Rogue waves in the sine-Gordon equation

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Overview

What is a rogue wave?

Rogue waves are gigantic waves that appear out of nowhere and then disappear without trace.

A wave is said to be rogue if its magnification, the ratio of the maximum of the rogue wave versus the mean of its background , exceeds two.

Laboratory Coordinates

The sine-Gordon equation in laboratory coordinates (x, t) is:

$$u_{tt} - u_{xx} + \sin(u) = 0,$$

with $u(x, t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The sine-Gordon equation models:

- magnetic flux in superconducting Josephson junctions
- fermions
- stability structure in galaxies
- ribbon pendulums
- much more ...

Characteristic Coordinates

The sine-Gordon equation in characteristic coordinates $(\xi, \eta) \in \mathbb{R}^2$ is:

$$u_{\xi\eta} = \sin(u),$$

where $x = \xi + \eta$ and $t = \xi - \eta$.

Lax pair in laboratory coordinates

The compatibility condition, $\chi_{xt} = \chi_{tx}$, of the following Lax pair of linear equations is the sine-Gordon equation:

$$\frac{\partial}{\partial x}\chi = \begin{bmatrix} \frac{-i\gamma}{2} + \frac{i\cos(u)}{8\gamma} & \frac{i\sin(u)}{8\gamma} - \frac{1}{4}(u_x + u_t) \\ \frac{i\sin(u)}{8\gamma} + \frac{1}{4}(u_x + u_t) & \frac{i\gamma}{2} - \frac{i\cos(u)}{8\gamma} \end{bmatrix} \chi$$

and

$$\frac{\partial}{\partial t}\chi = \begin{bmatrix} \frac{-i\gamma}{2} - \frac{i\cos(u)}{8\gamma} & -\frac{i\sin(u)}{8\gamma} - \frac{1}{4}(u_x + u_t) \\ -\frac{i\sin(u)}{8\gamma} + \frac{1}{4}(u_x + u_t) & \frac{i\gamma}{2} + \frac{i\cos(u)}{8\gamma} \end{bmatrix} \chi.$$

Here $\gamma \in \mathbb{C}$ is a spectral parameter and $\chi = (p, q)^T$ is an eigenfunction in variables (x, t) [Deconick et al, 2017].

Lax pair in characteristic coordinates

The following Lax pair is compatible for solutions of the sine-Gordon equation in characteristic coordinates :

$$\frac{\partial}{\partial \xi}\begin{bmatrix} p \\ q \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \lambda & -u_\xi \\ u_\xi & -\lambda \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$$

$$\frac{\partial}{\partial \eta}\begin{bmatrix} p \\ q \end{bmatrix} = \frac{1}{2\lambda}\begin{bmatrix} \cos(u) & \sin(u) \\ \sin(u) & -\cos(u) \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$$

where λ is the spectral parameter and $\chi = (p, q)^T$ is an eigenfunction in variables (ξ, η). The correspondence between the spectral parameters in both frames is

$$\lambda = -2i\gamma.$$

Why is this Lax structure important?

Compatibility of the Lax pair is really important in the analysis of the sine-Gordon equation. Applications of this Lax system include, but are not limited to:

- Analysis of modulation stability of periodic waves
- Application of the Darboux Transformation
- Solution to initial value problem
- Algebraic methods in constructing exact solutions [Pelinovsky and Chen,2018]

How to construct a rogue wave

The one-fold Darboux Transformation (DT) creates new solutions to the sine-Gordon equation with a solution u to the sine-Gordon equation and solution (λ_1, p_1, q_1) to the Lax pair,

$$-\hat{u}_\xi = -u_\xi + \frac{4\lambda_1 p_1 q_1}{p_1^2 + q_1^2}.$$

The two-fold DT creates new solutions to the sine-Gordon equation with a solution u to the sine-Gordon equation and two solutions (λ_1, p_1, q_1) and (λ_2, p_2, q_2) to the Lax system,

$$-\hat{u}_\xi = -u_\xi + \frac{4(\lambda_1^2 - \lambda_2^2)[\lambda_1 p_1 q_1 (p_2^2 + q_2^2) - \lambda_2 p_2 q_2 (p_1^2 + q_1^2)]}{(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 2\lambda_1\lambda_2[4p_1 q_1 p_2 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)]}.$$

How to construct a rogue wave

For our purposes we want:

- $-u_\xi$ to be periodic
- $-\hat{u}_\xi$ to be a rogue wave on the periodic background $-u_\xi$

Picking The Right Eigenfunctions

Question : What solutions to the linear Lax equation will make $-\hat{u}_\xi$ a proper rogue wave?

To be a proper rogue wave we want:

- The magnification factor to exceed 2.
- $\inf_{\xi_0,\eta_0,\theta_0\in\mathbb{R}} \sup_{x\in\mathbb{R}} |-\hat{u}_\xi(\xi,\eta) + u_\xi(\xi-\xi_0,\eta-\eta_0)e^{i\theta_0}| \to 0$ as $\eta \to \pm\infty$
- Magnification to be an isolated event

We considered traveling wave backgrounds $u(\xi,\eta) = f(\xi-\eta)$, where $f : \mathbb{R} \to \mathbb{R}$. The wave speed can be set to one because of the Lorentz transformation.

Traveling Periodic Background

Substituting the traveling wave $u(\xi, \eta) = f(\xi - \eta)$ into the sine-Gordon equation and then solving the resultant ordinary differential equation generates potentials

$$u_\xi = f'(\xi - \eta) = \frac{2}{k} dn(\frac{\xi - \eta}{k}; k)$$

called rotational waves and

$$u_\xi = f'(\xi - \eta) = 2k \; cn(\xi - \eta; k)$$

called librational waves. Here $k \in (0, 1)$ is the elliptic modulus parameter for Jacobi elliptic functions.

Floquet Theory for Lax Spectrum

Recall the linear equation from the Lax pair:

$$\varphi_\xi = U(\lambda, u)\varphi, \qquad U(\lambda, u) = \begin{pmatrix} \lambda & -u_\xi \\ u_\xi & -\lambda \end{pmatrix}$$

where $\varphi = (p, q)^T$ is the eigenfunction.
If u_ξ is L-periodic in ξ then Floquet theory tells us that there exists an eigenfunction of the form:

$$\varphi(\xi) = e^{i\mu\xi}\breve{\varphi}(\xi)$$

where $\breve{\varphi}(\xi) = \breve{\varphi}(\xi + L)$, and $\mu \in [-\frac{\pi}{L}, \frac{\pi}{L}]$.

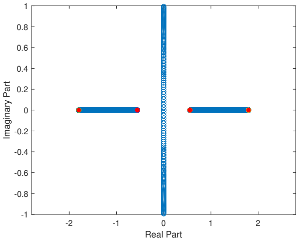
Re-writing The Linear Lax System

It follows from Floquet theory that $\breve{\varphi}_\mu(\xi)$ is a periodic eigenfunction of the spectral problem:

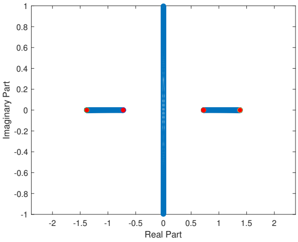
$$\begin{pmatrix} 2\frac{d}{d\xi} + 2i\mu & f' \\ f' & -2\frac{d}{d\xi} - 2i\mu \end{pmatrix} \breve{\varphi} = \lambda \breve{\varphi}$$

Approximating $\frac{d}{d\xi}$ with the 12-th order finite difference matrix and discretizing the domain of the eigenfunctions over one period allows us to approximate the spectral problem as a matrix eigenvector problem in MATLAB.

Spectral Pictures for Traveling Waves



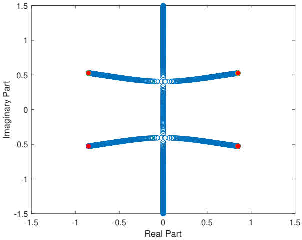
(a) k=0.85



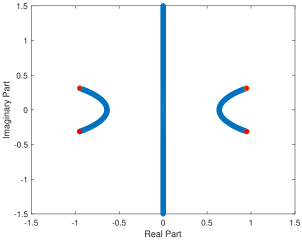
(b) k=0.95

Figure: Floquet Spectrum for potential $u_\xi(\xi,\eta) = \frac{2}{k} dn(\frac{\xi-\eta}{k}; k)$

Spectral Pictures for Traveling Waves



(a) k=0.85



(b) k=0.95

Figure: Floquet Spectrum for potential $u_\xi(\xi,\eta) = 2k cn(\xi - \eta; k)$

Eigenvalues extracted from algebraic method

End points of the Floquet Spectrum can be found analytically using a certain algebraic method:

$$\lambda_{1R} = \pm \frac{1 \pm \sqrt{1 - k^2}}{k}$$

for rotational waves and

$$\lambda_{1L} = \pm (k \pm i\sqrt{1 - k^2})$$

for librational waves. Here $k \in (0, 1)$ is the elliptic modulus parameter. These eigenvalues correspond to solutions of the Lax equation (λ_1, p_1, q_1) with potential $-f'$ that satisfies:

$$-f' = -u_\xi = p_1^2 + q_1^2.$$

Darboux Transformation

Applying the one fold DT

$$-\hat{u}_\xi = -u_\xi + \frac{4\lambda_1 p_1 q_1}{p_1^2 + q_1^2}$$

with the rotational wave and Lax pair solution (λ_{1R}, p_1, q_1) with $-f' = p_1^2 + q_1^2 = -\frac{2}{k} dn(\frac{\xi-\eta}{k}; k)$ generates

$$-\hat{u}_\xi = \pm\frac{2}{k} dn(\frac{\xi - \eta}{k} + K(k); k),$$

a translated and negated version of the rotational wave. Here K(k) is the complete elliptic integral of the first kind. We will need to consider new eigenfunctions in order to avoid a trivial transformation.

New eigenfunctions for rotational waves

Consider second linearly independent eigenfunctions for rotational waves of the form

$$\hat{p}_1 = p_1\phi_R - \frac{q_1}{p_1^2 + q_1^2}$$
$$\hat{q}_1 = q_1\phi_R + \frac{p_1}{p_1^2 + q_1^2},$$

where the Wronskian between (p_1, q_1) and $(\hat{p}_1, \hat{q}_1)$ is normalized to 1. I have introduced the function $\phi_R : (\xi, \eta) \to \mathbb{C}$ for rotational waves.

Algebraic Solitons

Applying the DT with Lax solution $(\lambda_{1R}, \hat{p}_1, \hat{q}_1)$ generates algebraic solitons on the background of rotational waves in the sine-Gordon equation

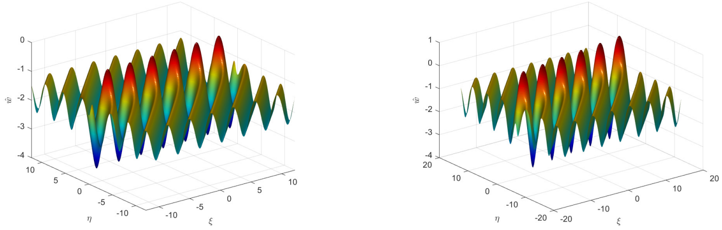
$$-\hat{u}_\xi = -u_\xi + \frac{4\lambda_1 \hat{p}_1 \hat{q}_1}{\hat{p}_1^2 + \hat{q}_1^2},$$

and

$$\lim_{|\phi_R|\to\infty} -\hat{u}_\xi = \frac{2}{k} dn(\frac{\xi - \eta}{k} + K(k); k).$$

$|\phi_R| \to \infty$ while (ξ, η) moves away from a particular line Ω in the (ξ, η)-plane. Algebraic solitons propogate along Ω.

Algebraic Soliton Pictures

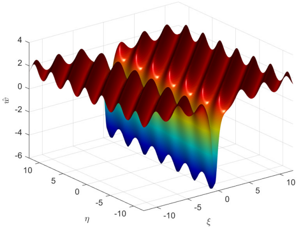


(a) k=0.85　　(b) k=0.95

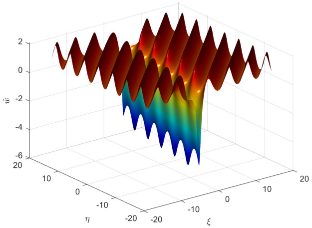
Figure: Algebraic solitons with k=0.85 (left) and k=0.95 (right)

$$\lambda_{1R} = \frac{1 - \sqrt{1 - k^2}}{k}$$

Algebraic Soliton Pictures



(a) k=0.85



(b) k=0.95

Figure: Algebraic solitons with k=0.85 (left) and k=0.95 (right)

$$\lambda_{1R} = \frac{1+\sqrt{1-k^2}}{k}$$

Magnification of solitons

The magnification of the algebraic solitons is

$$M := \frac{\sup_{(\xi,\eta)\in\mathbb{R}^2} |\hat{u}_\xi|}{\sup_{(\xi,\eta)\in\mathbb{R}^2} |u_\xi|} = 2 \mp \sqrt{1-k^2}.$$

The two thresold is definitely surpassed for the lower sign and $k \in (0,1)$.

Darboux Transformation

Applying the two fold DT

$$-\hat{u}_\xi = -u_\xi + \frac{4(\lambda_1^2-\lambda_2^2)[\lambda_1 p_1 q_1 (p_2^2+q_2^2) - \lambda_2 p_2 q_2 (p_1^2+q_1^2)]}{(\lambda_1^2+\lambda_2^2)(p_1^2+q_1^2)(p_2^2+q_2^2) - 2\lambda_1\lambda_2[4p_1q_1p_2q_2 + (p_1^2-q_1^2)(p_2^2-q_2^2)]}$$

with the librational wave and Lax pair solutions (λ_{1L}, p_1, q_1) and $(\lambda_2, p_2, q_2) = (\bar{\lambda}_{1L}, \bar{p}_1, \bar{q}_1)$ with $p_1^2 + q_1^2 = -f' = -2kcn(\xi - \eta; k)$ generates

$$-\hat{u}_\xi = 2k \; cn(\xi - \eta; k),$$

a negation of the librational wave potential. New eigenfunctions were also considered with the librational waves in order to avoid a trivial transformation.

New eigenfunctions

In order to avoid singularities in eigenfunctions for librational waves we will consider second linearly independent eigenfunctions of the form

$$\hat{p}_1 = \frac{\phi_L - 1}{q_1}$$
$$\hat{q}_1 = \frac{\phi_L + 1}{p_1},$$

where the Wronskian is normalized to 2. I have introduced the function $\phi_L(\xi, \eta) \to \mathbb{C}$.

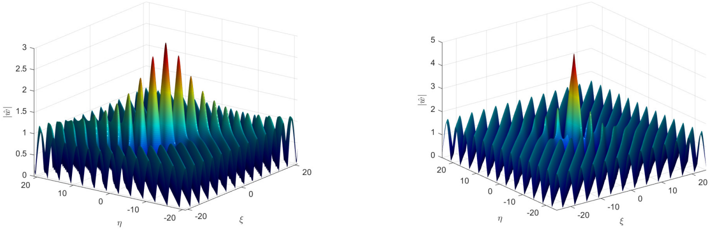
Rogue Waves

Applying the two fold DT with the librational wave and Lax pair solutions $(\lambda_{1L}, \hat{p}_1, \hat{q}_1)$ and $(\lambda_2, \hat{p}_2, \hat{q}_2) = (\bar{\lambda}_{1L}, \bar{\hat{p}}_1, \bar{\hat{q}}_1)$ generates an isolated rogue wave $-\hat{u}_\xi$ with

$$\lim_{|\phi_L| \to \infty} -\hat{u}_\xi = 2k \; cn(\xi - \eta; k).$$

The function $\phi_L(\xi, \eta)$ grows linearly in $|x| + |t|$ as $|x| + |t| \to \infty$ for every $k \in (0, 1)$

Rogue Wave Pictures

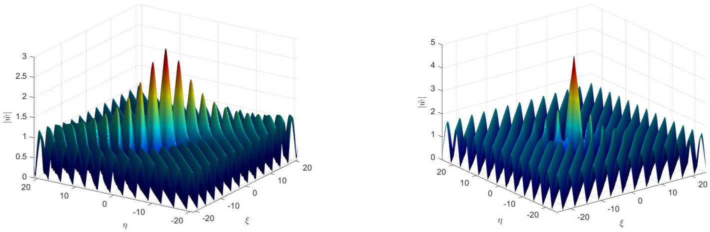


(a) k=0.5 (b) k=0.8

Figure: Rogue Waves with k=0.5 (left) and k=0.8 (right)

$$\lambda_{1L} = (k - i\sqrt{1-k^2})$$

Rogue Wave Pictures



(a) k=0.5 (b) k=0.8

Figure: Rogue Waves with k=0.5 (left) and k=0.8 (right)

$$\lambda_{1L} = (k + i\sqrt{1-k^2})$$

Magnification

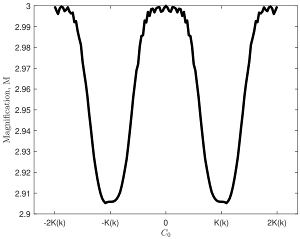
The magnification of the rogue waves is

$$M := \frac{\sup_{(\xi,\eta)\in\mathbb{R}^2} |\hat{u}_\xi|}{\sup_{(\xi,\eta)\in\mathbb{R}^2} |u_\xi|} = 3.$$

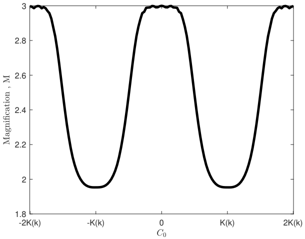
Modifying the growth

The growth term ϕ is determined by substituting the second eigenfunctions into the Lax equations and solving a transport like equation. ϕ is determined by integrating from a constant $C_0 \in \mathbb{R}$ to $\xi - \eta$. The figures and analysis above were performed with $C_0 = 0$ but changing C_0 can alter the magnification of the rogue wave.

Modifying the growth



(a) k=0.5



(b) k=0.8

Figure: Magnification of the rogue wave vs C_0 with k=0.5 (left) and k=0.8 (right)

Concluding Remarks

By finding solutions to the linear Lax equations and then applying the Darboux transformation we were able to successfully generate rogue waves occurring in the sine-Gordon equation.

References

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- D.E. Pelinovsky and J. Chen. Rogue periodic waves of the modified kdv equation. Nonlinearity,31(5):1955-1980, (2018).

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