_{j=1}^k |\alpha_j| \le s$ is bounded in $L^2(\mathbb{R}^n)$.*

PROOF: We sketch a proof for the reader's convenience. A similar estimate is obtained in [23, sec. 13.3]. Since $\partial_x^{\alpha_j}\rho \in L^2(\mathbb{R}^n)$ and $\partial_x^{\alpha_j}\rho \in H^{s-|\alpha_j|}(\mathbb{R}^n)$, with the norm bounded by $\|\rho\|_{H^s(\mathbb{R}^n)}$, the Sobolev embedding and interpolation with $\|\rho\|_{L^2(\mathbb{R}^n)} \le \|\rho\|_{H^s(\mathbb{R}^n)}$ shows that

$$\|\partial_x^{\alpha_j}\rho\|_{L^p(\mathbb{R}^n)} \le C_s \|\rho\|_{H^s(\mathbb{R}^n)}\,, \tag{B.2}$$

with some $C_s < \infty$ depending only on s as long as

$$2 \le p \le \frac{2}{1 - 2(s - |\alpha_j|)/n}\,, \quad 1 \le j \le k.$$

We are going to use these bounds together with Lemma B.3. Taking the smallest values $p_j = 2$ allowed in (B.2), we compute the minimal value of P for which Lemma B.3 becomes applicable:

$$\frac{1}{P_{\min}} \equiv \sum_{j=1}^{k} \frac{1}{2} = \frac{k}{2} \ge \frac{1}{2}\,. \tag{B.3}$$

Therefore, according to Lemma B.3,

$$\|\partial_x^{\alpha_1}\rho\cdots\partial_x^{\alpha_k}\rho\|_{L^{P_{\min}}} \le \|\partial_x^{\alpha_1}\|_{L^2}\cdots\|\partial_x^{\alpha_k}\rho\|_{L^2} \le \|\rho\|_{H^s}^k\,. \tag{B.4}$$

Now we take the largest values $p_j = \frac{2n}{n-2(s-|\alpha_j|)}$ allowed in (B.2):

$$\frac{1}{P_{\max}} \equiv \sum_{j=1}^k \frac{1}{p_j} = \sum_{j=1}^k \left(\frac{1}{2} - \frac{s-|\alpha_j|}{n}\right) = \frac{k}{2} - \frac{ks}{n} + \frac{\sum_{j=1}^k |\alpha_j|}{n}$$
$$\le \frac{k}{2} - \frac{ks}{n} + \frac{s}{n} = (k-1)\left(\frac{1}{2} - \frac{s}{n}\right) + \frac{1}{2}\,.$$

The right-hand side is not larger than $\frac{1}{2}$ since $s > \frac{n}{2}$ and $k \ge 1$; therefore, $P_{\max} \ge 2$. According to Lemma B.3,

$$\|\partial_x^{\alpha_1}\rho\cdots\partial_x^{\alpha_k}\rho\|_{L^{P_{\max}}} \le \|\partial_x^{\alpha_1}\|_{L^{p_1}}\cdots\|\partial_x^{\alpha_k}\rho\|_{L^{p_k}} \le C\|\rho\|_{H^s}^k\,. \tag{B.5}$$

Interpolation of (B.4) and (B.5) shows that

$$\|\partial_x^{\alpha_1}\rho\cdots\partial_x^{\alpha_k}\rho\|_{L^P(\mathbb{R}^n)} \le C\|\rho\|_{H^s}^k\,, \qquad P_{\min} \le P \le P_{\max}\,,$$

where $P_{\min} = \frac{2}{k} \le 2$ and $P_{\max} \ge 2$. This proves Lemma B.4. □

We completed the proof of Proposition B.1.

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