Nonlinear Waves in Weakly-Coupled Lattices

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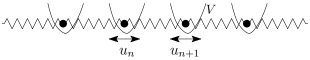
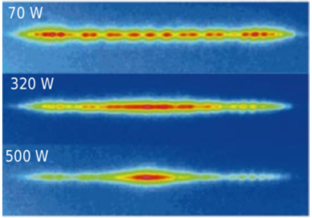
Lattice equations

Lattice equations are systems of coupled ODE with discrete translation invariance. They arise as

- models for physical processes
- discretization of PDEs



Mechanical vibrations.



Optical waveguides [CLS03].

We consider lattice equations which support localized time-periodic solutions – **discrete breathers**.

Equations

We consider the following one-dimensional lattices:

- Discrete Klein–Gordon (**dKG**) equation,

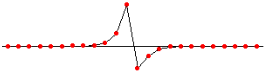
$$\ddot{u}_n + V'(u_n) = \epsilon(\Delta \mathbf{u})_n, \qquad u_n(t) : \mathbb{R} \to \mathbb{R},$$

- Discrete nonlinear Schrödinger (**dNLS**) equation,

$$i\dot{u}_n(t) \pm |u_n|^{2p} u_n = \epsilon(\Delta \mathbf{u})_n, \qquad u_n(t) : \mathbb{R} \to \mathbb{C}, \quad p \in \mathbb{N},$$

where $n \in \mathbb{Z}$, $\epsilon \in \mathbb{R}$, and $(\Delta \mathbf{u})_n = u_{n-1} - 2u_n + u_{n+1}$.

The main objective is to study existence and stability of discrete breathers near the **anti-continuum limit** ($\epsilon = 0$).



Thesis outline

dNLS

- existence of breathers
- linear stability & **internal modes**
- **dispersive decay** & asymptotic stability

dKG

- existence of breathers & **tail-to-tail interactions**
- **linear stability**
- numerical continuation of breathers
- **bifurcations**

dNLS breathers near the anti-continuum limit

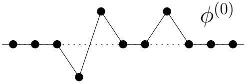
We look for discrete breathers in the dNLS equation,

$$i\dot{u}_n + |u_n|^{2p} u_n + \epsilon(\Delta \mathbf{u})_n = 0, \qquad n \in \mathbb{Z}, \quad p \in \mathbb{N},$$

using the rotating wave approximation, $u_n = \phi_n e^{it}$, where the discrete soliton ϕ satisfies

$$\left(1 - \phi_n^{2p}\right) \phi_n = \epsilon(\Delta \phi)_n.$$

Let $S_\pm$ be disjoint compact subsets of $\mathbb{Z}$, then, in the anti-continuum limit,



$$\phi^{(0)} = \sum_{n \in S_+} \mathbf{e}_n - \sum_{n \in S_-} \mathbf{e}_n, \qquad (\mathbf{e}_n)_m = \delta_{n,m}.$$

If ϵ is small enough there exists a unique solution $\phi \in l^2$ such that

$$\|\phi - \phi^{(0)}\|_{l^2} \leq C|\epsilon|.$$

Linear stability of dNLS breathers

Linear stability of a discrete breather ϕ is determined by the spectral problem

$$(\mathcal{L}-I\lambda)\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix}=0, \qquad \mathcal{L}=\begin{bmatrix}0 & L_-\\ -L_+ & 0\end{bmatrix}.$$

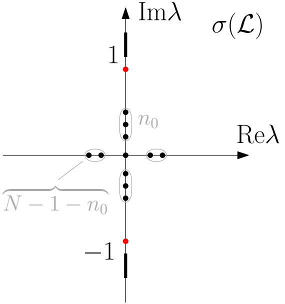
where $L_\pm$ are discrete Schrödinger operators given by

$$(L_+\mathbf{v})_n = -\epsilon(\Delta\mathbf{v})_n + (1-(2p+1)\phi_n^{2p})v_n,$$
$$(L_-\mathbf{v})_n = -\epsilon(\Delta\mathbf{v})_n + (1-\phi_n^{2p})v_n.$$

- Eigenvalue count: [PKF05], [CP10].
- Our focus: internal modes near the anti-continuum limit.

Limiting soliton $\phi^{(0)}$:

- N excited sites
- n_0 sign changes



Free resolvent

The free resolvent operator $R_0(\lambda) \equiv (-\Delta - \lambda)^{-1}$ can be written explicitly [KKK06]

$$\forall \mathbf{f} \in l^2 : \quad (R_0(\lambda)\mathbf{f})_n = \frac{1}{2i \sin z(\lambda)} \sum_{m\in\mathbb{Z}} e^{-iz(\lambda)|n-m|} f_m,$$

where $z(\lambda)$ is a unique solution of the transcendental equation

$$2(1 - \cos z(\lambda)) = \lambda, \quad \mathrm{Re} z(\lambda) \in [-\pi, \pi), \quad \mathrm{Im} z(\lambda) \leq 0.$$

Moreover,

- $R_0(\lambda) : l^2 \to l^2$ if $\lambda \notin [0, 4] \equiv \sigma_c(-\Delta)$,
- $R_0^{\pm}(\lambda) : l^1 \to l^\infty$ if $\lambda \notin \{0\} \cup \{4\}$ with poles at $\lambda = 0$ and $\lambda = 4$.

Spectral problem: leading order

Truncate the potential $\{\phi_n^{2p}\}$ to the leading order and rewrite the spectral problem as

$$(L - I\Omega)\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} = 0, \qquad L = \begin{bmatrix} -\epsilon\Delta + I - (1+p)V & -pV \\ pV & \epsilon\Delta - I + (1+p)V \end{bmatrix},$$

where

$$\Omega = i\lambda, \qquad \mathbf{a} = \mathbf{v} + i\mathbf{w}, \qquad \mathbf{b} = \mathbf{v} - i\mathbf{w},$$

and $V : l^2 \to l^2$ is a compact potential defined by

$$(V\mathbf{u})_n = \sum_{m \in S_- \cup S_+} \delta_{n,m} u_m, \quad n \in \mathbb{Z}.$$

Resolvent for the leading order operator

To study $R_L(\Omega) = (L - I\Omega)^{-1}$ we reduce the infinite system

$$(L - I\Omega)\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} = \begin{bmatrix}\mathbf{f}\\ -\mathbf{g}\end{bmatrix}, \qquad \mathbf{f},\mathbf{g} \in l^2$$

to a square system of $2N$ linear equations:

$$A(\Omega,\epsilon)c = h(\Omega,\epsilon), \quad h(\Omega,\epsilon) = \left\{ \begin{array}{c} \sum_{m\in\mathbb{Z}} e^{-iz(\lambda_+)|n-m|} f_m \\ \sum_{m\in\mathbb{Z}} e^{-iz(\lambda_-)|n-m|} g_m \end{array} \right\}_{n\in S},$$

where $c = (\mathbf{a},\mathbf{b})^T|_S$ and $\lambda_\pm = \frac{1}{\epsilon}(\pm\Omega - 1)$.

- The operator L has $2N$ small eigenvalues near $\Omega = 0$,
- The continuous spectrum is $\sigma_c(L) = [-1-4\epsilon, -1] \cup [1, 1+4\epsilon]$. Is the resolvent singular at the endpoints?

Resolvent on the continuous spectrum

Use asymptotic expansions to analyze the matrix $A(\Omega, \epsilon)$ along the continuous spectrum:

- Matrix $A(\Omega, \epsilon)$ is invertible for $\Omega \in (1, 1 + 4\epsilon)$ provided the limiting configuration $\phi^{(0)}$ has no "holes".
- At $\Omega \in \{1, 1 + 4\epsilon\}$, the ends of the continuum spectrum, get

$$n_g = \dim \mathrm{Null} A(\Omega, \epsilon) = N - 1 = n_a.$$

- Thank to some special properties of the system $A(\Omega, \epsilon)c = h(\Omega, \epsilon)$ there is a bounded solution in the limits $\Omega \to 1^+$ and $\Omega \to (1 + 4\epsilon)^-$.

No internal modes for small couplings

Perturbation arguments for the full resolvent lead to

Theorem (no internal modes)

Assume $S_+ \cup S_-$ is simply connected. Given $\mathbb{N} \ni p \geq 2$ and sufficiently small $\epsilon > 0$ there is $\delta > 0$ such that the resolvent operator

$$R(\Omega) = (\mathcal{L} - i\Omega\mathrm{I})^{-1} : l^2 \times l^2 \to l^2 \times l^2$$

- *bounded for any $\Omega \notin B_\delta(0) \cup \sigma_c(L)$ and has $2N$ poles inside $B_\delta(0)$,*
- *operators $R^\pm(\Omega) = \lim_{\mu\downarrow 0} R(\Omega \pm i\mu)$ admit the bounds*

$$\|R^\pm(\Omega)\|_{l^1_1 \times l^1_1 \to l^\infty \times l^\infty} \leq C\epsilon^{-1}, \qquad \forall \Omega \in [1, 1+4\epsilon].$$

Note: Solitons in cubic dNLS ($p = 1$) require at least second-order perturbation analysis.

Scattering of small solutions: dNLS example

Consider the dNLS equation

$$\begin{cases} i\dot{u}_n(t) = -(\Delta \mathbf{u})_n \pm |u_n|^{\beta-1} u_n, \\ u_n(0) = u_{0,n}, \end{cases} \qquad n \in \mathbb{Z}, \quad \beta \geq 3$$

- Using interpolation

$$\|e^{it\Delta}\mathbf{u}_0\|_{l^s} \leq C(s) t^{-\frac{s-2}{3s}} \|\mathbf{u}_0\|_{l^1}, \qquad s \in (2, \infty).$$

- Applying methods of [MP10] establish a sharper result

$$\|e^{it\Delta}\mathbf{u}_0\|_{l^s} \leq C(s)(1+t)^{-\alpha_s} \|\mathbf{u}_0\|_{l^1}, \qquad \alpha_s = \begin{cases} \frac{s-2}{2s}, & s \in [2,4), \\ \frac{s-1}{3s}, & s \in (4,\infty]. \end{cases}$$

This estimate also holds in the nonlinear case if $\beta > 4$.

Scattering of small solutions: dNLS example (contd)

We write the solution in a convolution form

$$u_n(t) = \sum_{k\in\mathbb{Z}} G_k(t) u_{0,n-k}, \qquad G_k(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(\omega(\theta)t - k\theta)} d\theta,$$

where $\omega(\theta) = 2(1-\cos\theta)$ is the dispersion relation. We also have

$$\|\mathbf{u}(t)\|_{l^s} \leq \|\mathbf{G}(t)\|_{l^s} \|\mathbf{u}_0\|_{l^1}$$

In the limit of $t \to \infty$, norm $\|\cdot\|_{l^s}^s$ is nearly a Riemann sum,

$$\|\mathbf{G}(t)\|_{l^s} = Ct^{1/s} \|g(t,\cdot)\|_{L^s} + \mathrm{Error}(t),$$

where the grid is $\{c_n = \frac{n}{t}\}$ and

$$g(t,c) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it\phi(\theta,c)} d\theta, \quad \phi(\theta,c) = \omega(\theta) - c\theta.$$

Klein–Gordon lattice

Consider the Klein–Gordon (KG) lattice

$$\ddot{u}_n + V'(u_n) = \epsilon(\Delta \mathbf{u})_n, \qquad n \in \mathbb{Z},$$

where $u_n(t) : \mathbb{R} \to \mathbb{R}$, ϵ is small, and V is even,

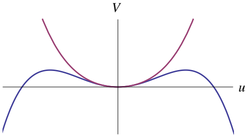
$$V'(u) = u \pm u^3 + \mathcal{O}(u^5), \qquad \text{(hard/soft)}.$$

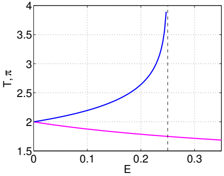
At $\epsilon = 0$ either $u_n \in \{0, \varphi\}$, where or φ satisfies

$$\ddot{\varphi} + V'(\varphi) = 0, \qquad \frac{1}{2}\dot{\varphi}^2 + V(\varphi) = E.$$

The period of φ is

$$T(E) = \frac{1}{\sqrt{2}} \oint \frac{dx}{\sqrt{E - V(x)}}.$$





Existence of breathers

Let $S \subset \mathbb{Z}$ be the set of excited sites so that

$$\mathbf{u}^{(0)}(t) = \sum_{k \in S} \sigma_k \varphi(t) \mathbf{e}_k \quad \in \quad l^2(\mathbb{Z}, H^2_{per}(0, T)),$$

where $\sigma_k \in \{-1, +1\}$, and $(\mathbf{e}_k)_n = \delta_{k,n}$.

Fix $T \neq 2\pi n$ such that $T'(E) \neq 0$. For ϵ small enough, there exists a unique solution $\mathbf{u}^{(\epsilon)} \in l^2(\mathbb{Z}, H^2_{per}(0, T))$ of the dKG equation satisfying

$$\left\| \mathbf{u}^{(\epsilon)} - \mathbf{u}^{(0)} \right\|_{l^2(\mathbb{Z}, H^2(0,T))} \leq C|\epsilon|.$$

Linear stability

Earlier results [MJKA02, ACSA03, KK09]: small amplitudes, no "holes".

Linearization about the breather $\mathbf{u}^{(\epsilon)}$ yields

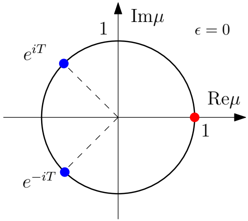
$$\ddot{w}_n + V''\left(u_n^{(\epsilon)}\right) w_n = \epsilon(\Delta \mathbf{w})_n, \qquad n \in \mathbb{Z}.$$

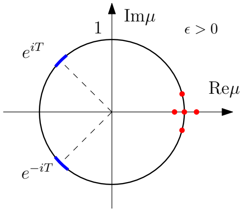
Stability is determined by Floquet multipliers, the eigenvalues of the monodromy matrix M:

$$\begin{bmatrix} w_n(T) \\ \dot{w}_n(T) \end{bmatrix}_{n\in\mathbb{Z}} = M \begin{bmatrix} w_n(0) \\ \dot{w}_n(0) \end{bmatrix}_{n\in\mathbb{Z}}.$$

At $\epsilon = 0$ the monodromy matrix M is block-diagonal with

- $\mu = e^{\pm iT}$ with $n_g = n_a = 1$ at $n \in \mathbb{Z}\backslash S$
- $\mu = 1$ with $n_g = 1$, $n_a = 2$ at $n \in S$.

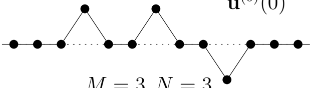




Splitting of the unit multiplier

Suppose in the anti-continuum limit

$$\mathbf{u}^{(0)}(t) = \sum_{j=1}^{N} \sigma_j \varphi(t) \mathbf{e}_{jM}.$$



Theorem (splitting of the unit multiplier)

For small $\epsilon > 0$ the linearized stability problem has $2N$ small Floquet exponents $\lambda = \epsilon^{M/2}\Lambda + \mathcal{O}\left(\epsilon^{(M+1)/2}\right)$, where Λ is determined from the eigenvalue problem

$$-\frac{T(E)^2}{T'(E)}\Lambda^2 c = K_M(T)\mathcal{M}c, \qquad c \in \mathbb{C}^N.$$

Where λ is Floquet exponent ($\mu = e^{\lambda T}$) and $\mathcal{M} \in \mathbb{R}^{N\times N}$ is the matrix

$$\mathcal{M}_{i,j} = -\sigma_j\left(\sigma_{j-1} + \sigma_{j+1}\right)\delta_{i,j} + \delta_{i,j-1} + \delta_{i,j+1}, \qquad i,j \in \{1,2,\ldots,N\}.$$

Stable breathers

Stability of multibreathers is determined by the sign of $T'(E)K_M(T)$ and the number n_0 of sign changes in $\{\sigma_k\}_{k=1}^{N-1}$, where

$$K_M(T) = \sum_{n\in\mathbb{N}_{odd}} \frac{4\pi^2 T^{2M-3}(E)\, n^2 |\hat{\varphi}_n(E)|^2}{[T^2(E) - (2\pi n)^2]^{M-1}}.$$

Depending on the sign of $T'(E)K_M(T)$ there are either n_0 or $N-1-n_0$ "unstable" pairs of Floquet multipliers. The following breathers are stable:

	M odd	M even
$V(u) = u + u^3$	in-phase	anti-phase
$V(u) = u - u^3$	anti-phase	anti: $2\pi < T < T_M^*$ in: $T_M^* < T < 6\pi$

Three-site model: numerics

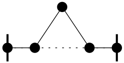
Consider existence and stability of T-periodic solutions in the truncated KG lattice

$$\begin{cases} \ddot{u}_0 + u_0 - u_0^3 = \epsilon(u_1 - 2u_0 + u_{-1}) \\ \ddot{u}_{\pm 1} + u_{\pm 1} - u_{\pm 1}^3 = \epsilon(u_0 - 2u_{\pm 1}) \end{cases}$$

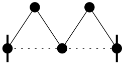
near the anti-continuum limit, $\epsilon = 0$.

Set initial conditions $\mathbf{u}(0) = \mathbf{a}$, $\dot{\mathbf{u}}(0) = 0$ and consider the following three-site symmetric breathers with $u_{+1}(t) = u_{-1}(t)$:

Fundamental breather $u_0^{(0)} = \varphi$, $u_{\pm 1}^{(0)} = 0$

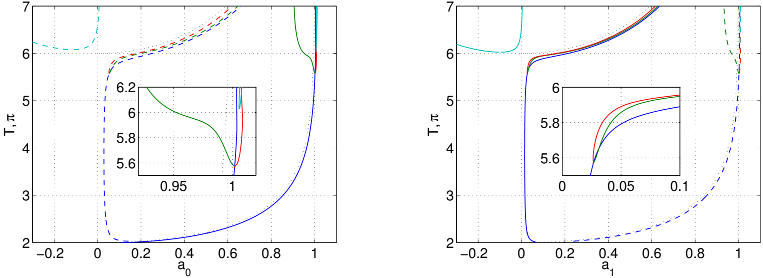


Breather with a "hole" $u_0^{(0)} = 0$, $u_{\pm 1}^{(0)} = \varphi$

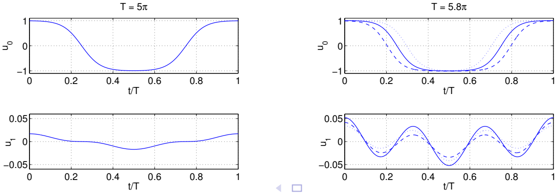


Existence of breathers and bifurcations ($\epsilon = 0.01$)

Fundamental breather (solid) and the breather with a hole (dashed):

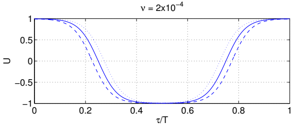


Symmetry-breaking bifurcation of the fundamental breather



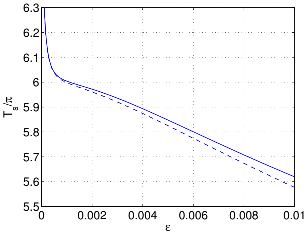
Normal form of the symmetry-breaking bifurcation

The symmetry-breaking bifurcation is described by the Duffing equation



$$\ddot{U} + U - U^3 = \beta U + \nu \cos\tau,$$

where $U(\tau)$ is 6π-periodic, $\beta = 1 - \frac{1+2\epsilon}{(1+\delta\epsilon^{2/3})^2}$, and $\nu = \frac{2\epsilon^{4/3} a(\delta)}{(1+\delta\epsilon^{2/3})^2}$.



Period of the breather at the symmetry breaking bifurcation: normal form (solid), KG lattice (dashed).

Main results

Main results near the anti-continuum limit:

- Non-existence of internal modes in the dNLS equation via resolvent analysis.
- General criteria for stability of multibreathers in the KG lattice via analysis of tail-to-tail interactions.
- Bifurcations and resonances in KG lattice.

dNLS: asymptotic stability

Assume $V \in l^1 \to l^1$ is exponentially decaying and $H = -\Delta + V$ has the only eigenvalue ψ_0 and $H\psi_0 = \omega_0\psi_0$. Consider

$$i\dot{u}_n = Hu_n + |u_n|^{2p}u_n, \qquad n \in \mathbb{Z}, \quad p > 0.$$

There is a soliton bifurcating from ω_0,

$$\mathbf{u}(t) = e^{-i\omega t}\phi(\omega) = e^{-i\omega t}\|\psi_0\|_{l^{2p+2}}^{-1-\frac{1}{p}}(\omega - \omega_0)^{\frac{1}{2p}}\psi_0 + \text{h.o.t.}$$

Theorem (MP12)

For any $p > 2.75$, there exist an $\epsilon_0 > 0$ and a $\delta > 0$ such that if $\epsilon = \omega_* - \omega_0 \in (0, \epsilon_0)$ and $\|\mathbf{u}_0 - \phi(\omega_*)\|_{l^1_1}$ is small enough, then there exist $(\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$ and a solution to dNLS equation

$$\mathbf{u}(t) = e^{-i\theta(t)}\phi(\omega(t)) + \mathbf{y}(t) \quad \in \quad C^1(\mathbb{R}_+, l^2)$$

such that $\lim_{t\to\infty}\left(\theta(t) - \int_0^t \omega(s)ds\right) = \theta_\infty$, $\sup_{t\geq 0}|\omega(t) - \omega_*| \leq C\delta\epsilon$, *and* $\lim_{t\to\infty}\omega(t) = \omega_\infty$. *Moreover, we have*

$$\|\mathbf{y}(t)\|_{l^s} \leq C_s\delta\epsilon^{\frac{1}{2p}}(1+t)^{-\alpha_s}, \qquad \forall s \in (2,4) \cup (4,\infty].$$

dKG: splitting of the unit multiplier

Linearizing with $\mathbf{u} \mapsto \mathbf{u} + \mathbf{w}$ and setting $\mathbf{w}(t) = \mathbf{W}(t)e^{\lambda t}$, where $\mathbf{W}$ is T-periodic, we get

$$\ddot{W}_n + V''(u_n)W_n + 2\lambda \dot{W}_n + \lambda^2 W_n = \epsilon(\Delta \mathbf{W})_n.$$

For the fundamental breather $\phi^{(\epsilon)}$ generated by $\mathbf{u}^{(0)} = \varphi(t)\mathbf{e}_0$ the expansion is

$$\mathbf{W} = \boldsymbol{\theta} + \lambda \boldsymbol{\mu} + \mathcal{O}(\lambda^2): \qquad \boldsymbol{\theta} = \dot{\phi}^{(\epsilon)}, \quad \ddot{\mu}_n + V''(u_n)\mu_n = \epsilon(\Delta \boldsymbol{\mu})_n - 2\dot{\theta}_n.$$

For the multibreather $\mathbf{u}^{(0)} = \sum_{j=1}^{N} \sigma_j \varphi(t)\mathbf{e}_{jM}$ we have $\lambda \approx \epsilon^{M/2}\tilde{\lambda}$ and

$$\mathbf{W} = \sum_{j=1}^{N} c_j \left(\tau_{jM} \boldsymbol{\theta}^{(\epsilon,M)} - \epsilon^M (\mathbf{e}_{(j-1)M} + \mathbf{e}_{(j+1)M})\dot{\varphi}_M \right) + \epsilon^{M/2}\tilde{\lambda} \sum_{j=1}^{N} c_j \tau_{jM} \boldsymbol{\mu}^{(\epsilon,M_*)} + \epsilon^M \tilde{\mathbf{W}},$$

where $(\partial_t^2 + 1)\varphi_m = \varphi_{m-1}$, $m = 1, \ldots, M$, and $\varphi_0 = \varphi$.

dKG Floquet multipliers

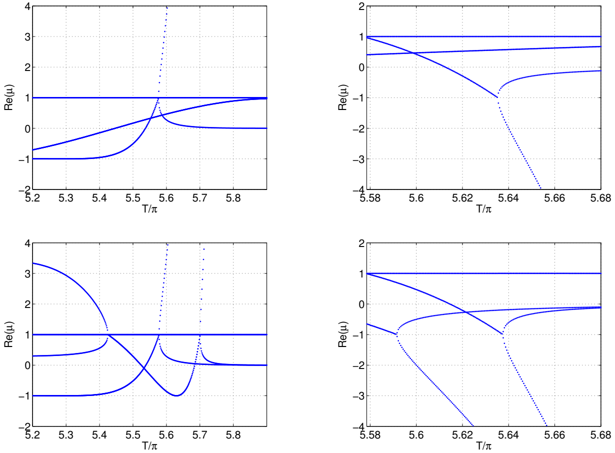


Figure: Floquet multipliers for a fundamental breather (top) and a breather with a hole (bottom).

dNLS: well posedness

Consider the initial value problem for the dNLS equation

$$\begin{cases} i\dot{u}_n(t) = -(\Delta \mathbf{u})_n + V_n u_n + f(|u_n|^2)u_n, \\ u_n(0) = u_{0,n}, \end{cases} \qquad n \in \mathbb{Z}.$$

- Global well posedness in l^2_α with $\alpha \geq 0$ if $\mathbf{V}$ is bounded and f is real analytic
- One can establish a bound

$$\|\mathbf{u}\|_{l^2_\alpha} \leq \|\mathbf{u}_0\|_{l^2_\alpha} e^{C_\alpha t}.$$

dKG: well posedness

Consider the initial value problem for the dKG equation

$$\begin{cases} \ddot{u}_n(t) = (\Delta \mathbf{u})_n - m u_n - \chi u_n^{2p+1}, \\ u_n(0) = u_{0,n}, \\ \dot{u}_n(0) = u_{1,n}, \end{cases} \qquad n \in \mathbb{Z},$$

where $m \in \mathbb{R}$, $p \in \mathbb{N}$ and $\chi = \pm 1$. This system admits conservation of energy

$$E(\mathbf{u}) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left((u_{n+1} - u_n)^2 + m u_n^2 + \dot{u}_n^2 \right) + \frac{\chi}{2p+2} \sum_{n \in \mathbb{Z}} u_n^{2p+2}.$$

- Case $\chi = +1$: global well-posedness in $l^2 \times l^2$.
- Case $\chi = -1$: blow-up on $[0, T^*]$ with

$$T^* = \frac{1}{p} \frac{\|\mathbf{u}_0\|_{l^2}^2}{\langle \mathbf{u}_0, \mathbf{u}_1 \rangle_{l^2}}$$

provided $E < 0$ and $\langle \mathbf{u}_0, \mathbf{u}_1 \rangle_{l^2} > 0$.

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