Stability of smooth periodic traveling waves in the Degasperis–Procesi equation

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Abstract

We derive a precise energy stability criterion for smooth periodic waves in the Degasperis–Procesi (DP) equation. Compared to the Camassa-Holm (CH) equation, the number of negative eigenvalues of an associated Hessian operator changes in the existence region of smooth periodic waves. We utilize properties of the period function with respect to two parameters in order to obtain a smooth existence curve for the family of smooth periodic waves with a fixed period. The energy stability condition is derived on parts of this existence curve, which correspond to either one or two negative eigenvalues of the Hessian operator. We show numerically that the energy stability condition is satisfied on either part of the curve and prove analytically that it holds in a neighborhood of the boundary of the existence region of smooth periodic waves.

1. Introduction

The Degasperis-Procesi (DP) equation

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + u u_{xxx} \tag{1.1}$$

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has a special role in the modeling of fluid motion. It was derived in [10] as a transformation of the integrable hierarchy of KdV equations, with the same asymptotic accuracy as the Camassa–Holm (CH) equation [2]. Although a more general family of model equations can also be derived by using this method [8,11], only the DP and CH equations are integrable with the use of the inverse scattering transform. It was shown in [4,26,28] that the DP and CH equations describe the horizontal velocity $u = u(t, x)$ for the unidirectional propagation of waves of a shallow water flowing over a flat bed at a certain depth. A review of the applicability of these model equations for approximations of peaked waves in fluids was recently given in [40].

In the present paper, we are concerned with smooth traveling wave solutions, for which the DP and CH equations have been justified as model equations in hydrodynamics [4]. Existence of smooth periodic traveling waves has been well understood by using ODE methods [34,35]. However, stability of smooth periodic traveling waves was considered to be a difficult problem in the functional-analytic framework, even though integrability implies their stability due to the structural stability of the Floquet spectrum of the associated linear system [36]. Only very recently in [19], we derived an energy stability criterion for the smooth periodic traveling waves of the CH equation by using its Hamiltonian formulation.

For smooth solitary waves, orbital stability was obtained for the CH equation in [5] and spectral and orbital stability for the DP equation was obtained in [37,38]. The energy stability criterion for the smooth solitary waves was derived for the entire family of the generalized CH equations [32] and was shown to be satisfied asymptotically and numerically. A recent work [39] used the period function to show that the energy stability criterion is satisfied analytically for the entire family of smooth solitary waves.

The purpose of this work is to derive an energy stability criterion for the smooth periodic traveling waves in the DP equation.

Let us briefly comment on the various Hamiltonian formulations which exist both for the CH and DP equations. These two equations belong to a larger class of generalized CH equations, the so-called b-family, which reduces to CH for $b = 2$ and to DP for $b = 3$. As far as we know, only one Hamiltonian formulation exists for general b, which was obtained in [9] and used in the stability analysis of smooth solitary waves for $b > 1$ in [32]. There exists one additional formulation for $b = 3$ and two additional formulations for $b = 2$. In [19], we used these two Hamiltonian formulations to study spectral stability of the smooth periodic waves in the CH equation. Here we will only use one Hamiltonian formulation to complete the same task for the DP equation. The universal Hamiltonian formulation from [9] can also be adopted to the study of spectral stability of smooth periodic waves for the b-family with $b > 1$ and this has recently been addressed in [12].

We consider the DP equation (1.1) in the periodic domain $\mathbb{T}_L := \mathbb{R} \backslash (L\mathbb{Z})$ of length $L > 0$. For notational simplicity, we write H^s_{per} instead of $H^s(\mathbb{T}_L)$ for the Sobolev space of L-periodic functions with index $s \geq 0$. The DP equation (1.1) on $\mathbb{T}_L$ formally conserves the mass, momentum, and energy given respectively by

$$M(u) = \oint u \, dx, \tag{1.2}$$

$$E(u) = \frac{1}{2} \oint u (1 - \partial_x^2)(4 - \partial_x^2)^{-1} u \, dx, \tag{1.3}$$

and

$$F(u) = \frac{1}{6} \oint u^3 dx. \tag{1.4}$$

We will be exploring the following Hamiltonian structure for the DP equation (1.1):

$$\frac{du}{dt} = J \frac{\delta F}{\delta u}, \quad J = -(1 - \partial_x^2)^{-1}(4 - \partial_x^2)\partial_x, \tag{1.5}$$

where J is a well-defined operator from H_{per}^{s+1} to H_{per}^s for every $s \geq 0$ and $\frac{\delta F}{\delta u} = \frac{1}{2}u^2$. The evolution problem (1.5) is well-defined for local solutions $u \in C((-t_0, t_0), H_{\text{per}}^s) \cap C^1((-t_0, t_0), H_{\text{per}}^{s-1})$ with $s > \frac{3}{2}$, see [14], where $t_0 > 0$ is the local existence time.

Smooth traveling waves of the form $u(t, x) = \phi(x - ct)$ with $c - \phi > 0$ are obtained from the critical points of the augmented energy functional

$$\Lambda_{c,b}(u) := cE(u) - F(u) - \frac{b}{4}M(u), \tag{1.6}$$

where b is a parameter obtained after integration of a third-order differential equation satisfied by the traveling wave profile ϕ, see Section 2. After two integrations with integration constants a and b, all smooth periodic wave solutions with the profile ϕ can be found from the first-order invariant

$$(c - \phi)^2(\phi'^2 - \phi^2 - b) + a = 0. \tag{1.7}$$

The second variation of the augmented energy functional (1.6) is determined by an associated Hessian operator $\mathcal{L} : L_{\text{per}}^2 \to L_{\text{per}}^2$ given by

$$\mathcal{L} := c - \phi - 3c(4 - \partial_x^2)^{-1}. \tag{1.8}$$

The operator $\mathcal{L}$ is self-adjoint and bounded as the sum of the bounded multiplication operator $(c - \phi)$ and the compact operator $-3c(4 - \partial_x^2)^{-1}$ in L_{per}^2. Since $c - \phi > 0$, the continuous spectrum of $\mathcal{L}$ is strictly positive, hence $\mathcal{L}$ has finitely many negative eigenvalues of finite algebraic multiplicities and a zero eigenvalue of finite algebraic multiplicity.

The first result of this paper is about the existence of smooth periodic traveling waves with profile ϕ satisfying the first-order invariant (1.7), and the number of negative eigenvalues of $\mathcal{L}$ given by (1.8), see Fig. 1.1.

Theorem 1.1. *For a fixed $c > 0$, smooth periodic solutions of the first-order invariant (1.7) with profile $\phi \in H_{\text{per}}^\infty$ satisfying $c - \phi > 0$ exist in an open, simply connected region on the (a, b) plane enclosed by three boundaries:*

- *$a = 0$ and $b \in (-c^2, 0)$, where the periodic solutions are peaked,*
- *$a = a_+(b)$ and $b \in (0, \frac{1}{8}c^2)$, where the solutions have infinite period,*
- *$a = a_-(b)$ and $b \in (-c^2, \frac{1}{8}c^2)$, where the solutions are constant,*
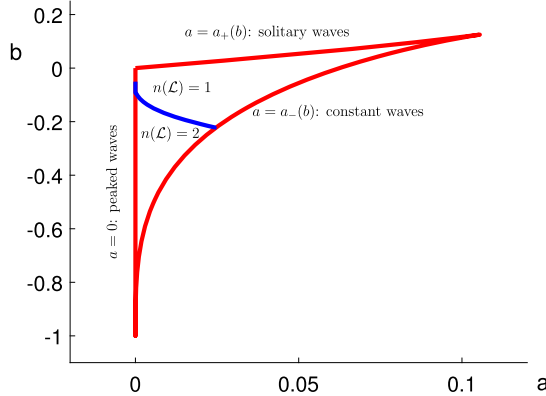
Fig. 1.1. The existence region of smooth periodic solutions of the first-order invariant (1.7) on the parameter plane (a, b) for $c = 1$ enclosed by three boundaries (red lines). The blue line shows the curve $a = a_0(b)$ which separates the cases of one ($n(\mathcal{L}) = 1$) and two ($n(\mathcal{L}) = 2$) negative eigenvalues of $\mathcal{L}$ in (1.8). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

where $a_+(b)$ and $a_-(b)$ are smooth functions of b. For every point inside the region, the periodic solutions are smooth functions of (a, b) and their period is strictly increasing in b for every fixed $a \in (0, \frac{27}{256}c^4)$. There exists a smooth curve $a = a_0(b)$ for $b \in (-\frac{2}{9}c^2, 0)$ in the interior of the existence region such that the Hessian operator $\mathcal{L}$ in L_{per}^2 has only one simple negative eigenvalue above the curve and two simple negative eigenvalues (or a double negative eigenvalue) below the curve. The rest of its spectrum for $a \neq a_0(b)$ includes a simple zero eigenvalue and a strictly positive spectrum bounded away from zero. Along the curve $a = a_0(b)$ the Hessian operator $\mathcal{L}$ has only one simple negative eigenvalue, a double zero eigenvalue, and the rest of its spectrum is strictly positive.

Remark 1.2. The three curves bounding the existence region of smooth periodic waves in Theorem 1.1 are shown in Fig. 1.1 for $c = 1$. The curve in the interior of the existence region is the curve $a = a_0(b)$, which was found numerically by plotting the period function of the periodic solutions of Theorem 1.1 versus a for fixed b and detecting its maximum if it exists, see Lemmas 3.2 and 4.4 below. The regions with one and two negative eigenvalues of $\mathcal{L}$ are located above and below the curve $a = a_0(b)$, respectively.

Remark 1.3. The double zero eigenvalue of $\mathcal{L}$ existing at the curve $a = a_0(b)$ does not imply any bifurcations of periodic solutions with the profile ϕ because the Hessian operator $\mathcal{L}$ in (1.8) is not related to the linearization of the differential equation (1.7), see also the discussion in Remark 2.5.

Remark 1.4. The transformation

$$\phi(x) = c\varphi(x), \quad b = c^2\beta, \quad a = c^4\alpha \tag{1.9}$$

normalizes the parameter c to unity with φ, β, and α satisfying the same equation (1.7) but with $c = 1$. Hence, the smooth periodic waves are uniquely determined by the free parameters

(a, b) and $c = 1$ can be used everywhere. Similarly, although we only consider the case of right-propagating waves with $c > 0$, all results can be extended to the left-propagating waves with $c < 0$ by using the scaling transformation (1.9).

Spectral stability of smooth periodic traveling waves with respect to co-periodic perturbations is determined from the spectrum of the linearized operator $J\mathcal{L}$ in L^2_{per}, with J given in (1.5). Since J is a skew-adjoint operator and $\mathcal{L}$ is self-adjoint, the spectrum of the linearized operator $J\mathcal{L}$ is symmetric with respect to $i\mathbb{R}$ [25,29]. Therefore, the periodic wave is spectrally stable if the spectrum of $J\mathcal{L}$ in L^2_{per} is located on $i\mathbb{R}$. The second result of this paper gives the energy criterion for the spectral stability of the smooth periodic waves in the DP equation (1.1).

Theorem 1.5. *For a fixed $c > 0$ and a fixed period $L > 0$, there exists a C^1 mapping $a \mapsto b = \mathcal{B}_L(a)$ for $a \in (0, a_L)$ with some L-dependent $a_L \in (0, \frac{27}{256}c^4)$ and a C^1 mapping $a \mapsto \phi = \Phi_L(\cdot, a) \in H^\infty_{\text{per}}$ of smooth L-periodic solutions along the curve $b = \mathcal{B}_L(a)$. Let*

$$\mathcal{M}_L(a) := M(\Phi_L(\cdot, a)) \quad \text{and} \quad \mathcal{F}_L(a) := F(\Phi_L(\cdot, a)),$$

where $M(u)$ and $F(u)$ are given by (1.2) and (1.4). The L-periodic wave with profile $\phi = \Phi_L(\cdot, a)$ such that $\mathcal{B}'_L(a) \neq 0$ for some $a \in (0, a_L)$ is spectrally stable if

$$\frac{d}{da} \frac{\mathcal{F}_L(a)}{\mathcal{M}_L(a)^3} < 0, \tag{1.10}$$

and, for $\mathcal{B}'_L(a) < 0$, if additionally $\mathcal{M}'_L(a) > 0$. The stability criterion holds true for every point in a neighborhood of the boundary $a = a_-(b)$.

Remark 1.6. Fig. 1.2 shows the numerically computed mappings $a \mapsto \mathcal{F}_L(a)/\mathcal{M}^3_L(a)$ and $a \mapsto \mathcal{M}_L(a)$ for four values of fixed L. The parameter a is chosen in $(0, a_L)$, where a_L depends on L. It follows that the stability criterion of Theorem 1.5 is satisfied for all cases. This property has been analytically proven only near the boundary $a = a_-(b)$ by means of the Stokes expansion, see Lemma 5.9.

It is difficult to check the stability criterion of Theorem 1.5 near the other two boundaries of the existence region of Theorem 1.1 where the waves are either peaked or solitary. The perturbation theory becomes singular in these two asymptotic limits because $c - \phi$ vanishes for the peaked periodic waves and the period function diverges for the solitary waves. Nevertheless, some relevant results are available in these two limits:

- For the boundary $a = 0$ and $b \in (-c^2, 0)$, where the periodic solutions are peaked, the spectral stability problem for the DP equation (1.1) needs to be set up by using a weak formulation of the evolution problem. This setup was elaborated for a generalized CH equation in [31], building on previous work in [44], to show spectral instability of peaked solitary waves. Linear and nonlinear instability of peaked periodic waves with respect to peaked periodic perturbations was shown for the CH equation in [43]. Spectral and linear instability of peaked periodic waves for the reduced Ostrovsky equation was proven in [20,21]. Instability of peaked periodic waves in the DP equation or in the generalized CH equation is still open for further studies.
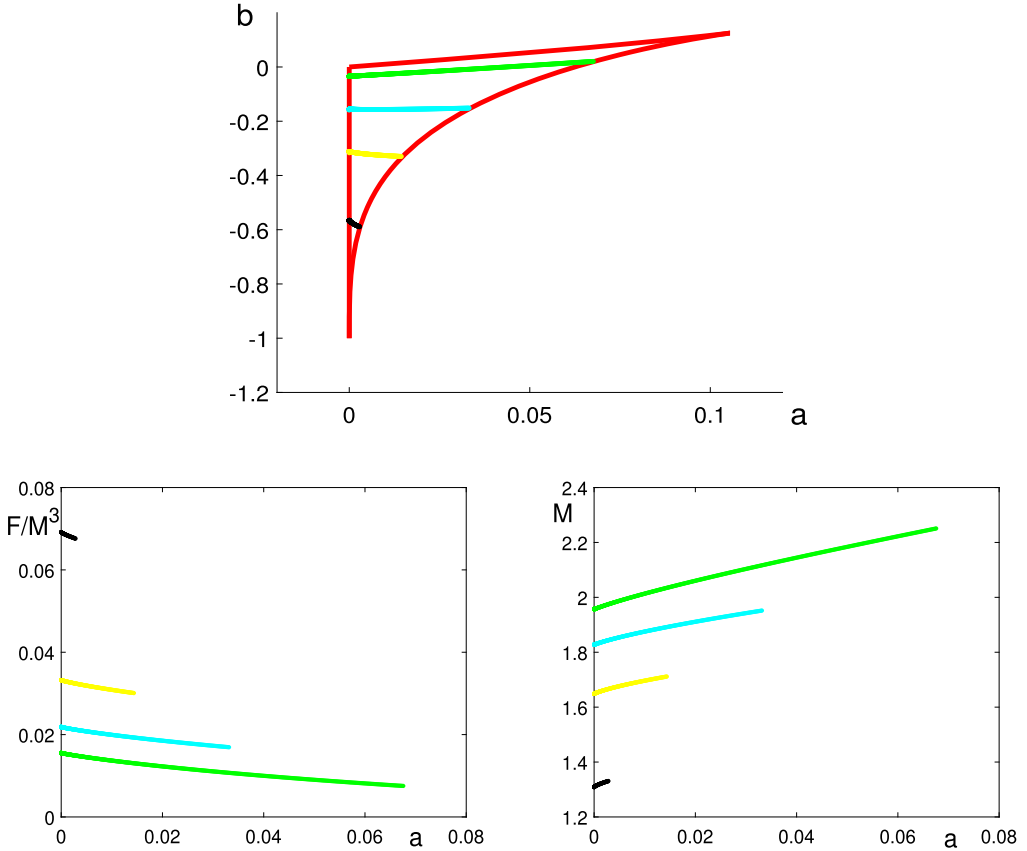
Fig. 1.2. Top: Existence region on the (a, b) plane with four curves $b = \mathcal{B}_L(a)$ for $c = 1$ and four values of period: $L = \pi/2$ (black), $L = 3\pi/4$ (yellow), $L = \pi$ (cyan), and $L = 3\pi/2$ (green). Bottom: $\mathcal{F}_L/\mathcal{M}_L^3$ versus a (left) and $\mathcal{M}_L$ versus a (right) along the four curves.

- For the boundary $a = a_+(b)$ and $b \in (0, \frac{1}{8}c^2)$, where the periodic solutions have infinite period, spectral stability of solitary waves over a nonzero background was shown for the general b-family in [32] and for the DP equation in [37]. The methods in [32,37] are not related to the energy stability criterion (1.10), and it remains open to show the equivalence of the three different stability criteria for smooth solitary waves over a nonzero background.

The analytical proof of the energy stability criterion of Theorem 1.5 in the interior of the bounded existence region is still open. Another interesting question is to explore the non-standard Hamiltonian formulation of the DP equation as a member of the b-family and to obtain a different energy stability criterion for the smooth periodic waves, see [12]. Finally, there may exist a deep connection between the energy stability criterion and the physical laws for fluids since the mapping (1.10) involves a homogeneous function of degree zero in terms of the wave profile ϕ. Similarly, the energy stability criterion for the CH equation obtained in [19] involves a homogeneous function of degree zero given by $\mathcal{E}_L(a)/\mathcal{M}_L(a)^2$, where $\mathcal{M}_L(a)$ is the same as in (1.2) and $\mathcal{E}_L(a)$ is obtained from $E(u) = \|u\|_{H^1_{\text{per}}}^2$, which is different from (1.3).

The paper is organized as follows.

- In Section 2 we state and prove the existence result for the smooth periodic wave with profile ϕ, similar to [19] and [32]. The proof is given here to set the foundation for establishing the connection between the monotonicity of the period function and the index of the Hessian operator.
- Section 3 details the monotonicity properties of the period function for the smooth periodic solutions of DP with respect to both parameters a and b. The proofs rely on the classical works [3,16] but involve more complicated details of computations compared to [19,23] for the CH equation.
- Section 4 describes the number of negative eigenvalues and the multiplicity of the zero eigenvalue of the Hessian operator $\mathcal{L}$. The count is obtained by a nontrivial adaptation of the Birman–Schwinger principle which is different from the study of a similar Hessian operator for solitary waves in [37]. The proof of Theorem 1.1 is achieved with the results obtained in Sections 2, 3, and 4.
- Finally, in Section 5 we extend the family of periodic waves with the profile ϕ along a curve with a fixed period $L > 0$ and give the proof of Theorem 1.5.
- Appendix A contains some auxiliary results on the period function which are used in the proofs of Section 3, based on previous work in [16,17,23].

2. Smooth traveling waves

Traveling waves of the form $u(t, x) = \phi(x - ct)$ with speed c and profile ϕ are found from the third-order differential equation

$$-(c - \phi)(\phi''' - \phi') - 3\phi\phi' + 3\phi'\phi'' = 0, \tag{2.1}$$

which is obtained from the DP equation (1.1). For notational convenience we denote $\phi = \phi(x)$ where x stands for the traveling coordinate $x - ct$. Integration of (2.1) in x gives the second-order equation

$$-(c - \phi)(\phi'' - \phi) + \phi'^2 - \phi^2 = b, \tag{2.2}$$

where b is an integration constant. Another second-order equation can be obtained after multiplying (2.1) by $(c - \phi)^2$ and integrating,

$$-(c - \phi)^3(\phi'' - \phi) = a, \tag{2.3}$$

where a is another integration constant. Both second-order equations (2.2) and (2.3) are compatible if and only if ϕ satisfies the first-order invariant (1.7), which can be viewed as the first-order invariant for either (2.2) or (2.3).

The following lemma characterizes the family of periodic waves by using phase plane analysis, and constitutes the existence part of Theorem 1.1.

Lemma 2.1. *For a fixed $c > 0$, smooth periodic solutions to the first-order invariant (1.7) with the profile $\phi \in H_{\mathrm{per}}^{\infty}$ satisfying $c - \phi > 0$ exist in an open, simply connected region on the (a, b) plane enclosed by three boundaries:*

- *$a = 0$ and $b \in (-c^2, 0)$, where the periodic solutions are peaked,*
- *$a = a_+(b)$ and $b \in (0, \frac{1}{8}c^2)$, where the solutions have infinite period,*
- *$a = a_-(b)$ and $b \in (-c^2, \frac{1}{8}c^2)$, where the solutions are constant,*

where $a_+(b)$ and $a_-(b)$ are smooth functions of b. The family of periodic solutions inside this region is smooth in (a, b).

Proof. For a fixed $c > 0$, the first-order invariant (1.7) represents the energy conservation $(\phi')^2 + U(\phi) = b$ for a Newtonian particle with mass $m = 2$ and the energy level b under a force with the potential energy

$$U(\phi) := -\phi^2 + \frac{a}{(c - \phi)^2}. \tag{2.4}$$

The critical points of U on $\mathbb{R}\backslash\{c\}$ are given by the roots of the algebraic equation

$$\phi(c - \phi)^3 = a. \tag{2.5}$$

The global maximum of $\phi \mapsto \phi(c - \phi)^3$ occurs at $\phi = \phi_c := c/4$ for which $a = a_c := \frac{27}{256}c^4$. We will now consider the three different cases $a \in (-\infty, 0]$, $a \in (0, a_c)$, and $a \in [a_c, \infty)$.

- If $a \in (0, a_c)$, the potential energy U has a local maximum ϕ_1 and a local minimum ϕ_2 which satisfy the ordering

$$0 < \phi_1 < \frac{c}{4} < \phi_2 < c, \tag{2.6}$$

  see the left panel of Fig. 2.1. The local maximum and minimum of U give the saddle point $(\phi_1, 0)$ and the center point $(\phi_2, 0)$ of the first-order planar system corresponding to the second-order equation (2.3). Smooth periodic solutions with the profile ϕ satisfying $c - \phi > 0$ correspond to periodic orbits inside a punctured neighborhood around the center $(\phi_2, 0)$ enclosed by the homoclinic orbit connecting at the saddle $(\phi_1, 0)$, see the right panel of Fig. 2.1. All other orbits are unbounded.
- If $a \in (-\infty, 0)$, the potential energy U has two local maxima, one is below the singularity at c and the other one is above the singularity at c with $U(\phi) \to -\infty$ as $\phi \to c$. All orbits are either unbounded or hit the singularity at c for which ϕ' is infinite. The same is true for $a = 0$, for which $U(\phi) = -\phi^2$.
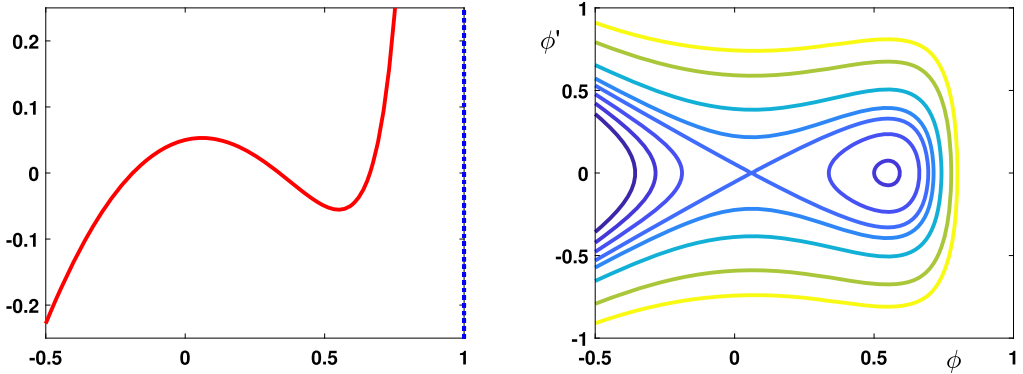
Fig. 2.1. Left: U versus ϕ for $c = 1$, and $a = 0.05$. Right: the phase portrait of the second-order equation (2.3) constructed from the level curves of $b = (\phi')^2 + U(\phi)$ on the phase plane (ϕ, ϕ') for the same parameter values.

- If $a \in [a_c, \infty)$, the potential energy does not have local extremal points and $U(\phi) \to +\infty$ as $\phi \to c$. All orbits of the second-order equation (2.3) are unbounded.

To summarize, bounded periodic solutions exist if and only if $a \in (0, a_c)$. Note that ϕ depends smoothly on the parameters a and b in view of smooth dependence of the first-order invariant (1.7) on ϕ, a, and b if $c - \phi > 0$. It remains to characterize the three boundaries of the existence region, see Fig. 1.1. This is done next.

- If $a = 0$, the second-order equation (2.3) becomes $\phi'' - \phi = 0$ and is solved explicitly by the L-periodic solution

$$\phi(x) = c \frac{\cosh\left(\frac{L}{2} - |x|\right)}{\cosh\left(\frac{L}{2}\right)}, \quad x \in \left[-\frac{L}{2}, \frac{L}{2}\right],$$

which attains the singularity $\phi = c$ placed at $x = 0$. The L-periodic wave is peaked at $x = 0$ and smooth at $x = \pm\frac{L}{2}$ with $\phi'\left(\pm\frac{L}{2}\right) = 0$. It follows from $b = (\phi')^2 - \phi^2$ that

$$b = -c^2 \text{sech}^2\left(\frac{L}{2}\right), \tag{2.7}$$

so that $b \in (-c^2, 0)$ for $L \in (0, \infty)$.

- If $a \in (0, a_c)$, the periodic orbit exists for the energy level $b \in (b_-, b_+)$, where $b_- = U(\phi_2)$ and $b_+ = U(\phi_1)$. On each respective boundary, a and b can be parameterized by $\phi_2 \in (\phi_c, c)$ and $\phi_1 \in (0, \phi_c)$, where $\phi_c = c/4$. The periodic solution along $b = b_-(a)$ is constant and we have

$$\begin{cases} b = c\phi_2 - 2\phi_2^2, \\ a = \phi_2(c - \phi_2)^3, \end{cases} \quad \Rightarrow \quad \begin{cases} \frac{db}{d\phi_2} = c - 4\phi_2 < 0, \\ \frac{da}{d\phi_2} = (c - \phi_2)^2(c - 4\phi_2) < 0. \end{cases} \tag{2.8}$$

Hence, in view of the chain rule, $b = b_-(a)$ is a monotonically increasing function, which can be inverted to obtain a function $a = a_-(b)$ for $b \in (-c^2, \frac{1}{8}c^2)$. Similarly, along $b = $

$b_+(a)$, the periodic solution degenerates into a homoclinic solution of infinite period and we have

$$\begin{cases} b = c\phi_1 - 2\phi_1^2, \\ a = \phi_1(c - \phi_1)^3, \end{cases} \qquad \Rightarrow \qquad \begin{cases} \frac{db}{d\phi_1} = c - 4\phi_1 > 0, \\ \frac{da}{d\phi_1} = (c - \phi_1)^2(c - 4\phi_1) > 0. \end{cases}$$

Hence $b = b_+(a)$ is a monotonically increasing function, which can be inverted to obtain a function $a = a_+(b)$ for $b \in (0, \frac{1}{8}c^2)$.

All together, all statements of Lemma 2.1 have been proven. $\quad\square$

Next we show that the periodic traveling wave with profile ϕ is a critical point of the augmented energy functional $\Lambda_{c,b}$ defined in (1.6).

Lemma 2.2. *Let $\phi \in H_{\mathrm{per}}^\infty$ be an L-periodic solution of the first-order invariant (1.7) for some (a, b) inside the existence region specified in Lemma 2.1 for fixed $c > 0$. Then, ϕ is a critical point of the augmented energy functional $\Lambda_{c,b}$.*

Proof. It follows from the second-order equation (2.3) that

$$c(1 - \partial_x^2)\phi - (4 - \partial_x^2)\left(\frac{1}{2}\phi^2\right) - b = 0.$$

Inverting the linear operator $(4 - \partial_x^2) : H_{\mathrm{per}}^2 \subset L_{\mathrm{per}}^2 \to L_{\mathrm{per}}^2$ yields

$$c(1 - \partial_x^2)(4 - \partial_x^2)^{-1}\phi - \frac{1}{2}\phi^2 - \frac{b}{4} = 0, \tag{2.9}$$

which is the Euler–Lagrange equation for $\Lambda_{c,b}$ in (1.6). By Lemma 2.1, the periodic solutions of the first-order invariant (1.7) are smooth, so that they are also smooth solutions of the Euler–Lagrange equation (2.9), and hence they are critical points of $\Lambda_{c,b}$. $\quad\square$

Remark 2.3. The statement of Lemma 2.2 does not work in the opposite direction, since critical points of $\Lambda_{c,b}$ are solutions of the Euler–Lagrange equation (2.9) which are only defined in the weak space $L^\infty(\mathbb{T}_L)$. In particular, the set of critical points of $\Lambda_{c,b}$ includes the peaked periodic waves which occur at the boundary $a = 0$ of the existence region for smooth periodic waves in Lemma 2.1.

Remark 2.4. In the variable $v := (4 - \partial_x^2)^{-1}u$, the DP equation (1.1) can be rewritten in local form as

$$v_t - v_{txx} + uu_x = 0.$$

The traveling wave reduction $v(t, x) = v(x - ct)$ with $v = (4 - \partial_x^2)^{-1}\phi$ satisfies

$$c(v' - v''') - \phi\phi' = 0.$$

Integration yields

$$c(v - v'') - \frac{1}{2}\phi^2 = d, \tag{2.10}$$

where d is an integration constant. Since $4v - v'' = \phi$, we obtain from (2.10) that

$$v = \frac{1}{3}\phi - \frac{1}{6c}\phi^2 - \frac{d}{3c}. \tag{2.11}$$

Substituting (2.11) into $\phi = 4v - v''$ and expressing ϕ'' and $(\phi')^2$ by using (1.7) and (2.3) gives us the relation $d = b/4$ between the integration constants. These considerations will be useful in the proof of Lemma 4.5 below.

Remark 2.5. The Hessian operator $\mathcal{L} = \Lambda''_{c,b}(\phi)$ given by (1.8) is not related to the linearization of the second-order equations (2.2) and (2.3). It is related to the linearization of the second-order equation (2.10) in the sense that the derivative of (2.10) in x yields

$$c(1 - \partial_x^2)v' - \phi\phi' = 0$$

which implies $\mathcal{L}\phi' = 0$ in view of the fact that $c(1 - \partial_x^2)(4 - \partial_x^2)^{-1} = c - 3c(4 - \partial_x^2)^{-1}$.

3. Period function

Here we shall study monotonicity properties of the period function for the smooth periodic solutions of Lemma 2.1 with respect to parameters a and b for fixed $c > 0$. For $a \in (0, a_c)$, where $a_c := \frac{27}{256}c^4$, we let ϕ_+ and ϕ_- be the turning points for which $U(\phi_\pm) = b$ for each $b \in (b_-, b_+)$. It follows from the proof of Lemma 2.1, see Fig. 2.1, that the turning points fit into the ordering (2.6) as follows:

$$0 < \phi_1 < \phi_- < \frac{c}{4} < \phi_2 < \phi_+ < c.$$

The period function $\mathcal{L}(a, b)$ assigns to each smooth periodic solution of the first-order invariant (1.7) its fundamental period $L = \mathcal{L}(a, b)$. Rewriting (1.7) in the form

$$(\phi')^2 + U(\phi) = b$$

with $U(\phi)$ given by (2.4) and integrating it along the periodic orbit ϕ, it follows that the period function is given by

$$\mathcal{L}(a, b) := 2\int_{\phi_-}^{\phi_+} \frac{d\phi}{\sqrt{b - U(\phi)}} \tag{3.1}$$

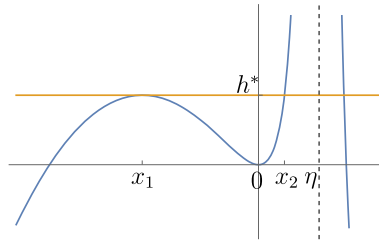for every point (a, b) inside the existence region of Lemma 2.1.

Fig. 3.1. The potential function $V(x)$ plotted for $\eta = 0.5$.

3.1. Monotonicity of the period function with respect to the parameter b

We shall prove that the period function $\mathfrak{L}(a, b)$ is a strictly increasing function of b for fixed $c > 0$ and $a \in (0, a_c)$. This gives the second result of Theorem 1.1.

Lemma 3.1. *Fix $c > 0$ and $a \in (0, a_c)$ The period function $\mathfrak{L}(a, b)$ is strictly increasing as a function of b.*

Proof. Recall that ϕ_2 is the local minimum of U and hence the second root of the algebraic equation $a = \phi_2(c - \phi_2)^3$, see the algebraic equation (2.5) and the ordering (2.6). We use ϕ_2 to replace the parameter a. Then, using the transformation $\{x = \frac{\phi - \phi_2}{\phi_2}, \ y = \frac{\phi'}{\phi_2}\}$, we can write the second-order equation (2.3) as the planar system

$$\begin{cases} x' = y, \\ y' = 1 + x - \frac{\eta^3}{(\eta - x)^3}, \end{cases} \tag{3.2}$$

associated with the Hamiltonian

$$H(x, y) = \frac{y^2}{2} + V(x), \quad V(x) := -\frac{x^2}{2} - x - \frac{\eta}{2} + \frac{\eta^3}{2(\eta - x)^2}, \tag{3.3}$$

where $\eta = \frac{c - \phi_2}{\phi_2} \in (0, 3)$. The potential V is smooth away from the singular line $x = \eta$, has a local minimum at $x = 0$ and a local maximum at $x_1 < 0$, see Fig. 3.1.

The center at the origin is surrounded by periodic orbits γ_h, which lie inside the level curves $H(x, y) = h$ with $h \in (0, h^*)$ and $h^* = V(x_1)$. Denote by x_2 the unique solution of $V(x_1) = V(x)$ such that $x_1 < 0 < x_2 < \eta$, see Fig. 3.1. Finally, define the period function of the center $(0, 0)$ of system (3.2) by

$$\ell(h) = \int\limits_{\gamma_h} \frac{dx}{y} \quad \text{for each } h \in (0, h^*).$$

Note that $b = 2\phi_2^2 h + \phi_2(c - 2\phi_2)$ and $\mathfrak{L}(a, b) = \ell(h)$ for fixed $a \in (0, a_c)$ and $c > 0$. Since ϕ_2 is fixed, we have $\partial_b \mathfrak{L}(a, b) > 0$ if and only if $\ell'(h) > 0$.

To prove that $\ell'(h) > 0$, we shall use a monotonicity criterion by Chicone [3] for planar systems with Hamiltonians of the form (3.3), where V is a smooth function on (x_1, x_2) with a

nondegenerate relative minimum at the origin. According to the main theorem in [3] the period function $\ell(h)$ is monotonically increasing in h if the function

$$W(x) := \frac{V(x)}{(V'(x))^2}$$

is convex in (x_1, x_2). Hence, we have to prove that $W''(x) > 0$ for every $x \in (x_1, x_2)$. A straight-forward computation shows that

$$W''(x) = -\frac{3(\eta - x)^2 R_\eta(x)}{(\eta^2(3 - \eta) + 3\eta(\eta - 1)x + (1 - 3\eta)x^2 + x^3)^4},$$

where

$$R_\eta(x) = (\eta - 1)x^6 + 10\eta(1 - \eta)x^5 + 5\eta^2(7\eta - 9)x^4 + 20\eta^3(5 - 3\eta)x^3$$
$$+ 5\eta^3(11\eta^2 - 22\eta - 1)x^2 + 2\eta^4(9 + 28\eta - 13\eta^2)x + 5\eta^5(\eta^2 - 2\eta - 3).$$

We need to show that $R_\eta(x) < 0$ for $x \in (x_1, x_2)$ and $\eta \in (0, 3)$. The case $\eta = 1$ has to be considered separately, for which we find that

$$R_1(x) = -10(x - 1)^4 - 2(5 - 4x) < 0 \text{ for } x \in (-\infty, 1),$$

and, in particular, $R_1(x) < 0$ on (x_1, x_2) since $x_1 < 0 < x_2 < \eta = 1$. For $\eta \in I := (0, 3) \setminus \{1\}$ we have to ensure that the polynomial R does not change sign on the interval $\Omega := (x_1, x_2)$ when varying the parameter η. To achieve this we will use a result which allows one to control the sign of univariate polynomials depending on a parameter, see Lemma A.1 in Appendix A. In what follows, $Res_x(f, g)$ stands for the *multipolynomial resultant* of two polynomials f and g in x (see for instance [6,15]).

The assumptions of Lemma A.1 essentially ensure that the number of roots of the one-parametric polynomial $R_\eta(x)$ are constant on (x_1, x_2) when varying $\eta \in (0, 3)$, since they depend continuously on the parameter η and prevent the bifurcation of roots at the boundary or appearance of double roots in the interior of the interval. We will now check the assumptions one by one.

- Assumption (i) clearly holds for all $\eta \in I$.
- For assumption (ii) we compute the discriminant of R_η with respect to x,

$$\text{Disc}_x(R_\eta) = -6400000\eta^{23}(\eta + 1)^4(\eta - 1)(27\eta^2 + 14\eta + 3)(\eta^3 - 5\eta^2 + 11\eta + 1)$$

  and see that it is different from zero on I since the term $\eta^3 - 5\eta^2 + 11\eta + 1$ has only a negative real root and all other terms do not have any real roots on I.
- For assumption (iv) we need to show that R_η does not vanish in the boundary points x_1, x_2 of Ω. Since we do not have explicit expressions for these points we will compare R_η with other polynomials with explicitly known roots since x_1 is the nontrivial zero of $V'(x)$ and x_2 is the unique zero of $V(x_1) - V(x)$ in (x_1, η). Since

$$\text{Res}_x(R_\eta, V') = 5\eta^{15}(\eta + 1)^2(\eta - 3)^2(27\eta^2 + 14\eta + 3)^2 \neq 0,$$

for $\eta \in I$, the polynomials R_η and V' do not have a common root, so in particular $R_\eta(x_1) \neq 0$ for $\eta \in I$. For the other boundary point x_2, we define

$$P_\eta(x) := \mathrm{Res}_y(V'(y), V(y) - V(x))$$

and compute

$$\mathrm{Res}_x(R_\eta, P_\eta) = 25\,\eta^{45}\,(\eta+1)^7\,(\eta-3)^6\,\left(9\,\eta^2 + 21\,\eta + 16\right)\left(27\,\eta^2 + 14\,\eta + 3\right)^2 Q(\eta),$$

where Q is a polynomial of degree 34 whose expression we omit. We use Sturm's method, see [48, Theorem 5.6.2], to prove that it has one root at $\eta = \eta_1 \approx 1.083$, and find the rational lower and upper bounds

$$\eta_1 \in [\underline{\eta}_1, \bar{\eta}_1] := \left[\frac{277}{256}, \frac{555}{512}\right] \subset I.$$

This proves that $R_\eta(x_2) \neq 0$ for $\eta \in I \setminus \{\bar{\eta}\}$. Therefore, the number of zeros of $R_\eta(x)$ on Ω is constant for $\eta \in I \setminus \{\eta_1\}$. The value $\eta = \eta_1$ is treated separately in the last item of the proof.

- For assumption (iii) we have to show that there exists some η in each of the subintervals of $I \setminus \{\eta_1\}$ such that $R_\eta(x) \neq 0$. For $\eta = \frac{1}{2} \in (0, 1)$,

$$R_{\frac{1}{2}}(x) = \frac{1}{128}\left(-64x^6 + 320x^5 - 880x^4 + 1120x^3 - 740x^2 + 316x - 75\right).$$

Using again Sturm's method we can show that $R_{\frac{1}{2}}$ has two real roots r_i, $i = 1, 2$, for which we can find rational lower and upper bounds such that $r_i \in [\underline{r}_i, \bar{r}_i] =: I_i$, $i = 1, 2$, for instance

$$r_1 \in I_1 = \left[\frac{327}{512}, \frac{655}{1024}\right] \quad \text{and} \quad r_1 < r_2 \in I_2 = \left[\frac{991}{1024}, \frac{31}{32}\right].$$

To show that the two roots are outside of Ω we use Sturm's method once more for the polynomial $P_{\frac{1}{2}}(x)$ to find rational bounds for

$$x_2 \in [\underline{x}_2, \bar{x}_2] := \left[\frac{94993}{131072}, \frac{23749}{32768}\right].$$

Then it is straightforward to see that $V(\bar{x}_2) - V(\underline{r}_1) < 0$, which implies that $x_2 < \bar{x}_2 < \underline{r}_1 < r_1 < r_2$ since V is monotone increasing for $x > 0$. Hence $R_{\frac{1}{2}}(x) \neq 0$. Similarly, we show that $R_\eta \neq 0$ for $\eta = \underline{\eta}_1 \in (1, \eta_1)$ and $\eta = 2 \in (\bar{\eta}_1, 3)$. Then, by Lemma A.1, $R_\eta(x) \neq 0$ on Ω for all $\eta \in I \setminus \{\eta_1\}$ and one can easily check that $R_\eta(x) < 0$ on Ω in each of the subintervals of $I \setminus \{\eta_1\}$.

- To ensure that also $R_{\eta_1}(x) < 0$ we prove that R_η is monotone in a neighborhood of η_1, i.e. we show that $R'_\eta(x) \neq 0$ on Ω for $\eta \in (\underline{\eta}_1, \bar{\eta}_1)$ using again Lemma A.1. Indeed, similarly as above we show that $R_\eta(x_1)R_\eta(x_2)\mathrm{Disc}_x(R'_\eta) \neq 0$ for $\eta \in (\underline{\eta}_1, \bar{\eta}_1)$ and evaluating $R'_\eta(x)$ in one value, for instance $\eta = 1083/1000 \in (\underline{\eta}_1, \bar{\eta}_1)$, we find using Sturm's method that $R'_\eta(x) \neq 0$ on Ω for $\eta \in (\underline{\eta}_1, \bar{\eta}_1)$.

This concludes the proof that $W''(x) > 0$ for $x \in (x_1, x_2)$ and $\eta \in I$, which yields $\ell'(h) > 0$ by the main theorem in [3]. $\quad\square$

3.2. Monotonicity of the period function with respect to the parameter a

We shall study monotonicity properties of the period function $\mathfrak{L}(a, b)$ as a function of a for fixed $c > 0$ and $b \in (-c^2, \frac{1}{8}c^2)$. This result will be used to prove the last assertion of Theorem 1.1, see Corollary 4.7.

Lemma 3.2. *Fix $c > 0$ and $b \in (-c^2, \frac{1}{8}c^2)$. The period function $\mathfrak{L}(a, b)$ satisfies the following properties:*

- *It is strictly monotonically increasing in a if $b \in (-c^2, -\frac{2}{9}c^2]$;*
- *It has a unique critical point in a, which is a maximum, if $b \in (-\frac{2}{9}c^2, 0)$;*
- *It is strictly monotonically decreasing in a if $b \in [0, \frac{1}{8}c^2)$.*

Remark 3.3. The proof of Lemma 3.2 follows very closely the one carried out in [23] for the period function of the CH equation and relies strongly on the tools developed in [16].

In contrast to the previous subsection, where periodic smooth traveling waves are characterized as solutions of the second-order equation (2.3), we now regard the traveling waves as solutions of the equivalent second-order equation (2.2). For convenience, we rewrite (2.2) as a planar system such that its center is located at the origin. This is obtained via the change of variables

$$\left\{ x = \frac{\phi - c}{\sqrt{\Delta}} + \theta, \qquad y = \frac{\phi'}{\sqrt{\Delta}} \right\},$$

where $\Delta := c^2 - 8b > 0$ and $\theta := \frac{1}{4}\left(\frac{3c}{\sqrt{\Delta}} - 1\right) > 0$. Periodic orbits are obtained from the planar system

$$\begin{cases} x' = y, \\ y' = \dfrac{x + 2x^2 - y^2}{x - \theta}, \end{cases} \tag{3.4}$$

which is analytic away from the singular line $x = \theta$ and has the analytic first integral

$$H(x, y) = A(x) + C(x)y^2,$$

with $A(x) = -\frac{1}{6}x^2(3x^2 + 2x(1 - 2\theta) - 3\theta)$ and $C(x) = \frac{1}{2}(x - \theta)^2$. The first integral satisfies the hypotheses in Proposition A.2 with $B(x) = 0$. Moreover, its integrating factor $K(x) = (x - \theta)^2$ depends only on x. The function $A(x)$ satisfies $A(0) = 0$ and has a minimum at $x = 0$, which yields a center at $(0, 0)$, and two local maxima at $x = \theta$ and $x = -\frac{1}{2}$, the latter one yielding a saddle point at $(-\frac{1}{2}, 0)$. The period function associated to the center of the differential system (3.4) can be written as
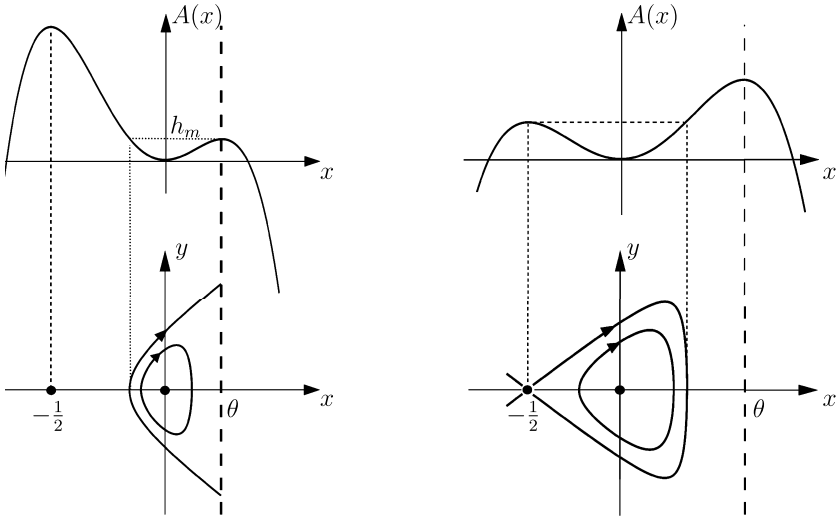
Fig. 3.2. A sketch of the period annulus $\mathscr{P}$ of the center at the origin of system (3.4) for $\theta \in (0, \frac{1}{2})$ (left) and $\theta \in [\frac{1}{2}, \infty)$ (right).

$$\ell(h) = \int_{\gamma_h} \frac{dx}{y} \quad \text{for } h \in (0, h^*),$$

where γ_h is the periodic orbit inside the energy level $\{(x, y) : H(x, y) = h\}$ with either $h^* = A(-\frac{1}{2})$ for $\theta \in [\frac{1}{2}, \infty)$ or $h^* = A(\theta)$ for $\theta \in (0, \frac{1}{2})$.

When $\theta \geq \frac{1}{2}$, we find that $A(\theta) \geq A(-\frac{1}{2})$, in which case the period annulus $\mathscr{P}$ is bounded by the homoclinic orbit at the saddle point, see the right panel of Fig. 3.2. When $\theta \in (0, \frac{1}{2})$ the outer boundary of $\mathscr{P}$ consists of a trajectory with α and ω-limit in the straight line $\{x = \theta\}$ and the segment between these two points, see the left panel of Fig. 3.2. In view of these structural differences, we will study the monotonicity of the period function separately for $\theta \in (0, \frac{1}{2})$ and for $\theta \geq \frac{1}{2}$.

Recall that a mapping σ is said to be an involution if $\sigma \circ \sigma = \text{Id}$. The function A defines an *involution* σ satisfying $A = A \circ \sigma$. We find that

$$A(x) - A(z) = -\frac{1}{6}(x - z)S(x, z), \tag{3.5}$$

where

$$S(x, z) := 3x^3 + (3z + 2 - 4\theta)x^2 + (3z^2 + 2z - 4\theta z - 3\theta)x + 3z^3 + 2z^2 - 4\theta z^2 - 3\theta z,$$

such that $S(x, \sigma(x)) = 0$. Let (x_ℓ, x_r) be the projection onto the x-axis of the period annulus $\mathscr{P}$ around the center at the origin of the differential system (3.4). Given an analytic function f on $(x_\ell, x_r) \setminus \{0\}$ one can define its σ-*balance* to be

$$\mathscr{B}_\sigma(f)(x) := \frac{f(x) - f(\sigma(x))}{2}.$$

The number of zeros of the sigma balance of certain polynomials gives upper bounds for the number of critical points of the period function, see [16] and Appendix A.2, as we will study below. The proof of the following auxiliary result is a straightforward computation of the first coefficients in the Taylor expansion of the period function $\ell(h)$ using standard techniques (see for example [18]).

Lemma 3.4. *The first two period constants of the period function $\ell(h)$ are given by*

$$\Delta_1 = \frac{\pi(4\theta + 1)(5\theta - 1)}{6\theta},$$

$$\Delta_2 = -\frac{\pi}{288\theta^2}\left(-48\theta^6 - 144\theta^5 - 1808\theta^4 + 1152\theta^{5/2} + 1096\theta^3 + 741\theta^2 + 322\theta + 13\right),$$

such that the expansion of ℓ is given by $\ell(h) = 2\pi\sqrt{\theta} + \Delta_1 h^2 + \Delta_2 h^4 + \mathcal{O}(h^5)$.

We are now in position to prove monotonicity of the period function $\ell(h)$ for $\theta \geq \frac{1}{2}$.

Lemma 3.5. *If $\theta \geqslant \frac{1}{2}$, then the period function $\ell(h)$ is monotonically increasing.*

Proof. For $\theta \geqslant \frac{1}{2}$, the projection of the period annulus on the x-axis is $(-\frac{1}{2}, x_r)$, where $A(x_r) = A(-\frac{1}{2})$. To prove that the period function $\ell(h)$ has no critical points we will use Proposition A.2, which gives an upper bound on the number of critical periods in terms of the number of zeros of the sigma balance $\mathscr{B}_\sigma(\ell_1)$ of ℓ_1. Our goal is to prove that $\mathscr{B}_\sigma(\ell_1) \neq 0$ on $(0, x_r)$. Note that since $\mathscr{B}_\sigma(f) \circ \sigma = -\mathscr{B}_\sigma(f)$ and σ maps $(0, x_r)$ to $(x_\ell, 0)$, we may for convenience study the latter interval, which in our case is $(-\frac{1}{2}, 0)$. The function ℓ_1 is defined in terms of A, C and K, and takes the form

$$\ell_1(x) = \frac{\sqrt{2}}{6}\frac{(4\theta + 1)(x + 1)}{(2x + 1)^3(x - \theta)}.$$

We find that $L\big(x, \ell_1(x)\big) \equiv 0$ with

$$
\begin{aligned}
L(x, y) := &\, (4\theta + 1)(x - y)(-8x - 8)y^3 + \left(8x\theta - 8x^2 + 8\theta - 20x - 12\right)y^2 \\
&+ \left(8\theta x^2 - 8x^3 + 20x\theta - 20x^2 + 12\theta - 18x - 6\right)y \\
&+ 8\theta x^2 - 8x^3 + 12x\theta - 12x^2 + 5\theta - 6x - 1.
\end{aligned}
$$

We find that

$$\mathrm{Res}_z\big(L(x, z), L(y, z)\big) = 8192(\theta + 1)(4\theta + 1)^8(x - y)^4 T(x, y)^4,$$

with T a bivariate polynomial of degree 12 in x and y, which also depends polynomially on θ. Finally

$$\mathscr{R}(x) := \mathrm{Res}_y\big(S(x, y), T(x, y)\big) = (2x + 1)^{12}(x - \theta)^4(\theta + 1)^3(4\theta + 1)^4 R(x)^4,$$

where R is a univariate polynomial of degree 10 in x depending polynomially on θ, and S was defined in (3.5).

Let us denote by $\mathcal{Z}(\theta)$ the number of roots of R on $(-\frac{1}{2}, 0)$ counted with multiplicities. We claim that $\mathcal{Z}(\theta) = 0$ for all $\theta \geqslant \frac{1}{2}$. For $\theta = \frac{1}{2}$ this can be easily verified by applying Sturm's method, see [48, Theorem 5.6.2]. To prove it for $\theta > \frac{1}{2}$ note that

$$R(0) = (5\theta - 1)(\theta + 1)(64\theta^3 + 48\theta^2 + 21\theta + 1)$$

and

$$R\left(-\frac{1}{2}\right) = \frac{9}{2}(2\theta - 1)(1 + 2\theta)^4,$$

which do not vanish for $\theta > \frac{1}{2}$. The discriminant of R with respect to x, $\mathrm{Disc}_x(R)$, is a polynomial of degree 70 in θ with no real roots for $\theta > \frac{1}{2}$. Choosing one value of $\theta > \frac{1}{2}$ and applying Sturm's method, we find that $\mathcal{Z}(\theta) = 0$ for all $\theta \in (\frac{1}{2}, +\infty)$ using Lemma A.1. Therefore, $\mathcal{R}$ does not vanish on $(-\frac{1}{2}, 0)$ for any $\theta \geqslant \frac{1}{2}$. In view of (a) in Proposition A.3 this implies that $\mathscr{B}_\sigma(\ell_1) \neq 0$ on $(-\frac{1}{2}, 0)$. Since by Proposition A.2 the number of critical periods is bounded above by the number of roots of $\mathscr{B}_\sigma(l_1)$ it follows that the period function is monotonous. Finally, the result follows by noting that, thanks to Lemma 3.4, the first period constant Δ_1 is positive for $\theta \geqslant \frac{1}{2}$. $\quad\square$

Now we study the period function $\ell(h)$ for $\theta \in (0, \frac{1}{2})$. The following lemma describes the behavior of the period function near its outer boundary.

Lemma 3.6. *If $\theta \in (0, \frac{1}{2})$, then the period function $\ell(h)$ satisfies $\lim\limits_{h \to h_m} \ell'(h) = -\infty$, where $h_m = A(\theta)$ is the energy level of the outer boundary of $\mathscr{P}$.*

Proof. It was proven in [23] that the derivative of the period function $\ell(h)$ can be written as

$$\ell'(h) = \frac{1}{h} \int\limits_{\gamma_h} R(x) \frac{dx}{y},$$

where

$$R = \frac{1}{2C}\left(\frac{KA}{A'}\right)' - \frac{K(AC)'}{4A'C^2}.$$

Taking into account the respective definitions of these quantities, we find that

$$R(x) = \frac{x(x+1)(4\theta + 1)}{6(2x+1)^2(x-\theta)}.$$

For $\theta \in (0, \frac{1}{2})$ and $h \in (0, h_m)$, we have that $h - A(x) = (x - x_h^-)(x - x_h^+)(x - x_h^\ell)(x - x_h^r)$, where $x_h^\ell < -\frac{1}{2} < x_\ell < x_h^- < 0 < x_h^+ < \theta < x_h^r$, see Fig. 3.3. In particular, the projection of γ_h
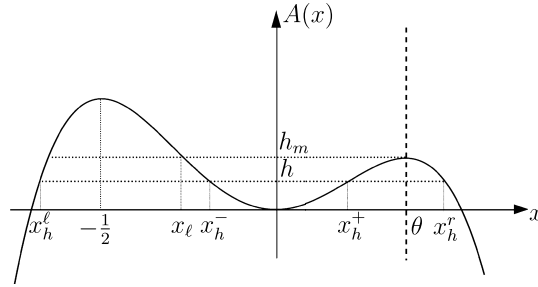
Fig. 3.3. The distribution of roots of the function $h - A(x)$ in the proof of Lemma 3.6 for $\theta \in (0, \frac{1}{2})$.

onto the x-axis is $[x_h^-, x_h^+]$ so that we split the integral into two parts,

$$\ell'(h) = \frac{2}{h} \left[I_1(h) + I_2(h) \right],$$

where

$$I_1(h) = \int\limits_{x_h^-}^{0} f(x, h) \, dx \ \text{ and } \ I_2(h) = \int\limits_{0}^{x_h^+} f(x, h) \, dx$$

with

$$f(x, h) = \frac{R(x)\sqrt{C(x)}}{\sqrt{h - A(x)}} = \frac{-(4\theta + 1)x(x + 1)}{6\sqrt{2}(2x + 1)^2 \sqrt{(x - x_h^-)(x - x_h^+)(x - x_h^\ell)(x - x_h^r)}}.$$

In order to study I_1, let us write $f(x, h) = \frac{g_1(x, h)}{\sqrt{x - x_h^-}}$, where

$$g_1(x, h) := \frac{-(4\theta + 1)x(x + 1)}{6\sqrt{2}(2x + 1)^2 \sqrt{(x - x_h^+)(x - x_h^\ell)(x - x_h^r)}}.$$

Note that g_1 is a continuous function on $(-\frac{1}{2}, 0] \times (0, h_m)$. Consequently there exists $M_1 \in \mathbb{R}$ such that $M_1 := \sup\{g_1(x, h) : (x, h) \in [x_r, 0] \times [\frac{1}{2}h_m, h_m]\}$. In addition, observe that $M_1 > 0$ for $-\frac{1}{2} < x < 0$. Thus for $h \in (\frac{1}{2}h_m, h_m)$ we have that

$$I_1(h) = \int\limits_{x_h^-}^{0} \frac{g_1(x, h) dx}{\sqrt{x - x_h^-}}$$

$$\leqslant M_1 \int\limits_{x_h^-}^{0} \frac{dx}{\sqrt{x - x_h^-}}$$

$$= 2M_1\sqrt{-x_h^-}$$
$$< \sqrt{2}M_1.$$

In order to study I_2 let us write $f(x,h) = \dfrac{g_2(x,h)}{\sqrt{(x-x_h^+)(x-x_h^r)}}$, where

$$g_2(x,h) := \frac{-(4\theta+1)x(x+1)}{6\sqrt{2}(2x+1)^2\sqrt{(x-x_h^-)(x-x_h^\ell)}}.$$

Since g_2 is continuous on $[0,\infty) \times (0, h_m)$, there exists $M_2 := \sup\{g_2(x,h) : (x,h) \in [0,\theta] \times [\frac{1}{2}h_m, h_m]\}$ and we observe that $M_2 < 0$ for $0 < x < \theta$. Consequently, if $h \in (\frac{1}{2}h_m, h_m)$, then

$$I_2(h) = \int_0^{x_h^+} \frac{g_2(x,h)dx}{\sqrt{(x-x_h^+)(x-x_h^r)}}$$

$$\leqslant M_2 \int_0^{x_h^+} \frac{dx}{\sqrt{(x-x_h^+)(x-x_h^r)}}$$

$$= M_2 \ln\left(\frac{\sqrt{x_h^r/x_h^+}+1}{\sqrt{x_h^r/x_h^+}-1}\right),$$

where the upper bound diverges to $-\infty$ as $h \to h_m$ since $M_2 < 0$ and $x_h^+, x_h^r \to \theta$ as $h \to h_m$. Since $\ell'(h) = \frac{2}{h}\big(I_1(h) + I_2(h)\big)$, the bound on $I_1(h)$ and the divergence of $I_2(h)$ as $h \to h_m$ imply the result. $\quad\square$

We are now ready to prove the monotonicity result of the period function $\ell(h)$ in the case that $\theta \in (0, \frac{1}{2})$.

Lemma 3.7. *For $\theta \in (0, \frac{1}{2})$ the period function $\ell(h)$*

(a) *is monotonically decreasing for $\theta \in (0, \frac{1}{5}]$,*

(b) *has a unique critical period, which is a maximum, for $\theta \in (\frac{1}{5}, \frac{1}{2})$.*

Proof. For $\theta \in (0, \frac{1}{2})$ the projection of the period annulus onto the x-axis is (x_ℓ, θ), where $A(x_\ell) = A(\theta) = h_m$, see Fig. 3.3. We proceed in exactly the same way as we did for $\theta > \frac{1}{2}$, i.e. applying Propositions A.2 and A.3 which bound the number of critical periods from above by the number of roots of the sigma balance of certain functions.

Let us now denote by $\mathcal{Z}(\theta)$ the number of roots of R on $(0, \theta)$ counted with multiplicities and let R be defined as in the proof of Lemma 3.5. We find that $R(0)$ has a root at $\theta = \frac{1}{5} \in (0, \frac{1}{2})$, while

$$R(\theta) = (\theta + 1)^3 (2\theta - 1)^3 (1 + 2\theta)^4$$

does not have a real root for $\theta \in (0, \frac{1}{2})$. The discriminant of R with respect to x, $\mathrm{Disc}_x(R)$, is a polynomial of degree 70 that has only one real root on $(0, \frac{1}{2})$ in $\theta = \frac{1}{5}$. Therefore, $\mathcal{Z}(\theta)$ is constant on $I_1 := (0, \frac{1}{5})$ and on $I_2 := (\frac{1}{5}, \frac{1}{2})$. Choosing $\theta = \frac{1}{10} \in I_1$ we find that R does not vanish on $(0, \frac{1}{10})$ and hence $\mathcal{Z}(\theta) = 0$ for all $\theta \in (0, \frac{1}{5})$. For $\theta = \frac{1}{5}$, we find that $R \neq 0$ on $(0, \frac{1}{5})$ as well by applying Sturm's method. Hence, it follows from Proposition A.2 that $\mathscr{B}_\sigma(\ell_1) \neq 0$ on $(0, \theta)$, and we may conclude that the period function is monotonous for $\theta \in (0, \frac{1}{5}]$. On the other hand, choosing $\theta = \frac{3}{10} \in I_2$ we find that R vanishes once, which implies that the criterion in Proposition A.2 does not apply. Therefore, we move on to studying $\mathscr{B}_\sigma(\ell_2)$, where

$$\ell_2 = \frac{\sqrt{2}}{36} \frac{(4\theta + 1) P(x)}{(x - \theta)^3 (2x + 1)^5},$$

and $L(x, \ell_2(x)) \equiv 0$ with P and L polynomials which we omit for the sake of brevity. As before, we compute $\mathrm{Res}_z\big(L(x, z), L(y, z)\big) = T(x, y)$, with T a bivariate polynomial which also depends polynomially on θ, and $\mathscr{R}(x) := \mathrm{Res}_y\big(S(x, y), T(x, y)\big) = 12\sqrt{2}x(-x + \theta)^3(4\theta + 1)^3(2x + 1)^5 R(x)$, where R is a univariate polynomial of degree 30 in x depending polynomially on θ, and S was defined in (3.5).

Let us denote by $\mathcal{Z}(\theta)$ the number of roots of R on $(0, \theta)$ counted with multiplicities. We claim that $\mathcal{Z}(\theta) = 1$ for all $\theta \in (\frac{1}{5}, \frac{1}{2})$. We find that in that parameter interval, $R(0)R(\theta)\mathrm{Disc}_x(R) \neq 0$ and hence $\mathcal{Z}(\theta)$ is constant. Evaluating in $\theta = \frac{3}{10} \in (\frac{1}{5}, \frac{1}{2})$ and using Sturm's method we find that $R(x)$ has exactly one real root in $(0, \frac{3}{10})$ and hence $\mathcal{Z}(\theta) = 1$ for all $\theta \in (\frac{1}{5}, \frac{1}{2})$. In view of Proposition A.2 for $i = 2 > 1 = n$ we may conclude that the number of critical periods is at most 1.

Recall from Lemma 3.6 that $\ell'(h) \to -\infty$ as h tends to h_m for all $\theta \in (0, \frac{1}{2})$. Since the first period constant Δ_1 computed in Lemma 3.4 is negative for $\theta \in (0, \frac{1}{5})$ and positive for $\theta \in (\frac{1}{5}, \frac{1}{2})$, we conclude that the period function $\ell(h)$ is monotonous decreasing near both endpoints of $(0, h_m)$ for all $\theta \in (0, \frac{1}{5})$, while it is increasing near $h = 0$ and decreasing near $h = h_m$ for $\theta \in (\frac{1}{5}, \frac{1}{2})$. For $\theta = \frac{1}{5}$ we have that $\Delta_1 = 0$ and $\Delta_2 < 0$, and hence the period function is decreasing near the endpoint $h = 0$. Taking into account the upper bounds derived above, we may conclude that the period function $\ell(h)$ is monotonous decreasing for $\theta \in (0, \frac{1}{5}]$ and it has a unique critical period which is a maximum for $\theta \in (\frac{1}{5}, \frac{1}{2})$. $\quad\square$

Remark 3.8. For the sake of completeness we give the limiting value of the integral defining the period function at the right endpoint of its interval of definition:

$$\ell_{h_m} = \frac{1}{2} \ln\left(\frac{(\theta + 1)(1 - 2\theta)}{4\theta + 1 + 3\sqrt{\theta(1 + 2\theta)}}\right).$$

Hence ℓ_{h_m} is positive and finite on $(0, \frac{1}{2})$.

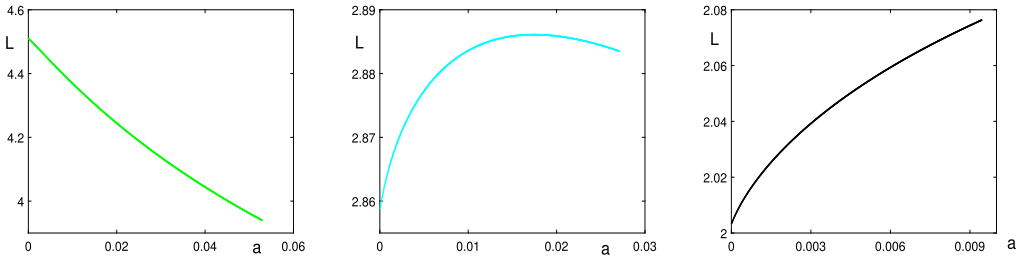We finish this section with the proof of Lemma 3.2.

Fig. 3.4. The period function $\mathfrak{L}(a, b)$ versus a for $c = 1$ and three values of b: $b = 0$ (left), $b = -0.2$ (middle), and $b = -0.4$ (right).

Proof of Lemma 3.2. The smooth periodic solutions of the second-order equation (2.2) are periodic orbits of system (3.4), which are parametrized by $h \in (0, h_m)$ and whose periods are assigned by the period function $\ell(h)$. A straightforward computation shows that $\mathfrak{L}(a, b) = \ell(h)$ and

$$a = -\left(\frac{\sqrt{\Delta} - 3c}{4}\right)^2 \left(2h\frac{\Delta}{\theta} + \frac{\sqrt{\Delta} - 3c}{4}\left(\frac{\sqrt{\Delta} - 3c}{4} + c\right)\right).$$

Therefore, $\frac{da}{dh} < 0$ and so for fixed $b \in (-c^2, \frac{1}{8}c^2)$ and $c > 0$ we have that

$$\text{sign}(\partial_a \mathfrak{L}(a, b)) = -\text{sign}(\ell'(h)),$$

which means that the monotonicity properties of $\ell(h)$ imply those of $\mathfrak{L}(a, b)$. More precisely, in view of the definition of θ, the parameter regime $\theta \geq \frac{1}{2}$ corresponds to values $b \in [0, \frac{1}{8}c^2)$ for which $\partial_a \mathfrak{L}(a, b) < 0$ in view of Lemma 3.5. On the other hand, the value $\theta = \frac{1}{5}$ corresponds to $b = -\frac{2}{9}c^2$ and we infer from Lemma 3.7 that $\partial_a \mathfrak{L}(a, b) > 0$ for $b \in (-c^2, -\frac{2}{9}c^2)$ whereas $\mathfrak{L}(a, b)$ has a unique critical point in a, which is a maximum, for $b \in (-\frac{2}{9}c^2, \frac{1}{8}c^2)$. This concludes the proof. $\quad\square$

Remark 3.9. Fig. 3.4 illustrates the result of Lemma 3.2 for $c = 1$. The period function $\mathfrak{L}(a, b)$ is monotonically decreasing in a for $b = 0$, is non-monotone in a with a single maximum for $b = -0.2$, and is monotonically increasing in a for $b = -0.4$. The range of a values depends on the values of b as is clear from Fig. 1.1. Note that the colors do not correspond to the colors of Fig. 1.2, where the values of $L = \mathfrak{L}(a, b)$ are fixed.

4. Spectral properties of the Hessian operator $\mathcal{L}$

Here we shall consider the spectral properties of the Hessian operator $\mathcal{L}$ given by (1.8). Since $\mathcal{L} : L^2_{\text{per}} \to L^2_{\text{per}}$ is self-adjoint, its spectrum $\sigma(\mathcal{L})$ consists of the absolutely continuous part, denoted by $\sigma_c(\mathcal{L})$, and the point spectrum, denoted by $\sigma_p(\mathcal{L})$. Since $c - \phi$ is a bounded multiplicative operator in L^2_{per} and $-3c(4 - \partial_x^2)^{-1}$ is a compact operator in L^2_{per}, Kato's theorem [30] implies that

$$\sigma_c(\mathcal{L}) = \sigma(c - \phi) = \text{Range}(c - \phi)$$

Since $c - \phi > 0$ by Lemma 2.1, there exists

$$\lambda_0 := c - \max_{x \in \mathbb{T}_L} \phi(x) > 0$$

such that $\sigma_p(\mathcal{L})$ admits only finitely many eigenvalues of finite multiplicities below λ_0.

The following lemma gives an efficient technique to count the negative and zero eigenvalues of $\mathcal{L}$. It is an analogue of the Birman–Schwinger principle used in quantum mechanics [24, Section 5.6]. A similar criterion was developed in our previous work [22].

Lemma 4.1. *For every $\lambda \in (-\infty, \lambda_0)$ with $\lambda_0 > 0$, let the Schrödinger operator $\mathcal{K}(\lambda) : H^2_{\mathrm{per}} \subset L^2_{\mathrm{per}} \to L^2_{\mathrm{per}}$ be defined by*

$$\mathcal{K}(\lambda) := -\partial_x^2 + \frac{c - 4\phi - 4\lambda}{c - \phi - \lambda}. \tag{4.1}$$

Then, we have

$$\#\{\lambda < 0: \ \mathcal{L}w = \lambda w, \ w \in L^2_{\mathrm{per}}\} = \#\{\mu < 0: \ \mathcal{K}(0)v = \mu v, \ v \in H^2_{\mathrm{per}}\}, \tag{4.2}$$

where $\#\{\cdot\}$ denotes the number of eigenvalues, taking into account their multiplicities.

Proof. The spectral problem $\mathcal{L}w = \lambda w$ with $w \in L^2_{\mathrm{per}}$ can be rewritten in the variable $v := (4 - \partial_x^2)^{-1} w$ as the spectral problem $\mathcal{K}(\lambda)v = 0$ with $v \in H^2_{\mathrm{per}}$. Since the operator

$$(4 - \partial_x^2) : H^2_{\mathrm{per}} \subset L^2_{\mathrm{per}} \to L^2_{\mathrm{per}}$$

is invertible with a bounded inverse in L^2_{per}, the correspondence $v = (4 - \partial_x^2)^{-1} w$ implies that if $\lambda < 0$ is an eigenvalue of $\mathcal{L}$, then $\mathcal{K}(\lambda)$ admits a zero eigenvalue of the same multiplicity. Because of the compact embedding of H^2_{per} into L^2_{per}, we have

$$\sigma(\mathcal{K}(\lambda)) = \sigma_p(\mathcal{K}(\lambda)) \quad \text{for } \lambda \in (-\infty, \lambda_0),$$

that is, the spectrum of $\mathcal{K}(\lambda)$ consists of eigenvalues as long as

$$A(x, \lambda) := \frac{c - 4\phi(x) - 4\lambda}{c - \phi(x) - \lambda}$$

is bounded in x. Since

$$\partial_\lambda A(x, \lambda) = -\frac{3c}{(c - \phi(x) - \lambda)^2} < 0,$$

the eigenvalues of $\mathcal{K}(\lambda)$ are monotonically decreasing functions of λ. Since
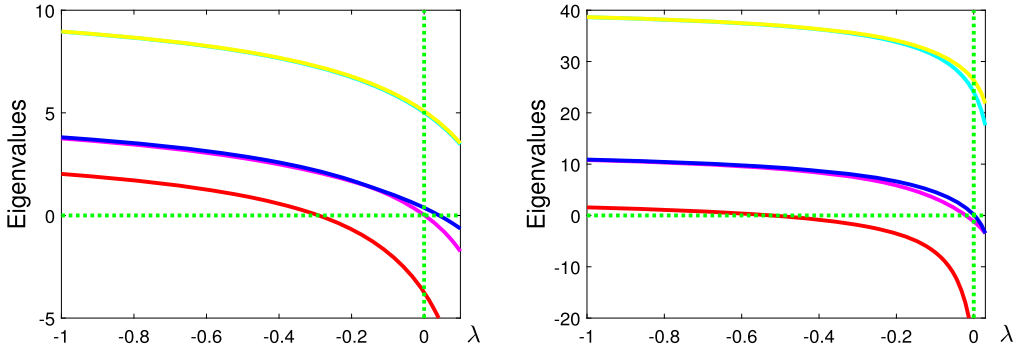
$$\lim_{\lambda \to -\infty} A(x, \lambda) = 4,$$

Fig. 4.1. The lowest five eigenvalues of $\mathcal{K}(\lambda)$ versus λ for $a = 0.04$, $b = 0$ (left) and $a = 0.001$, $b = -0.3$ (right) with $c = 1$. Eigenvalues are strictly decreasing in λ.

there exists $\lambda_\infty \in (-\infty, 0)$ such that $A(x, \lambda) > 0$ for all $x \in \mathbb{R}$ and $\lambda < \lambda_\infty$, and hence $\sigma_p(\mathcal{K}(\lambda)) > 0$ for $\lambda \in (-\infty, \lambda_\infty)$. Each eigenvalue of $K(\lambda)$, say $\mu(\lambda)$, is decreasing and positive for large negative λ, and therefore crosses the horizontal axis at most once in $(-\infty, 0)$. If there exists $\lambda \in (-\infty, 0)$ such that $\mu(\lambda) = 0$, then $\mu(0) < 0$, i.e. it corresponds to a negative eigenvalue of $K(0)$. Therefore, the number of negative eigenvalues of $K(0)$ equals the number of λ for which $K(\lambda)v = 0$. In view of the correspondence with the number of negative eigenvalues of $\mathcal{L}$, this proves the equality (4.2). $\quad\square$

Remark 4.2. Because $\mathcal{L}\phi' = 0$, we have $\mathcal{K}(0)v' = 0$ so that 0 is an eigenvalue of $\mathcal{K}(0)$.

Remark 4.3. Fig. 4.1 illustrates the criterion in Lemma 4.1 with numerical approximations of the eigenvalues of $\mathcal{K}(\lambda)$ in L^2_{per} versus λ for two different values of (a, b) with $c = 1$. The left panel corresponds to the choice $(a, b) = (0.04, 0)$ above the curve $a = a_0(b)$ shown on Fig. 1.1. Only the first eigenvalue of $\mathcal{K}(\lambda)$ crosses the zero level (dotted line) in $(-\infty, 0)$, whereas the second eigenvalue crosses the zero level at $\lambda = 0$. The right panel corresponds to the choice $(a, b) = (0.001, -0.3)$ below the curve $a = a_0(b)$ shown on Fig. 1.1. The first two eigenvalues of $\mathcal{K}(\lambda)$ cross the zero level in $(-\infty, 0)$ and the third eigenvalue, which is close to the second eigenvalue, crosses the zero level at $\lambda = 0$. The zero eigenvalue of $\mathcal{K}(0)$ exists in both cases, in accordance with Remark 4.2.

The next result uses the criterion in Lemma 4.1 to relate the number of negative eigenvalues and the multiplicity of the zero eigenvalue of $\mathcal{L}$ to the monotonicity of the period function $\mathfrak{L}(a, b)$ defined in (3.1).

Lemma 4.4. *The linearized operator $\mathcal{L} : L^2_{\mathrm{per}} \to L^2_{\mathrm{per}}$ given by (1.8) admits*

- *two negative eigenvalues and a simple zero eigenvalue if $\partial_a \mathfrak{L} > 0$;*
- *a simple negative eigenvalue and a double zero eigenvalue if $\partial_a \mathfrak{L} = 0$;*
- *a simple negative eigenvalue and a simple zero eigenvalue if $\partial_a \mathfrak{L} < 0$,*

where $\mathfrak{L}(a, b)$ is given by (3.1), and the rest of its spectrum in L^2_{per} is strictly positive.

The idea of the proof of Lemma 4.4 is to view the operator $\mathcal{K}(0)$ as a Schrödinger operator with an even L-periodic smooth potential and to analyze the solutions of $\mathcal{K}(0)v = 0$. In the final

step of the proof we use the following result from Floquet theory, see for instance [27, Lemma 4.2], [42, Theorem 3.1], and also the classical results in [13,41].

Proposition 4.5. *Let $\mathcal{M} := -\partial_x^2 + Q(x)$ be the Schrödinger operator with the even, L-periodic, smooth potential Q. Assume that $\mathcal{M}w = 0$ is satisfied by a linear combination of two solutions φ_1 and φ_2 satisfying*

$$\varphi_1(x + L) = \varphi_1(x) + \theta\varphi_2(x)$$

and

$$\varphi_2(x + L) = \varphi_2(x)$$

with some $\theta \in \mathbb{R}$. Assume that φ_2 has two zeros on the period of Q. The zero eigenvalue of $\mathcal{M}$ in L_{per}^2 is simple if $\theta \neq 0$ and double if $\theta = 0$. It is the second eigenvalue of $\mathcal{M}$ if $\theta \geq 0$ and the third eigenvalue of $\mathcal{M}$ if $\theta < 0$.

Proof of Lemma 4.4. By Lemma 4.1, we need to control the negative and zero eigenvalues of the linear operator $\mathcal{K}(0) : H_{\mathrm{per}}^2 \subset L_{\mathrm{per}}^2 \to L_{\mathrm{per}}^2$ given by (4.1). Using the change of variables $w = (4 - \partial_x^2)v$, the second-order differential equation $\mathcal{K}(0)v = 0$ can be written as

$$c(v - v'') - \phi w = 0.$$

This equation has the two solutions $v_1 = v'$ and $v_2 = \partial_a v$, which follows by differentiating (2.10) in x and a since c and $d = b/4$ are independent of x and a. In other words,

$$\mathcal{L}\phi' = \partial_x \Lambda_{c,b}(\phi) = 0 \quad \text{and} \quad \mathcal{L}\partial_a\phi = \partial_a\Lambda_{c,b}(\phi) = 0$$

yield

$$\mathcal{K}(0)v' = 0 \quad \text{and} \quad \mathcal{K}(0)\partial_a v = 0.$$

Let $\{y_1, y_2\}$ be the fundamental set of solutions associated to the equation $\mathcal{K}(0)v = 0$ in $H^2(0, L)$ such that

$$\begin{cases} y_1(0) = 1, \\ y_1'(0) = 0, \end{cases} \quad \begin{cases} y_2(0) = 0. \\ y_2'(0) = 1. \end{cases} \tag{4.3}$$

We set $\phi(0) = \phi(L) = \phi_+$ and $\phi'(0) = \phi'(L) = 0$, where ϕ_+ is the turning point for the maximum of ϕ in x satisfying the equation

$$(c - \phi_\pm)^2(b + \phi_\pm^2) = a, \tag{4.4}$$

in view of the first-order invariant (1.7). It follows from (2.11) that we can define

$$v_\pm := \frac{1}{3}\phi_\pm - \frac{1}{6c}\phi_\pm^2 - \frac{b}{12c} \tag{4.5}$$

as the corresponding turning points of $v = (4 - \partial_x^2)^{-1}\phi$. We compute from (2.10) and (4.5) that

$$v''(0) = \frac{1}{3c}(c\phi_+ - 2\phi_+^2 - b) = \frac{1}{3c}(c - \phi_+)\phi''(0)$$

and

$$\partial_a v_+ = \frac{1}{3c}(c - \phi_+)\partial_a \phi_+,$$

which are both nonzero since $c - \phi_+ > 0$, $\partial_a \phi_+ \neq 0$, and $\phi''(0) \neq 0$. Moreover, differentiating (4.4) in a yields

$$2(c - \phi_+)(c\phi_+ - 2\phi_+^2 - b)\partial_a \phi_+ = 1,$$

from which, together with (2.2), we obtain

$$\phi''(0)\partial_a \phi_+ = \frac{1}{2(c - \phi_+)^2} > 0.$$

Due to the normalization (4.3), we can then define

$$y_1(x) := \frac{\partial_a v(x)}{\partial_a v_+}, \quad y_2(x) := \frac{v'(x)}{v''(0)},$$

and obtain $y_1(L) = y_1(0) = 1$, $y_1'(0) = 0$, and

$$y_1'(L) = -\frac{\partial_a \mathfrak{L}}{\partial_a v_+}v''(0) = -\frac{\partial_a \mathfrak{L}}{\partial_a \phi_+}\phi''(0) = -\frac{\partial_a \mathfrak{L}}{2(c - \phi_+)^2(\partial_a \phi_+)^2},$$

where we have differentiated $v'(L) = 0$ with respect to a and used that $L = \mathcal{L}(a, b)$. On the other hand, $y_2(L) = y_2(0) = 0$ and $y_2'(L) = y_2'(0) = 1$. If we denote $\theta := y_1'(L)$, then $y_1(x + L) = y_1(x) + \theta y_2(x)$. Note that the sign of θ is opposite to that of $\partial_a \mathfrak{L}$. Hence, by Proposition 4.5, the zero eigenvalue of $\mathcal{K}(0)$ is simple if $\partial_a \mathfrak{L} \neq 0$ and double if $\partial_a \mathfrak{L} = 0$. Moreover, it is the second eigenvalue if $\partial_a \mathfrak{L} \leq 0$ and the third eigenvalue if $\partial_a \mathfrak{L} > 0$. This proves the assertion of the Lemma. $\quad\square$

Remark 4.6. In the case of smooth solitary waves on a constant background, the Schrödinger operator $\mathcal{K}(\lambda) : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ for $\lambda \in (-\infty, \lambda_0)$ admits a finite number of simple isolated eigenvalues and an absolutely continuous spectrum located in $[\mu_\infty, \infty)$, where $\mu_\infty := \lim_{|x| \to \infty} A(x, \lambda)$. Since $\phi(x) \to \phi_1$ as $|x| \to \infty$ on the top boundary of the region of Lemma 2.1, we have

$$\lim_{|x| \to \infty} A(x, 0) = \frac{c - 4\phi_1}{c - \phi_1} > 0$$

and $\mathcal{K}(0)v' = 0$ with $v' \in H^2(\mathbb{R})$. By Sturm's nodal theorem, since v' has only one zero on $\mathbb{R}$, we have

$$\#\{\mu < 0: \; \mathcal{K}(0)v = \mu v, \; v \in H^2(\mathbb{R})\} = 1$$

so that

$$\#\{\lambda < 0: \; \mathcal{L}w = \lambda w, \; w \in L^2(\mathbb{R})\} = 1$$

by the criterion in Lemma 4.1. Thus, $\mathcal{L}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ has a simple negative eigenvalue and a simple zero eigenvalue in the case of smooth solitary waves. This yields a much simpler argument compared to the theory developed in [37].

We use the monotonicity properties of the period function in Lemma 3.2 and the criterion in Lemma 4.4 to prove the last assertion of Theorem 1.1 stated as the following corollary.

Corollary 4.7. *For a fixed $c > 0$, there exists a smooth curve $a = a_0(b)$ for $b \in (-\frac{2}{9}c^2, 0)$ inside the existence region of smooth periodic waves in Lemma 2.1 such that the linear operator $\mathcal{L}$ in L^2_{per} has only one simple negative eigenvalue above the curve and two simple negative eigenvalues (or a double negative eigenvalue) below the curve, the rest of its spectrum for $a \neq a_0(b)$ includes a simple zero eigenvalue and a strictly positive spectrum bounded away from zero. Along the curve $a = a_0(b)$, the linear operator $\mathcal{L}$ in L^2_{per} has only one simple negative eigenvalue, a double zero eigenvalue, and the rest of its spectrum is strictly positive.*

Proof. Let $n(\mathcal{L})$ denote the number of negative eigenvalues of $\mathcal{L}$, taking into account their multiplicities. By Lemma 3.2, $\partial_a \mathfrak{L} > 0$ for every a if $b \in (-c^2, -\frac{2}{9}c^2]$ so that $n(\mathcal{L}) = 2$ by Lemma 4.4. Similarly, $\partial_a \mathfrak{L} < 0$ for every a if $b \in [0, \frac{1}{8}c^2)$ so that $n(\mathcal{L}) = 1$. For $b \in (-\frac{2}{9}c^2, 0)$, there exists exactly one $a = a_0(b)$ for which the mapping $a \mapsto \mathcal{L}(a, b)$ has the maximum point. This curve is shown on Fig. 1.1. Hence, $\partial_a \mathfrak{L} > 0$ for $a < a_0(b)$ with $n(\mathcal{L}) = 2$ and $\partial_a \mathfrak{L} < 0$ for $a > a_0(b)$ with $n(\mathcal{L}) = 1$. Combining the results in these three regions, we conclude that $n(\mathcal{L}) = 1$ above the curve and $n(\mathcal{L}) = 2$ below the curve inside the existence region. Along the curve $a = a_0(b)$, $\partial_a \mathfrak{L} = 0$ so that $n(\mathcal{L}) = 1$ and the zero eigenvalue of $\mathcal{L}$ is double. $\quad\square$

Proof. The proof of Theorem 1.1 is complete with the results of Lemma 2.1 Lemma 3.1, and Corollary 4.7. $\quad\square$

5. Energy stability criterion

To study the stability of the smooth periodic traveling waves with the profile ϕ with respect to co-periodic perturbations, we consider the decomposition

$$u(t, x) = \phi(x - ct) + w(t, x - ct).$$

When this is substituted into the DP equation (1.1) and quadratic terms in w are neglected, we obtain the linearized equation in the form

$$w_t - w_{txx} - cw_x + cw_{xxx} + 4\phi w_x + 4w\phi' = 3\phi' w_{xx} + 3w_x \phi'' + \phi w_{xxx} + w\phi''',$$

where x stands for the traveling wave coordinate $x - ct$. The linearized equation can be written in the Hamiltonian form

$$w_t = -J\mathcal{L}w, \tag{5.1}$$

where J is the same as in (1.5) and $\mathcal{L}$ is the same as in (1.8). Indeed, the equivalence of the linearized equations follows from the relation

$$\partial_x (4 - \partial_x^2)\left[(c - \phi)w - 3c(4 - \partial_x^2)^{-1}w\right]$$
$$= (c - 4\phi)w_x - (c - \phi)w_{xxx} - 4\phi' w + 3\phi' w_{xx} + 3\phi'' w_x + \phi''' w.$$

Linearization of the mass and energy functionals (1.2) and (1.4) at the traveling wave with the profile ϕ using a co-periodic perturbation with the profile w yields the constrained subspace of L_{per}^2 of the form

$$X_0 := \left\{w \in L_{\text{per}}^2 : \quad \langle 1, w \rangle = 0, \quad \langle \phi^2, w \rangle = 0\right\}. \tag{5.2}$$

The following lemma shows that the two constraints are invariant in the time evolution of the linearized equation (5.1).

Lemma 5.1. *Let $w \in C(\mathbb{R}, H_{\text{per}}^s) \cap C^1(\mathbb{R}, H_{\text{per}}^{s-1})$ be the global solution to the linearized equation (5.1) with $s > \frac{3}{2}$ for initial data $w(0, \cdot) = w_0 \in H_{\text{per}}^s$. If $w_0 \in X_0$, then $w(t, \cdot) \in X_0$ for every $t \in \mathbb{R}$.*

Proof. Since J is skew-adjoint and $J1 = 0$, we obtain

$$\frac{d}{dt}\langle 1, w \rangle = -\langle 1, J\mathcal{L}w \rangle = \langle J1, \mathcal{L}w \rangle = 0.$$

Similarly, since $\mathcal{L}^* = \mathcal{L}$, $J\phi^2 = -2c\phi'$, and $\mathcal{L}\phi' = 0$, we obtain

$$\frac{d}{dt}\langle \phi^2, w \rangle = -\langle \phi^2, J\mathcal{L}w \rangle = \langle J\phi^2, \mathcal{L}w \rangle = -2c\langle \phi', \mathcal{L}w \rangle = -2c\langle \mathcal{L}\phi', w \rangle = 0.$$

It follows from the invariance of the two constraints under the time evolution of the linearized equation (5.1) that if $w_0 \in X_0$, then $w(t, \cdot) \in X_0$ for every $t \in \mathbb{R}$. $\square$

Remark 5.2. Formal differentiation of the second-order equation (2.10) with $d = b/4$ in b and c yields

$$\mathcal{L}\partial_b \phi = \frac{1}{4}, \quad \mathcal{L}\partial_c \phi = -\frac{b}{4c} - \frac{\phi^2}{2c}. \tag{5.3}$$

The relations (5.3) allow us to characterize $1 \in \text{Range}(\mathcal{L})$ and $\phi^2 \in \text{Range}(\mathcal{L})$ in L_{per}^2 provided that we can take derivatives in b and c of the family of periodic waves with the profile $\phi \in H_{\text{per}}^\infty$ along a curve with fixed period $L = \mathcal{L}(a, b)$.

   The following lemma uses the fact that the period function is monotone in b, see Lemma 3.1, to guarantee the existence of a unique curve in the (a, b) parameter space for which the smooth periodic solutions with the profile $\phi \in H_{\text{per}}^\infty$ have a fixed period L for every $L \in (0, \infty)$.

Lemma 5.3. *Fix $c > 0$ and $L > 0$. There exists a C^1 mapping $a \mapsto b = \mathcal{B}_L(a)$ for $a \in (0, a_L)$ with some $a_L \in (0, \frac{27}{256}c^4)$ and a C^1 mapping $a \mapsto \phi = \Phi_L(\cdot, a) \in H^\infty_{\text{per}}$ of smooth L-periodic solutions along the curve $b = \mathcal{B}_L(a)$.*

Proof. It follows from (2.7) that the mapping $b \mapsto L = \mathfrak{L}(0, b) \in (0, \infty)$ is one-to-one and onto at the boundary $a = 0$, where $b \in (-c^2, 0)$. The limiting L-periodic wave has a peaked profile ϕ on the boundary $a = 0$.

Similarly, at the boundary $b = b_-(a)$, the limiting L-periodic wave corresponds to the constant wave $\phi = \phi_2$ and the period L is found from the linearization of the second-order equation (2.3) at $\phi = \phi_2$. A simple computation for $\varphi := \phi - \phi_2$ yields the linearized equation in the form

$$\varphi'' + \left(\frac{3a}{(c - \phi_2)^4} - 1\right)\varphi = 0.$$

Since $a = \phi_2(c - \phi_2)^3$ on the boundary $b = b_-(a)$, see (2.8), it follows that the mapping

$$\phi_2 \mapsto \omega^2 := \frac{3\phi_2}{c - \phi_2} - 1 \in (0, \infty) \tag{5.4}$$

is one-to-one and onto for $\phi_2 \in (c/4, c)$. Therefore, the mapping $a \mapsto L = \mathfrak{L}(a, b_-(a)) \in (0, \infty)$ is one-to-one and onto at the boundary $b = b_-(a)$. For every $L \in (0, \infty)$, there exists a unique root of $L = \mathfrak{L}(a, b_-(a))$, which we denote by a_L.

Thus, for every fixed $c > 0$ and $L > 0$, there exists exactly one L-periodic solution on the left and right boundaries of the existence region, see Lemma 2.1. Since $\mathfrak{L}(a, b)$ is smooth in (a, b) and it is strictly increasing in b by Lemma 3.1, the existence of the C^1 mapping $a \mapsto b = \mathcal{B}_L(a)$ for $a \in (0, a_L)$ follows by the implicit function theorem for $\mathfrak{L}(a, b) = L$ for every fixed $L > 0$. Indeed, $\partial_a \mathfrak{L} + \mathcal{B}'_L(a)\partial_b \mathfrak{L} = 0$ and since $\partial_b \mathfrak{L} > 0$, $\mathcal{B}'_L(a)$ is uniquely defined for every $a \in (0, a_L)$. Since ϕ is smooth with respect to parameters by Lemma 2.1, the mapping $a \mapsto \phi = \Phi_L(\cdot, a) \in H^\infty_{\text{per}}$ is C^1 along the curve $b = \mathcal{B}_L(a)$. $\square$

Remark 5.4. The mapping $b \mapsto \phi = \Psi_L(\cdot, b) \in H^\infty_{\text{per}}$ may not be C^1 along the curve $b = \mathcal{B}_L(a)$ because of the non-monotonicity of $\mathfrak{L}(a, b)$ with respect to a shown in Lemma 3.2. In particular, the mapping $b \mapsto \phi = \Psi_L(\cdot, b) \in H^\infty_{\text{per}}$ is not C^1 at the point where $\mathcal{B}'_L(a) = 0$, which corresponds to $\partial_a \mathfrak{L} = 0$.

We next characterize the negative and zero eigenvalues of the Hessian operator $\mathcal{L}$ under the two constraints defining X_0 given by (5.2). The restriction of $\mathcal{L}$ onto X_0 is denoted by $\mathcal{L}|_{X_0}$ with the corresponding notations $n(\mathcal{L}|_{X_0})$ for the number of negative eigenvalues, taking into account their multiplicities, and $z(\mathcal{L}|_{X_0})$ for the multiplicity of the zero eigenvalue. The following lemma gives the count of negative and zero eigenvalues under the two constraints.

Lemma 5.5. *Let $a \mapsto b = \mathcal{B}_L(a)$ and $a \mapsto \phi = \Phi_L(\cdot, a) \in H^\infty_{\text{per}}$ be the C^1 mappings of Lemma 5.3. Assume that $\mathcal{B}'_L(a) \neq 0$ and denote*

$$\mathcal{M}_L(a) := M(\Phi_L(\cdot, a)) \quad \text{and} \quad \mathcal{F}_L(a) := F(\Phi_L(\cdot, a)),$$

where $M(u)$ and $F(u)$ are given by (1.2) and (1.4). Then, $n(\mathcal{L}|_{X_0}) = 0$ and $z(\mathcal{L}|_{X_0}) = 1$ if and only if

$$\frac{d}{da} \frac{\mathcal{F}_L(a)}{\mathcal{M}_L(a)^3} < 0 \tag{5.5}$$

and, for $\mathcal{B}'_L(a) < 0$, if additionally, $\mathcal{M}'_L(a) > 0$.

Proof. Recall that the counting formulas for the negative and zero eigenvalues of $\mathcal{L}|_{X_0}$, see e.g. [33,45,46] and references therein, are given by

$$\begin{cases} n(\mathcal{L}|_{X_0}) = n(\mathcal{L}) - n_0 - z_0, \\ z(\mathcal{L}|_{X_0}) = z(\mathcal{L}) + z_0, \end{cases} \tag{5.6}$$

where n_0 and z_0 are the numbers of negative and zero eigenvalues (counting their multiplicities) of the matrix of projections

$$S := \begin{bmatrix} \langle \mathcal{L}^{-1}1, 1 \rangle & \langle \mathcal{L}^{-1}\phi^2, 1 \rangle \\ \langle \mathcal{L}^{-1}1, \phi^2 \rangle & \langle \mathcal{L}^{-1}\phi^2, \phi^2 \rangle \end{bmatrix}. \tag{5.7}$$

It follows from (5.3) with $\phi = \Phi_L(\cdot, a)$ being the smooth L-periodic solution along the curve $b = \mathcal{B}_L(a)$ with $\mathcal{B}'_L(a) \neq 0$ that

$$\mathcal{L}^{-1}1 = 4\partial_b\phi, \quad \mathcal{L}^{-1}\phi^2 = -2c\partial_c\phi - 2b\partial_b\phi. \tag{5.8}$$

For each part of the curve $b = \mathcal{B}_L(a)$ for which $\mathcal{B}'_L(a) \neq 0$ we introduce the inverse mapping $a = \mathcal{B}_L^{-1}(b)$ and redefine $\Phi_L(\cdot, \mathcal{B}_L^{-1}(b)) \equiv \Psi_L(\cdot, b)$, $\mathcal{M}_L(\mathcal{B}_L^{-1}(b)) \equiv \mathcal{M}_L(b)$, and $\mathcal{F}_L(\mathcal{B}_L^{-1}(b)) \equiv \mathcal{F}_L(b)$. Due to (1.2), (1.4), and (5.8), matrix S in (5.7) can be rewritten in the form

$$S = \begin{bmatrix} 4\partial_b\mathcal{M}_L & -2c\partial_c\mathcal{M}_L - 2b\partial_b\mathcal{M}_L \\ 8\partial_b\mathcal{F}_L & -4c\partial_c\mathcal{F}_L - 4b\partial_b\mathcal{F}_L \end{bmatrix},$$

so that we obtain

$$\det(S) = 16c \left[\partial_c\mathcal{M}_L \partial_b\mathcal{F}_L - \partial_b\mathcal{M}_L \partial_c\mathcal{F}_L \right]. \tag{5.9}$$

Due to the scaling transformation (1.9), we can write

$$b = c^2\beta, \quad \Phi_L(\cdot; b) = c\hat{\Phi}(\cdot; \beta), \quad \mathcal{M}_L(b) = c\hat{\mathcal{M}}_L(\beta), \quad \mathcal{F}_L(b) = c^3\hat{\mathcal{F}}_L(\beta), \tag{5.10}$$

where β and the hat functions are c-independent. Substituting the transformation (5.10) into (5.9) yields

$$\det(S) = 16c^2 \left[\hat{\mathcal{M}}_L(\beta)\hat{\mathcal{F}}'_L(\beta) - 3\hat{\mathcal{F}}_L(\beta)\hat{\mathcal{M}}'_L(\beta) \right]$$

$$= 16c^2 \hat{\mathcal{M}}_L(\beta)^4 \frac{d}{d\beta} \left[\frac{\hat{\mathcal{F}}_L(\beta)}{\hat{\mathcal{M}}_L(\beta)^3} \right].$$

Recall that $\partial_a \mathfrak{L} + \mathcal{B}'_L(a) \partial_b \mathfrak{L} = 0$ and $\partial_b \mathfrak{L} > 0$ by Lemma 3.1. For the part of the curve $b = \mathcal{B}_L(a)$ with $\mathcal{B}'_L(a) > 0$, we have $\partial_a \mathfrak{L} < 0$ so that $n(\mathcal{L}) = 1$ and $z(\mathcal{L}) = 1$ by Lemma 4.4. If

$$\frac{d}{d\beta}\left[\frac{\hat{\mathcal{F}}_L(\beta)}{\hat{\mathcal{M}}_L(\beta)^3}\right] < 0, \tag{5.11}$$

then $\det(S) < 0$ so that S has one positive and one negative eigenvalue. Then, $n_0 = 1$ and $z_0 = 0$ so that the counting formulas (5.6) give $n(\mathcal{L}|_{X_0}) = 0$ and $z(\mathcal{L}|_{X_0}) = 1$. Since $\mathcal{B}'_L(a) > 0$ for this part of the curve $b = \mathcal{B}_L(a)$, the criterion (5.11) is equivalent to (5.5).

For the part of the curve $b = \mathcal{B}_L(a)$ with $\mathcal{B}'_L(a) < 0$, we have $\partial_a \mathfrak{L} > 0$ so that $n(\mathcal{L}) = 2$ and $z(\mathcal{L}) = 1$ by Lemma 4.4. If

$$\frac{d}{d\beta}\left[\frac{\hat{\mathcal{F}}_L(\beta)}{\hat{\mathcal{M}}_L(\beta)^3}\right] > 0 \quad \text{and} \quad \frac{d}{d\beta}\hat{\mathcal{M}}_L(\beta) < 0, \tag{5.12}$$

then $\det(S) > 0$. Since the first diagonal entry of S is negative, the symmetric matrix S with $\det(S) > 0$ has two negative eigenvalues. Then, $n_0 = 2$ and $z_0 = 0$ so that the counting formulas (5.6) give $n(\mathcal{L}|_{X_0}) = 0$ and $z(\mathcal{L}|_{X_0}) = 1$. Since $\mathcal{B}'_L(a) < 0$ for this part of the curve $b = \mathcal{B}_L(a)$, the criterion (5.12) is equivalent to the conditions (5.5) and $\mathcal{M}'_L(a) > 0$. □

Remark 5.6. It is well-known (see, e.g., [25,29]) that if $n(\mathcal{L}|_{X_0}) = 0$ and $z(\mathcal{L}|_{X_0}) = 1$, then the spectrum of $J\mathcal{L}$ in L^2_{per} is located on the imaginary axis, which implies that the L-periodic wave is spectrally stable. Indeed, let $w \in \mathrm{Dom}(J\mathcal{L}) \subset L^2_{\mathrm{per}}$ be the eigenvector of the spectral problem $J\mathcal{L}w = \lambda w$ for the eigenvalue $\lambda \in \mathbb{C}$. By the same computations as in Lemma 5.1, we have $w \in X_0$ if $\lambda \neq 0$. For every $w \in \mathrm{Dom}(J\mathcal{L}) \cap X_0$, we obtain

$$\lambda \langle \mathcal{L}w, w \rangle = \langle \mathcal{L}J\mathcal{L}w, w \rangle = -\langle \mathcal{L}w, J\mathcal{L}w \rangle = -\bar{\lambda}\langle \mathcal{L}w, w \rangle,$$

so that

$$(\lambda + \bar{\lambda})\langle \mathcal{L}w, w \rangle = 0.$$

Since $w \in X_0 \subset L^2_{\mathrm{per}}$, the conditions $n(\mathcal{L}|_{X_0}) = 0$ and $z(\mathcal{L}|_{X_0}) = 1$ imply that $\langle \mathcal{L}w, w \rangle = 0$ can be satisfied if and only if $w \in \mathrm{Ker}(\mathcal{L})$ which contradicts $\lambda \neq 0$. Hence, $\langle \mathcal{L}w, w \rangle > 0$, which implies that $\lambda + \bar{\lambda} = 0$ and so $\lambda \in i\mathbb{R}$.

Remark 5.7. In the context of the generalized KdV equation, it was shown in [1] that the spectral stability of the periodic waves can be determined by the variation of conserved quantities with respect to the free parameters of the traveling wave solutions. This is very similar to the stability characterization in Lemma 5.5, where the matrix S is shown to be related to the variation of the two conserved quantities $M(u)$ and $F(u)$ of the DP equation (1.1) with respect to the free parameters b and c of the traveling wave solution with the same period $L > 0$, for which the third parameter a is determined along the existence curve on Fig. 1.2.

Remark 5.8. Spectral stability of traveling periodic waves of the generalized KdV equation was addressed in [7,47]. We note that the structure of the Hamiltonian formulation of the DP equation (1.1) given by (1.5) is the same as in the Hamiltonian formulation of the generalized KdV equation, where just like here the operator J is skew-adjoint and has a finite-dimensional kernel, see also [29, Chapter 6.1.2].

Finally, we confirm the validity of the stability criterion of Lemma 5.5 for every point in a neighborhood of the boundary $a_-(b)$ where the periodic solution is constant.

Lemma 5.9. *Fix $c > 0$ and denote $a_L := a_-(b)$ for fixed period $L = \frac{2\pi}{\omega}$, where b is given by (2.8) and ω^2 is given by (5.4). There exists $\varepsilon > 0$ such that for every $a \in (a_L - \varepsilon, a_L)$, the conditions (5.5) and $\mathcal{M}'_L(a) > 0$ are satisfied.*

Proof. Let us parameterize the boundary $a = a_-(b)$ by $\phi_2 \in (c/4, c)$ as in (2.8). We substitute

$$\phi = \phi_2(1 + \varphi), \quad a = \phi_2(c - \phi_2)^3(1 + \alpha) \tag{5.13}$$

for a function φ and a scalar α into (2.3) and obtain

$$\varphi'' - \varphi + \frac{1 + \alpha}{(1 - \eta^{-1}\varphi)^3} - 1 = 0, \tag{5.14}$$

where $\eta := (c - \phi_2)/\phi_2 \in (0, 3)$ as in the proof of Lemma 3.1. The period $L > 0$ is fixed for fixed $c > 0$ and $\phi_2 \in (c/4, c)$ by $L = \frac{2\pi}{\omega}$, where ω^2 is given by (5.4). We use the Stokes expansion for even, L-periodic solutions with their maximum at $x = 0$, see also [45,46],

$$\varphi(x) = A\cos(\omega x) + A^2\varphi_2(x) + A^3\varphi_3(x) + \mathcal{O}(A^4), \quad \alpha = A^2\alpha_2 + \mathcal{O}(A^4),$$

where $A > 0$ and φ_2, φ_3 are even, L-periodic functions. Substituting this expansion into the linearization of (5.14) and using the definition of ω in (5.4), we obtain a sequence of compatibility conditions at each order,

$$\mathcal{O}(A^2): \quad \varphi_2'' + \omega^2\varphi_2 + 6\eta^{-2}\cos^2(\omega x) + \alpha_2 = 0,$$
$$\mathcal{O}(A^3): \quad \varphi_3'' + \omega^2\varphi_3 + 12\eta^{-2}\cos(\omega x)\varphi_2 + 10\eta^{-3}\cos^3(\omega x) + 3\alpha_2\eta^{-1}\cos(\omega x) = 0,$$

from which the correction terms can be found. The solution to the inhomogeneous equation at the order $\mathcal{O}(A^2)$ is given by

$$\varphi_2(x) = -\frac{3\eta^{-2} + \alpha_2}{\omega^2} + \frac{\eta^{-2}}{\omega^2}\cos(2\omega x),$$

where the solutions of the homogeneous equation $\varphi_2'' + \omega^2\varphi_2 = 0$ have been set to zero due to the arbitrariness of the parameter A. To ensure that the solution to the inhomogeneous equation at the order $\mathcal{O}(A^3)$ is L-periodic and not unbounded, we have to remove the term $\cos(\omega x)$ from the source term. After substituting the solution φ_2 found in the previous step and recalling that

$$2\cos(\theta)\cos(2\theta) = \cos(3\theta) + \cos(\theta),$$

$$4\cos^3(\theta) = \cos(3\theta) + 3\cos(\theta),$$

we find that this is the case if and only if $\alpha_2 = -\frac{5}{2\eta^2}$.

Note that if $a \in (a_L - \varepsilon, a_L)$ and the period $L > 0$ is fixed along a curve in the (a, b)-plane, see Fig. 1.2, then the small deviation in a implies a small deviation in $\alpha = A^2\alpha_2 + \mathcal{O}(A^4)$ in view of (5.13), which yields the corresponding value of A since $\alpha_2 = -\frac{5}{2\eta^2}$ is fixed. The value of b also deviates from the boundary value in (2.8) but this deviation is not needed for computations.

The next step is to expand $\mathcal{M}_L(a)$ and $\mathcal{F}_L(a)$ in terms of A^2. Note that we can write these expressions in terms of the new variable φ given by (5.13) as follows:

$$\mathcal{M}_L(a) = \phi_2 \left(L + \oint \varphi \, dx \right),$$

$$\mathcal{F}_L(a) = \frac{1}{6} \phi_2^3 \left(L + 3 \oint \varphi \, dx + 3 \oint \varphi^2 \, dx + \oint \varphi^3 \, dx \right).$$

After straightforward computations we obtain that the expansions of $\mathcal{M}_L(a)$ and $\mathcal{F}_L(a)$ in terms of small A^2 are given by

$$\mathcal{M}_L(a) = \phi_2 L \left(1 - \frac{1}{2\eta^2\omega^2} A^2 + \mathcal{O}(A^4) \right),$$

$$\mathcal{F}_L(a) = \frac{1}{6} \phi_2^3 L \left(1 + \frac{3}{2}(1 - \eta^{-2}\omega^{-2}) A^2 + \mathcal{O}(A^4) \right),$$

so that

$$\frac{\mathcal{F}_L(a)}{\mathcal{M}_L(a)^3} = \frac{1}{6L^2} \left[1 + \frac{3}{2} A^2 + \mathcal{O}(A^4) \right].$$

Since $\alpha_2 < 0$, we have $\frac{da}{dA^2} < 0$. It follows from $\frac{d}{dA^2}\mathcal{M}_L(a) < 0$ so that $\mathcal{M}'_L(a) > 0$. Similarly, $\frac{d}{dA^2}\frac{\mathcal{F}_L(a)}{\mathcal{M}_L(a)^3} > 0$ so that the condition (5.5) is satisfied. □

Remark 5.10. The proof of Theorem 1.5 is complete with the results of Lemma 5.3, Lemma 5.5, Remark 5.6, and Lemma 5.9.

Remark 5.11. It is tempting to conjecture, similarly to what was proven for the CH equation [19], that the monotonicity of the mapping (5.5) along the entire curve with $b = \mathcal{B}_L(a)$ is the only energy stability criterion needed for Theorem 1.5, whereas the information on the monotonicity of the mapping $a \mapsto \mathcal{M}_L(a)$ is unnecessary and the exceptional point $\mathcal{B}'_L(a) \neq 0$ is irrelevant. However, we are not able to prove this conjecture by only using properties of the Hessian operator $\mathcal{L}$, which is related to the differential equation (2.10). The successful strategy in [19] relies on the linearized operator for the second-order equation (2.3). However, this linearized operator is not related to the Hamiltonian formulation of the DP equation. As a result, positivity of this operator under two constraints no longer implies spectral stability of smooth periodic waves. For this reason we have not replicated the strategy of [19] here, but instead rely on the Hamiltonian formulation (1.5) of the DP equation (1.1).

Data availability

No data was used for the research described in the article.

Appendix A. Some auxiliary results

We summarize some auxiliary results used in the proofs of Lemmas 3.1, 3.2, 3.5 and 3.7.

A.1. Controlling the number of roots of parametric polynomials

The following result allows one to control the number of roots of a family of one-parametric polynomials $G_b(x)$. A variant of this can be found in [17, Lemma 8.1].

Lemma A.1. *Consider an interval $\Omega \subset \mathbb{R}$ and a family of real polynomials whose coefficients depend continuously on a real parameter b,*

$$G_b(x) = g_n(b)x^n + g_{n-1}(b)x^{n-1} + \cdots + g_1(b)x + g_0(b).$$

Suppose there exists an open interval $I \subset \mathbb{R}$ such that:

(i) *For all $b \in I$, $g_n(b) \neq 0$.*
(ii) *For all $b \in I$, the discriminant of G_b with respect to x is different from zero.*
(iii) *There is some $b_0 \in I$, such that $G_{b_0}(x)$ has exactly k simple roots on Ω.*

Then for all $b \in I$, $G_b(x)$ has exactly k simple roots on Ω.
Moreover, if $\Omega = \Omega_b := ((c(b), \infty) \subset \mathbb{R}$ for some continuous function $c(b)$ the same result holds if we add the hypothesis:

(iv) *For all $b \in I$, $G_b(c(b)) \neq 0$.*

The key idea of the proof is that the roots of $G_b(x)$ depend continuously on the parameter b since $g_n(b) \neq 0$ in view of assumption (i). The hypothesis (ii) prevents the appearance of double roots in Ω. Hypotheses (i) and (iv) ensure that no roots enter into Ω from infinity or from the boundary of Ω. Therefore, the number of real roots of G_b is constant for any $b \in I$. Since $G_{b_0}(x)$ has exactly k simple roots on Ω by hypothesis (iii), this is true for all $b \in I$.

A.2. Tools to determine the number of critical periods

The proof of monotonicity of the period function in Lemma 3.5 follows closely the one in [23] and strongly relies on the tools developed in [16]. In this paper the authors consider analytic planar differential systems

$$\begin{cases} x' = p(x, y), \\ y' = q(x, y), \end{cases} \tag{A.1}$$

satisfying the following *hypothesis*:

(H)   The differential system (A.1) has a center at the origin and an analytic first integral of the form $H(x, y) = A(x) + B(x)y + C(x)y^2$ with $A(0) = 0$. Moreover its integrating factor K depends only on x.

Let (x_ℓ, x_r) be the projection onto the x-axis of the period annulus $\mathscr{P}$ around the center at the origin of the differential system (A.1). Note that $x_\ell < 0 < x_r$. Then, by Lemma 3.1 in [16], the hypothesis **(H)** implies that $M := \frac{4AC - B^2}{4|C|}$ is a well defined analytic function on (x_ℓ, x_r) with $M(0) = 0$ and $xM'(x) > 0$ for all $x \in (x_\ell, x_r) \setminus \{0\}$. Accordingly, there exists a unique analytic function σ on (x_ℓ, x_r) with $\sigma(x) = -x + o(x)$ such that $M \circ \sigma = M$. Note that σ is an *involution* with $\sigma(0) = 0$. (Recall that a mapping σ is said to be an involution if $\sigma \circ \sigma = \mathrm{Id}$.) Given an analytic function f on $(x_\ell, x_r) \setminus \{0\}$ we define its σ-*balance* to be

$$\mathscr{B}_\sigma(f)(x) := \frac{f(x) - f(\sigma(x))}{2}.$$

Taking these definitions into account, the statement (b) of [16, Theorem A] asserts the following:

Proposition A.2. *Suppose that the analytic differential system* (A.1) *satisfies the hypothesis **(H)**. Setting $\mu_0 = -1$, define recursively*

$$\mu_i := \left(\frac{1}{2} + \frac{1}{2i - 3} \right) \mu_{i-1} + \frac{\sqrt{|C|}M}{(2i - 3)K} \left(\frac{K\mu_{i-1}}{\sqrt{|C|}M'} \right)' \quad and \quad \ell_i := \frac{K\mu_i}{\sqrt{|C|}M'} \quad for\ i \geqslant 1.$$

If the number of zeros of $\mathscr{B}_\sigma(\ell_i)$ on $(0, x_r)$, counted with multiplicities, is $n \geqslant 0$ and it holds that $i > n$, then the number of critical periods of the center at the origin, counted with multiplicities, is at most n.

In particular, we note that the period function is monotonous if $n = 0$.

We point out that it can be difficult to apply Proposition A.2 to determine the number of zeros of $\mathscr{B}_\sigma(l_i)$, due to the fact that it is in general not possible to compute the involution σ explicitly. If σ and ℓ_i are algebraic functions, then one can overcome this difficulty by using *multipolynomial resultants Res* (see for instance [6,15]). More specifically, we use the following result, see [16, Theorem B].

Proposition A.3. *Let σ be an analytic involution on (x_ℓ, x_r) with $\sigma(0) = 0$ and let ℓ be an analytic function on $(x_\ell, x_r) \setminus \{0\}$. Assume that ℓ and σ are algebraic, i.e., that there exist $L, S \in \mathbb{C}[x, y]$ such that $L(x, \ell(x)) \equiv 0$ and $S(x, \sigma(x)) \equiv 0$. Let us define $T(x, y) := Res_z(L(x, z), L(y, z))$ and $\mathscr{R}(x) := Res_y(S(x, y), T(x, y))$. Finally let $s(x)$ and $t(x)$ be, respectively, the leading coefficients of $S(x, y)$ and $T(x, y)$ with respect to y. Then the following hold:*

(a) *If $\mathscr{B}_\sigma(\ell)(x_0) = 0$ for some $x_0 \in (x_\ell, x_r) \setminus \{0\}$, then $\mathscr{R}(x_0) = 0$.*
(b) *If $s(x)$ and $t(x)$ do not vanish simultaneously at x_0, then the multiplicity of $\mathscr{B}_\sigma(\ell)$ at x_0 is not greater than the multiplicity of $\mathscr{R}$ at x_0.*

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