Received: 27 March 2021

DOI: 10.1111/sapm.12430

ORIGINAL ARTICLE

STUDIES IN APPLIED MATHEMATICS WILEY

Stability of smooth periodic travelling waves in the Camassa–Holm equation

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Funding information
Fundacao Araucaria, Grant/Award Number: grant 002/2017; Ministry of Science and Higher Education of the Russian Federation, Grant/Award Number: Task No. FSWE-2020-0007; Russian Federation for the leading scientific schools, Grant/Award Number: grant No. NSH-2485.2020.5; CNPq, Grant/Award Number: grant 304240/2018-4; CAPES Math-AmSud, Grant/Award Number: grant 88881.520205/2020-01

Abstract

We solve the open problem of spectral stability of smooth periodic waves in the Camassa–Holm equation. The key to obtaining this result is that the periodic waves of the Camassa–Holm equation can be characterized by an alternative Hamiltonian structure, different from the standard formulation common to the Korteweg-de Vries equation. The standard formulation has the disadvantage that the period function is not monotone and the quadratic energy form may have two rather than one negative eigenvalues. We prove that the nonstandard formulation has the advantage that the period function is monotone and the quadratic energy form has only one simple negative eigenvalue. We deduce a precise condition for the spectral and orbital stability of the smooth periodic travelling waves and show numerically that this condition is satisfied in the open region where the smooth periodic waves exist.

KEYWORDS
Camassa–Holm equation, periodic travelling waves, spectral stability

2000 MATHEMATICS SUBJECT CLASSIFICATION
35Q53, 37K45, 76B15

1 | INTRODUCTION

The Camassa–Holm (CH) equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1}$$

was derived in Refs. 1 and 2 and justified in Refs. 3 and 4 as a model for the propagation of unidirectional shallow water waves. A generalized version of this equation also models propagation of nonlinear waves inside a cylindrical hyperelastic rod with a small diameter.[5] The CH equation can be interpreted geometrically in terms of geodesic flows on the diffeomorphism group.[6,7]

We consider the CH equation (1) on the periodic domain $\mathbb{T}_L := [0, L]$ of length $L > 0$. For notational simplicity, we write H^s_{per} instead of $H^s_{\text{per}}(\mathbb{T}_L)$. The CH equation (1) on $\mathbb{T}_L$ conserves formally the mass, momentum, and energy given, respectively, by

$$M(u) = \int_0^L u dx, \tag{2}$$

$$E(u) = \frac{1}{2}\int_0^L (u_x^2 + u^2) dx, \tag{3}$$

and

$$F(u) = \frac{1}{2}\int_0^L (u^3 + uu_x^2) dx. \tag{4}$$

Many results are available for the CH equation (1) in the periodic domain $\mathbb{T}_L$. The initial-value problem is locally well posed in the space H^3_{per},[8,9] H^s_{per} with $s > \frac{3}{2}$,[10–12] C^1_{per},[13] and $H^1_{\text{per}} \cap \text{Lip}$,[14] where Lip stands for Lipshitz continuous functions suitable for peaked periodic waves.

Smooth, peaked, and cusped periodic travelling waves were classified in Ref. 15. Cusped periodic waves were constructed in Refs. 16 and 17 to show that local solutions in H^1_{per} are not uniformly continuous with respect to the initial data. More recently, the period function for the smooth periodic waves was analyzed in Ref. 18.

Orbital stability of the smooth periodic travelling waves in H^1_{per} was obtained in Ref. 19 with the inverse scattering transform for initial data u_0 in H^3_{per} such that $m_0 := u_0 - u_0''$ is strictly positive. Thanks to the Lax representation, the orbital stability of the smooth periodic waves in the time evolution of the CH equation (1) follows from the structural stability of the Floquet spectrum in the associated Lax equations.

Orbital stability of peaked periodic waves in H^1_{per} was proven in Refs. 20 and 21 by using two different variational methods, each uses the three conserved quantities (2)–(4). The recent work [22] shows that the perturbations to the peaked periodic waves grow exponentially in $H^1_{\text{per}} \cap W^{1,\infty}$ and may blow up in finite time.

Stability of cusped periodic waves is an open problem due to the lack of continuity with respect to initial data in H^1_{per}.

Regarding the limit to the solitary waves, orbital stability of smooth solitary waves in $H^1(\mathbb{R})$ was obtained in Ref. 23 for a modified version of the CH equation, where the standard orbital stability for solitary waves hold. Orbital stability of peaked solitary waves in $H^1(\mathbb{R})$ was obtained in Refs. 24 and 25 but the recent work[26] showed that the perturbations to the peaked solitary waves actually grow in $W^{1,\infty}(\mathbb{R})$.

The main purpose of this paper is to address the spectral and orbital stability of the smooth periodic waves by using the analytic theory used for stability of periodic waves in the nonlinear evolution equations of the Korteweg–de Vries (KdV) type.[27–34]

In the standard spectral stability theory, we identify the smooth periodic travelling wave as a critical point of the action functional and compute the number of negative eigenvalues in the linearized operator that represents the quadratic energy form. The number of negative eigenvalues is typically controlled by the monotonicity of the period function for the smooth periodic waves.[33] If the action functional is a linear combination of the mass, momentum, and energy, the spectral stability is determined by positivity of the linearized operator under the constraints of fixed mass and momentum. If the periodic waves of a fixed period are extended smoothly with respect to parameters, derivatives of the mass and momentum computed at the periodic waves with respect to their parameters determine the number of negative eigenvalues of the linearized operator under the constraints.

The technique is straightforward in the case when the linearized operator has only one simple negative eigenvalue, for example, in Ref. 30. Moreover, orbital stability of such periodic waves in the energy space can be easily concluded from its spectral stability.[27] However, computations become messy when the linearized operator has two negative eigenvalues and the periodic wave has limited smoothness with respect to its parameters.[32,34]

The main novelty of this paper is to show that stability of the smooth periodic waves of the CH equation (1) can be characterized in two different ways by using two different Hamiltonian structures.[35] The two equivalent formulations are related to two different parameters a and b of the periodic travelling wave solutions in addition to the wave speed c. The standard formulation has many disadvantages, whereas the nonstandard formulation is suitable for the proof of spectral and orbital stability of the smooth periodic waves.

We now describe *the main results of this paper*.

Smooth travelling waves of the form $u(x,t) = \phi(x-ct)$ with speed c and profile ϕ satisfy the third-order differential equation

$$-(c-\phi)(\phi''' - \phi') - 2\phi\phi' + 2\phi'\phi'' = 0. \tag{5}$$

This equation can be integrated in two different ways. The standard integration of (5) in x gives the second-order equation:

$$-(c-\phi)\phi'' + c\phi - \frac{3}{2}\phi^2 + \frac{1}{2}\phi'^2 = b, \tag{6}$$

where b is the integration constant. However, another integration is obtained after multiplying (5) by $(c-\phi)$, which yields the second-order equation:

$$-(c-\phi)^2(\phi''-\phi)=a, \tag{7}$$

where a is another integration constant. Both second-order equations (6) and (7) are compatible if and only if ϕ satisfies the first-order invariant:

$$(c-\phi)(\phi'^2-\phi^2-2b)+2a=0. \tag{8}$$

It is easy to verify that one of the three equations (6)–(8) is satisfied if and only if the other two equations are satisfied. We consider the smooth L-periodic travelling wave solutions of (1), which means that we are looking for solutions $\phi \in H^{\infty}_{\text{per}}$ of the system (6)–(8).

Let us connect the second-order equations (6) and (7) with two different Hamiltonian structures of the CH equation (1).

The standard Hamiltonian structure for the CH equation (1) is given by

$$\frac{du}{dt}=J\frac{\delta F}{\delta u},\quad J=-(1-\partial_x^2)^{-1}\partial_x,\quad \frac{\delta F}{\delta u}=\frac{3}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2, \tag{9}$$

where J is a well-defined operator from H^s_{per} to H^{s+1}_{per} for every $s\in\mathbb{R}$ and $\frac{\delta F}{\delta u}$ is an operator from H^s_{per} to H^{s-2}_{per} for $s>\frac{3}{2}$ thanks to Sobolev's embedding of H^s_{per} into C^1_{per}. The evolution problem (9) is well defined for local solutions $u\in C((-t_0,t_0),H^s_{\text{per}})\cap C^1((-t_0,t_0),H^{s-1}_{\text{per}})$ with some $t_0>0$ and $s>\frac{3}{2}$, see Refs. 10, 11, and 12.

The second-order equation (6) is the Euler–Lagrange equation for the action functional

$$\Lambda_{c,b}(u):=cE(u)-F(u)-bM(u). \tag{10}$$

The linearized operator for the second-order equation (6) is given by

$$\mathcal{L}:=-\partial_x(c-\phi)\partial_x+(c-3\phi+\phi''), \tag{11}$$

which is related to the action functional (10) as $\mathcal{L}=\Lambda''_{c,b}(\phi)$. The linearized operator $\mathcal{L}:H^2_{\text{per}}\subset L^2_{\text{per}}\mapsto L^2_{\text{per}}$ is a self-adjoint, unbounded operator in L^2_{per} equipped with the standard inner product $\langle\cdot,\cdot\rangle$.

The alternative Hamiltonian structure for the CH equation (1) is given by

$$\frac{dm}{dt}=J_m\frac{\delta E}{\delta m},\quad J_m=-(m\partial_x+\partial_x m),\quad \frac{\delta E}{\delta m}=u, \tag{12}$$

where $m:=u-u_{xx}$ and $E(u)$ can be written equivalently as

$$E(u)=\frac{1}{2}\int_0^L(u_x^2+u^2)dx=\frac{1}{2}\int_0^L umdx=\frac{1}{2}\int_0^L m(1-\partial_x^2)^{-1}mdx. \tag{13}$$

By using (12), the CH equation (1) is rewritten in the local differential form

$$m_t + um_x + 2mu_x = 0. \tag{14}$$

The second-order equation (7) is related to the action functional

$$\Lambda_c(m) := E(u) - cM(u), \quad u := (1-\partial_x^2)^{-1} m. \tag{15}$$

Indeed, the Euler–Lagrange equation $J_m \frac{\delta\Lambda_c}{\partial m} = 0$ gives the differential equation

$$\mu'(\phi - c) + 2\mu\phi' = 0, \tag{16}$$

where $\mu = \phi - \phi''$ or $\phi = (1-\partial_x^2)^{-1}\mu$. Integration of (16) multiplied by $(\phi - c)$ yields $(c-\phi)^2\mu = a$ that is equivalent to (7). The linearized operator for $(c-\phi)^3\mu = a(c-\phi)$ acting on μ is given by

$$\mathcal{K} := (c-\phi)^3 - 2a(1-\partial_x^2)^{-1}, \tag{17}$$

The linearized operator $\mathcal{K} : L^2_{\text{per}} \mapsto L^2_{\text{per}}$ is the sum of a bounded and a compact self-adjoint operator in L^2_{per}.

Let us now give the definitions of spectral and orbital stability of the smooth periodic travelling waves in the CH equation (1).

Definition 1. We say that the smooth periodic travelling wave $\phi \in H^\infty_{\text{per}}$ is spectrally stable in the evolution problem (1) if the spectrum of $J\mathcal{L}$ in L^2_{per} is located on the imaginary axis.

Definition 2. We say that the smooth periodic travelling wave $\phi \in H^\infty_{\text{per}}$ is orbitally stable in the evolution problem (1) in H^1_{per} if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $u_0 \in H^s_{\text{per}}$ with $s > \frac{3}{2}$ satisfying

$$\|u_0 - \phi\|_{H^1_{\text{per}}} < \delta,$$

the global solution $u \in C(\mathbb{R}, H^s_{\text{per}})$ with the initial data u_0 satisfies

$$\inf_{r\in\mathbb{R}} \|u(t,\cdot) - \phi(\cdot + r)\|_{H^1_{\text{per}}} < \varepsilon$$

for all $t \geq 0$.

The following two theorems represent the main results of this paper.

Theorem 1. *For a fixed $c > 0$, smooth periodic solutions of the system (6)–(8) exist in an open, simply connected region on the (a, b) plane closed by three boundaries:*

- $a = 0$ *and* $b \in (-\frac{1}{2}c^2, 0)$ *(where the periodic solutions are peaked),*
- $a = a_+(b)$ *and* $b \in (0, \frac{1}{6}c^2)$ *(where the solutions have infinite period),*
- $a = a_-(b)$ *and* $b \in (-\frac{1}{2}c^2, \frac{1}{6}c^2)$ *(where the solutions are constant),*

where $a_+(b)$ and $a_-(b)$ are smooth functions of b specified in Lemmas 2 and 3. For every point inside the region, the periodic solutions are smooth functions of (a,b,c) and their period is strictly increasing in b for every fixed $a \in (0, \frac{4}{27}c^3)$ and $c > 0$. Moreover, $\mathcal{K}$ has exactly one simple negative eigenvalue, a simple zero eigenvalue, and the rest of its spectrum in L^2_{per} is strictly positive and bounded away from zero.

Theorem 2. *For a fixed $c > 0$ and a fixed period $L > 0$, there exists a C^1 mapping $a \mapsto b = \mathcal{B}_L(a)$ for $a \in (0, a_L)$ with some $a_L \in (0, \frac{4}{27}c^3)$ and a C^1 mapping $a \mapsto \phi = \Phi_L(\cdot, a) \in H^\infty_{\mathrm{per}}$ of smooth L-periodic solutions along the curve $b = \mathcal{B}_L(a)$. Let*

$$\mathcal{M}_L(a) := M(\Phi_L(\cdot, a)) \quad \text{and} \quad \mathcal{E}_L(a) := E(\Phi_L(\cdot, a)).$$

The L-periodic wave with profile $\Phi_L(\cdot, a)$ is spectrally and orbitally stable in the sense of Definitions 1 and 2, respectively, if the mapping

$$a \mapsto \frac{\mathcal{E}_L(a)}{\mathcal{M}_L(a)^2} \tag{18}$$

is strictly decreasing.

Remark 1. In comparison with Theorem 1, it follows from the results in Ref. 18 that the period of the periodic solutions of the system (6)–(8)

- is monotonically increasing in a if $b \in (-\frac{1}{2}c^2, -(1 - \frac{\sqrt{2}}{\sqrt{3}})c^2]$;
- has a single maximum point in a if $b \in (-(1 - \frac{\sqrt{2}}{\sqrt{3}})c^2, 0)$;
- is monotonically decreasing in a if $b \in [0, \frac{1}{6}c^2)$.

We will show in Theorem 4 that $\mathcal{L}$ has two simple negative eigenvalues if the period is increasing in a and one simple negative eigenvalue if the period is decreasing in a, in addition to the simple zero eigenvalue in L^2_{per}.

Remark 2. The spectrum of $\mathcal{L}$ in L^2_{per} is purely discrete, whereas the spectrum of $\mathcal{K}$ in L^2_{per} includes both continuous and discrete parts, see Lemma 7. However, the continuous part of $\mathcal{K}$ is strictly positive and does not contribute to the count of negative eigenvalues.

Remark 3. In the context of Theorem 2, we show numerically that the stability criterion (18) is satisfied for every periodic solution of Theorem 1, see Figure 7.

Remark 4. We also show in Lemma 13 and Remark 19 without appealing to the numerical verification of the stability criterion (18) that the smooth periodic travelling waves of Theorem 1 are orbitally stable for $b \leq 0$ and also in a neighborhood of the boundary $a = a_-(b)$ where the solutions are constant.

Remark 5. The following transformation

$$\phi(x) = c\varphi(x), \quad b = c^2\beta, \quad a = c^3\alpha \tag{19}$$

normalizes the parameter c to unity, so that φ, β, and α satisfy the same system (6)–(8) but with $c = 1$. Hence, the smooth periodic waves are uniquely determined by the free parameters (a, b) and $c = 1$ can be used everywhere. For clarity of presentation, we will keep the parameter c in all equations until Lemma 10, where we will use the scaling transformation (19).

Remark 6. We only consider the case of right-propagating waves with $c > 0$; however, all results are extended to the left-propagating waves with $c < 0$ by simply flipping the signs in the scaling transformation (19).

Remark 7. Peaked periodic waves were shown to be stable in $H^1_{\rm per}$[20,21] and unstable in $W^{1,\infty}$.[22] For the smooth periodic waves, it was proven that if $m = u - u_{xx}$ is positive initially, then it remains positive for all times with the unique global continuation of the solutions with the global bound $\|u_x\|_{L^\infty} \le \|u\|_{L^\infty}$, where $\|u\|_{L^\infty}$ is controlled by the conserved momentum $E(u)$.[19,36] Therefore, the $W^{1,\infty}$-norm of the solution does not grow during the evolution of the smooth periodic waves.

Remark 8. For the limit to the smooth solitary waves at the boundary $a = a_+(b)$ specified in Lemma 3, we can show that this case corresponds to the solitary waves considered in Ref. 23 within a modified version of the CH equation given by

$$v_t - v_{txx} + 2kv_x + 3vv_x = 2v_x v_{xx} + vv_{xxx}, \tag{20}$$

where k is an arbitrary parameter. Indeed, if u is a solution of (1) such that $u(x,t) \to k$ as $|x| \to \infty$, then $v(x,t) = u(x + kt, t) - k$ is a solution of (20) such that $v(x,t) \to 0$ as $|x| \to \infty$. The smooth solitary waves of the form $v(x,t) = \hat{\phi}(x - \hat{c}t)$ exist for $\hat{c} > 2k$,[23] which coincides with the constraint $c = \hat{c} + k > 3k$ in (22) since $k = \phi_1$ from Lemma 3. The main result of Ref. 23 that $\mathcal{L}$ at the smooth solitary waves has exactly one simple negative eigenvalue coincides with the result of Remark 1 because the boundary $a = a_+(b)$ is located in a region, where $b > 0$ and the period function is decreasing in a.

Theorems 1 and 2 and Remark 1 are illustrated in Figure 1, where the existence region of the smooth periodic solutions enclosed by the three boundaries is shown on the (a, b) plane for a fixed $c > 0$. The blue curve shows the location of the single maximum of the period function $\mathfrak{L}(a, b, c)$ with respect to a, where $\partial_a \mathfrak{L} = 0$. Since $\partial_b \mathfrak{L} > 0$, it follows from the relation $\partial_a \mathfrak{L} + \mathcal{B}_L'(a)\partial_b \mathfrak{L} = 0$ that $\mathcal{B}_L'(a) = 0$ on this curve, while $\mathcal{B}_L'(a) > 0$ above and $\mathcal{B}_L'(a) < 0$ below the curve.

The black, green, cyan, and magenta curves in the (a, b) plane show the curves where smooth L-periodic solutions exist for four different values of L. The black curve with smaller L does not intersect the blue curve and the family of L-periodic solutions remains smooth both in a and b. However, the other three curves with larger periods intersect the blue curve and the family of L-periodic solutions is smooth in a but is not smooth in b at the blue curve where $\mathcal{B}_L'(a) = 0$.

The paper is organized as follows. In Section 2, we study the existence of the smooth periodic solutions and give the proof of the first two assertions in Theorem 1. In Section 3, we study the eigenvalues of the linearized operators $\mathcal{L}$ and $\mathcal{K}$ and give the proof of the last assertion of Theorem 1. In parallel, we also show the assertions in Remarks 1 and 2. In Section 4, we study the

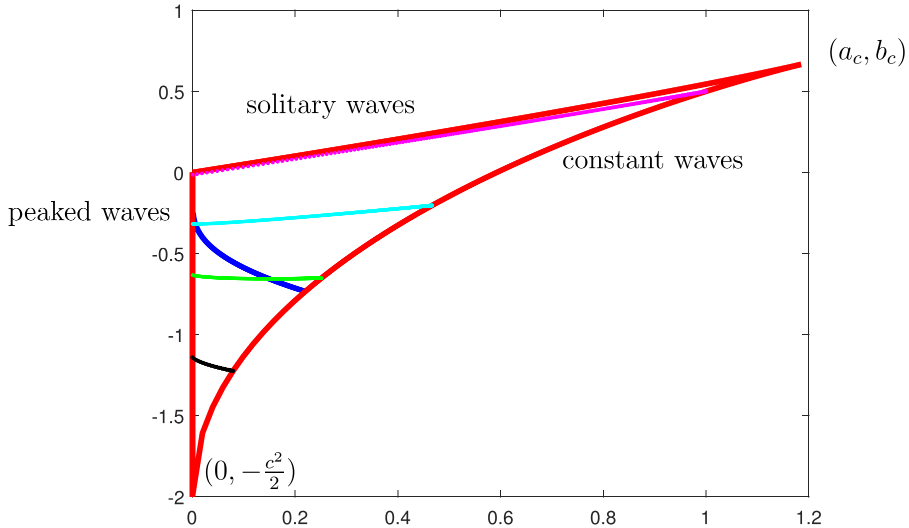

FIGURE 1 The existence region of smooth L-periodic solutions on the parameter plane (a, b) for $c = 2$ enclosed by three boundaries (red lines). The blue line shows the values of (a, b) for which the period function $\mathfrak{L}(a, b, c)$ has a maximum point in a at fixed (b, c). The black, green, cyan, and magenta lines show curves of the fixed period at $L = \pi/2$, $L = 3\pi/4$, $L = \pi$, and $L = 2\pi$, respectively

linearized evolution under two constraints and give the proof of spectral stability in Theorem 2 and Remark 3. In Section 5, we prove the orbital stability in Theorem 2 and also obtain orbital stability directly for $b \leq 0$ as in Remark 4. Section 6 concludes the paper with a summary and a discussion of open directions. Appendix A describes the approach used for numerical approximations of the smooth periodic waves.

2 | EXISTENCE OF SMOOTH PERIODIC TRAVELLING WAVES

Here we study existence of smooth periodic solutions of the system (6)–(8) and provide the proof of the first two assertions of Theorem 1.

Let us rewrite (8) as the total energy b of Newton's particle of unit mass with the coordinate ϕ in "time" x with the potential energy $U(\phi)$:

$$b = \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + U(\phi), \qquad U(\phi) = -\frac{1}{2}\phi^2 + \frac{a}{c-\phi}. \tag{21}$$

Critical points of U are given by roots of the cubic equation $a = \phi(c-\phi)^2$. The local maximum of $\phi \mapsto \phi(c-\phi)^2$ occurs at $\phi = \frac{c}{3}$, from which we define $a_c := \frac{4c^3}{27}$. For $a \in (-\infty, 0) \cup (a_c, \infty)$, there exists only one critical (maximum) point of U, whereas for $a \in (0, a_c)$, there exist three critical points of U, two are local maximum, and one is local minimum. In addition, $\phi = c$ is the pole singularity of $U(\phi)$ if $a \neq 0$. See Figure 2 for illustration of the three different cases of U.

It follows from dynamics of the Newton particle with the total energy in (21) that all smooth mappings $x \mapsto \phi(x)$ for $a \in (-\infty, 0] \cup [a_c, \infty)$ are unbounded. Although peaked and cusped

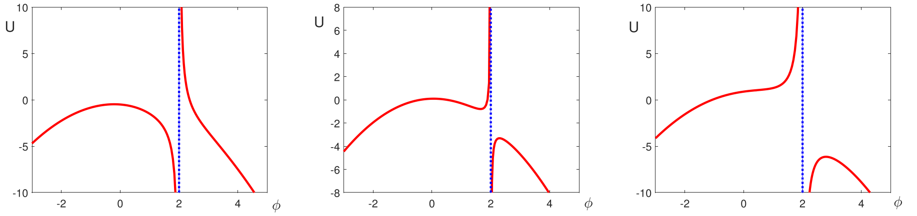

FIGURE 2 The graph of U versus ϕ in (21) for $a \in (-\infty, 0)$ (left), $a \in (0, a_c)$ (middle), and $a \in (a_c, \infty)$ (right) for $c = 2$

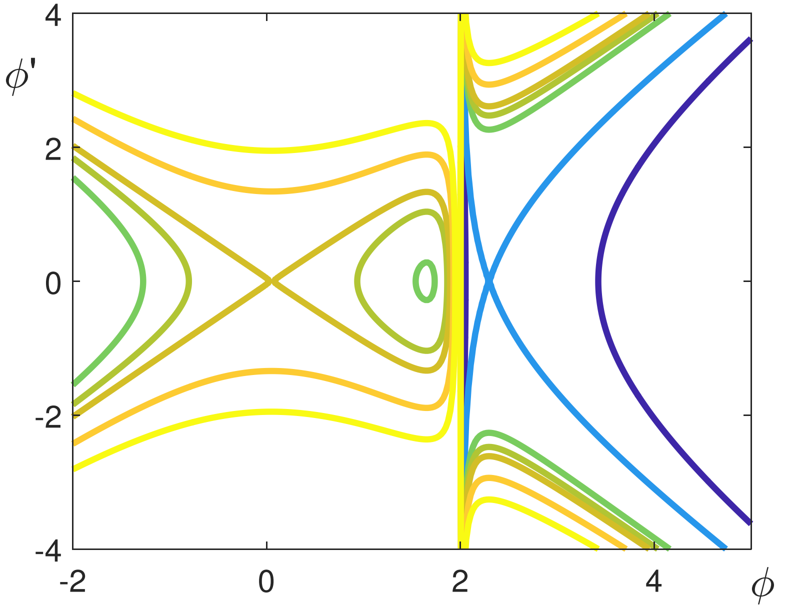

FIGURE 3 Phase portrait of the second-order equation (7) constructed from the level curves of the first-order invariant (21) on the phase plane (ϕ, ϕ') for $a = 0.4$ and $c = 2$

periodic solutions exist in this case,[15,22] we are only concerned with the smooth periodic solutions here.

For $a \in (0, a_c)$, we label the critical points of U as $\phi_1 < \phi_2 < \phi_3$. Roots of the cubic equation $a = \phi(c - \phi)^2$ satisfy the ordering

$$0 < \phi_1 < \frac{c}{3} < \phi_2 < c < \phi_3. \tag{22}$$

The local minimum of U at ϕ_2 gives the center of the second-order equation (7) at $(\phi_2, 0)$. This implies that the smooth periodic solutions form a *period annulus*, that is, a punctured neighborhood of the center $(\phi_2, 0)$ enclosed by the homoclinic orbit connecting the saddle $(\phi_1, 0)$. The phase portrait on the phase plane (ϕ, ϕ') with the period annulus around the center $(\phi_2, 0)$ is illustrated in Figure 3.

The smooth periodic solutions for fixed $c > 0$ and $a \in (0, a_c)$ are parameterized by the parameter b in (b_-, b_+), where $b_- = U(\phi_2)$ and $b_+ = U(\phi_1)$. The following result summarizes the existence of smooth periodic solutions.

Lemma 1. *Fix $c > 0$. For a fixed $a \in (0, a_c)$ with $a_c := \frac{4c^3}{27}$, there exists a family of smooth periodic solutions ϕ of the second-order equation (7) closed with the first-order invariant (21) parameterized by $b \in (b_-, b_+)$, where $b_- = U(\phi_2)$ and $b_+ = U(\phi_1)$. The solution ϕ is smooth with respect to parameters a, b, and c and satisfy $\phi \in (0, c)$.*

Proof. Every periodic solution ϕ of the second-order equation (7) corresponds to a periodic orbit of the planar system with the first integral given by (21). Its level set parametrized by $b \in (b_-, b_+)$ defines the periodic orbits inside the period annulus around the center $(\phi_2, 0)$, which exists if $a \in (0, a_c)$. Due to the ordering (22), the periodic solutions satisfy $\phi \in (\phi_1, c)$, which implies that $\phi \in (0, c)$. As the first-order invariant (21) is smooth with respect to parameters a, b, and c, the periodic orbits inside the period annulus are also smooth with respect to parameters. ■

Let us now define the period function $\mathfrak{L}(a, b, c)$ for the smooth L-periodic solutions of Lemma 1. For fixed $a \in (0, a_c)$, $b \in (b_-, b_+)$, and $c > 0$, let ϕ_+ and ϕ_- be turning points of the Newton's particle satisfying the ordering

$$0 < \phi_1 < \phi_- < \phi_2 < \phi_+ < c < \phi_3. \tag{23}$$

The turning points are roots of the algebraic equation

$$(c - \phi_\pm)(2b + \phi_\pm^2) = 2a. \tag{24}$$

Without loss of generality, we place the maximum of ϕ at $x = 0$ and the minimum of ϕ at $x = \pm L/2$ so that $\phi(0) = \phi_+$ and $\phi(\pm L/2) = \phi_-$. As the extremal values of ϕ are nondegenerate if $b \in (b_-, b_+)$, then $\phi''(0) < 0$ and $\phi''(\pm L/2) > 0$. It follows from (24) that

$$(c\phi_\pm - \frac{3}{2}\phi_\pm^2 - b)\partial_a \phi_\pm = 1, \tag{25}$$

$$(c\phi_\pm - \frac{3}{2}\phi_\pm^2 - b)\partial_b \phi_\pm = -(c - \phi_\pm) \tag{26}$$

and since $c - \phi_\pm > 0$ and

$$\phi''(0) = \frac{c\phi_+ - \frac{3}{2}\phi_+^2 - b}{c - \phi_+} < 0, \quad \phi''(\pm L/2) = \frac{c\phi_- - \frac{3}{2}\phi_-^2 - b}{c - \phi_-} > 0, \tag{27}$$

we have $\partial_a \phi_\pm, \partial_b \phi_\pm \neq 0$ with

$$\text{sign}(\partial_a \phi_\pm) = -\text{sign}(\partial_b \phi_\pm). \tag{28}$$

The period function $\mathfrak{L}(a, b, c)$ is defined by integrating the quadrature (21)

$$\mathfrak{L}(a, b, c) := \int_{\phi_-}^{\phi_+} \frac{2\sqrt{c-\phi}\,d\phi}{\sqrt{(c-\phi)(2b+\phi^2)-2a}}. \tag{29}$$

The following three lemmas clarify how the smooth periodic solutions transform when (a, b) approach each boundary of the existence region shown in Figure 1 for a fixed $c > 0$.

Lemma 2. *Fix $c > 0$ and $a \in (0, a_c)$. The smooth periodic solutions of Lemma 1 transform as $b \to b_-(a)$ to the constant solutions. The limiting period function*

$$\mathfrak{L}_-(a) := \mathfrak{L}(a, b_-(a), c)$$

satisfies $\mathfrak{L}'_-(a) > 0$ with $\mathfrak{L}_-(a) \to 0$ as $a \to 0$ and $\mathfrak{L}_-(a) \to \infty$ as $a \to a_c$. The mapping $a \mapsto b_-(a)$ is C^1 and invertible with the inverse $a = a_-(b)$ for $b \in (-\frac{1}{2}c^2, \frac{1}{6}c^2)$.

Proof. It follows from the ordering (23) that the boundary $b = b_-(a)$ corresponds to the center $\phi_- = \phi_+ = \phi_2$. Hence, $\phi(x) = \phi_2$ is constant in x. Linearization of the second-order equation (7) at the center point $(\phi_2, 0)$ determines the period $\mathfrak{L}_-(a) := \mathfrak{L}(a, b_-(a), c)$ in the form:

$$\mathfrak{L}_-(a) = \frac{2\pi}{\omega}, \quad \omega = \sqrt{\frac{2a}{(c-\phi_2)^3} - 1}. \tag{30}$$

Along the curve $b = b_-(a)$, a and b can be parametrized by ϕ_2 as

$$\begin{cases} b = c\phi_2 - \frac{3}{2}\phi_2^2, \\ a = \phi_2(c-\phi_2)^2, \end{cases} \tag{31}$$

which follow from Equations (6) and (7) using that $\phi = \phi_2$ is constant.

Solving the first (quadratic) equation in (31) for ϕ_2 as

$$\phi_2 = \frac{c}{3} + \frac{\sqrt{c^2-6b}}{3}$$

and substituting the second (cubic) equation in (31) for a into (30) yields

$$\omega^2 = \frac{2a}{(c-\phi_2)^3} - 1 = \frac{3\phi_2 - c}{c - \phi_2} = \frac{3\sqrt{c^2-6b}}{2c - \sqrt{c^2-6b}}.$$

This allows us to express b explicitly in terms of $\mathfrak{L}_-(a)$ by

$$b = \frac{c^2}{6}\left[1 - \frac{64\pi^4}{(4\pi^2 + 3\mathfrak{L}_-(a)^2)^2}\right]. \tag{32}$$

It follows from (32) that $\mathfrak{L}_-(a)$ increases in b along the curve $b = b_-(a)$ and satisfies $\mathfrak{L}_-(a) \to 0$ as $b \to -\frac{1}{2}c^2$ (or equivalently, $a \to 0$) and $\mathfrak{L}_-(a) \to \infty$ as $b \to \frac{1}{6}c^2$ (or equivalently, $a \to a_c$). As the parametrization (31) implies that

$$\frac{db}{d\phi_2} = c - 3\phi_2, \quad \frac{da}{d\phi_2} = (c-\phi_2)(c-3\phi_2) \quad \Rightarrow \quad \frac{da}{db} = c - \phi_2, \tag{33}$$

and $\phi_2 < c$, the mapping $a \mapsto b_-(a)$ is C^1, invertible, and monotonically increasing from $(a,b) = (0, -\frac{1}{2}c^2)$ to $(a,b) = (\frac{4}{27}c^3, \frac{1}{6}c^2)$. Hence $\mathfrak{L}_-(a)$ is also increasing in a along the curve $b = b_-(a)$. ■

Lemma 3. *Fix $c > 0$ and $a \in (0, a_c)$. The smooth periodic solutions of Lemma 1 transform as $b \to b_+(a)$ to the solitary wave solutions with*

$$\mathfrak{L}_+(a) := \mathfrak{L}(a, b_+(a), c) = \infty.$$

The mapping $a \mapsto b_+(a)$ is C^1 and invertible with the inverse $a = a_+(b)$ for $b \in (0, \frac{1}{6}c^2)$.

Proof. It follows from ordering (23) that the boundary $b = b_+(a)$ corresponds to $\phi_- = \phi_1$. Hence, $\phi(x)$ is the solitary wave solution satisfying $\phi(x) \to \phi_1$ as $x \to \pm\infty$ so that $\mathfrak{L}_+(a) := \mathfrak{L}(a, b_+(a), c) = \infty$. Along the curve $b = b_+(a)$, a and b can be parametrized by ϕ_1 as

$$\begin{cases} b = c\phi_1 - \frac{3}{2}\phi_1^2, \\ a = \phi_1(c-\phi_1)^2, \end{cases} \tag{34}$$

which follow from Equations (6) and (7) using that $\phi = \phi_1$ is a constant solution if $b = U(\phi_1)$. By the same argument as in (33) but with ϕ_2 replaced by ϕ_1, the mapping $a \mapsto b_+(a)$ is C^1, invertible, and monotonically increasing from $(a,b) = (0,0)$ to $(a,b) = (\frac{4}{27}c^3, \frac{1}{6}c^2)$. ■

Lemma 4. *Fix $c > 0$ and $b \in (-\frac{1}{2}c^2, 0)$. The smooth periodic solutions of Lemma 1 transform as $a \to 0$ to the peaked periodic solutions and the period function*

$$\mathfrak{L}_0(b) := \mathfrak{L}(0, b, c)$$

satisfies $\mathfrak{L}_0'(b) > 0$ with $\mathfrak{L}_0(b) \to 0$ as $b \to -\frac{1}{2}c^2$ and $\mathfrak{L}_0(b) \to \infty$ as $b \to 0$.

Proof. If $a = 0$, then ϕ satisfies the equation $\phi'' - \phi = 0$ with

$$\max_{x \in [-\frac{L}{2}, \frac{L}{2}]} \phi(x) = \phi(0) = c$$

since $\phi_+ = c$ and $\phi_- = \sqrt{2|b|}$. This equation can be solved explicitly

$$\phi(x) = c\frac{\cosh\left(\frac{L}{2} - |x|\right)}{\cosh\left(\frac{L}{2}\right)}, \quad x \in \left[-\frac{L}{2}, \frac{L}{2}\right]. \tag{35}$$

The periodic wave is peaked at $x = 0$ and smooth at $x = \pm\frac{L}{2}$ with $\phi'(\pm\frac{L}{2}) = 0$. It follows from (8) and (35) that

$$b = \frac{1}{2}\left[(\phi')^2 - \phi^2\right] = -\frac{c^2}{2\cosh^2\left(\frac{L}{2}\right)}, \tag{36}$$

in agreement with $\phi_- = \sqrt{2|b|}$. Hence, $b \in (-\frac{1}{2}c^2, 0)$ and it follows from (36) that $L = \mathfrak{L}_0(b)$ increases in b and satisfies $\mathfrak{L}_0(b) \to 0$ as $b \to -\frac{1}{2}c^2$ and $\mathfrak{L}_0(b) \to \infty$ as $b \to 0$. ■

Remark 9. The two boundaries of Lemmas 2 and 3 intersect at $a = a_c = \frac{4c^3}{27}$, where the two critical points coallesce: $\phi_1 = \phi_2 = \frac{c}{3}$. This corresponds to $b = b_c = \frac{c^2}{6}$. The two boundaries intersect with the third boundary $a = 0$ of Lemma 4 at $b = 0$ and $b = -\frac{1}{2}c^2$, respectively.

Finally, we prove the main result of this section that the period function $\mathfrak{L}(a, b, c)$ is a strictly increasing function of b for any fixed $a \in (0, a_c)$ and $c > 0$.

Theorem 3. *Fix $c > 0$ and $a \in (0, a_c)$, where $a_c := \frac{4c^3}{27}$. The period function $\mathfrak{L}(a, b, c)$ is strictly increasing in b.*

Proof. Let $a = \phi_2(c - \phi_2)^2$, where ϕ_2 is the second root in the ordering (22). Using the transformation $\{x = \frac{\phi - \phi_2}{\phi_2}, y = \frac{\phi'}{\phi_2}\}$, we can write the second-order equation (7) as the planar system

$$\begin{cases} x' = y, \\ y' = 1 + x - \frac{\eta^2}{(\eta - x)^2}, \end{cases} \tag{37}$$

associated with the Hamiltonian

$$H(x, y) = \frac{y^2}{2} + V(x), \quad V(x) := -\frac{x^2}{2} - x - \eta + \frac{\eta^2}{\eta - x}, \tag{38}$$

where $\eta = \frac{c - \phi_2}{\phi_2} \in (0, 2)$.

The potential V is smooth away from the singular line $x = \eta$, and has a local minimum at $x = 0$ and two maxima at $x_1 := \eta - \frac{1}{2} - \frac{\sqrt{4\eta+1}}{2}$ and $x_3 := \eta - \frac{1}{2} + \frac{\sqrt{4\eta+1}}{2}$. The center at the origin is surrounded by periodic orbits γ_h, which lie inside the level curves $H(x, y) = h$ with $h \in (0, h^*)$ and $h^* = V(x_1)$. Denote by x_2 the unique solution of $V(x_1) = V(x)$ such that $x_1 < 0 < x_2 < \eta < x_3$, see Figure 4. Finally, define the period function of the center (0,0) of system (37) by

$$\ell(h) = \int_{\gamma_h} \frac{dx}{y} \quad \text{for each } h \in (0, h^*). \tag{39}$$

Note that $b = \phi_2^2(h + \eta - \frac{1}{2})$ and $\mathfrak{L}(a, b, c) = \ell(h)$ for fixed $a \in (0, a_c)$ and $c > 0$. Since ϕ_2 is fixed, we have $\partial_b \mathfrak{L}(a, b, c) > 0$ if and only if $\ell'(h) > 0$.

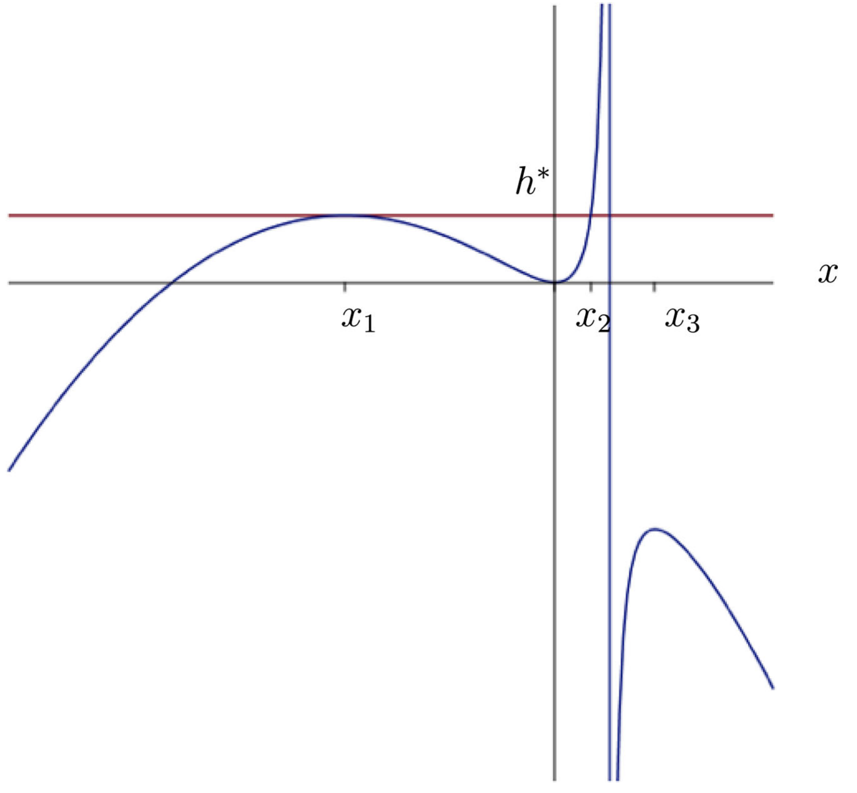

FIGURE 4 The potential function $V(x)$ plotted for $\eta = \frac{1}{4}$

To prove that $\ell'(h) > 0$, we use a monotonicity criterion by Chicone[37] for planar systems with Hamiltonians of the form (38), where V is a smooth function on (x_1, x_2) with a nondegenerate relative minimum at the origin. The period function $\ell(h)$ is monotonically increasing in h if the function

$$W(x) := \frac{V(x)}{[V'(x)]^2}$$

is convex in (x_1, x_2). Hence, we have to prove that $W''(x) > 0$ for every $x \in (x_1, x_2)$. A straightforward computation shows that

$$W''(x) = \frac{-3(\eta - x)R(x)}{(x^2 + (1-2\eta)x + \eta(\eta-2))^4}, \tag{40}$$

where

$$R(x) = (-2\eta + 1)x^3 + \eta(6\eta - 7)x^2 - 3\eta^2(2\eta - 3)x + \eta^2(2\eta + 1)(\eta - 2).$$

Since $x_2 < \eta$, we need to show that $R(x) < 0$ for $x \in [x_1, x_2]$ and $\eta \in (0, 2)$. Note that $R(0) = \eta^2(2\eta + 1)(\eta - 2) < 0$ for $\eta \in (0, 2)$.

The discriminant of R with respect to x is given by

$$\mathrm{Disc}_x(R) = -4(4\eta + 1)(4\eta^2 - 16\eta + 27)\eta^4, \tag{41}$$

which is strictly negative for $\eta \in (0, 2)$. Hence, for $\eta \neq \frac{1}{2}$, the cubic polynomial R has exactly one real root, say x_0.

For $\eta < \frac{1}{2}$, it follows from the dominant behavior of R that $R(x) \to -\infty$ as $x \to -\infty$. Since $R(\eta) = -2\eta^2 < 0$, it is clear that the only real root x_0 is located for $x_0 > \eta$. Therefore, $R(x) < 0$ for $x \in [x_1, x_2]$ with $x_2 < \eta$.

For $\eta > \frac{1}{2}$, we have $R(x) \to -\infty$ as $x \to +\infty$. We claim that

$$R(x_1) = \frac{1}{2}((\eta - 1)\sqrt{4\eta + 1} - \eta - 1)(4\eta + 1) < 0 \tag{42}$$

for $\eta \in (0, 2)$. Therefore, the only real root x_0 is located for $x_0 < x_1$ and $R(x) < 0$ for $x \in [x_1, x_2]$. To prove (42), we substitute $\eta = (w^2 - 1)/4$ into $R(x_1)$ and obtain $R(x_1) = \frac{1}{4}(w - 3)(w + 1)^2 w^2$, which is negative for $w \in (1, 3)$.

Finally, for $\eta = \frac{1}{2}$ we have that $R(x) = -2x^2 + 3x/2 - 3/4$, which is strictly negative for all x.

Hence, $R(x) < 0$ for $x \in [x_1, x_2]$ if $\eta \in (0, 2)$. Therefore, $W''(x) > 0$ for $x \in (x_1, x_2)$ and $\ell'(h) > 0$ follows by the theorem proven in Ref. 37. ■

Remark 10. The result of Theorem 3 can also be verified using the tools from Ref. 38 and 39, where Hamiltonian systems with Hamiltonian in the form $H(x, y) = \frac{1}{2}y^2 + V(x)$ are considered with $V(x) = \frac{1}{2m}x^{2m} + o(x^{2m})$, which is analytic in a neighborhood of $x = 0$.

Remark 11. As claimed in Remark 1, the period function $\mathfrak{L}(a, b, c)$ has different monotonicity properties in a for fixed $b \in (-\frac{1}{2}c^2, \frac{1}{6}c^2)$ and $c > 0$. To be precise, the period function

- is monotonically increasing in a if $b \in (-\frac{1}{2}c^2, -(1 - \frac{\sqrt{2}}{\sqrt{3}})c^2]$;
- has a single maximum point in a if $b \in (-(1 - \frac{\sqrt{2}}{\sqrt{3}})c^2, 0)$;
- is monotonically decreasing in a if $b \in [0, \frac{1}{6}c^2)$.

This result was obtained in Theorem 2.5 of Ref. 18, where the second-order equation (6) with the first-order invariant (8) was reformulated into the system

$$\dot{x} = y, \quad \dot{y} = -\frac{y^2 + x - 3x^2}{2(x + \nu)}, \tag{43}$$

where

$$\nu := \frac{1}{6}\left[\frac{2c}{\sqrt{c^2 - 6b}} - 1\right].$$

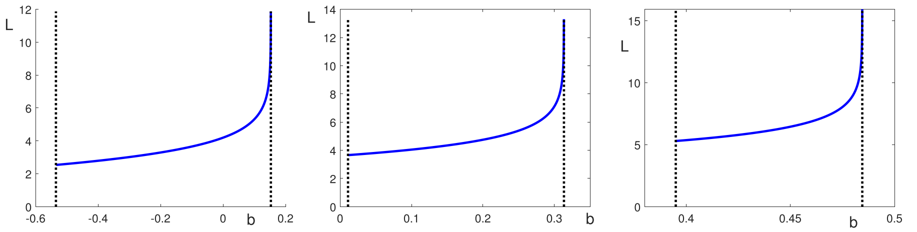

FIGURE 5 The period function $\mathfrak{L}(a, b, c)$ versus parameter b for the smooth periodic solutions with $c = 2$ and three values of a: (left) $a = 0.3$, (middle) $a = 0.6$, and (right) $a = 0.9$

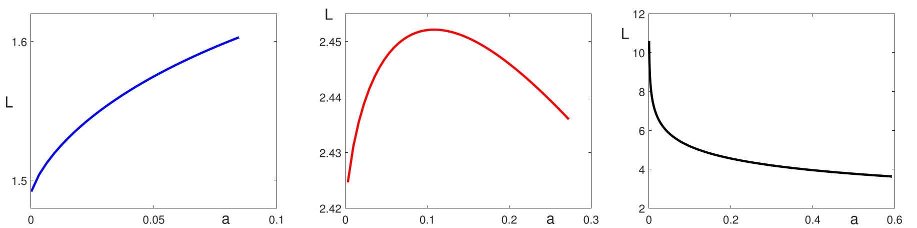

FIGURE 6 The period function $\mathfrak{L}(a, b, c)$ versus parameter a for the smooth periodic solutions with $c = 2$ and three values of b: (left) $b = -1.2$, (middle) $b = -0.6$, and (right) $b = 0$

The value $b = -\frac{1}{2}c^2$ corresponds to $\nu = 0$, the value $b = -(1 - \frac{\sqrt{2}}{\sqrt{3}})c^2$ corresponds to $\nu = -\frac{1}{10} + \frac{\sqrt{6}}{15}$, the value $b = 0$ corresponds to $\nu = \frac{1}{6}$, and the value $b = \frac{1}{6}c^2$ corresponds to the limit $\nu \to \infty$.

Figure 5 shows the graphs of $\mathfrak{L}(a, b, c)$ versus b for three cases of a. The period function is monotonically increasing in b in agreement with Theorem 3. Figure 6 shows the graphs of $\mathfrak{L}(a, b, c)$ versus a for three representative cases of b. The period function is increasing in a for $b \in (-\frac{1}{2}c^2, -(1 - \frac{\sqrt{2}}{\sqrt{3}})c^2]$ (left), has a single maximum point in a if $b \in (-(1 - \frac{\sqrt{2}}{\sqrt{3}})c^2, 0)$ (middle), and is monotonically decreasing in a if $b \in [0, \frac{1}{6}c^2)$ (right), in agreement with Remark 11. The numerical method used to generate Figures 5 and 6 is described in Appendix A.

For the study of spectral stability of periodic solutions in Section 4, it is important to fix the period L and consider the family of L-periodic solutions along a curve in the (a, b) plane for a fixed $c > 0$. The following result provides this characterization of the L-periodic solutions.

Lemma 5. *Fix $c > 0$ and $L > 0$. There exists a C^1 mapping $a \mapsto b = \mathcal{B}_L(a)$ for $a \in (0, a_L)$ with some $a_L \in (0, \frac{4}{27}c^3)$ and a C^1 mapping $a \mapsto \phi = \Phi_L(\cdot, a) \in H^\infty_{\rm per}$ of smooth L-periodic solutions along the curve $b = \mathcal{B}_L(a)$.*

Proof. It follows from the monotonicity results in Lemmas 2 and 4 that for every $c > 0$ and $L > 0$, there exists exactly one L-periodic solution on the left and right boundaries of the existence domain on the (a, b)-plane. The left boundary corresponds to $a = 0$ and the right boundary corresponds to $a = a_L$, where a_L is uniquely defined from the equation $\mathfrak{L}(a_L, b_-(a_L), c) = L$ with $b_-(a)$ defined in Lemma 1.

Since $\mathfrak{L}(a,b,c)$ is smooth in (a,b,c) and it is strictly increasing in b by Theorem 3, the existence of the C^1 mapping $a \mapsto b = \mathcal{B}_L(a)$ for $a \in (0,a_L)$ follows by the implicit function theorem for $\mathfrak{L}(a,b,c) = L$ with fixed $c > 0$ and $L > 0$. Indeed, $\partial_a \mathfrak{L} + \mathcal{B}_L'(a)\partial_b \mathfrak{L} = 0$ and since $\partial_b \mathfrak{L} > 0$, $\mathcal{B}_L'(a)$ is uniquely defined for every $a \in (0,a_L)$. Since ϕ is smooth with respect to parameters by Lemma 1, the mapping $a \mapsto \phi = \Phi_L(\cdot,a) \in H^\infty_{\rm per}$ is C^1 along the curve $b = \mathcal{B}_L(a)$. ■

Remark 12. The mapping $b \mapsto \phi = \Psi_L(\cdot,b) \in H^\infty_{\rm per}$ may not be C^1 along the curve $b = \mathcal{B}_L(a)$ because of the nonmonotonicity of $\mathfrak{L}(a,b,c)$ with respect to a. It follows from Remark 11 that there exists at most one point where $\mathcal{B}_L'(a) = 0$ and this is the minimum of the mapping $a \mapsto b = \mathcal{B}_L(a)$. The mapping $b \mapsto \phi = \Psi_L(\cdot,b) \in H^\infty_{\rm per}$ is not C^1 at the minimum point.

3 | SPECTRAL PROPERTIES OF THE LINEARIZED OPERATOR

Here, we study the linearization of the CH equation (1) at the smooth periodic solutions of the system (6)–(8) and provide the proof of the last assertion of Theorem 1.

Adding a perturbation v to the smooth travelling wave ϕ propagating with the same fixed speed c in

$$u(x,t) = \phi(x-ct) + v(x-ct,t) \tag{44}$$

gives the perturbation equation derived from the CH equation (1):

$$(1-\partial_x^2)(v_t - cv_x) + 3\partial_x(\phi v) + 2vv_x = \partial_x(\phi v_{xx} + \phi' v_x + \phi'' v) + 2v_x v_{xx} + v v_{xxx}. \tag{45}$$

Dropping the quadratic terms in v yields the linearized evolution equation

$$v_t = -J\mathcal{L}v, \tag{46}$$

where J is defined in (9) and $\mathcal{L}$ is the linearized operator given by (11).

Recall that the travelling periodic wave is spectrally stable in the sense of Definition 1 if the spectrum of the linearized operator $J\mathcal{L}$ in $L^2_{\rm per}$ is located on $i\mathbb{R}$. The following lemma reformulates the spectral stability criterion in terms of the linearized operator $\mathcal{K}$ introduced in (17).

Lemma 6. *The spectrum of $J\mathcal{L}$ in $L^2_{\rm per}$ is located on the imaginary axis if and only if the spectrum of $(c-\phi)^{-1}\partial_x(c-\phi)^{-1}\mathcal{K}$ in $L^2_{\rm per}$ is located on the imaginary axis.*

Proof. Consider the time evolution of the CH equation in the form (14), where $m := u - u_{xx}$. We add a perturbation p to the smooth travelling wave μ in

$$m(x,t) = \mu(x-ct) + p(x-ct,t). \tag{47}$$

It follows from the decompositions (44) and (47) that $\mu = \phi - \phi''$ and $p = v - v_{xx}$. Substituting (44) and (47) into (14) gives the perturbation equation

$$p_t + (\phi - c)p_x + 2p\phi' + v\mu' + 2\mu v_x + vp_x + 2pv_x = 0. \tag{48}$$

Dropping the quadratic terms in p and v yields the linearized evolution equation

$$p_t + (\phi - c)p_x + 2p\phi' + v\mu' + 2\mu v_x = 0. \tag{49}$$

It follows from (7) that $(c-\phi)^2\mu = a$ and hence

$$v\mu' + 2\mu v_x = \frac{2a\phi'}{(c-\phi)^3}v + \frac{2a}{(c-\phi)^2}v_x. \tag{50}$$

Multiplying (49) by $c - \phi$ and using (50) yield the equivalent evolution form:

$$\frac{\partial}{\partial t}[(c-\phi)p] = \frac{\partial}{\partial x}\left[(c-\phi)^2 p - \frac{2a}{c-\phi}v\right], \tag{51}$$

which can be written in the form

$$p_t = (c-\phi)^{-1}\partial_x (c-\phi)^{-1}\mathcal{K}p, \tag{52}$$

where $\mathcal{K}$ is the linearized operator given by (17). It follows from the equivalence of (46) and (52) under the transformation $v = (1-\partial_x^2)^{-1}p$ that $\lambda \in \sigma(J\mathcal{L})$ in $L^2_{\rm per}$ if and only if $\lambda \in \sigma[(c-\phi)^{-1}\partial_x(c-\phi)^{-1}\mathcal{K}]$ in $L^2_{\rm per}$, where $\sigma(A)$ denotes the spectrum of a linear operator A in $L^2_{\rm per}$. ■

In what follows we study the spectra of the linearized operators $\mathcal{L}$ and $\mathcal{K}$ in $L^2_{\rm per}$. The following lemma shows that the spectra of these operators are different.

Lemma 7. *The spectrum of $\mathcal{L}$ in $L^2_{\rm per}$ is purely discrete. The spectrum of $\mathcal{K}$ in $L^2_{\rm per}$ consists of the strictly positive continuous spectrum at*

$$\text{image}[(c-\phi)^3] = [(c-\phi_+)^3, (c-\phi_-)^3]$$

and the discrete spectrum outside image$[(c-\phi)^3]$, *where $\phi_\pm$ are the turning points defined in (24).*

Proof. Since $c - \phi > 0$ and $\phi \in H^\infty_{\rm per}$, the linearized operator $\mathcal{L}$ with the dense domain $H^2_{\rm per} \subset L^2_{\rm per}$ is a self-adjoint, unbounded operator in $L^2_{\rm per}$. Consequently, $\sigma(\mathcal{L}) \subset \mathbb{R}$ is purely discrete in $L^2_{\rm per}$ due to the compact embedding of $H^2_{\rm per}$ into $L^2_{\rm per}$.

Since $c - \phi > 0$, the linearized operator $\mathcal{K}$ is a self-adjoint, bounded operator in $L^2_{\rm per}$, which is the sum of a bounded and a compact operator in $L^2_{\rm per}$. Consequently, $\sigma(\mathcal{K}) \subset \mathbb{R}$ includes both the continuous and discrete spectra in $L^2_{\rm per}$ denoted by σ_c and σ_d, respectively. Since the compact operator $-2a(1-\partial_x^2)^{-1}$ is in the trace class in $L^2_{\rm per}$, Kato's theorem (Theorem 4.4 in Ref. 40) gives

$$\sigma_c(\mathcal{K}) = \sigma_c((c-\phi)^3) = \text{image}[(c-\phi)^3] = [(c-\phi_+)^3, (c-\phi_-)^3].$$

Since $\phi_+ < c$, $\sigma_c(\mathcal{K})$ is strictly positive. ■

The following two theorems describe the nonpositive part of the spectrum of $\mathcal{L}$ and $\mathcal{K}$ in $L^2_{\rm per}$. The proofs rely on Theorem 3.1 in Ref. 41 (see also the classical Floquet theory in Refs. 42 and 43) and on Sylvester's inertial law theorem (see Theorem 2.2 in Ref. 44). These auxilary results are formulated in the following two propositions.

Proposition 1 (Ref. 41). *Let $\mathcal{M} := -\partial_x^2 + Q(x)$ be the Schrödinger operator with the even, $L-$periodic, smooth potential Q. Assume that $\mathcal{M}w = 0$ is satisfied by a linear combination of two solutions φ_1 and φ_2 such that*

$$\varphi_1(x+L) = \varphi_1(x) + \theta\varphi_2(x)$$

and

$$\varphi_2(x+L) = \varphi_2(x)$$

with some $\theta \in \mathbb{R}$. Assume that φ_2 has two zeros on the period of Q. The zero eigenvalue of $\mathcal{M}$ in $L^2_{\rm per}$ is simple if $\theta \neq 0$ and double if $\theta = 0$. It is the second eigenvalue of $\mathcal{M}$ if $\theta \geq 0$ and the third eigenvalue of $\mathcal{M}$ if $\theta < 0$.

Remark 13. Compared to Ref. 41, we have interchanged the order of nonperiodic φ_1 and periodic φ_2 so that our θ is negative relative to θ used in Ref. 41.

Proposition 2 (Ref. 44). *Let L be a self-adjoint operator in a Hilbert space H and S be a bounded invertible operator in H. Then, SLS^* and L have the same inertia, which is the dimension of the negative, null, and positive invariant subspaces of H.*

We can now formulate and prove two theorems on the nonpositive part of the spectrum of $\mathcal{L}$ and $\mathcal{K}$ in $L^2_{\rm per}$.

Theorem 4. *The linearized operator $\mathcal{L} : H^2_{\rm per} \subset L^2_{\rm per} \to L^2_{\rm per}$ admits*

- *two negative eigenvalues and a simple zero eigenvalue if $\partial_a \mathfrak{L} > 0$;*
- *one negative eigenvalue and a double zero eigenvalue if $\partial_a \mathfrak{L} = 0$;*
- *one negative eigenvalue and a simple zero eigenvalue if $\partial_a \mathfrak{L} < 0$,*

where $\mathfrak{L}(a,b,c)$ is the period function for the smooth periodic wave ϕ of Lemma 1. The rest of the spectrum of $\mathcal{L}$ in $L^2_{\rm per}$ is strictly positive and bounded away from zero.

Proof. Due to the invariance of the CH equation (1) with respect to spatial translations, the third-order equation (5) is equivalent to $\mathcal{L}\phi' = 0$, which means that $\phi' \in \ker(\mathcal{L}) \subset H^2_{\rm per}$. On the other hand, differentiating of the second-order equation (6) in a is equivalent to $\mathcal{L}\partial_a\phi = 0$, which means that $\partial_a\phi$ is the second, linearly independent solution of $\mathcal{L}v = 0$. Note that $\partial_a\phi$ is well defined by Lemma 1 but may not be L-periodic in x.

Let $\{y_1, y_2\}$ be the fundamental set of solutions associated with the equation $\mathcal{L}v = 0$ in $H^2(0,L)$ such that

$$\begin{cases} y_1(0) = 1, \\ y_1'(0) = 0, \end{cases} \qquad \begin{cases} y_2(0) = 0, \\ y_2'(0) = 1. \end{cases} \tag{53}$$

As previously, we set $\phi(0) = \phi(L) = \phi_+$ for the smooth L-periodic solution of Lemma 1, where ϕ_+ is the turning point for the maximum of ϕ in x. Hence, we have $\phi'(0) = \phi'(L) = 0$ so that we define

$$y_1(x) := \frac{\partial_a \phi(x)}{\partial_a \phi_+}, \quad y_2(x) := \frac{\phi'(x)}{\phi''(0)}, \tag{54}$$

where $\partial_a \phi_+ \neq 0$ and $\phi''(0) \neq 0$ as follows from (25) and (27). Differentiating of the boundary conditions $\phi(L) = \phi_+$ and $\phi'(L) = 0$ for $L = \mathfrak{L}(a, b, c)$ in a yields $y_1(L) = y_1(0) = 1$ and

$$y_1'(L) = -\frac{\partial_a \mathfrak{L}}{\partial_a \phi_+} \phi''(0) =: \theta, \tag{55}$$

which implies that

$$y_1(x + L) = y_1(x) + \theta y_2(x). \tag{56}$$

Since $c - \phi_+ > 0$, it follows from (25) and (27) that $\text{sign}(\theta) = -\text{sign}(\partial_a \mathfrak{L})$.

To transform the spectral problem $\mathcal{L}v = \lambda v$ to the spectral problem $\mathcal{M}w = \lambda w$ for the Schrödinger operator $\mathcal{M}$ in Proposition 1, we write $\mathcal{L}v = \lambda v$ as the second-order differential equation

$$p(x)v'' + q(x)v' + (r(x) + \lambda)v = 0, \tag{57}$$

with $p(x) := c - \phi(x)$, $q(x) := -\phi'(x)$, and $r(x) := -\phi''(x) + 3\phi(x) - c$. The Liouville transformation

$$D(x) = -\int_0^x \frac{\phi'(s)}{c - \phi(s)} ds = \ln\left(\frac{c - \phi(x)}{c - \phi(0)}\right) \tag{58}$$

is nonsingular since $c - \phi > 0$. Substituting the change of variables

$$v(x) = w(x) e^{-\frac{1}{2}D(x)} = w(x)\sqrt{\frac{c - \phi(0)}{c - \phi(x)}}. \tag{59}$$

into the second-order equation (57), we obtain the equivalent equation

$$-w''(x) + Q(x)w(x) = \lambda(c - \phi(x))^{-1} w(x), \tag{60}$$

where

$$Q(x) := \frac{c - 3\phi(x)}{c - \phi(x)} + \frac{\phi''(x)}{2(c - \phi(x))} + \frac{1}{4}\left(\frac{\phi'(x)}{c - \phi(x)}\right)^2.$$

With the transformation $w = (c - \phi)^{1/2}\hat{w}$, the spectral problem (60) is equivalent to the spectral problem for the operator $S\mathcal{M}S$, where $\mathcal{M} := -\partial_x^2 + Q(x)$ is self-adjoint in L^2_{per} and $S = (c - \phi)^{1/2}$

is a bounded and invertible multiplication operator in L^2_{per}. By Proposition 2, the numbers of negative and zero eigenvalues of the spectral problem (60) coincide with those of the operator $\mathcal{M}$.

The operator $\mathcal{M}$ satisfies the condition of Proposition 1 because Q is even, L−periodic, and smooth. As the set $\{y_1, y_2\}$ is a fundamental set for the equation $\mathcal{L}v = 0$ and the initial conditions $v(0) = w(0)$ and $v'(0) = w'(0)$ are preserved in the transformation (59), it follows that

$$\{\varphi_1, \varphi_2\} := \left\{ \left(\frac{c-\phi(0)}{c-\phi}\right)^{-1/2} y_1, \left(\frac{c-\phi(0)}{c-\phi}\right)^{-1/2} y_2 \right\} \tag{61}$$

is the fundamental set of solutions to $\mathcal{M}w = 0$. It follows from (56) and (61) that

$$\varphi_1(x+L) = \varphi_1(x) + \theta\varphi_2(x). \tag{62}$$

where θ is given by the same expression (55). Furthermore, since ϕ' has two zeros in $\mathbb{T}_L$, the same is true for y_2 and φ_2. By the standard Floquet theory in Refs. 42 and 43, it follows that $\lambda = 0$ is the second or third eigenvalue of $\mathcal{M}$ in L^2_{per}. If $\theta = 0$, then $\lambda = 0$ is the double eigenvalue so that it is the second eigenvalue of $\mathcal{M}$. If $\theta \neq 0$, then $\lambda = 0$ is a simple eigenvalue of $\mathcal{M}$. By Proposition 1, it is the second eigenvalue if $\theta \geq 0$ and the third eigenvalue if $\theta < 0$. Due to the equivalence provided by the nonsingular transformation (59), the same is true for the operator $\mathcal{L}$ in L^2_{per}, which yields the assertion of the theorem since $\text{sign}(\theta) = -\text{sign}(\partial_a \mathfrak{L})$. ■

Theorem 5. *The linearized operator $\mathcal{K} : L^2_{\text{per}} \to L^2_{\text{per}}$ admits*

- *two negative eigenvalues and a simple zero eigenvalue if $\partial_b \mathfrak{L} < 0$;*
- *one negative eigenvalue and a double zero eigenvalue if $\partial_b \mathfrak{L} = 0$;*
- *one negative eigenvalue and a simple zero eigenvalue if $\partial_b \mathfrak{L} > 0$,*

where $\mathfrak{L}(a, b, c)$ is the period function for the smooth periodic wave ϕ of Lemma 1. The rest of the spectrum of $\mathcal{K}$ is strictly positive and bounded away from zero.

Proof. The linear operator $\mathcal{K}$ is congruent to another operator $\mathcal{K}_0$ by the transformation

$$\mathcal{K} = (1-\partial_x^2)^{-1/2}\mathcal{K}_0(1-\partial_x^2)^{-1/2}, \tag{63}$$

where

$$\mathcal{K}_0 = (1-\partial_x^2)^{1/2}(c-\phi)^3(1-\partial_x^2)^{1/2} - 2a. \tag{64}$$

Since $S := (1-\partial_x^2)^{-1/2}$ is a bounded and invertible operator in L^2_{per} and $\mathcal{K}_0$ is self-adjoint in L^2_{per}, it follows by Proposition 2 that $\mathcal{K}$ and $\mathcal{K}_0$ in (63) have the same inertia, that is, the dimension of the negative, null, and positive invariant subspaces of L^2_{per}. By Lemma 7, the positive invariant subspace of $\mathcal{K}$ is infinite-dimensional. Hence, we study the nonpositive spectrum of $\mathcal{K}_0$.

It follows from (64) that $\mathcal{K}_0$ is an unbounded self-adjoint operator defined in L^2_{per} with densely defined domain $H^2_{\text{per}} \subset L^2_{\text{per}}$. The spectrum of $\sigma(\mathcal{K}_0)$ is given by the union of the continuous and discrete spectra. However, as the embedding of H^2_{per} into L^2_{per} is compact, the continuous spectrum is an empty set. Hence, we consider the spectral problem for the discrete spectrum:

$$\mathcal{K}_0 w = \lambda w, \quad w \in H^2_{\mathrm{per}}, \tag{65}$$

where $\lambda \in \mathbb{R}$ is an isolated eigenvalue of $\mathcal{K}_0$ and $w \not\equiv 0$ is the corresponding eigenfunction. Considering the change of variables $w := (1-\partial_x^2)^{1/2} v$, it follows from (65) that

$$(1-\partial_x^2)^{1/2}\left[(c-\phi)^3(1-\partial_x^2)v - (2a+\lambda)v\right] = 0. \tag{66}$$

Since $(1-\partial_x^2)^{1/2}$ is invertible in L^2_{per}, the spectral problem (66) is equivalent to the spectral problem

$$\mathcal{M} v = \lambda (c-\phi)^{-3} v, \tag{67}$$

where $\mathcal{M}$ is the Schrödinger operator given by

$$\mathcal{M} := -\partial_x^2 + 1 - \frac{2a}{(c-\phi)^3}. \tag{68}$$

With the transformation $v = (c-\phi)^{3/2}\hat{v}$, the spectral problem (67) is equivalent to that for the operator $S\mathcal{M}S$, where $S := (c-\phi)^{3/2}$ is a bounded and invertible operator in L^2_{per} and $\mathcal{M}$ is a self-adjoint operator in L^2_{per}. By Proposition 2, operators $\mathcal{M}$ and $S\mathcal{M}S$ have the same inertia in L^2_{per}.

Finally, we study the nonpositive spectrum of $\mathcal{M}$. It follows from the differential equation (7) that

$$\mathcal{M}\phi' = 0, \quad \mathcal{M}\partial_b\phi = 0. \tag{69}$$

Therefore, the general solution of $\mathcal{M}v = 0$ is given by a linear combination of two linearly independent solutions

$$y_1(x) := \frac{\partial_b\phi(x)}{\partial_b\phi_+}, \quad y_2(x) := \frac{\phi'(x)}{\phi''(0)}, \tag{70}$$

where $\partial_b\phi_+ \neq 0$ and $\phi''(0) \neq 0$ as follows from (25) and (27). Differentiating of the boundary conditions $\phi(L) = \phi_+$, and $\phi'(L) = 0$ for $L = \mathfrak{L}(a,b,c)$ in b yields $y_1(L) = y_1(0) = 1$ and

$$y_1'(L) = -\frac{\partial_b\mathfrak{L}}{\partial_b\phi_+}\phi''(0),$$

so that

$$y_1(x+L) = y_1(x) + \theta y_2(x), \quad \theta := y_1'(L) = -\frac{\partial_b\mathfrak{L}}{\partial_b\phi_+}\phi''(0). \tag{71}$$

Since $c - \phi_+ > 0$, $\phi''(0) < 0$ and $\phi''(0)\partial_b\phi_+ = -1$, as follows from (25) and (27), we obtain $\mathrm{sign}(\theta) = \mathrm{sign}(\partial_b\mathfrak{L})$. The assertion of the theorem follows by Proposition 1 due to equivalence of the negative and null subspaces of $\mathcal{K}_0$ and $\mathcal{M}$ and the inertial law between $\mathcal{K}$ and $\mathcal{K}_0$ and between $\mathcal{M}$ and $S\mathcal{M}S$. ■

Remark 14. By Remark 11, we have $\partial_a \mathfrak{L} > 0$ for every point (a, b) below the blue curve in the existence region of Figure 1 and $\partial_a \mathfrak{L} < 0$ for every point (a, b) above the blue curve. Therefore, the count of negative eigenvalues of the linearized operator $\mathcal{L}$ in Theorem 4 changes depending on the point (a, b). However, by Theorem 3, $\partial_b \mathfrak{L} > 0$ for every point (a, b) inside the existence region, hence the linearized operator $\mathcal{K}$ in Theorem 5 admits a simple negative eigenvalue and a simple zero eigenvalue for every (a, b) in the existence region.

4 | SPECTRAL STABILITY OF PERIODIC WAVES

Here, we study the linearized CH equations (46) and (52) and prove the spectral stability of periodic waves stated in Theorem 2. We start by deducing the constraints on the perturbations $v \in H^2_{\rm per}$ and $p \in L^2_{\rm per}$ satisfying these linearized equations.

Lemma 8. *Let $v_0 \in H^2_{\rm per} \cap X_0$, where X_0 is given by*

$$X_0 := \left\{ v \in L^2_{\rm per} : \quad \langle 1, v\rangle = 0, \quad \langle \phi - \phi'', v\rangle = 0 \right\}. \tag{72}$$

If $v \in C^0(\mathbb{R}, H^2_{\rm per}) \cap C^1(\mathbb{R}, H^1_{\rm per})$ is a solution to the linearized CH equation (46) with initial data v_0, then $v(t, \cdot) \in H^2_{\rm per} \cap X_0$ for all $t \in \mathbb{R}$.

Proof. Conservation of the two orthogonality conditions in X_0 in the time evolution of the linearized CH equation (46) is checked directly using integration by parts:

$$\frac{d}{dt}\langle 1, v\rangle = \langle 1, \partial_x (1 - \partial_x^2)^{-1} \mathcal{L} v\rangle = 0$$

and

$$\frac{d}{dt}\langle \phi - \phi'', v\rangle = \langle \phi - \phi'', (1 - \partial_x^2)^{-1} \partial_x \mathcal{L} v\rangle = \langle \phi', \mathcal{L} v\rangle = \langle \mathcal{L}\phi', v\rangle = 0,$$

where we recall that $\mathcal{L}\phi' = 0$. Integrations by parts are justified since $\phi \in H^\infty_{\rm per}$ and $v(t, \cdot) \in H^2_{\rm per}$ is in the domain of $\mathcal{L}$. ■

Corollary 1. *Let $p_0 \in L^2_{\rm per} \cap Y_0$, where Y_0 is given by*

$$Y_0 := \left\{ p \in L^2_{\rm per} : \quad \langle 1, p\rangle = 0, \quad \langle \phi, p\rangle = 0 \right\}. \tag{73}$$

If $p \in C^0(\mathbb{R}, L^2_{\rm per}) \cap C^1(\mathbb{R}, H^{-1}_{\rm per})$ is a solution to the linearized CH equation (52) with initial data p_0, then $p(t, \cdot) \in L^2_{\rm per} \cap Y_0$ for all $t \in \mathbb{R}$.

Proof. Orthogonality conditions in (73) follow from those in (72) by using the relation $v = (1 - \partial_x^2)^{-1} p$ between solution $v \in C^0(\mathbb{R}, H^2_{\rm per}) \cap C^1(\mathbb{R}, H^1_{\rm per})$ of (46) and the corresponding solution

$p \in C^0(\mathbb{R}, L^2_{\rm per}) \cap C^1(\mathbb{R}, H^{-1}_{\rm per})$ of (52). In particular,

$$\langle 1, v \rangle = \langle 1, v - v_{xx} \rangle = \langle 1, p \rangle$$

and

$$\langle \phi - \phi'', v \rangle = \langle \phi, v - v_{xx} \rangle = \langle \phi, p \rangle.$$

The linearized equations (46) and (52) are equivalent by Lemma 6. ■

Remark 15. The two orthogonality conditions in (72) are related to the conservation of mass (2) and energy (3) by adding a perturbation of v to the smooth periodic wave ϕ and truncating the quadratic terms in v. The third orthogonality condition related to the higher order energy (4) is redundant due to the other two conditions:

$$\langle \frac{3}{2}\phi^2 - \phi\phi'' - \frac{1}{2}(\phi')^2, v \rangle = c\langle \phi - \phi'', v \rangle - b\langle 1, v \rangle = 0, \tag{74}$$

where the second-order equation (6) has been used.

The following lemma together with Lemma 6 gives the sufficient condition for spectral stability of the periodic wave in Definition 1.

Lemma 9. *Let $\mathcal{K}|_{Y_0}$ be the restriction of $\mathcal{K}$ on $Y_0 \subset L^2_{\rm per}$. If*

$$\mathcal{K}|_{Y_0} \geq 0 \quad \text{and} \quad \ker(\mathcal{K}|_{Y_0}) = \ker(\mathcal{K}), \tag{75}$$

then the spectrum of $(c-\phi)^{-1}\partial_x(c-\phi)^{-1}\mathcal{K}$ in $L^2_{\rm per}$ is located on the imaginary axis.

Proof. Consider the spectral problem

$$J_\phi \mathcal{K} p = \lambda p, \quad p \in H^1_{\rm per}, \tag{76}$$

where $J_\phi := (c-\phi)^{-1}\partial_x(c-\phi)^{-1}$ satisfies $J_\phi^* = -J_\phi$ in $L^2_{\rm per}$. The spectrum of $J_\phi\mathcal{K}$ is purely discrete due to compact embedding of $H^1_{\rm per}$ into $L^2_{\rm per}$.

If λ_0 is an eigenvalue of $J_\phi\mathcal{K}$ in $L^2_{\rm per}$ and $\lambda_0 \neq 0$, then the corresponding eigenfunction p_0 satisfies $p_0 \in H^1_{\rm per} \cap Y_0$. Indeed, since $p = p_0 e^{\lambda_0 t}$ is a solution of $p_t = J_\phi \mathcal{K} p$ and

$$\frac{d}{dt}\langle 1, p \rangle = 0, \quad \frac{d}{dt}\langle \phi, p \rangle = 0,$$

by Corollary 1, then $\langle 1, p_0 \rangle = 0$ and $\langle \phi, p_0 \rangle = 0$ if $\lambda_0 \neq 0$.

A simple computation shows that for the eigenfunction $p_0 \in H^1_{\rm per} \cap Y_0$, we have

$$\lambda_0 \langle \mathcal{K} p_0, p_0 \rangle = \langle \mathcal{K} J_\phi \mathcal{K} p_0, p_0 \rangle = -\langle \mathcal{K} p_0, J_\phi \mathcal{K} p_0 \rangle = -\bar{\lambda}_0 \langle \mathcal{K} p_0, p_0 \rangle,$$

so that

$$(\lambda_0 + \bar{\lambda}_0)\langle \mathcal{K} p_0, p_0 \rangle = 0.$$

Since $p_0 \in H^1_{\text{per}} \cap Y_0$, then $\langle \mathcal{K} p_0, p_0 \rangle = 0$ if and only if $p_0 \in \ker(\mathcal{K})$ due to assumptions of the lemma. However, this is a contradiction with $\lambda_0 \neq 0$. Hence, $\langle \mathcal{K} p_0, p_0 \rangle > 0$, which implies that $\lambda_0 \in i\mathbb{R}$. This proves the assertion of the lemma. ∎

For the proof of spectral stability in Theorem 2, it remains to justify the sufficient condition (75) for the operator $\mathcal{K}$. The following proposition from Theorem 4.1 in Ref. 45 formulates the useful result.

Proposition 3 (Ref. 45). *Let L be a self-adjoint operator in a Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ such that L has $n(L)$ negative eigenvalues (counting their multiplicities) and $z(L)$ multiplicity of the zero eigenvalue bounded away from the positive spectrum of L. Let $\{v_j\}_{j=1}^N$ be a linearly independent set in H and define*

$$H_0 := \{f \in H : \quad \langle f, v_1 \rangle = \langle f, v_2 \rangle = \cdots = \langle f, v_N \rangle = 0\}.$$

Let $A(\lambda)$ be the matrix-valued function defined by its elements

$$A_{ij}(\lambda) := \langle (L - \lambda I)^{-1} v_i, v_j \rangle, \quad 1 \leq i, j \leq N, \quad \lambda \notin \sigma(L).$$

Then,

$$\begin{cases} n(L|_{H_0}) = n(L) - n_0 - z_0, \\ z(L|_{H_0}) = z(L) + z_0 - z_\infty, \end{cases} \tag{77}$$

where n_0, z_0, and p_0 are the numbers of negative, zero, and positive eigenvalues of $\lim_{\lambda \uparrow 0} A(\lambda)$ (counting their multiplicities) and $z_\infty = N - n_0 - z_0 - p_0$ is the number of eigenvalues of $A(\lambda)$ diverging in the limit $\lambda \uparrow 0$.

By Lemma 5, for a fixed $c > 0$ and $L > 0$, there exists a C^1 mapping $a \mapsto b = \mathcal{B}_L(a)$ and a C^1 mapping $a \mapsto \phi = \Phi_L(\cdot, a) \in H^\infty_{\text{per}}$ of smooth L-periodic solutions along the curve $b = \mathcal{B}_L(a)$. Along this curve, we define

$$\mathcal{M}_L(a) := M(\Phi_L(\cdot, a)) \quad \text{and} \quad \mathcal{E}_L(a) := E(\Phi_L(\cdot, a)), \tag{78}$$

where $M(u)$ and $E(u)$ are given by (2) and (3). To include the dependence on c, we will now write $\mathcal{M}_L(a, c)$ and $\mathcal{E}_L(a, c)$. The following lemma provides the criterion for positivity of $\mathcal{K}|_{Y_0}$ based on Proposition 3.

Lemma 10. *For fixed $c > 0$ and $L > 0$, the condition (75) is satisfied if and only if*

$$\frac{d}{da} \frac{\mathcal{E}_L(a)}{\mathcal{M}_L(a)^2} < 0 \tag{79}$$

along the curve $b = \mathcal{B}_L(a)$.

Proof. Since $\Phi_L(\cdot, a) \in H^\infty_{\text{per}}$ is also C^1 with respect to c as follows from the scaling transformation (19), we are allowed to differentiate the second-order equation (7) in a and c. Writing this equation as $(c-\phi)^3\mu = a(c-\phi)$ for $\mu = \phi - \phi''$ and differentiating it in a and c, we obtain

$$\mathcal{K}\partial_a\mu = c - \phi, \quad \mathcal{K}\partial_c\mu = -2a. \tag{80}$$

Since $a > 0$, we express

$$\mathcal{K}^{-1}1 = -\frac{1}{2a}\partial_c\mu, \quad \mathcal{K}^{-1}\phi = -\partial_a\mu - \frac{c}{2a}\partial_c\mu.$$

By Proposition 3, we construct the bounded 2×2 matrix $P := \lim_{\lambda\uparrow 0} A(\lambda)$ in

$$P = \begin{bmatrix} \langle \mathcal{K}^{-1}1, 1\rangle & \langle \mathcal{K}^{-1}\phi, 1\rangle \\ \langle \mathcal{K}^{-1}1, \phi\rangle & \langle \mathcal{K}^{-1}\phi, \phi\rangle \end{bmatrix} = \begin{bmatrix} -\frac{1}{2a}\partial_c\mathcal{M}_L & -\partial_a\mathcal{M}_L - \frac{c}{2a}\partial_c\mathcal{M}_L \\ -\frac{1}{2a}\partial_c\mathcal{E}_L & -\partial_a\mathcal{E}_L - \frac{c}{2a}\partial_c\mathcal{E}_L \end{bmatrix}, \tag{81}$$

where $\mathcal{M}_L$ and $\mathcal{E}_L$ in (78) are C^1 functions in a and c. It follows from (81) that

$$\det(P) = \frac{1}{2a}[\partial_c\mathcal{M}_L\partial_a\mathcal{E}_L - \partial_a\mathcal{M}_L\partial_c\mathcal{E}_L]. \tag{82}$$

By using the scaling transformation (19), we write

$$\mathcal{M}_L(a,c) = c\hat{\mathcal{M}}_L(\alpha), \quad \mathcal{E}_L(a,c) = c^2\hat{\mathcal{E}}_L(\alpha), \quad a = c^3\alpha, \tag{83}$$

where $\hat{\mathcal{M}}_L$ and $\hat{\mathcal{E}}_L$ can be computed by formally setting $c = 1$. Substituting the transformation (83) into (82), we obtain

$$\begin{aligned} \det(P) &= \frac{1}{2\alpha c^4}\left[\hat{\mathcal{M}}_L(\alpha)\hat{\mathcal{E}}_L'(\alpha) - 2\hat{\mathcal{E}}_L(\alpha)\hat{\mathcal{M}}_L'(\alpha)\right] \\ &= \frac{\hat{\mathcal{M}}_L^3(\alpha)}{2\alpha c^4}\frac{d}{d\alpha}\left(\frac{\hat{\mathcal{E}}_L(\alpha)}{\hat{\mathcal{M}}_L(\alpha)^2}\right). \end{aligned} \tag{84}$$

Thus, $\det(P) < 0$ if and only if the condition (79) is satisfied for a given $c > 0$. Since $n(\mathcal{K}) = 1$ and $z(\mathcal{K}) = 1$ by Theorem 1 independently of a and c, we use the count formulas (77) to get $n(\mathcal{K}|_{Y_0}) = 0$ and $z(\mathcal{K}|_{Y_0}) = 1$ since $n_0 = 1$, $z_0 = z_\infty = 0$. Hence, the conditions (75) are satisfied if and only if the condition (79) is satisfied. ∎

Numerical results show that the condition (79) is satisfied for every $c > 0$ and $L > 0$ along the curve $b = \mathcal{B}_L(a)$ for $a \in (0, a_L)$ in Lemma 5. Figure 7 shows that the mapping $a \mapsto \frac{\mathcal{E}_L(a)}{\mathcal{M}_L^2(a)}$ is monotonically decreasing for four values of L shown also in Figure 1. The numerical method used to generate Figure 7 is described in Appendix A.

In the rest of this section, we will first explain why the linearized CH equation (46) associated with the operator $J\mathcal{L}$ is not convenient for the proof of spectral stability of the smooth periodic

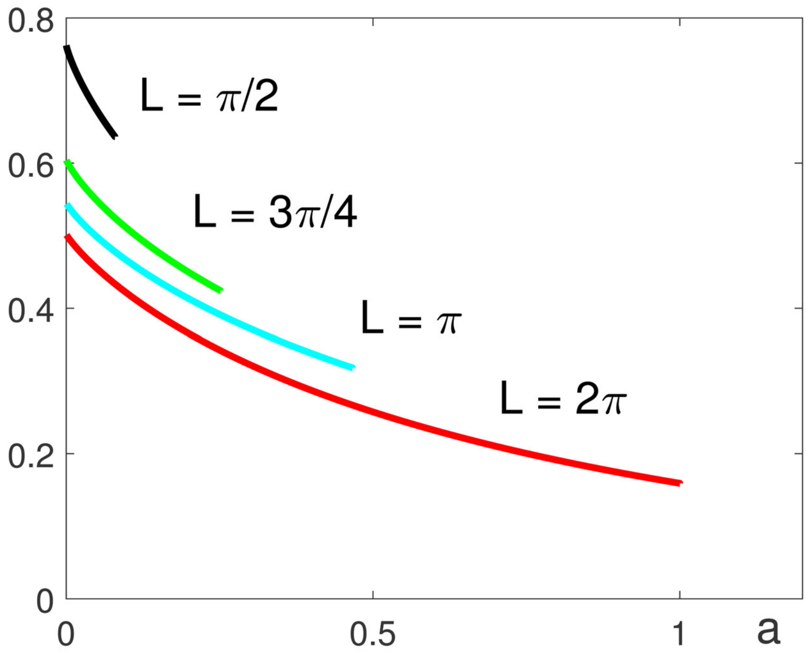

FIGURE 7 The dependence of $\mathcal{E}_L/\mathcal{M}_L^2$ versus a along the curve $b = \mathcal{B}_L(a)$ for $c = 2$ and five values of period: $L = \pi/2$ (black), $L = 3\pi/4$ (green), $L = \pi$ (cyan), and $L = 2\pi$ (red)

waves. One of the problems is that the count of negative eigenvalues of the linearized operator $\mathcal{L}$ depends on the point (a, b) in the existence region, see Remark 14. Moreover, the C^1 continuation of the smooth periodic waves with respect to parameter b is not possible at the points where $\partial_a \mathfrak{L} = 0$, see Lemma 11. Nevertheless, we will show in Remark 16 that we can still obtain the spectral stability of the smooth periodic waves from the linearized operator $\mathcal{L}$ by using numerical computations.

Lemma 11. *For fixed $c > 0$ and $L > 0$, there exists a C^1 mapping $b \mapsto \phi = \Psi_L(\cdot, b) \in H^\infty_{\rm per}$ of smooth L-periodic solutions of Lemma 1 if and only if $\partial_a \mathfrak{L} \neq 0$, where $\mathfrak{L}(a, b, c)$ is the period function.*

Proof. If $\partial_a \mathfrak{L} \neq 0$, then arguments of the proof of Lemma 5 based on the implicit function theorem and smoothness of periodic solutions of Lemma 1 with respect to parameters give existence of the C^1 mapping $b \mapsto \phi = \Psi_L(\cdot, b) \in H^\infty_{\rm per}$.

In the converse direction, we assume existence of the C^1 mapping $b \mapsto \phi = \Psi_L(\cdot, b) \in H^\infty_{\rm per}$ and prove that $\partial_a \mathfrak{L} \neq 0$. Due to the C^1 smoothness, it follows by differentiating the second-order equation (6) in c and b that

$$\mathcal{L}\partial_c \phi = \phi'' - \phi \quad \text{and} \quad \mathcal{L}\partial_b \phi = 1. \tag{85}$$

Let $\{y_1, y_2\}$ be the fundamental set of solutions associated with the equation $\mathcal{L}v = 0$ in $H^2(0, L)$ as in (53) and (54). By Liouville's theorem, the associated Wronskian is given by

$$\mathcal{W}(y_1, y_2)(x) = e^{\int_0^x \frac{\phi'(s)}{c-\phi(s)} ds} = \frac{c - \phi(0)}{c - \phi(x)} > 0. \tag{86}$$

Now, since $\mathcal{W}(y_1, y_2)(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ for all $x \in \mathbb{R}$ and $y_2(x) = \phi'(x)/\phi''(0)$, we obtain by (86) that

$$\phi''(0) \int_0^L \frac{c - \phi(0)}{c - \phi(x)} dx = \int_0^L \left[y_1(x)\phi''(x) - y_1'(x)\phi'(x) \right] dx. \tag{87}$$

By contradiction, assume that $\partial_a \mathfrak{L} = 0$, then y_1 is L-periodic similar to y_2. Integration by parts in (87) yields

$$\phi''(0) \int_0^L \frac{c - \phi(0)}{c - \phi(x)} dx = 2\langle \phi'', y_1 \rangle. \tag{88}$$

It follows from (85) that

$$\langle \phi'', y_1 \rangle = \langle \phi, y_1 \rangle, \qquad \langle 1, y_1 \rangle = 0. \tag{89}$$

On the other hand, we also have $\mathcal{L}1 = c - 3\phi + \phi''$, hence

$$\langle \phi'', y_1 \rangle - 3\langle \phi, y_1 \rangle + c\langle 1, y_1 \rangle = 0. \tag{90}$$

Substituting (89) into (90) yields $\langle \phi'', y_1 \rangle = 0$, which is a contradiction with the nonzero left-hand side in (88). Hence, $\partial_a \mathfrak{L} = 0$ leads to the contradiction with the C^1 smoothness of the mapping $b \mapsto \phi = \Psi_L(\cdot, b) \in H^\infty_{\text{per}}$. ■

Remark 16. We claim that

$$\mathcal{L}|_{X_0} \geq 0 \quad \text{and} \quad \ker(\mathcal{L}|_{X_0}) = \ker(\mathcal{L}), \tag{91}$$

for each point in the existence region.

Let us first consider points (a, b) in the parameter space where $\partial_a \mathfrak{L} \neq 0$. In this case, the 2×2 matrix of projections in Proposition 3 can be constructed and evaluated for the operator $\mathcal{L}$ under the two orthogonality conditions in X_0 given by (72) as follows:

$$S := \begin{bmatrix} \langle \mathcal{L}^{-1}1, 1 \rangle & \langle \mathcal{L}^{-1}(\phi - \phi''), 1 \rangle \\ \langle \mathcal{L}^{-1}1, (\phi - \phi'') \rangle & \langle \mathcal{L}^{-1}(\phi - \phi''), (\phi - \phi'') \rangle \end{bmatrix} = \begin{bmatrix} \partial_b \mathcal{M}_L & -\partial_b \mathcal{E}_L \\ \partial_c \mathcal{M}_L & -\partial_c \mathcal{E}_L \end{bmatrix}, \tag{92}$$

where $\mathcal{M}_L$ and $\mathcal{E}_L$ are computed at $\phi = \Psi_L(\cdot, b) \in H^\infty_{\text{per}}$ and extended in both b and c. By using the scaling transformation (19), we write

$$\mathcal{M}_L(c, b) = c\hat{\mathcal{M}}_L(\beta), \quad \mathcal{E}_L(c, b) = c^2 \hat{\mathcal{E}}_L(\beta), \quad b = c^2\beta. \tag{93}$$

Substituting the transformation (93) into (92) yields

$$\det(S) = \hat{\mathcal{M}}_L(\beta)\hat{\mathcal{E}}_L'(\beta) - 2\hat{\mathcal{E}}_L(\beta)\hat{\mathcal{M}}_L'(\beta) = \hat{\mathcal{M}}_L^3(\beta) \frac{d}{d\beta} \left(\frac{\hat{\mathcal{E}}_L(\beta)}{\hat{\mathcal{M}}_L(\beta)^2} \right). \tag{94}$$

Here, the derivative is computed along the curve $b = \mathcal{B}_L(a)$, where $n(\mathcal{L}) = 1$ if $\mathcal{B}_L'(a) > 0$ and $n(\mathcal{L}) = 2$ if $\mathcal{B}_L'(a) < 0$, see Remark 14. In the former case, numerical results show that $\det(S) < 0$ so that $n_0 = 1$, $z_0 = z_\infty = 0$ and by Proposition 3, we have $n(\mathcal{L}|_{X_0}) = 0$ and $z(\mathcal{L}|_{X_0}) = 1$. In the latter case, numerical results give $\det(S) > 0$ and $\hat{\mathcal{M}}_L'(\beta) < 0$ so that the 2×2 matrix S is negative with $n_0 = 2$, $z_0 = z_\infty = 0$, and by Proposition 3, we still have $n(\mathcal{L}|_{X_0}) = 0$ and $z(\mathcal{L}|_{X_0}) = 1$.

At the points (a, b) where $\partial_a \mathfrak{L} = 0$, we see by Lemma 11 that the relation $\mathcal{L}\partial_b\phi = 1$ in (85) cannot be used to express the projection matrix because $\partial_b\phi$ and hence $\mathcal{L}^{-1}1$ are not well defined. The other entries are well defined, and so S is unbounded with $z_\infty = 1$. Note that $n_0 = 1$ by continuity because there is at least one negative eigenvalue in the regions where $\mathcal{B}_L'(a) \neq 0$. Since $n(\mathcal{L}) = 1$ and $z(\mathcal{L}) = 2$, this gives $n(\mathcal{L}|_{X_0}) = 0$ and $z(\mathcal{L}|_{X_0}) = 1$.

Thus, we obtain the conditions (91) for the linearized operator $\mathcal{L}$ but with three different computations depending on whether $\partial_a \mathfrak{L} < 0$, $\partial_a \mathfrak{L} = 0$, and $\partial_a \mathfrak{L} > 0$.

5 | ORBITAL STABILITY OF PERIODIC WAVES

Here, we prove the orbital stability of periodic waves stated in Theorem 2. We follow the approach in Ref. 27, where the following useful result was proven in Proposition 3.8 and Theorem 4.2.

Proposition 4 (Ref. 27)**.** *Let $V(u)$ be a conserved quantity in the time evolution of the Hamiltonian system (9). Assume that the linearized operator $\mathcal{L}$ at the periodic travelling wave with profile ϕ admits a simple negative and a simple zero eigenvalue with* $\ker(\mathcal{L}) = \mathrm{span}(\phi')$ *satisfying $\langle V'(\phi), \phi' \rangle = 0$. Assume that there exists* $\Upsilon \in H^2_{\mathrm{per}}$ *such that $\langle \mathcal{L}\Upsilon, v \rangle = 0$ for every $v \in L^2_{\mathrm{per}}$ such that $\langle V'(\phi), v \rangle = 0$. If* $\langle \mathcal{L}\Upsilon, \Upsilon \rangle < 0$*, then the periodic travelling wave is orbitally stable in the time evolution of (9) in H^1_{per}.*

Remark 17. The notion of orbital stability in Definition 2 prescribes the existence of global solutions $u \in C(\mathbb{R}, H^s_{\mathrm{per}})$ for $s > \frac{3}{2}$. The local solutions $u \in C((-t_0, t_0), H^s_{\mathrm{per}})$ for some $t_0 > 0$ exist due to the local well-posedness theory in Refs. 10, 11, and 12. As M, E, and F are conserved quantities, one can combine the local solution with the standard a priori estimates $M(u(t)) = M(u_0)$, $E(u(t)) = E(u_0)$, and $F(u(t)) = F(u_0)$ for all $t \geq 0$ to extend the local to global solutions near the smooth periodic waves in the case when they are stable by Proposition 4.

Remark 18. The result of Proposition 4 can be equivalently written for the Hamiltonian system (12) with the conserved quantity $V(m)$ written in variable $m = u - u_{xx}$ and with the linearlized operator $\mathcal{K}$ at the periodic travelling wave $\mu = \phi - \phi''$.

The following lemma transfers the spectral stability criterion in Lemma 10 to the orbital stability criterion.

Lemma 12. *For fixed $c > 0$ and $L > 0$, the smooth L-periodic wave with profile $\phi = \Phi_L(\cdot, a) \in H^\infty_{\mathrm{per}}$ is orbitally stable in H^1_{per} if the mapping*

$$a \mapsto \frac{\mathcal{E}_L(a)}{\mathcal{M}_L(a)^2} \tag{95}$$

is strictly decreasing along the curve $b = \mathcal{B}_L(a)$.

Proof. For $\mu = \phi - \phi''$, let us rewrite (80) in the form:

$$\mathcal{K}\left(\frac{1}{2a}\partial_c \mu\right) = -1, \quad \mathcal{K}\left(\partial_a \mu + \frac{c}{2a}\partial_c \mu\right) = -\phi. \tag{96}$$

For $m = u - u_{xx}$, we define a linear superposition of the two conserved quantities (2) and (3):

$$V(m) := rM(u) + sE(u), \quad u = (1-\partial_x^2)^{-1} m, \tag{97}$$

where r and s are real coefficients, $M(u)$ is given by (2), and $E(u)$ is given by (13). Since $V'(\mu) = r + s\phi$ and $\ker(\mathcal{K}) = \mathrm{span}(\mu')$, we check that $\langle V'(\mu), \mu' \rangle = 0$ since $\phi \in H^\infty_{\text{per}}$. By Theorem 1, the linearized operator $\mathcal{K}$ satisfies the assumption of Proposition 4. We then proceed by constructing Υ. Letting

$$\Upsilon := \frac{r}{2a}\partial_c \mu + s\left(\partial_a \mu + \frac{c}{2a}\partial_c \mu\right),$$

it follows from (96) that $\mathcal{K}\Upsilon = -r - s\phi$ and $\langle \mathcal{K}\Upsilon, p \rangle = 0$, for all $p \in Y_0$ defined in (73). A straightforward calculation gives us that

$$\langle \mathcal{K}\Upsilon, \Upsilon \rangle = r^2 \langle \mathcal{K}^{-1} 1, 1 \rangle + 2rs \langle \mathcal{K}^{-1}\phi, 1 \rangle + s^2 \langle \mathcal{K}^{-1}\phi, \phi \rangle. \tag{98}$$

The quadratic form (98) in r and s is defined by the same 2×2 symmetric matrix P as in (81). If condition (95) is satisfied, we have that $\det(P) < 0$ and there exists a choice of real coefficients r and s such that $\langle \mathcal{K}\Upsilon, \Upsilon \rangle < 0$. Hence, the orbital stability of the periodic waves in the time evolution of the Hamiltonian system (12) follows from Proposition 4 and Remark 18. ■

In what follows, we show that the orbital stability condition is satisfied for every $b \le 0$. The main advantage of this result is that we do not need to verify the criterion (95) by using numerical computations. The following lemma reports the relevant result.

Lemma 13. *For fixed $c > 0$ and $b \le 0$, the smooth periodic wave with profile $\phi \in H^\infty_{\text{per}}$ is orbitally stable in H^1_{per}.*

Proof. It follows from (7) and (17) that

$$\mathcal{K}\mu = (c-\phi)^3 \mu - 2a(1-\partial_x^2)^{-1}\mu = a(c - 3\phi). \tag{99}$$

We can define $V(m) := cM(u) - 3E(u)$ so that if $\langle V'(\mu), p \rangle = 0$, then $\langle \mathcal{K}\mu, p \rangle = 0$. Thus, we can take $\Upsilon := \mu$ and compute

$$\langle \mathcal{K}\mu, \mu \rangle = a[cM(\phi) - 6E(\phi)]. \tag{100}$$

Since $a > 0$, we check the sign of $cM(\phi) - 6E(\phi)$:

$$cM(\phi) - 6E(\phi) = \int_0^L \left[c\phi - 3\phi^2 - 3(\phi')^2\right] dx$$

$$= \int_0^L \left[2b + c\phi'' - c\phi - 2(\phi')^2\right] dx$$

$$= 2bL - cM(\phi) - 2\int_0^L (\phi')^2 dx,$$

where we have used $\mathcal{L}1 = c - 3\phi + \phi''$ and $\mathcal{L}\phi = 2b + c(\phi'' - \phi)$ and $\langle \mathcal{L}1, \phi\rangle = \langle 1, \mathcal{L}\phi\rangle$. If $b \leq 0$ and $c > 0$, then $\langle \mathcal{K}\mu, \mu\rangle < 0$ since $M(\phi) > 0$, so that the periodic waves are orbitally stable in the time evolution of the Hamiltonian system (12) by Proposition 4 and Remark 18. ■

Remark 19. The criterion $\langle \mathcal{K}\mu, \mu\rangle < 0$ of the orbital stability is satisfied near the boundary $a = a_-(b)$ in Lemma 2 both for $b \leq 0$ and $b > 0$. Indeed, we can write

$$\langle \mathcal{K}\mu, \mu\rangle = a^2 \int_0^L \frac{c - 3\phi}{(c-\phi)^2} dx,$$

where $c - 3\phi < 0$ because the constant solution $\phi = \phi_2$ satisfies the ordering (22).

Remark 20. The criterion $\langle \mathcal{K}\mu, \mu\rangle < 0$ is not satisfied near the boundary $a = a_+(b)$ in Lemma 3 for $b > 0$. Indeed, since $\phi = \phi_1 + \hat{\phi}$, where $\hat{\phi}(x) \to 0$ as $|x| \to \infty$, we derive

$$cM(\phi) - 6E(\phi) = L\phi_1(c - 3\phi_1) + \mathcal{O}(1),$$

where $\mathcal{O}(1)$ denotes bounded terms in the limit $L \to \infty$. Since $c - 3\phi_1 > 0$ by the ordering (22) and $a > 0$, we have $\langle \mathcal{K}\mu, \mu\rangle > 0$ near the boundary $a = a_+(b)$. Thus, the criterion for orbital stability in Lemma 13 is not as sharp as the criterion in Lemma 12.

Remark 21. The result of Lemma 13 can be established directly for the linearized operator $\mathcal{L}$. Since $\mathcal{L}\phi = 2b + c(\phi'' - \phi)$, we can define $V(u) := 2bM(u) - cE(u)$ so that if $\langle V'(u), v\rangle = 0$, then $\langle \mathcal{L}\phi, v\rangle = 0$. Thus, we can take $\mathrm{Y} := \phi$ and compute

$$\langle \mathcal{L}\phi, \phi\rangle = 2bM(\phi) - 2cE(\phi). \tag{101}$$

If $b \leq 0$ and $c > 0$, then $\langle \mathcal{L}\phi, \phi\rangle < 0$ since $M(\phi) > 0$. However, Proposition 4 can only be used if $\mathcal{L}$ has a simple negative eigenvalue, which is only true in a subset of $b \leq 0$, where $\partial_a \mathfrak{L} < 0$. Similar to Remark 16, we can see that the linearized operator $\mathcal{K}$ provides wider region for orbital stability compared to the linearized operator $\mathcal{L}$.

6 | CONCLUSION

We have studied spectral and orbital stability of smooth periodic travelling waves in the CH equation by using functional-analytic tools. We showed that the standard Hamiltonian formulation of the CH equation has several shortcomings, for example, the number of negative eigenvalues in the linearized operator is either one or two depending on the parameters of the periodic travelling wave. On the other hand, the nonstandard Hamiltonian formulation based on the momentum quantity $m := u - u_{xx}$ provides a better framework for analysis with only one simple negative eigenvalue of the associated linearized operator.

The criterion for spectral and orbital stability has been derived by using the nonstandard Hamiltonian formulation. The stability criterion has been checked numerically and it is an open problem to prove analytically that this criterion is satisfied in the entire existence region for the smooth periodic travelling waves. We proved analytically that the stability criterion is satisfied in a subset of the existence region.

As the CH equation is a prototypical example of a more general class of nonlinear evolution equations, it is expected that our methods will be useful for the analysis of spectral and orbital stability in the systems where other methods based on the inverse scattering transform are not applicable.

ACKNOWLEDGMENT

F. Natali is supported by Fundação Araucária (grant 002/2017), CNPq (grant 304240/2018-4), and CAPES MathAmSud (grant 88881.520205/2020-01). D.E. Pelinovsky acknowledges financial support from the state task program in the sphere of scientific activity of the Ministry of Science and Higher Education of the Russian Federation (Task No. FSWE-2020-0007) and from the grant of the president of the Russian Federation for the leading scientific schools (grant No. NSH-2485.2020.5).

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon request.

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REFERENCES

1. Camassa R, Holm DD. An integrable shallow water equation with peaked solitons. *Phys Rev Lett*. 1993; 71:1661-1664.
2. Camassa R, Holm D, Hyman J. A new integrable shallow water equation. *Adv Appl Math*. 1994;31:1-33.
3. Constantin A, Lannes D. The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations. *Arch Ration Mech Anal*. 2009;192:165-186.
4. Johnson RS. Camassa–Holm, Korteweg–de Vries and related models for water waves. *J Fluid Mech*. 2002;455:63-82.
5. Dai HH. Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod. *Acta Mech*. 1998;127:193-207.
6. Kouranbaeva S. The Camassa–Holm equation as geodesic flow on the diffeomorphism group. *J Math Phys*. 1999;40:857-868.
7. Misiolek G. Shallow water equation as a geodesic flow on the Bott–Virasoro group. *J Geom Phys*. 1998;24:203-208.
8. Constantin A, Escher J. Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation. *Comm Pure Appl Math*. 1998; 51:475-504.
9. Constantin A, Escher J. On the blow-up rate and the blow-up set of breaking waves for a shallow water equation. *Math Z*. 2000;233:75-91.
10. Constantin A, Escher J. Wave breaking for nonlinear nonlocal shallow water equations. *Acta Math*. 1998; 181:229-243.

11. Hakkaev S, Iliev ID, Kirchev K. Stability of periodic travelling shallow-water waves determined by Newton's equation. *J Phys A: Math Theor*. 2008;41:085203.
12. Himonas A, Misiolek G. The Cauchy problem for an integrable shallow water equation. *Diff Int Eqs*. 2001;14:821-831.
13. Misiolek G. Classical solutions of the periodic Camassa–Holm equation. *Geom Funct Anal*. 2002;12:1080-1104.
14. De Lellis C, Kappeler T, Topalov P. Low-regularity solutions of the periodic Camassa–Holm equation. *Comm PDEs*. 2007;32:87-126.
15. Lenells J. Traveling wave solutions of the Camassa–Holm equation. *J Diff Equat*. 2005;217:393-430.
16. Byers P. Existence time for the Camassa-Holm equation and the critical Sobolev index. *Indiana Univ Math J*. 2006;55:941-954.
17. Himonas A, Misiolek G, Ponce G. Non-uniform continuity in H^1 of the solution map of the CH equation. *Asian J Math*. 2007; 11:141-150.
18. Geyer A, Villadelprat J. On the wave length of smooth periodic traveling waves of the Camassa–Holm equation. *J Diff Equat*. 2015;259:2317-2332.
19. Lenells J. Stability for the periodic Camassa–Holm equation. *Math Scand*. 2005; 97:188-200.
20. Lenells J. Stability of periodic peakons. *Int Math Res Not*. 2004;2004:485-499.
21. Lenells J. A variational approach to the stability of periodic peakons. *J Nonlinear Math Phys*. 2004;11:151-163.
22. Madiyeva A, Pelinovsky DE. Growth of perturbations to the peaked periodic waves in the Camassa-Holm equation. *SIAM J Math Anal*. 2021;53:3016-3039.
23. Constantin A, Strauss WA. Stability of the Camassa–Holm solitons. *J Nonlinear Sci*. 2002;12:415-422.
24. Constantin A, Molinet L. Orbital stability of solitary waves for a shallow water equation. *Physica D*. 2001;157:75-89.
25. Constantin A, Strauss WA. Stability of peakons. *Comm Pure Appl Math*. 2000;53:603-610.
26. Natali F, Pelinovsky DE. Instability of H^1-stable peakons in the Camassa–Holm equation. *J Diff Eqs*. 2020;268:7342-7363.
27. Alves G, Natali F, Pastor A. Sufficient conditions for orbital stability of periodic traveling waves. *J Diff Equat*. 2019;267:879-901.
28. Angulo J, Natali F. Positivity properties and stability of periodic traveling waves solutions. *SIAM J Math Anal*. 2008;40:1123-1151.
29. Angulo J, Natali F. Stability and instability of periodic traveling-wave solutions for the critical Korteweg-de Vries and non-linear Schrödinger equations. *Physica D*. 2009;238:603-621.
30. Geyer A, Pelinovsky DE. Spectral stability of periodic waves in the generalized reduced Ostrovsky equation. *Lett Math Phys*. 2017; 107:1293-1314.
31. Hărăguş M, Kapitula T. On the spectra of periodic waves for infinite-dimensional Hamiltonian systems. *Physica D*. 2008;237:2649-2671.
32. Hur VM, Johnson M. Stability of periodic traveling waves for nonlinear dispersive equations. *SIAM J Math Anal*. 2015;47:3528-3554.
33. Johnson M. Nonlinear stability of periodic traveling wave solutions of the generalized Korteweg-de Vries equation. *SIAM J Math Anal*. 2009;41:1921-1947.
34. Natali F, Le U, Pelinovsky DE. New variational characterization of periodic waves in the fractional Korteweg-de Vries equation. *Nonlinearity*. 2020;33:1956-1986.
35. Constantin A. The Hamiltonian structure of the Camassa–Holm equation. *Expo Math*. 1997;15:53-85.
36. Brandolese L. Local-in-space criteria for blowup in shallow water and dispersive rod equations. *Commun Math Phys*. 2014;330:401-414.
37. Chicone C. The monotonicity of the period function for planar Hamiltonian vector fields. *J Diff Equat*. 1987;69:310-321.
38. Mañosas F, Villadelprat J. Criteria to bound the number of critical periods. *J Dif Equat*. 2009;246:24153.
39. Garijo A, Villadelprat J. Algebraic and analytical tools for the study of the period function. *J Differ Equ*. 2014;257:2464-2484.
40. Kato T. *Perturbation Theory for Linear Operators*. Springer-Verlag; 1995.
41. Neves A. Floquet's Theorem and the stability of periodic waves. *J Dyn Diff Equat*. 2009;21:555-565.
42. Eastham MSP. *The Spectral Theory of Periodic Differential Equations*. Scottish Academic Press; 1973.
43. Magnus W, Winkler S. *Hill's Equation*. Wiley; 1966.

44. Lopes O. A class of isoinertial one parameter families of selfadjoint operators. In: Lupo Daniela, Pagani Carlo D., Ruf Bernhard, *Nonlinear Differential Equations: Methods, Models and Applications*. Birkhäuser; 2003.
45. Pelinovsky DE. *Localization in Periodic Potentials: from Schrödinger Operators to the Gross–Pitaevskii Equation*. LMS Lecture Note Series 390. Cambridge University Press; 2011.
46. Lenells J. The correspondence between KdV and Camassa-Holm. *Int Math Res Not*. 2004;2004:3797-3811.

How to cite this article: Geyer A, Martins RH, Natali F, Pelinovsky DE. Stability of smooth periodic travelling waves in the Camassa–Holm equation. *Stud Appl Math*. 2022;148:27–61. https://doi.org/10.1111/sapm.12430

APPENDIX A: SMOOTH PERIODIC WAVES IN THE EXPLICIT FORM

Here, we derive the explicit expressions for the smooth periodic wave with the profile ϕ satisfying the system (6)–(8). Since $\phi < c$, we can transform the variables

$$\phi(x) = \psi(z(x)), \quad z(x) = \int_0^x \frac{ds}{\sqrt{c - \phi(s)}} \tag{A1}$$

and rewrite the second-order equation (6) with the chain rule to the form

$$-\psi'' + c\psi - \frac{3}{2}\psi^2 = b, \tag{A2}$$

which is the stationary KdV equation. Note that this reduction of the travelling periodic waves of the CH equation (1) to the travelling periodic waves of the KdV equation is different from the previously explored connection between the CH and KdV equations in Ref. 46. A similar transformation was used in Ref. 23 in the context of the solitary waves.

The second-order equation (A2) has the following explicit periodic solution (see, e.g., Ref. 34):

$$\psi(z) = \frac{1}{3}c + \frac{4}{3}\gamma^2\left[1 - 2k^2 + 3k^2\mathrm{cn}^2(\gamma z; k)\right], \tag{A3}$$

where $\gamma > 0$ and $k \in (0, 1)$ are arbitrary parameters, and cn is the Jacobian elliptic function. The period of the periodic solution ψ in z is $P = 2\gamma^{-1}K(k)$. The free parameters γ and k parametrize the turning points ϕ_- and ϕ_+ in (23) and (24):

$$\begin{cases} \phi_+ = \frac{1}{3}c + \frac{4}{3}\gamma^2(1 + k^2), \\ \phi_- = \frac{1}{3}c + \frac{4}{3}\gamma^2(1 - 2k^2), \end{cases} \tag{A4}$$

which can be inverted as follows:

$$\gamma^2 = \frac{1}{4}(2\phi_+ + \phi_- - c), \quad k^2 = \frac{\phi_+ - \phi_-}{2\phi_+ + \phi_- - c}. \tag{A5}$$

Parameters a and b are related to parameters γ and k by

$$a = \frac{1}{2}(\phi_+ + \phi_-)(c - \phi_+)(c - \phi_-)$$

$$= \frac{4}{27}(c + 2\gamma^2(2 - k^2))(c - 2\gamma^2(1 + k^2))(c - 2\gamma^2(1 - 2k^2)) \tag{A6}$$

and

$$\begin{aligned} b &= \frac{1}{2}c(\phi_+ + \phi_-) - \frac{1}{2}(\phi_+^2 + \phi_+\phi_- + \phi_-^2) \\ &= \frac{1}{6}c^2 - \frac{8}{3}\gamma^4(1 - k^2 + k^4), \end{aligned} \tag{A7}$$

where Equations (24) have been used. When a or b are fixed, for example, for numerical results obtained on Figures 5 and 6, we express γ from the roots of either (A6) or (A7) and parameterize the family by the only parameter k in a subset of (0,1).

The period function $L = \mathfrak{L}(a, b, c)$ can be expressed by (A1) and (A3) in the form:

$$L = \frac{\sqrt{2}}{\gamma\sqrt{3}} \int_0^{2K(k)} \sqrt{c - 2\gamma^2(1 - 2k^2 + 3k^2\mathrm{cn}^2(z; k))}dz. \tag{A8}$$

If L is fixed, for example, for numerical results obtained in Figure 7, then γ can be found from a root finding algorithm for equation (A8), after which the periodic solutions are parameterized by the only parameter k in a subset of (0,1). The mass and energy integrals in (2) and (3) are evaluated at the periodic wave (A1) with the chain rule:

$$M(\phi) = \int_0^{2\gamma^{-1}K(k)} \psi(z)\sqrt{c - \psi(z)}dz \tag{A9}$$

and

$$E(\phi) = \int_0^{2\gamma^{-1}K(k)} \left[b + \psi(z)^2 - \frac{a}{c - \psi(z)}\right]\sqrt{c - \psi(z)}dz, \tag{A10}$$

where we have used $(\phi')^2 = \phi^2 + 2b - 2a/(c - \phi)$.

The limit $k \to 0$ corresponds to the constant solution

$$\psi = \frac{1}{3}c + \frac{4}{3}\gamma^2 \tag{A11}$$

in Lemma 2. The limit $k \to 1$ corresponds to the solitary wave solution

$$\psi(z) = \frac{1}{3}c + \frac{4}{3}\gamma^2\left[-1 + 3\mathrm{sech}^2(\gamma z)\right] \tag{A12}$$

in Lemma 3. The curve $c = 2\gamma^2(1 + k^2)$ corresponds to the peaked periodic wave

$$\psi(z) = 2\gamma^2\left[1 - k^2 + 2k^2\mathrm{cn}^2(\gamma z; k)\right], \tag{A13}$$

in Lemma 4.