NONLINEAR WAVES IN WEAKLY-COUPLED LATTICES

NONLINEAR WAVES IN WEAKLY-COUPLED LATTICES

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Doctor of Philosophy

McMaster University

DOCTOR OF PHILOSOPHY (2013) (Mathematics)	McMaster University Hamilton, Ontario

TITLE:	Nonlinear Waves In Weakly-Coupled Lattices
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SUPERVISOR:	Dr. Dmitry Pelinovsky
NUMBER OF PAGES:	vii, 133

Abstract

We consider existence and stability of breather solutions to discrete nonlinear Schrödinger (dNLS) and discrete Klein–Gordon (dKG) equations near the anti-continuum limit, the limit of the zero coupling constant. For sufficiently small coupling, discrete breathers can be uniquely extended from the anti-continuum limit where they consist of periodic oscillations on excited sites separated by "holes" (sites at rest).

In the anti-continuum limit, the dNLS equation linearized about its discrete breather has a spectrum consisting of the zero eigenvalue of finite multiplicity and purely imaginary eigenvalues of infinite multiplicities. Splitting of the zero eigenvalue into stable and unstable eigenvalues near the anti-continuum limit was examined in the literature earlier. The eigenvalues of infinite multiplicity split into bands of continuous spectrum, which, as observed in numerical experiments, may in turn produce internal modes, additional eigenvalues on the imaginary axis. Using resolvent analysis and perturbation methods, we prove that no internal modes bifurcate from the continuous spectrum of the dNLS equation with small coupling.

Linear stability of small-amplitude discrete breathers in the weakly-coupled KG lattice was considered in a number of papers. Most of these papers, however, do not consider stability of discrete breathers which have "holes" in the anti-continuum limit. We use perturbation methods for Floquet multipliers and analysis of tail-to-tail interactions between excited sites to develop a general criterion on linear stability of multi-site breathers in the KG lattice near the anti-continuum limit. Our criterion is not restricted to small-amplitude oscillations and it allows discrete breathers to have "holes" in the anti-continuum limit.

Acknowledgements

I express my sincere gratitude to Dr. Dmitry Pelinovsky for posing interesting research problems, sharing his knowledge, as well as for his constant guidance and supervision. I would also like to thank Dr. Walter Craig and Dr. Stanley Alama for their comments on my research and continuous support during my Ph.D. studies.

I express my gratitude to McMaster University and Department of Mathematics & Statistics for the financial support and help they provided throughout my graduate studies.

I am especially grateful to my wife Katya, my sister Anna, and my parents Sergei and Lyudmila for their continuous care, constant encouragement, understanding and love on my way to this degree. My warm and sincere thanks also go to my friends and colleagues: Diego Ayala, Vladislav Bukshtynov, Ekaterina Nehay, Dmitry Ponomarev, and Yusuke Shimabukuro, for all the assistance in my research, fruitful discussions and encouragement during my stay at McMaster.

To my wife, Katya

Contents

Chapter 1

Introduction

1.1 Nonlinear weakly-coupled lattices

We study lattice equations, infinite systems of ordinary differential equations, describing dynamics in networks of coupled oscillators. These equations arise as spatial discretizations of partial differential equations or independently as models for physical processes, such as vibrations in crystals or interactions of pulses in networks of coupled optical waveguides. As each of the lattice sites naturally supports time-periodic solutions, the whole nonlinear lattice may allow for time-periodic solutions as well. In addition, spatial discreteness and the presence of a nonlinear potential often make spatial localization of time-periodic solutions possible. Existence and stability of spatially localized time-periodic solutions, called *discrete breathers*, is the main subject of this work.

Mathematical studies on localized solutions in nonlinear lattices were spurred by the work of Sievers & Takeno [83] where existence of a discrete breather was established in a chain of coupled oscillators interacting through a harmonic and quartic anharmonic potentials. Later, Page [65] constructed two types of discrete breathers for a chain of oscillators with purely anharmonic coupling. Many interesting analytical and numerical works followed after these pioneering papers. It was observed that discrete breathers tend to emerge from thermal equilibrium in the process of spontaneous localization. In such a process, breathers of larger amplitudes grow at the expense of smaller breathers, which results in appearance of stationary breathers that are trapped by the lattice (e.g. [27, 93]). On the other hand, travelling breather solutions do not generally exist in lattice equations. This is in sharp contrast to continuous wave equations, where breathers are structurally unstable and travelling waves are more abundant.

In lattice equations, the strength of the inter-site interaction is governed by a coupling constant. This constant, or its inverse, can be conveniently used as a perturbation parameter for analysis of existence and stability of solutions to the lattice equations. When the coupling constant approaches infinity, i.e. in the limit of the *continuous approximation,* one can study solutions to a lattice equation using perturbation analysis for the solutions to the corresponding partial differential equation. Alternatively, one can consider the lattice equation in the limit of the small coupling constant, a so-called *anti-continuum limit.* In the context of existence of discrete breathers, this method was proposed for the first time by MacKay & Aubry [56]. Since then, the anti-continuous limit became quite popular for construction of discrete breathers, and analysis of their stability.

In this thesis we study existence and stability of discrete breathers in nonlinear lattices near the anti-continuum limit. For a broad mathematical consideration of discrete breathers and other waves in nonlinear lattices, we refer the reader to recent review papers [6, 33] and books [46, 69].

1.2 Some lattice systems

Below we introduce three fundamental lattice equations: the Fermi–Pasta–Ulam (FPU) chain, the Klein–Gordon (KG) lattice, and the discrete nonlinear Schrödinger (dNLS) equation. We describe the FPU chain because of its historical importance. The KG lattice and the dNLS equation will be the main subjects of this thesis. Note that we do not consider completely integrable lattices, such as Toda and Ablowitz–Ladik lattices, which exhibit some special remarkable properties.

1.2.1 The Fermi–Pasta–Ulam (FPU) lattice

The Fermi–Pasta–Ulam (FPU) lattice, a toy model for vibrations in a perfect crystal, can be written in the form

$$\ddot{u}_n + V'(u_n - u_{n-1}) - V'(u_{n+1} - u_n) = 0, \qquad n \in \mathbb{Z}, \quad u_n(t) : \mathbb{R} \to \mathbb{R}, \tag{1.1}$$

where $V'(x) = x + \mathcal{O}(x^\gamma)$ and $\gamma > 1$. This equation is a Hamiltonian system which admits conservation of energy,

$$H = \sum_{n\in\mathbb{Z}} \left[\frac{1}{2}\dot{u}_n^2 + V(u_{n+1} - u_n) \right].$$

Since $\sum_{n\in\mathbb{Z}} \ddot{u}_n = 0$, this equation also admits conservation of the total momentum, $P = \sum_{n\in\mathbb{Z}} \dot{u}_n$.

Equation (1.1) became very famous after a numerical experiment performed about sixty years ago by Fermi, Pasta & Ulam [31] on the nonlinear lattice with the potential $V(x) = \frac{1}{2}x^2 + \frac{1}{4}\beta x^4$. Counterintuitively to scientists at that time, the experiment did not demonstrate the expected equipartition of energy between the normal modes of the nonlinear lattice. The numerical solution, in fact, demonstrated localization and quasiperiodicity in Fourier space. These observations gave a start to new research directions in nonlinear science, such as inverse scattering method for integrable systems and KAM theory for periodic orbits.

Emergence of discrete quasiperiodic breathers in the FPU lattice has been demonstrated in a number of numerical experiments (e.g. [22, 59]). Despite this, only fragments of rigorous existence and stability theory for discrete breathers exists to date. For example, it is not possible to derive discrete breather solutions to the FPU chain using the technique of the anti-continuum limit approach: only constant solution is available in that limit. Existence of discrete breathers was, however, studied by Aubry *et al.* [5, 7] using variational techniques, and by James [38] using centre manifold reductions. While some breather configurations are linearly stable in the FPU lattice, no indication of nonlinearly stable breathers is available to date [32].

1.2.2 The discrete Klein–Gordon (dKG) equation

The KG chain, frequently referred to as the discrete Klein–Gordon (dKG) equation, can be written in the form

$$\ddot{u}_n + V'(u_n) = \epsilon(u_{n-1} - 2u_n + u_{n+1}), \qquad n \in \mathbb{Z}, \quad u_n(t) : \mathbb{R} \to \mathbb{R}, \tag{1.2}$$

where $V'(x) = x + \mathcal{O}(x^\gamma)$ and $\gamma > 1$. This equation admits a Hamiltonian

$$H = \sum_{n\in\mathbb{Z}} \left[\frac{1}{2}\dot{u}_n^2 + V(u_n) + \frac{\epsilon}{2}(u_{n+1} - u_n)^2 \right]. \tag{1.3}$$

The dKG equation is a version of the Frenkel–Kontorova model for dislocations in crystals [12] with a non-periodic on-site potential. In [28], this equation was used to study oscillations in the DNA molecule. This equation is a great toy model for discrete breathers in nonlinear lattices. In particular, the anti-continuum limit can be realized by taking small values of ϵ.

1.2.3 The discrete nonlinear Schrödinger (dNLS) equation

The discrete nonlinear Schrödinger equation (dNLS) became popular almost thirty years ago after a numerical work of Scott & MacNeil [82] on existence of a single-peak breather. This study was in turn inspired by Davydov soliton on proteins [29]. Since then, this equation has appeared in a number of applied problems, such as those related to coupled optical waveguides (e.g. [20, 30]) or Bose–Einstein condensate trapped in a periodic potential (e.g. [18, 92]).

The dNLS equation,

$$i\dot{u}_n = \epsilon(u_{n-1} - 2u_n + u_{n+1}) \pm |u_n|^{2p} u_n, \qquad n \in \mathbb{Z}, \quad u_n(t) : \mathbb{R} \to \mathbb{C}, \tag{1.4}$$

where $p \in \mathbb{N}$, arises from the Hamiltonian

$$H = \sum_{n\in\mathbb{Z}} \left(\mp \frac{1}{p+1} |u_n|^{2p+2} + \frac{\epsilon}{2} |u_{n+1} - u_n|^2 \right)$$

written in canonically conjugated variables $\{u_n, \bar{u}_n\}$ such that $i\dot{u}_n = -\frac{\partial H}{\partial \bar{u}_n}$. The plus and minus signs in the dNLS equation (1.4) correspond to the *focusing* and *defocusing* nonlinearities respectively. It is easy to check that the focusing dNLS equation for $\{u_n\}_{n\in\mathbb{Z}}$ is related to the defocusing one for $\{w_n\}_{n\in\mathbb{Z}}$ via the staggering transformation

$$u_n = (-1)^n \bar{w}_n e^{4i\epsilon t}.$$

Thanks to the gauge invariance, $u_n \mapsto u_n e^{i\theta}$ with $\theta \in [0, 2\pi)$, the system also admits conservation of the "power",

$$N = \sum_{n\in\mathbb{Z}} |u_n|^2. \tag{1.5}$$

It is important to note that the dNLS equation arises in the small-amplitude limit for the KG lattice [62]. The anti-continuum limit is again related to the small values of ϵ.

1.3 Main results

This thesis is primarily concerned with existence and stability of discrete breathers in the dNLS equation (1.4) and KG chain (1.2) near the anti-continuum limit. We are also interested in dispersive decay of small time-dependent solutions in these lattices. Let us discuss the main results obtained in this thesis.

In Chapter 2, we review well-posedness of the initial value problem in the dNLS and dKG equations.

- We prove global existence of time-dependent solutions to the dNLS equation (1.4) in algebraically weighted l^2 spaces by invoking the Banach Fixed-Point Theorem and conservation of the "power" (1.5). Another version of this result was established by Pacciani, Konotop & Menzala [64] who also considered dNLS lattices with saturable nonlinearities and long-range interactions. In addition, a recent paper by N'Guérékata & Pankov [63] provides a proof of global well-posedness in exponentially-weighted l^2 spaces.

- We review results on existence of time-dependent solutions in the energy space of the KG lattice (1.2) with the potential

$$V'(x) = x + \beta x^{2\sigma+1}, \qquad \beta \in \mathbb{R}, \quad \sigma \in \mathbb{N}, \tag{1.6}$$

where the potential with $\beta > 0$ ($\beta < 0$) is often called a *hard* (*soft*) potential. For the case of the hard potential, Hamiltonian (1.3) is convex in $\mathbf{u}$ and $\dot{\mathbf{u}}$, which immediately implies global existence of l^2 solutions. In the case of the soft potential, both global existence and finite-time blow up are possible and we review relevant results based on the recent work of Karachalios [44] and Achilleos *et al.* [1].

- Dispersive decay of small l^1 initial data in the dKG and dNLS equations was examined by Stefanov & Kevrekidis [89] using Strichartz estimates and by Mielke & Patz [58] using pointwise dispersive decay estimates. We discuss the techniques from [58] in the context of the dNLS equation, and derive decay estimates in l^q spaces with $q \in [2, \infty]$. These techniques rely on approximation of l^q norms with integrals in the asymptotic limit of $t \to \infty$ and application of the Van der Corput lemma to the resulting oscillatory integrals. In contrast to [89], where the nonlinearity exceeds quintic ($p > 2$), the method in [58] allows us to consider the dNLS equation with the nonlinearity higher than quartic ($p > 3/2$).

In Chapter 3, we prove existence of breather solutions to the dNLS and dKG equations near the anti-continuum limit by an application of the Implicit Function Theorem. In the context of weakly-coupled lattices, this approach originates from the work of MacKay & Aubry [55].

Chapters 4 and 5 contain the original results of this Ph.D. dissertation. These results were published in papers [74] and [75]. In Chapter 4, we study linear and nonlinear stability of discrete breathers in the dNLS equation.

- In the anti-continuum limit, the spectrum of the linear stability problem for the dNLS equation consists of a zero eigenvalue of finite multiplicity and eigenvalues of infinite multiplicity on the imaginary axis. Splitting of the zero eigenvalue into stable and unstable eigenvalues was examined by Pelinovsky, Kevrekidis & Frantzeskakis [72]. While splitting of the eigenvalues of infinite multiplicity always results in creation of spectral bands, bifurcation of purely imaginary discrete eigenvalues from these bands is also possible. For the case of one-site discrete breather, such eigenvalues, also known as *internal modes*, were observed numerically by Johansson & Aubry [43] and Kevrekidis [46] near the continuous limit of large coupling constant. Internal modes play a crucial role in asymptotic stability of discrete breathers. We confirm the numerical observations in [43] and [46], by proving that no internal modes bifurcate from the continuous spectrum of the dNLS equation if the coupling constant is sufficiently small. Our method relies on resolvent techniques for the discrete Laplacian operator developed by Komech, Kopylova & Kunze [50] and Pelinovsky & Stefanov [76]. Derived in the leading order of perturbation theory, our results are generally restricted to simply-connected discrete breathers and to quintic or higher nonlinearities ($p \geq 2$).

- Orbital stability of discrete breathers in the dNLS equation was proved by Weinstein [94] using a variational method. Early works [50, 76, 89] on dispersive decay estimates for the discrete Schödinger operator $H = -\Delta + V$ with a localized potential V spurred progress on asymptotic stability analysis of dNLS breathers. For the case of septic or higher nonlinearity ($p \geq 3$), asymptotic stability of small breathers bifurcating from the unique eigenvalue of the operator H was recently studied by Kevrekidis, Pelinovsky & Stefanov [49], as well as by Cuccagna & Tarulli [25]. More recently, Mizumachi & Pelinovsky [61] extended the asymptotic stability result to the case of $p \geq 2.75$ using pointwise dispersive decay estimates of Mielke & Patz [58]. We follow up on these estimates, by giving a proof of the asymptotic stability in the spirit of [61]. It is worth mentioning here that very recently Bambusi [9] proved asymptotic stability of breathers in KG lattices using the normal form methods of Giorgilli [37] and dispersive decay estimates from [49, 60].

In Chapter 5, we develop a stability theory for multi-site breathers in the KG lattice near the anti-continuum limit. A general method for linear stability analysis of discrete breathers in time-reversible Hamiltonian systems was developed by Aubry [4]. His method relies on properties of spectral band structure for the problem linearized about discrete breathers. The first criterion for stability of small-amplitude multi-site breathers in the dKG equation was established by Morgante *et al.* [62] with the help of numerical computations. The stability criterion from [62] was later confirmed analytically in the work of Archilla *et al.* [3], where the Aubry's method was applied to multi-site breathers in the KG lattice. More recently, Koukouloyannis & Kevrekidis [52] recovered exactly the same stability criterion using the averaging theory for Hamiltonian systems in action–angle variables. The stability results in [3, 52, 62] are restricted to small-amplitude breathers which contain no sites at rest in the anti-continuum limit.

- We study stability of multi-site breathers in the KG lattice (1.2) with potential (1.6) using perturbation methods for Floquet multipliers and analysis of the leading-order interactions between the neighbouring sites of the lattice. We develop a general criterion for linear stability of multi-site breathers in the KG lattice. In this criterion, linear stability depends on hardness/softness of the potential (1.6), the period of the breather, as well as on phase differences and distances between the excited sites. We mention here a recent application of our technique by Pelinovsky & Rothos [73], where linear stability of discrete breathers is considered for the dKG equation with a coupling term of $\epsilon(\ddot{u}_{n-1} + \ddot{u}_{n+1})$ for the nth lattice site.

- In the case of soft potentials, we find that breathers of the dKG equation cannot be continued far away from the small-amplitude limit because of the resonances between the nonlinear oscillators at the excited sites and the linear oscillators at the other sites. It turns out that branches of breather solutions continued from the anti-continuum limit above and below the resonance are disconnected. At these resonance points, the stability conclusion changes to the opposite.

- In the case of soft potentials, we also discover a symmetry-breaking (pitchfork) bifurcation of one-site and multi-site breathers that occur near the points of resonances. We analyze the symmetry-breaking bifurcation by using asymptotic expansions and a reduction of the dKG equation to a normal form, which coincides with the nonlinear Duffing oscillator perturbed by a small harmonic forcing. The normal form equation for the 1:3 resonance described in this thesis is different from the normal form equations derived in a neighbourhood of equilibrium points in earlier works [14, 84, 85].

1.4 Future research

Let us mention some questions, directly related to the topics in this thesis, that will require more work in the future.

- We show in Section 4.2 that one-site soliton in the cubic dNLS equation has no internal modes near the anti-continuum limit. The case of multi-site solitons in the cubic dNLS equation is still to be examined.

- Asymptotic stability of a small-amplitude soliton supported by the dNLS equation with exponentially decaying potential is examined in Section 4.3. Beside this result, very little is known about nonlinear stability of solitons that exist in dNLS models near the anti-continuum limit.

- Dispersive decay estimates for linear parts of the dKG and dNLS equations are now available in the form of Strichartz and pointwise estimates. In full nonlinear problems, however, these decay estimates are only known to work provided the nonlinearity is sufficiently large (see Sections 2.3 and 4.3). New methods are to be developed in order to push the nonlinearity down for the proof of asymptotic stability of discrete solutions.

- Stability of small-amplitude multi-site breathers in the KG lattice with an asymmetric potential has been addressed by both the Hamiltonian averaging method [52] and the Aubry's band theory [3]. It is worth to apply the method of tail-to-tail interactions described in Section 5.1 to stability of multi-site breathers in asymmetric potentials.

There are many problems of current interest concerned with weakly-coupled nonlinear lattices that are not discussed in this thesis. Let us mention some of these in the context of the dNLS and dKG equations.

- For the dNLS equation in two or higher dimensions, discrete breathers can also be derived from the anti-continuum limit. Discrete breathers localized on a closed contour in two-dimensional cubic dNLS equation were considered by Pelinovsky, Kevrekidis & Frantzeskakis [71]. With the method of Lyapunov–Schmidt reductions, persistence and stability was studied for discrete solitons, which have phase differences of 0 or π between the adjacent sites, and discrete vortices, which have the phase differences measured in fractions of π. A similar study was performed for the dNLS equations in three dimensions by Lukas, Pelinovsky & Kevrekidis [54], and for a coupled dNLS system in two dimensions by Kevrekidis & Pelinovsky [48].

- Discrete solitons of the one-dimensional dNLS equation can persist in its two-dimensional counterpart as line solitons with a repeating profile in one of the spatial directions. In a recent work of Yang [97], such solitons bifurcating from the continuous spectrum (Bloch bands) of the two-dimensional cubic dNLS equation were considered. It was shown numerically that there are some configurations of line solitons which are stable for sufficiently high values of the conserving l^2 norm. This observation was later justified by Pelinovsky & Yang [77], who counted eigenvalues in the linearized stability problem and proved the earlier numerical observations.

- Some fascinating findings on persistence and stability of multi-site breathers in Hamiltonian lattices with non-nearest-neighbour interactions have been recently reported by Kevrekidis [47] and Koukouloyannis *et al.* [53]. For instance, in [53], it is shown that the KG lattice with non-nearest-neighbour interactions supports not only discrete breathers with in-phase and anti-phase oscillators, but also phase-shift discrete breathers, which have phase differences other than 0 or π for the lattice neighbours. Using Hamiltonian averaging methods it is shown that some of the phase-shift breathers are actually stable near the anti-continuum limit. Bifurcations of new breather configurations are demonstrated for a critical ratio of coupling constants. Let us note that stability of multi-site breathers in KG chains with non-nearest-neighbour interactions has been also recently examined by Rapti [79]. She extended the method of Archilla et al. [3], which is based on Aubry's band theory, to include multi-site breathers with non-nearest neighbour interactions.

- Continuation of large-amplitude discrete breathers from infinity has been recently studied by James, Levitt & Ferreira [40] and James & Pelinovsky [41]. In these papers, the KG chain with a saturable potential is considered. When the diagonal term of the discrete Laplacian is incorporated into the on-site potential, the oscillators are trapped but have infinite amplitude in the anti-continuum limit of small lattice couplings. Using the contraction mapping techniques, large-amplitude discrete breathers oscillating outside the potential well [40] or above the potential barrier [41] are constructed.

1.5 Preliminaries

In this thesis we adopt the following notations.

- For the sequence $\{u_n\}_{n\in\mathbb{Z}}$, we define the discrete Laplacian operator Δ by

$$(\Delta \mathbf{u})_n = u_{n-1} - 2u_n + u_{n+1}.$$

- The $l^p(\mathbb{Z})$ space with $p \in \mathbb{R}$ is defined by the norm

$$\|\mathbf{u}\|_{l^p(\mathbb{Z})} = \left(\sum_{n\in\mathbb{Z}} |u_n|^p \right)^{1/p}.$$

- The space $l^p_s(\mathbb{Z})$ for the sequence $\{u_n\}_{n\in\mathbb{Z}}$ is equivalent to the space $l^p(\mathbb{Z})$ for the sequence $\{(1+n^2)^{s/2}u_n\}_{n\in\mathbb{Z}}$:

$$\|\mathbf{u}\|_{l^p_s(\mathbb{Z})} = \left(\sum_{n\in\mathbb{Z}}(1+n^2)^{ps/2}|u_n|^p\right)^{1/p}.$$

- Since we work only on one-dimensional problems, we simplify the notations for the function spaces by writing l^p_s in the place of $l^p_s(\mathbb{Z})$.
- For sequences $\{u_n\}_{n\in\mathbb{Z}}$ and $\{v\}_{n\in\mathbb{Z}}$ we define

$$(\mathbf{uv})_n := u_n v_n.$$

We are also going to use the embeddings of l^p spaces,

- $l^p \subset l^q$ with $p < q$, such that

$$\|\mathbf{u}\|_{l^q} \leq \|\mathbf{u}\|_{l^p},$$

- $l^p_\sigma \subset l^p \subset l^p_{-\sigma}$, such that

$$\|\mathbf{u}\|_{l^p_{-\sigma}} \leq \|\mathbf{u}\|_{l^p} \leq \|\mathbf{u}\|_{l^p_\sigma}.$$

To interpolate the norm in Lebesgue spaces we can use the Riesz–Thorin formula:

$$\|\mathbf{u}\|_{l^p} \leq \|\mathbf{u}\|^{\theta}_{l^r}\|\mathbf{u}\|^{1-\theta}_{l^s}, \qquad \frac{1}{p} = \frac{\theta}{r} + \frac{1-\theta}{s}, \qquad \theta \in (0,1).$$

Chapter 2

Local and global existence of time-dependent solutions

This chapter is concerned with existence of time-dependent solutions to lattice equations, as well as the decay rates of small solutions. In Sections 2.1 and 2.2, we review local and global well-posedness of the dNLS and dKG equations. Then, in Section 2.3, we use an example of the dNLS equation, to study scattering of small initial data.

2.1 Well-posedness of the dNLS equation

In this section, we consider well-posedness of the initial-value problem for the dNLS equation on a one-dimensional lattice:

$$\begin{cases} i\dot{u}_n(t) = -(\Delta \mathbf{u})_n + V_n u_n + f(|u_n|^2)u_n, \\ u_n(0) = u_{0,n}, \end{cases} \qquad n \in \mathbb{Z}, \tag{2.1}$$

where $\{u_n(t)\}_{n\in\mathbb{Z}} : \mathbb{R}_+ \to \mathbb{C}^{\mathbb{Z}}$ represents a vector of amplitude functions, $\mathbf{V}$ is a bounded potential, and f is a real analytic function that can be expanded into convergent power series

$$f(x) = \sum_{k=1}^{\infty} f_k x^k, \qquad x \in \mathbb{R}.$$

Although the focusing NLS equation with supercritical nonlinearity admits solutions that blow up in finite time (see [91] for a review), the dNLS equation admits global solutions for initial data in l_s^2 with $s \geq 0$ no matter what the sign or the power of the nonlinearity is [63, 64, 69]. We begin by proving local well-posedness of the initial value problem (2.1) in the Banach space $C([0,T], l_s^2)$ with $s \geq 0$.

Theorem 2.1. *Fix $s \geq 0$ and let $\mathbf{u}_0 \in l_s^2$. Assume $\mathbf{V} \in l^\infty$ and f is a real analytic function. There exists $T \in (0, +\infty)$ and a unique solution $\mathbf{u}(t) \in C^1([0,T], l_s^2)$ to the initial-value problem (2.1). The solution $\mathbf{u}(t)$ depends continuously on the initial data $\mathbf{u}_0$.*

Proof. Let us rewrite the Cauchy problem (2.1) in its equivalent integral form

$$\mathbf{u}(t) = \mathbf{A}(\mathbf{u}(t)), \tag{2.2}$$

where the operator in the right-hand side is defined by

$$A_n(\mathbf{u}(t)) := u_{0,n} - i \int_0^t \left(-(\Delta \mathbf{u}(t'))_n + V_n u_n(t') + f(|u_n|^2) u_n \right) dt', \quad n \in \mathbb{Z}.$$

We are going to prove that for any $\mathbf{u}_0 \in l_s^2$ there is $T > 0$ and a unique fixed point of (2.2) in the Banach space $X = C([0,T], l_s^2)$ with the norm

$$\|\mathbf{u}\|_X = \sup_{t \in [0,T]} \|\mathbf{u}(t)\|_{l_s^2}.$$

To achieve this, let us show that for sufficiently small $T > 0$ the map $\mathbf{A}$ satisfies conditions of the Banach Fixed-Point Theorem. Given a closed ball

$$\overline{B_\delta} = \left\{ \mathbf{x} \in X | \ \|\mathbf{x}\|_X \leq \delta \right\},$$

we need to show that

(i) $\mathbf{A}$ maps $\overline{B_\delta}$ to $\overline{B_\delta}$,

(ii) $\mathbf{A}$ is a contractive map on $\overline{B_\delta}$, i.e. there is $q \in (0,1)$ such that for all $\mathbf{u}, \mathbf{v} \in \overline{B_\delta}$ we have

$$\|\mathbf{A}(\mathbf{u}) - \mathbf{A}(\mathbf{v})\|_X \leq q \|\mathbf{u} - \mathbf{v}\|_X.$$

Since $\mathbf{V} \in l^\infty$ and $\Delta : l_s^2 \to l_s^2$ is a bounded operator, we get

$$\forall \mathbf{u} \in l_s^2 : \qquad \|\Delta \mathbf{u}\|_{l_s^2} \leq C_\Delta \|\mathbf{u}\|_{l_s^2}, \qquad \|\{V_n u_n\}\|_{l_s^2} \leq C_V \|\mathbf{u}\|_{l_s^2}.$$

Also, by the Banach algebra property of the l_s^2 space, there is a constant $C_s > 0$ such that

$$\forall \mathbf{u}, \mathbf{v} \in l_s^2 : \qquad \|\mathbf{u}\mathbf{v}\|_{l_s^2} \leq C_s \|\mathbf{u}\|_{l_s^2} \|\mathbf{v}\|_{l_s^2} \tag{2.3}$$

Therefore, given $\mathbf{u} \in \overline{B_\delta}$ we can estimate the nonlinear term by

$$\|\{f(|u_n|^2)u_n\}\|_{l^2_s} \leq \sum_{k=1}^{\infty} |f_k| C_s^{2k} \|\mathbf{u}\|_{l^2_s}^{2k+1} \leq C_f(\delta)\|\mathbf{u}\|_{l^2_s},$$

where $C_f(\delta) = \sum_{k=1}^{\infty} |f_k| C_s^{2k} \delta^{2k}$. Using the above estimates we obtain the following bound

$$\|\mathbf{A}(\mathbf{u})\|_X \leq \delta_0 + T\delta(C_\Delta + C_V + C_f(\delta)),$$

where $\delta_0 = \|\mathbf{u}_0\|_{l^2_s}$. Finally, if $\delta_0 = \delta/2$ and

$$T(C_\Delta + C_V + C_f(\delta)) \leq \frac{1}{2}, \tag{2.4}$$

the condition (i) is satisfied.

To satisfy condition (ii), we need to show that there is a constant $\tilde{C}_f(\delta)$ such that

$$\forall \mathbf{u}, \mathbf{v} \in \overline{B_\delta}: \qquad \|\{f(|u_n|^2)u_n - f(|v_n|^2)v_n\}\|_{l^2_s} \leq \tilde{C}_f(\delta)\|\mathbf{u} - \mathbf{v}\|_{l^2_s}. \tag{2.5}$$

Using an elementary algebraic estimate

$$\begin{aligned} \left| |u|^{2k}u - |v|^{2k}v \right| &= \left| u^{k+1}(\bar{u}^k - \bar{v}^k) + \bar{v}^k(u^{k+1} - v^{k+1}) \right| \\ &\leq \Big(|u|^{k+1}|u^{k-1} + u^{k-2}v + \cdots + v^{k-1}| \\ &\quad + |v|^k |u^k + u^{k-1}v + \cdots + v^k| \Big) |u - v| \end{aligned}$$

and the Banach algebra property (2.3) we show that the map $\{u_n\} \mapsto \{|u_n|^{2k}u_n\}$ is Lipschitz-continuous in l^2_s and

$$\|\{|u_n|^{2k}u_n - |v_n|^{2k}v_n\}\|_{l^2_s} \leq (2k+1)C_s^{2k}\delta^{2k}\|\mathbf{u} - \mathbf{v}\|_{l^2_s}.$$

This estimate allows us to give an explicit formula for the constant $\tilde{C}_f(\delta)$ in (2.5):

$$\tilde{C}_f(\delta) = \sum_{k=1}^{N} (2k+1)|f_k| C_s^{2k} \delta^{2k}.$$

Therefore, in order to satisfy condition (ii) we must require

$$T(C_\Delta + C_V + \tilde{C}_f(\delta)) < 1. \tag{2.6}$$

By the Banach Fixed-Point Theorem, given $\|\mathbf{u}_0\|_{l^2_s} \leq \delta/2$, there exists a unique solution $\mathbf{u} \in X = C(([0,T], l^2_s)$ with $\|\mathbf{u}\|_X \leq \delta$ provided the existence time T satisfies conditions (2.4) and (2.6) simultaneously. Since the right-hand side of the dNLS equation (2.1) belongs to $C(([0,T],l^2_s)$, we immediately conclude that $\mathbf{u} \in C^1(([0,T],l^2_s)$. The continuous dependence of the solution $\mathbf{u}(t)$ on initial data $\mathbf{u}_0$ also follows from the Banach Fixed-Point Theorem. □

One may want to iterate the above local arguments to achieve global existence results. This, however, may not be possible as the local existence time is inversely proportional to the solution's norm (see (2.4) and (2.6)). Since at each iteration the upper bound on the solution's norm grows while the local existence time shrinks, the norm $\|\mathbf{u}(t)\|_{l^2_s}$ may diverge before we reach the limit $t \to \infty$.

Fortunately, the proof of global well-posedness can be achieved because of l^2 norm conservation in the dNLS equation. We give the proof of global well-posedness in l^2_s below.

Theorem 2.2. *Fix $s \geq 0$ and let $\mathbf{u}_0 \in l^2_s$. Assume $\mathbf{V} \in l^\infty$ is a bounded operator and f is a real analytic function. The initial-value problem (2.1) admits a unique global solution $\mathbf{u}(t) \in C(\mathbb{R}_+, l^2_s)$ that depends continuously on the initial data $\mathbf{u}_0$.*

Proof. Since local well-posedness for $\mathbf{u}(t) \in C^1([0,T],l^2_s)$ with the existence time $T = T\left(\|\mathbf{u}_0\|_{l^2_s}\right) \in (0,+\infty)$ was proved in Theorem 2.1, it is enough to show that T can be extended to infinity.

Multiplying the n^{th} component of the dNLS equation by $\overline{u}_n$ we obtain

$$i\dot{u}_n\overline{u}_n = -\left(\Delta\mathbf{u}\right)_n\overline{u}_n + V_n|u_n|^2 + f(|u_n|^2)|u_n|^2.$$

From this equation and its complex conjugate, we immediately obtain the following identity, which is independent of both potential and nonlinear terms:

$$i\frac{d}{dt}|u_n|^2 = -(\Delta\mathbf{u})_n\overline{u}_n + (\Delta\overline{\mathbf{u}})_n u_n.$$

We can use the Cauchy–Schwarz inequality and the boundedness of the discrete Laplacian, $\|\Delta\mathbf{u}\|_{l^2_s} \leq C_\Delta\|\mathbf{u}\|_{l^2_s}$, to obtain the following estimate

$$\frac{d}{dt}\|\mathbf{u}\|^2_{l^2_s} \leq 2\left|\langle\Delta\mathbf{u},\mathbf{u}\rangle_{l^2_s}\right| \leq 2C_\Delta\|\mathbf{u}\|^2_{l^2_s}.$$

Hence, by the Gronwall's lemma, it follows that

$$\|\mathbf{u}\|_{l^2_s} \leq \|\mathbf{u}_0\|_{l^2_s}e^{C_\Delta t}. \tag{2.7}$$

This inequality provides a global bound on the solution's norm and concludes the proof of the theorem. □

Remark 2.3. For $s = 0$, we actually have l^2 norm conservation, i.e. $\|\mathbf{u}(t)\|_{l^2} = \|\mathbf{u}_0\|_{l^2}$ for all $t \geq 0$. However, it is shown in [61] (see also Lemma 4.31) that the weighted norm may grow algebraically,

$$\|\mathbf{u}\|_{l^2_s} \leq C_s\big(1 + |t|\big)^s \|\mathbf{u}_0\|_{l^2_1}, \qquad s \in [0, 1].$$

This estimate provides a sharper alternative to the exponential growth in (2.7).

Remark 2.4. Since the discrete Laplacian operator Δ has a bounded multi-dimensional extension, Theorems 2.1 and 2.2 also hold true for the dNLS equation on multi-dimensional lattices.

Remark 2.5. As pointed out by Pacciani *et al.* [64] and N'Guérékata & Pankov [63], Theorem 2.2 on global well-posedness can be generalized to dNLS lattices with long-range interactions and gauge-invariant, uniformly locally Lipschitz continuous nonlinearities.

2.2 Well-posedness and blow up in the dKG equation

In this section, we consider well-posedness of the initial value problem for the discrete Klein–Gordon (dKG) equation on a one-dimensional lattice:

$$\begin{cases} \ddot{u}_n(t) + u_n + \beta u_n^{2\sigma+1} = (\Delta \mathbf{u})_n, \\ u_n(0) = u_{0,n}, \quad \dot{u}_n(0) = u_{1,n}, \end{cases} \qquad n \in \mathbb{Z}, \tag{2.8}$$

where $\{u_n(t)\}_{n\in\mathbb{Z}} : \mathbb{R}_+ \to \mathbb{R}^{\mathbb{Z}}$ is the set of displacement functions, $\beta \neq 0$, and $\sigma \in \mathbb{N}$. This is a Hamiltonian system with a conserving energy functional

$$H = \frac{1}{2}\sum_{n\in\mathbb{Z}} \big(\dot{u}_n^2 + u_n^2 + (u_{n+1} - u_n)^2\big) + \frac{\beta}{2\sigma + 2}\sum_{n\in\mathbb{Z}} u_n^{2\sigma+2}. \tag{2.9}$$

Let us look into intuitive arguments on well-posedness of the KG lattice (2.8). When the system is decoupled, the oscillator at each lattice site is described by the Duffing equation

$$\ddot{\varphi} + V'(\varphi) = 0, \qquad V(\varphi) = \frac{1}{2}\varphi^2 + \frac{\beta}{2\sigma + 2}\varphi^{2\sigma+2}, \tag{2.10}$$

where $\beta \neq 0$ and $\sigma \in \mathbb{N}$.

Definition 2.6. We call the on-site potentials V in (2.10) with $\beta < 0$ and $\beta > 0$ as a *soft* and *hard potentials* respectively.

As shown on Figure 2.1, in the case of hard potential, all solutions to equation (2.10)

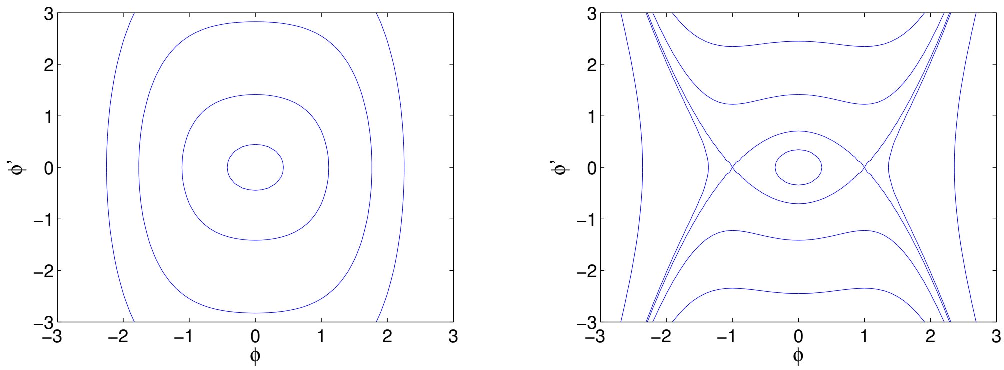

Figure 2.1: The phase plane $(\varphi, \dot{\varphi})$ for the Duffing oscillator (2.10) with $\beta = +1$ (left), and $\beta = -1$ (right).

are periodic, while in the case of soft potential there are both periodic and unbounded trajectories.

In the case of KG lattice (2.8) with hard potential ($\beta > 0$), every oscillator is trapped into a confining potential, which suggests global existence of time-dependent solutions. On the contrary, in the case of soft potential ($\beta < 0$), one or more oscillators may be outside the potential well and the coupling to the rest of the lattice may not be strong enough to hold them back. Thus, under some initial conditions, we can expect a finite-time blow up to occur in the dKG equation (2.8) with soft potential.

2.2.1 Local and global existence

Let us first look into existence of local solutions to the dKG equation (2.8). Introducing $\mathbf{v} = \dot{\mathbf{u}}$, we rewrite the initial-value problem (2.8) in its equivalent integral form

$$\begin{cases} u_n(t) = u_{0,n} + \displaystyle\int_0^t v_n(t')dt', \\ v_n(t) = u_{1,n} + \displaystyle\int_0^t \left(\left(\Delta \mathbf{u}(t')\right)_n - u_n(t') - \beta u_n^{2\sigma+1}(t') \right) dt'. \end{cases}$$

The techniques described in the Section 2.1 can also be applied here to prove local well-posedness of the dKG equation in the Banach space $X = C^1([0,T], l^2)$ with the norm

$$\|\mathbf{u}\|_X^2 = \sup_{t\in[0,T]} \left(\|\mathbf{u}(t)\|_{l^2}^2 + \|\dot{\mathbf{u}}(t)\|_{l^2}^2 \right).$$

The local well-posedness result can be stated as follows:

Theorem 2.7. *Fix $\sigma \in \mathbb{N}$, $s \geq 0$ and let $\mathbf{u}_0, \mathbf{u}_1 \in l_s^2$. There exists $T \in (0, +\infty)$ and a unique solution $\mathbf{u}(t) \in C^2([0,T], l_s^2)$ to the initial-value problem (2.8) such that*

$\mathbf{u}(0) = \mathbf{u}_0$ and $\dot{\mathbf{u}}(0) = \mathbf{u}_1$. The solution depends continuously on the initial data $(\mathbf{u}_0, \mathbf{u}_1)$.

For the hard potential, the energy in (2.9) is positive and global existence is easy to show (see e.g. Menzala & Konotop [57]):

Theorem 2.8. *Fix $\beta > 0$, $\sigma \in \mathbb{N}$, and let $\mathbf{u}_0, \mathbf{u}_1 \in l^2$. The initial-value problem (2.8) admits a unique global solution $\mathbf{u}(t) \in C^1(\mathbb{R}_+, l^2)$ that depends continuously on the initial data.*

Proof. Using the conservation of energy H defined in (2.9) and the Banach algebra property for the l^2 space, we obtain a global bound on the solution's norm:

$$\|\mathbf{u}\|_{l^2}^2 + \|\dot{\mathbf{u}}\|_{l^2}^2 \leq 2H < \infty.$$

Continuous dependence on initial data follows from the local well-posedness result above. □

Global existence in the dKG equation with a soft potential and $\sigma = 1$ was recently examined by Achilleos *et. al.* [1], who established the following result:

Theorem 2.9. *Consider the initial value problem (2.8) with $\sigma = 1$ and $\beta < 0$. Assume that the functional*

$$E(t) := \sum_{n \in \mathbb{Z}} \left(\dot{u}_n^2 + u_n^2 + (u_{n+1} - u_n)^2 \right) \tag{2.11}$$

satisfies

$$E(0) < \min \left\{ 1, \frac{1}{|\beta|(2 + |\beta|)} \right\}.$$

Then the solution exists globally in time, and for all $t \in [0, \infty)$ functional (2.11) satisfies the bound

$$E(t) < \frac{1}{|\beta|} \left[1 - \sqrt{1 - E(0)|\beta|\,(2 + |\beta|)} \right].$$

Remark 2.10. As pointed out in [1], this global existence result persists if a dissipative term $\gamma \dot{\mathbf{u}}$ with $\gamma > 0$ is added to the left hand side of the dKG equation (2.8).

To complement Theorems 2.8 and 2.9 on global existence, we would like to mention that the dKG equation with sufficiently large nonlinearity scatters small initial data independently of the sign of β. This fact was proved by Stefanov & Kevrekidis [89] and Mielke & Patz [58] who established the decay rates for the dKG equation with $\sigma > 2$ and $\sigma > 3/2$ respectively. We demonstrate the technique of pointwise decay estimates from [58] in the context of the dNLS equation in Section 2.3.

2.2.2 Finite-time blow up

Let us now discuss collapse of solutions in the dKG equation (2.8). As was pointed out above using the example of Duffing oscillator (2.10), solutions to the KG lattice (2.8) with soft potential ($\beta < 0$) may blow up in finite time. The blow up was examined by Karachalios [44] using the abstract framework for hyperbolic partial differential equations developed by Galaktionov & Pohozaev in [34]. The following result comes from the paper of Karachalios:

Theorem 2.11. *Consider the initial value problem (2.8) with $\beta < 0$, $H < 0$, and $\sigma \in (0, \infty)$. Then if $\langle \mathbf{u}_0, \mathbf{u}_1 \rangle_{l^2} > 0$, the solution becomes unbounded on a finite time interval $[0, T^*]$ with*

$$T^* = \frac{\|\mathbf{u}_0\|_{l^2}^2}{\sigma \langle \mathbf{u}_0, \mathbf{u}_1 \rangle_{l^2}}.$$

Proof. The proof is based on the analysis of a differential inequality for

$$\mu(t) = \|\mathbf{u}(t)\|_{l^2}^2.$$

The Hamiltonian (2.9) is written as $H = \frac{1}{2}\|\dot{\mathbf{u}}\|_{l^2}^2 + P(\mathbf{u})$ where

$$P(\mathbf{u}) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(u_n^2 + (u_{n+1} - u_n)^2 \right) - \frac{|\beta|}{2\sigma + 2} \sum_{n \in \mathbb{Z}} u_n^{2\sigma + 2}$$

is $C^1(l^2, \mathbb{R})$ and its Gâteaux derivative given by

$$\langle P'(\mathbf{u}), \mathbf{v} \rangle_{l^2} = \sum_{n \in \mathbb{Z}} \left(u_n v_n + (u_{n+1} - u_n)(v_{n+1} - v_n) - |\beta| u_n^{2\sigma+1} v_n \right), \qquad \forall \mathbf{v} \in l^2.$$

Hence, we obtain

$$\langle P'(\mathbf{u}), \mathbf{u} \rangle_{l^2} - (2\sigma + 2) P(\mathbf{u}) = -\sigma \sum_{n \in \mathbb{Z}} \left((u_{n+1} - u_n)^2 + u_n^2 \right) \leq 0,$$

which allows us to get the following estimate

$$\langle \mathbf{u}, \ddot{\mathbf{u}} \rangle_{l^2} = -\langle P'(\mathbf{u}), \mathbf{u} \rangle_{l^2} \geq -2(\sigma + 1) P(\mathbf{u}) = -(\sigma + 1) \left(2H - \|\dot{\mathbf{u}}\|_{l^2}^2 \right).$$

We notice that $\mu'(t) = 2\langle \mathbf{u}, \dot{\mathbf{u}} \rangle_{l^2}$ and $\mu'(0) = 2\langle \mathbf{u}_0, \mathbf{u}_1 \rangle_{l^2}$. The Cauchy–Schwarz inequality yields

$$\|\dot{\mathbf{u}}\|_{l^2}^2 \geq \frac{(\mu'(t))^2}{4\mu(t)}.$$

Since $H < 0$, $\mu(t)$ satisfies the differential inequality

$$\begin{aligned}\mu''(t) &= 2\|\dot{\mathbf{u}}\|_{l^2}^2 + 2\langle \mathbf{u}, \ddot{\mathbf{u}}\rangle_{l^2} \\ &\geq (4+2\sigma)\|\dot{\mathbf{u}}\|_{l^2}^2 - 4(\sigma+1)H \\ &\geq \frac{\sigma+2}{2}\frac{(\mu'(t))^2}{\mu(t)} \geq 0\end{aligned} \tag{2.12}$$

which tells us that $\mu'(t)$ is an increasing function.

If $\mu'(0)$ is positive, so is $\mu'(t)$ for $t > 0$. In this case, we rewrite the last inequality as

$$\frac{\mu''(t)}{\mu'(t)} \geq \frac{\sigma+2}{2}\frac{\mu'(t)}{\mu(t)}$$

and perform integrations twice to obtain the bound

$$\mu(t) \geq \mu(0)\left(1 - \frac{\sigma}{2}\frac{\mu'(0)}{\mu(0)}t\right)^{-\frac{2}{\sigma}} \tag{2.13}$$

which explains the finite-time blow up of $\mu(t)$. □

Remark 2.12. The estimates in the proof can be modified for the case $\mu(0) < 0$. The result, however, would neither prove global existence nor finite-time blow up. Assuming $\mu(0) < 0$, we can prove that also $\mu'(t) < 0$ for $t > 0$. To show this, notice that (2.12) implies

$$\frac{\mu''(t)}{(\mu'(t))^2} \geq \frac{\sigma+2}{2}\frac{1}{\mu(t)}$$

and thus

$$\frac{1}{\mu'(t)} - \frac{1}{\mu'(0)} \leq -\frac{\sigma+2}{2}\int_0^t \frac{ds}{\mu(s)} < 0.$$

This last inequality can be rewritten as $\mu'(0) < \mu'(t) < 0$. Now, knowing that $\mu'(t)$ is negative we rewrite (2.12) as

$$\frac{\mu''(t)}{\mu'(t)} \leq \frac{\sigma+2}{2}\frac{\mu'(t)}{\mu(t)}.$$

After one integration we obtain

$$\frac{\mu'(t)}{\mu'(0)} \leq \left(\frac{\mu(t)}{\mu(0)}\right)^{\frac{\sigma+2}{2}}.$$

Then, the next integration provides a formula analogous to (2.13):

$$\mu(t) \geq \mu(0)\left(1 + \frac{2}{\sigma}\frac{|\mu'(0)|}{\mu(0)}t\right)^{-\frac{2}{\sigma}}. \tag{2.14}$$

This last inequality does not tell whether $\mu(t)$ blows up or tends to zero as $t \to \infty$. It simply shows that $\mu(t)$ admits the time-dependent lower bound (2.14).

In [44], inequality (2.14) has the opposite sign, which leads to erroneous conclusion that solutions to the dKG equation (2.8) with $\beta < 0$, H <0, and $\langle \mathbf{u}_0, \mathbf{u}_1 \rangle_{l^2} < 0$ decay to zero as $t \to \infty$. The proof in [44] is derived from Lemma 2.1 in [34], where an analogous error was made. The erroneous results in [34, 44] were recently shown to contradict a numerical experiment of Achilleos *et al.* [1].

The recent paper [1] also offers a new result on collapse of solutions in the dKG equation with soft cubic potential. The result, which we state here without a proof, shows that the solutions with positive energy H in (2.9) can also undergo blow up.

Theorem 2.13. *Consider the initial value problem (2.8) with $\beta < 0$ and $\sigma = 1$. The solution blows up in finite time provided the Hamiltonian (2.9) and the initial data $\mathbf{u}_0$ satisfy the following conditions:*

$$H < \frac{1}{4|\beta|} \quad \textit{and} \quad \|\mathbf{u}_0\|_{l^4} > \frac{1}{\sqrt{|\beta|}}.$$

Of course, this is not the only blow up scenario. For other initial data leading to blow up in the dKG equation with soft cubic potential we refer the reader to numerical results and discussion in [1].

2.3 Scattering of small solutions to the dNLS equation

In this section, we study temporal rates of decay of solutions to nonlinear lattice equations. We rely on the novel techniques developed by Mielke & Patz [58] which include approximating l^p norms by oscillatory integrals and careful estimations of the latter using the Van der Corput lemma. We illustrate the abstract results using the dNLS equation as an example. The techniques presented here can be applied to a wide class of lattice equations which includes dKG equation and FPU lattice.

The initial value problem for the dNLS equation can be written as

$$\begin{cases} i\dot{u}_n(t) = -(\Delta \mathbf{u})_n \pm |u_n|^{\beta-1}u_n, \\ u_n(0) = u_{0,n}, \end{cases} \qquad n \in \mathbb{Z}, \tag{2.15}$$

where $\{u_n(t)\}_{n\in\mathbb{Z}} : \mathbb{R}_+ \to \mathbb{C}^{\mathbb{Z}}$ and $\beta > 1$. The decay of small initial data in this equation occurs independently of the sign of the nonlinearity provided β is large enough.

One can quickly derive a dispersive decay estimate for the linearized Schrödinger equation

$$i\dot{\mathbf{u}}(t) = -\Delta\mathbf{u}, \qquad \mathbf{u}(0) = \mathbf{u}_0,$$

using Riesz–Thorin interpolation formula

$$\|\mathbf{u}\|_{l^p} \leq \|\mathbf{u}\|_{l^r}^{\theta}\|\mathbf{u}\|_{l^s}^{1-\theta}, \qquad \frac{1}{p} = \frac{\theta}{r} + \frac{1-\theta}{s}, \qquad \theta \in (0,1).$$

Running interpolation between l^2 norm conservation

$$\|e^{it\Delta}\mathbf{u}_0\|_{l^2} = \|\mathbf{u}_0\|_{l^2}$$

and decay in l^∞ norm that was proved in Stefanov & Kevrekidis [89] (also see Section 2.3.1 for the proof),

$$\|e^{it\Delta}\mathbf{u}_0\|_{l^\infty} \leq C(1+t)^{-1/3}\|\mathbf{u}_0\|_{l^1}, \tag{2.16}$$

we obtain the dispersive decay estimate,

$$\|e^{it\Delta}\mathbf{u}_0\|_{l^p} \leq C(p)t^{-\frac{p-2}{3p}}\|\mathbf{u}_0\|_{l^{p'}},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $2 \leq p \leq \infty$. As shown in [89], a similar estimate applies to solutions of the full dNLS equation (2.15). Namely, if $\beta > 5$ and the initial data $\mathbf{u}_0 \in l^{p'}$ is small enough, then

$$\|\mathbf{u}(t)\|_{l^p} \leq C(p)t^{-\frac{p-2}{3p}}\|\mathbf{u}_0\|_{p'}, \tag{2.17}$$

where $2 \leq p \leq 5$.

In [58], Mielke & Patz proved an improved decay estimate for the linear dNLS equation:

$$\|e^{it\Delta}\mathbf{u}_0\|_{l^p} \leq C(p)(1+t)^{-\alpha_p}\|\mathbf{u}_0\|_{l^1}, \tag{2.18}$$

where the exponent α_p is given by

$$\alpha_p = \begin{cases} \dfrac{p-2}{2p}, \ p \in [2,4), \\ \dfrac{p-1}{3p}, \ p \in (4,\infty]. \end{cases} \tag{2.19}$$

Since $\alpha_p \geq \frac{p-2}{3p}$ this result implies faster decay than inequality (2.17). In the same article, the estimate (2.18) was used to prove scattering for a general class of lattice equations which includes dNLS, dKG, and FPU equations. For instance, it was shown that the decay estimate (2.18) extends to the dNLS equation (2.15) with $\beta > 4$. A precise statement of this result is as follows:

Theorem 2.14. *Let $\beta > 4$, then for each $p \in [2,4) \cup (4,\infty]$ there exists $\epsilon > 0$ such that all solutions to dNLS equation (2.15) with $\|\mathbf{u}_0\|_{l^1} \le \epsilon$ satisfy the estimate*

$$\|\mathbf{u}(t)\|_{l^p} \le C(p,\beta,\epsilon)(1+t)^{-\alpha_p}\|\mathbf{u}_0\|_{l^1}, \tag{2.20}$$

where the decay exponent α_p is given in (2.19).

This section is devoted to discussion of techniques and ideas involved in the proof of this theorem. These techniques and ideas are used later in Section 4.3 to prove asymptotic stability of solitons in the dNLS equation.

2.3.1 Linear decay

In this section we study the dispersive decay of solutions to an abstract linear lattice system

$$\begin{cases} \dot{\mathbf{u}}(t) = L\mathbf{u}, \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \qquad \mathbf{u}(t) = \{u_n(t)\}_{n\in\mathbb{Z}} : \mathbb{R}_+ \to \mathbb{C}^{\mathbb{Z}}, \tag{2.21}$$

where $\mathbf{u}_0 \in l^1$. We are going to employ Fourier transform to solve (2.21) and then apply Van der Corput lemma together with some ideas from numerical integration to analyze the asymptotics of the resulting oscillatory integrals in the limit $t \to \infty$.

Let us first introduce the notion of dispersion relation associated with a linear lattice operator L, the key concept in establishing the rates of dispersive decay.

Definition 2.15. The *dispersion relation* associated with the linear lattice operator L is a 2π-periodic function $\omega : \mathbb{T} \to \mathbb{C}$ defined by the identity

$$Le^{in\theta} = -i\omega(\theta)e^{in\theta}, \qquad \theta \in \mathbb{T},\ n \in \mathbb{Z}, \tag{2.22}$$

where $\mathbb{T} = [-\pi,\pi]$. We also call

$$\Theta_{\text{cr}} = \{\theta \in \mathbb{T} :\ \omega''(\theta) = 0\},$$

the *set of critical points* of the dispersion relation ω.

To simplify the analysis, we are going to impose some restrictions on the dispersion relations we encounter.

Assumption 2.16. *We require that the dispersion relation ω associated with the linear operator L in (2.21) satisfies the following conditions:*

- $\omega \in C^3(\mathbb{T})$,
- *ω has finitely many critical points,*
- *the critical points are not degenerate in the sense that*

$$\omega'''(\theta_0) \neq 0, \qquad \forall \theta_0 \in \Theta_{\text{cr}}.$$

Example 2.17. For the linear operator $L = i\Delta$ in the dNLS equation (2.15) the dispersion relation is given by $\omega(\theta) = 2(1 - \cos\theta)$. The set of critical points where ω'' turns to zero is then $\Theta_{\text{cr}} = \{-\frac{\pi}{2}, \frac{\pi}{2}\}$. This dispersion relation clearly satisfies all of the above assumptions.

We write a solution to initial value problem (2.21) using Fourier transform $\mathcal{F} : L^2(\mathbb{T}) \to l^2(\mathbb{Z})$ as

$$u_n(t) = \frac{1}{2\pi} \int_{\mathbb{T}} \hat{u}(\theta, 0) e^{i(n\theta - \omega(\theta)t)} d\theta, \qquad n \in \mathbb{Z}, \tag{2.23}$$

where $\hat{u}(\theta, 0) = \sum_{m\in\mathbb{Z}} u_{0,m} e^{-im\theta}$ is the inverse Fourier transform of initial data. According to the Van der Corput lemma, the dispersion relation ω determines the decay properties of the oscillatory integral in (2.23). Let us recall this important lemma (see e.g. [90] p. 334):

Lemma 2.18 (Van der Corput)**.** *Fix $k \in \mathbb{N}$ and assume that $\psi \in C^1(a, b)$, $\varphi \in C^k(a, b)$, and $|\varphi^{(k)}(x)| \geq M > 0$ for all $x \in (a, b)$. Then, for all $t \geq 1$ we have*

$$\left| \int_a^b \psi(x) e^{it\varphi(x)} dx \right| \leq C_k (M\, t)^{-1/k} \left(|\psi(b)| + \int_a^b |\psi'(x)| dx \right), \tag{2.24}$$

provided either $k \geq 2$ or $k = 1$ and φ' is monotonic on (a, b).

To obtain the time decay of $\{u_n(t)\}_{n\in\mathbb{Z}}$ along the rays with slopes $\frac{n}{t}$ in (n, t)-plane let us apply Van der Corput lemma directly to (2.23) where the phase function of the oscillatory integral is

$$\varphi(\theta) = \frac{n}{t}\theta - \omega(\theta).$$

If $\varphi'(\theta) \neq 0$ for all $\theta \in \mathbb{T}$, which happens provided

$$\frac{n}{t} \in \mathbb{R} \backslash [\min_{\theta\in\mathbb{T}} \omega'(\theta), \max_{\theta\in\mathbb{T}} \omega'(\theta)],$$

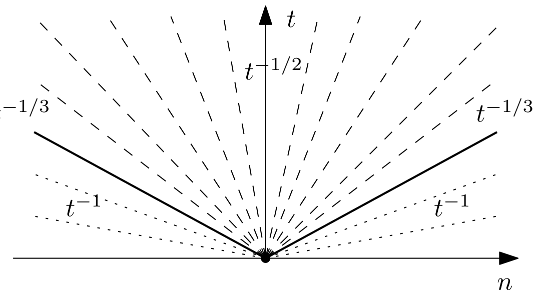

Figure 2.2: Decay rates of solution $\{u_n(t)\}_{n\in\mathbb{Z}}$ to the linear dNLS equation (2.15) along the rays with slopes $\frac{n}{t}$ in (n,t)-plane.

the solution decays like t^{-1} as $t \to \infty$. If for some $\theta \in \mathbb{T}$ we have $\varphi'(\theta) = 0$ and $\varphi''(\theta) \equiv -\omega''(\theta) \neq 0$ then the decay is $t^{-1/2}$. These conditions on φ are satisfied along the rays with slopes

$$\frac{n}{t} \in [\min_{\theta\in\mathbb{T}} \omega'(\theta), \max_{\theta\in\mathbb{T}} \omega'(\theta)] \backslash \{\omega'(\theta) | \, \theta \in \Theta_{\rm cr}\}.$$

Finally, the decay of order $t^{-1/3}$ occurs along the *critical rays*, i.e. the rays with $\omega''(\theta) = 0$ and $\omega'''(\theta) \neq 0$. For such rays we have

$$\frac{n}{t} \in \{\omega'(\theta) | \, \theta \in \Theta_{\rm cr}\}.$$

Thus the the norm $\|\mathbf{u}(t)\|_\infty$ follows the slowest decay which has the order of $t^{-1/3}$ as $t \to \infty$. In particular, this leads to formula (2.16) which imply that this slow decay occurs for the Schrödinger equation with $L = i\Delta$.

Example 2.19. Consider the discrete Schrödinger equation $\dot{\mathbf{u}} = i\Delta\mathbf{u}$ and recall that the associated dispersion relation is $\omega(\theta) = 2(1 - \cos\theta)$. The phase function in the oscillatory integral (2.24) is $\varphi(\theta) = \frac{n}{t}\theta - \omega(\theta)$ where $\omega(\theta) = 2(1 - \cos\theta)$. Since $\Theta_{\rm cr} = \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ the rate of decay is of the order of $t^{-1/3}$ along the critical rays which have slopes $\frac{n}{t} = \omega'(\theta)|_{\theta\in\Theta_{\rm cr}} = \pm 2$. If $\varphi'(\theta) = 0$ and $\varphi''(\theta) \neq 0$, which happens along the rays with $\frac{n}{t} = \omega'(\theta) \in (-2, 2)$, the decay rate is $t^{-1/2}$. Finally, if $\varphi'(\theta) \neq 0$, i.e. $\frac{n}{t} \in \mathbb{R}\backslash[-2, 2]$, the decay is t^{-1}. We summarize these results on Figure 2.2.

The above consideration only gives us the decay in l^∞ norm. In order to study the decay in l^p norm with $p \in [2, \infty]$ let us rewrite solution (2.23) to linear system (2.21) in the convolution form:

$$u_n(t) \equiv \left(e^{tL}\mathbf{u}_0\right)_n = \sum_{k\in\mathbb{Z}} G_k(t) u_{0,n-k},$$

where

$$G_k(t) = \frac{1}{2\pi}\int_{\mathbb{T}} e^{i(k\theta-\omega(\theta)t)}d\theta. \tag{2.25}$$

Thanks to Young's inequality we can obtain the following bound on the solution's l^p norm:

$$\|\mathbf{u}(t)\|_{l^p} \leq \|\mathbf{G}(t)\|_{l^p}\|\mathbf{u}_0\|_{l^1}. \tag{2.26}$$

Hence to study the decay of $\|\mathbf{u}(t)\|_{l^p}$ in time t it is enough to analyze the decay of the oscillatory integral $\mathbf{G}(t)$.

Let us now convert the l^p norm $\|\mathbf{G}(t)\|_{l^p}$ into a L^p norm using the Fundamental Theorem of Calculus in the form $f(x_{n+1}) - f(x_n) = \int_{x_n}^{x_{n+1}} f'(s)ds$. Assuming that $x_{n+1} - x_n = h$ we obtain an identity

$$\begin{aligned}\int_{x_n}^{x_{n+1}} f(x)dx - hf(x_n) &= \int_{x_n}^{x_{n+1}} dx \int_{x_n}^{x} ds\, f'(s)\\ &= \int_{x_n}^{x_{n+1}} (x_{n+1}-s)f'(s)ds,\end{aligned}$$

which immediately gives an estimate

$$\left|\int_{\mathbb{R}} f(x)dx - h\sum_{n\in\mathbb{Z}} f(x_n)\right| \leq h\int_{\mathbb{R}} |f'(s)|ds.$$

Let us now consider the norm $\|\mathbf{G}(t)\|_{l^p}^p$ as a Riemann sum for a one-dimensional integral with the grid points $\{c_n = \frac{n}{t}\}_{n\in\mathbb{Z}}$ and the grid step size $h = c_{n+1} - c_n = \frac{1}{t}$, namely

$$\frac{1}{t}\|\mathbf{G}(t)\|_{l^p}^p = \sum_{n\in\mathbb{Z}} h\,|g(t,c_n)|^p = \int_{\mathbb{R}} |g(t,c)|^p dc + \mathcal{O}\left(\frac{1}{t}\int_{\mathbb{R}}\left|\frac{\partial|g|^p}{\partial c}(t,c)\right|dc\right), \tag{2.27}$$

where

$$g(t,c) = \frac{1}{2\pi}\int_{\mathbb{T}} e^{it\phi(\theta,c)}d\theta, \qquad \phi(\theta,c) = c\theta - \omega(\theta). \tag{2.28}$$

In the proof of Theorem 2.24 below we are going to justify approximation (2.27). An application of the Young inequality (2.26) to the leading-order term in this approximation yields

$$\|\mathbf{u}(t)\|_{l^p} \leq Ct^{1/p}\|g(t,\cdot)\|_{L^p}\|\mathbf{u}_0\|_{l^1}, \tag{2.29}$$

where the function g comes from (2.28). We are going to prove three lemmas on decay of $g(t,c)$ for different values of parameter $c \in \mathbb{R}$. Lemma 2.20 gives the decay of order t^{-1} for $g(t,c)$ provided with c is outside the set $\{\omega'(\theta)|\,\theta\in\mathbb{T}\}$. According to Lemma 2.23, if c is outside the sectors of angular width $t^{-2/3}$ about the points in the discrete set $\{\omega'(\theta)|\,\theta\in\Theta_{\text{cr}}\}$, the function $g(t,c)$ decays like $t^{-1/2}$. For all other values of c, the upper bound on the decay of $g(t,c)$ provided by Lemma 2.22 is $t^{-1/3}$.

It is important to note that although we know the decay rate of $g(t,c)$ along different rays $c \approx \frac{n}{t}$, the formula (2.29) does not tell the specifics of the decay along different rays in the (n,t)-plane. This information has been lost once we passed from the solution $\{u_n(t)\}_{n\in\mathbb{Z}}$ to its norm in (2.26).

The following lemma specifies the range of c that gives the fastest decay of $g(t,c)$:

Lemma 2.20. *Let $\omega \in C^2(\mathbb{T})$ be a dispersion relation in (2.22) and set a_ω and b_ω be such that*

$$a_\omega < \min_{\theta\in\mathbb{T}} \omega'(\theta), \qquad b_\omega > \max_{\theta\in\mathbb{T}} \omega'(\theta). \tag{2.30}$$

There is a constant C_ω, that depends on ω, a_ω, and b_ω such that for all $t \geq 0$ the oscillatory integral (2.28) satisfies the following bound:

$$|g(t,c)| \leq \frac{C_\omega}{(1+t)c^2}, \qquad \forall c \in \mathbb{R}\backslash[a_\omega, b_\omega]. \tag{2.31}$$

Proof. Recall that

$$g(t,c) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it\phi(\theta,c)}d\theta, \qquad \phi(\theta,c) = c\theta - \omega(\theta).$$

Because ϕ' is monotonic on $\mathbb{T}$ if $c \in \mathbb{R}\backslash[a_\omega, b_\omega]$ the Van der Corput Lemma 2.18 immediately gives the decay rate of t^{-1}. However, this lemma does not give the appropriate scaling of the bound in parameter c. We have to use an explicit integration to prove the assertion of this Lemma.

Integrating by parts and using 2π-periodicity of $\exp(it\phi(\theta,c))/\partial_\theta\phi(\theta,c)$ we obtain

$$\begin{aligned} g(t,c) &= \frac{1}{2\pi it\partial_\theta\phi(\theta,c)} e^{it\phi(\theta,c)}\Big|_{-\pi}^{\pi} + \frac{1}{2\pi it}\int_{-\pi}^{\pi} \frac{\partial_\theta^2\phi(\theta,c)}{(\partial_\theta\phi(\theta,c))^2} e^{it\phi(\theta,c)}d\theta \\ &= \frac{i}{2\pi tc^2}\int_{-\pi}^{\pi} \frac{\omega''(\theta)}{(1-\omega'(\theta)/c)^2} e^{it(c\theta-\omega(\theta))}d\theta, \end{aligned}$$

where the last integral is bounded for all $c \in \mathbb{R}\backslash[a_\omega, b_\omega]$. Recalling that $g(0,c)$ is bounded, we obtain the bound (2.31). $\square$

Remark 2.21. By periodicity of ω we have $\min_{\theta\in\mathbb{T}} \omega'(\theta) < 0$ and $\max_{\theta\in\mathbb{T}} \omega'(\theta) > 0$, so that c does not attain the zero value outside $[a_\omega, b_\omega]$ and the right hand side in (2.31) is bounded.

The next lemma provides a uniform bound on the decay of $g(t,\cdot)$.

Lemma 2.22. *Consider the oscillatory integral (2.28) with ω satisfying Assumption 2.16. For all $t \geq 0$ and $c \in \mathbb{R}$ there exists a constant $C_\omega > 0$ such that the integral*

(2.28) satisfies the following decay estimate:

$$|g(t,c)| \leq \frac{C_\omega}{(1+t)^{1/3}}. \tag{2.32}$$

Proof. Let δ be a small positive number. We introduce

$$U_\delta = \{\theta \in \mathbb{T} : \ |\theta - \theta_0| < \delta, \ \theta_0 \in \Theta_{cr}\},$$

a small neighbourhood about critical points of the dispersion relation. Since $\omega \in C^3(\mathbb{T})$, there exist positive constants λ_1 and λ_2 such that $|\omega''(\theta)||_{\theta \in \mathbb{T}\setminus U_\delta} \geq \lambda_1$ and $|\omega'''(\theta)||_{\theta \in U_\delta} \geq \lambda_2$. Therefore, we can estimate oscillatory integral (2.28) using the Van der Corput lemma as follows:

$$\begin{aligned} |g(t,c)| &\leq \left| \int_{T\setminus U_\delta} e^{it\phi(\theta,c)} d\theta \right| + \left| \int_{U_\delta} e^{it\phi(\theta,c)} d\theta \right| \\ &\leq C_1(\lambda_1 t)^{-1/2} + C_2(\lambda_2 t)^{-1/3}, \end{aligned} \tag{2.33}$$

where C_1 and C_2 are constants given in (2.24). The bound (2.32) holds because $g(0,c)$ is bounded and $g(t,c)$ decays as $t \to \infty$ according to (2.33). □

Although in Lemma 2.22 we have established a bound that is independent of parameter c, we can take advantage of the fact that the decay rate is better than that in Lemma 2.22 if c is outside a small neighbourhood of the discrete set $\{\omega'(\theta) | \, \theta \in \Theta_{\rm cr}\}$.

Lemma 2.23. *Consider the oscillatory integral (2.28) with ω satisfying Assumption 2.16. For all $c \in \{x : \ |\omega'(\theta) - x| \geq t^{-2/3}, \ \theta \in \Theta_{\rm cr}\}$ there exist a constant $C_\omega > 0$ such that for all $t \geq 0$ the integral in (2.28) satisfies*

$$|g(t,c)| \leq \frac{C_\omega}{(1+t)^{1/2}} \left(1 + \sum_{\theta \in \Theta_{\rm cr}} \frac{1}{|\omega'(\theta) - c|^{1/4}} \right).$$

Proof. Let us first show the main idea of the proof for the case of $\Theta_{\rm cr} = \{0\}$. For simplicity of notations we set $c_0 = \omega'(0)$. Consider

$$I(t,c) = \int_0^\pi e^{it\phi(\theta,c)} d\theta, \qquad \phi(\theta,c) = c\theta - \omega(\theta). \tag{2.34}$$

To get a sharp estimate on this integral we are going to split it in several pieces and then estimate each piece separately using the Van der Corput lemma. The choice of the splitting may depend on both c and t.

Using Taylor series expansions for $\omega'(\theta)$ and $\omega''(\theta)$ we find that there is $\delta \in (0,1)$ such that

$$\forall \theta \in [0,\delta]: \qquad |\omega'(\theta) - c_0| \le \overline{A}\theta^2, \qquad |\omega''(\theta)| \ge \underline{A}\theta.$$

Since $\partial_c^2 \phi(\theta, c) = -\omega''(\theta)$ and δ is fixed by the dispersion relation, we get

$$\begin{aligned} \forall \theta \in [\delta,\pi]: &\qquad |\partial_\theta^2 \phi(\theta,c)| \ge B, \\ \forall \theta \in [\tilde{\delta},\delta]: &\qquad |\partial_\theta^2 \phi(\theta,c)| \ge \underline{A}\tilde{\delta}, \end{aligned}$$

To obtain a similar estimate on the phase function $\phi(\theta, c)$ with $\theta \in \left[0, \tilde{\delta}\right]$ we assume

$$\tilde{\delta}^2 \le \frac{|c_0 - c|}{\overline{A}+1}. \tag{2.35}$$

This allows us to find the following bound:

$$\begin{aligned} |\partial_\theta \phi(\theta,c)| &= |c_0 - c + \partial_\theta \phi(\theta, c_0)| \\ &\ge |c_0 - c| - |\partial_\theta \phi(\theta, c_0)| \\ &\ge |c_0 - c| - \overline{A}\tilde{\delta}^2 \ge \tilde{\delta}^2. \end{aligned}$$

The way we estimate the integral $I(t,c)$ in (2.34) depends on how big t is:

(i) If $\delta^2 \le \left(\overline{A}+1\right)^{-1} |c_0 - c|$, which can happen when t is small enough, we choose $\tilde{\delta} = \delta$ and find that

$$|I(t,c)| = \left| \int_0^\delta e^{it\phi(t,c)} d\theta + \int_\delta^\pi e^{it\phi(t,c)} d\theta \right| \le \frac{3}{\delta^2 t} + \frac{8}{(Bt)^{1/2}}.$$

(ii) If $\delta^2 > \left(\overline{A}+1\right)^{-1} |c_0 - c|$, we fix $\tilde{\delta} < \delta$ by setting the equality in (2.35):

$$\tilde{\delta}^2 = \frac{|c_0 - c|}{\overline{A}+1}.$$

In this case, we have $\tilde{\delta} t^{1/3} \ge \left(\overline{A}+1\right)^{1/2}$ due to inequality $|c_0 - c| \ge t^{-3/2}$. We then obtain

$$\begin{aligned}
|I(t,c)| &= \left| \int_0^{\tilde{\delta}} e^{it\phi(t,c)} d\theta + \int_{\tilde{\delta}}^{\delta} e^{it\phi(t,c)} d\theta + \int_{\delta}^{\pi} e^{it\phi(t,c)} d\theta \right| \\
&\le \frac{3}{\tilde{\delta}^2 t} + \frac{8}{(\underline{A}\tilde{\delta}t)^{1/2}} + \frac{8}{(Bt)^{1/2}} \\
&= \frac{1}{\tilde{\delta}^{1/2} t^{1/2}} \left(\frac{3}{\tilde{\delta}^{3/2} t^{1/2}} + \frac{8}{\underline{A}^{1/2}} \right) + \frac{8}{(Bt)^{1/2}} \\
&\le \frac{\left(\overline{A}+1\right)^{1/4}}{|c_0 - c|^{1/4} t^{1/2}} \left(3\left(\overline{A}+1\right)^{3/4} + \frac{8}{\underline{A}^{1/2}} \right) + \frac{8}{(Bt)^{1/2}}.
\end{aligned}$$

Putting the arguments in *(i)* and *(ii)* together we conclude that

$$|I(t,c)| \le \frac{C_\omega}{(1+t)^{1/2}} \left(1 + \frac{1}{|c_0 - c|^{1/4}} \right).$$

If the dispersion relation ω has several critical points, we need to consider a neighbourhood of each critical point separately. For any $\theta^* \in \Theta_{\rm cr}$ the integral of $e^{it\phi(\theta,c)}$ in θ over $[\theta^* - \delta^*, \theta^* + \delta^*]$ decays like $|\omega'(\theta^*) - c|^{-1/4} t^{-1/2}$. The integrals over $[0,\pi] \setminus \cup_{\theta^* \in \Theta_{\rm cr}} [\theta^* - \delta^*, \theta^* + \delta^*]$ decay like $t^{-1/2}$. $\square$

Now, we can use Lemmas 2.20–2.23 to prove a general result on dispersive decay of solutions to linear lattice system (2.21).

Theorem 2.24. *Suppose L is a linear operator and ω is its dispersion relation satisfying Assumption 2.16. For $p \in [2,4) \cup (4,\infty]$ there exists a constant $C_{\omega,p} > 0$ such that for all $t \ge 0$ we have*

$$\left\| e^{Lt} \right\|_{l^1 \to l^p} \le \frac{C_{\omega,p}}{(1+t)^{\alpha_p}}, \quad \text{where} \quad \alpha_p = \begin{cases} \dfrac{p-2}{2p}, & \text{for } p \in [2,4), \\ \dfrac{p-1}{3p}, & \text{for } p \in (4,\infty]. \end{cases}$$

Proof. To prove this theorem, we are going to examine the bound

$$\left\| e^{Lt} \right\|_{l^1 \to l^p} \le \left[t \|g(t,\cdot)\|_{L^p}^p + \mathcal{O}\left(\int_{\mathbb{R}} \left| \frac{\partial |g|^p}{\partial c}(t,c) \right| dc \right) \right]^{1/p} \tag{2.36}$$

which follows from formulae (2.26) and (2.27). As in Lemma 2.20, let us fix the constants a_ω and b_ω such that

$$a_\omega < \min_{\theta \in \mathbb{T}} \omega'(\theta), \qquad b_\omega > \max_{\theta \in \mathbb{T}} \omega'(\theta).$$

For simplicity of presentation let us assume that the dispersion relation ω has only one critical point θ_0. Let us also set $c_0 = \omega'(\theta_0)$. To estimate $\|g(t,\cdot)\|_{L^p(a_\omega,b_\omega)}$ we are going to use the decay estimate of Lemma 2.22 for $c \in [c_0 - t^{-2/3}, c_0 + t^{-2/3}]$ and that of Lemma 2.23 for $c \in U := [a_\omega, c_0 - t^{-2/3}] \cup [c_0 + t^{-2/3}, b_\omega]$. For $p \neq 4$, we obtain the estimate

$$\begin{aligned}\int_{[a_\omega,b_\omega]} |g(t,c)|^p dc &\leq \int_{c_0-t^{-2/3}}^{c_0+t^{-2/3}} \frac{C_1}{(1+t)^{p/3}} dc \\ &\quad + \int_U \frac{C_2}{(1+t)^{p/2}} \left(1 + \frac{1}{|c_0-c|^{1/4}}\right)^p dc \\ &\leq \frac{2t^{-2/3}C_1}{(1+t)^{p/3}} + \frac{\widetilde{C_2}}{(1+t)^{p/2}} \left(1 + \int_{a_\omega}^{c_0-t^{-2/3}} \frac{1}{|c_0-c|^{p/4}} dc\right) \\ &\leq \frac{\widetilde{C_1}}{(1+t)^{(p+2)/3}} + \frac{\widetilde{\widetilde{C_2}}}{(1+t)^{p/2}} \left(1 + t^{(p-4)/6}\right) \\ &\leq C\left(\frac{1}{(1+t)^{(p+2)/3}} + \frac{1}{(1+t)^{p/2}}\right),\end{aligned} \tag{2.37}$$

where the constant C depends on the dispersion relation ω and p. We notice that if $p \in (2,4)$ the bound (2.37) decays like $t^{-p/2}$, while if $p \in (4,\infty)$ it decays like $t^{-(p+2)/3}$. Using Lemma 2.20, we find that

$$\int_{\mathbb{R}\backslash[a_\omega,b_\omega]} |g(t,c)|^p dc \leq C \int_{\mathbb{R}\backslash[a_\omega,b_\omega]} \frac{dc}{t^p c^{2p}} \leq \frac{\widetilde{C}}{t^p}, \tag{2.38}$$

which gives next-to-leading-order correction to (2.37).

Now let us show that the error term in (2.36) is smaller than the leading-order term in the limit of $t \to \infty$. Thanks to Lemma 2.20, for all $c \in \mathbb{R}\backslash[a_\omega,b_\omega]$ and $t \geq 1$ we have $|g(t,c)| \leq \frac{C}{tc^2}$. Using integration by parts one can also show that $|\partial_c g(t,c)| \leq \frac{C}{tc^3}$ for all $c \in \mathbb{R}\backslash[a_\omega,b_\omega]$ and $t \geq 1$. This estimate and the uniform bound (2.32) imply

$$\int_{\mathbb{R}} \left|\frac{\partial |g|^p}{\partial c}(t,c)\right| dc \leq \int_{[a_\omega,b_\omega]} \frac{C_1}{t^{p/3}} dc + \int_{\mathbb{R}\backslash[a_\omega,b_\omega]} \frac{C_2}{t^p c^{2p+1}} dc = \mathcal{O}\left(t^{-\frac{p}{3}}\right). \tag{2.39}$$

Therefore, for sufficiently large values of t under estimates (2.37)–(2.39) the bound (2.36) simplifies to

$$\|e^{Lt}\|_{l^1\to l^p} \leq Ct^{1/p}\|g(t,\cdot)\|_{L^p(a_\omega,b_\omega)} \leq \widetilde{C}\left((1+t)^{-\frac{p-1}{3p}} + (1+t)^{-\frac{p-2}{2p}}\right).$$

If the dispersion relation has more than one critical point, the integration has to be split into the union of balls with radius $t^{-2/3}$ centred at the critical points, and its complement in $[a_\omega,b_\omega]$. Similar to (2.37), the bounds on the two resulting integrals can be established using Lemmas 2.22 and 2.23. □

Remark 2.25. If $p = 4$, the integral on the third line in (2.37) results in a logarithmic term so that

$$\left\|e^{Lt}\right\|_{l^1\to l^p} \le C_\omega \left(\frac{\ln(2+t)}{1+t}\right)^{1/4}.$$

2.3.2 Nonlinear decay

In this section, we prove Theorem 2.14 on scattering of small solutions in the dNLS equation with sufficiently high nonlinearity. We first prove a theorem that guarantees that the rate of scattering in the nonlinear system

$$\begin{cases} \dot{\mathbf{u}}(t) = L\mathbf{u} + \mathbf{N}(\mathbf{u}), \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \qquad \mathbf{u}(t) = \{u_n(t)\}_{n\in\mathbb{Z}} : \mathbb{R}_+ \to \mathbb{C}^{\mathbb{Z}}, \tag{2.40}$$

is the same as that in the underlying linear problem (2.21). We then apply the theorem to find decay rates of small solutions in the dNLS equation.

Let us first prove the following lemma:

Lemma 2.26. *Suppose* $\alpha_1, \alpha_2 \in [0,1)\cup(1,\infty)$*, then there exists a constant* $C > 0$ *such that*

$$\frac{t}{C(1+t)^{\gamma+1}} \le \int_0^t \frac{1}{(1+t-s)^{\alpha_1}(1+s)^{\alpha_2}} ds \le \frac{C}{(1+t)^\gamma} \quad \text{for all } t > 0,$$

where $\gamma = \min\{\alpha_1, \alpha_2, \alpha_1+\alpha_2-1\}$

Proof. On the interval $s \in [0, t/2]$ we have $(1+t)/2 \le 1+t-s \le 1+t$, so that

$$\frac{1}{(1+t)^{\alpha_1}} M_1(t) \le \int_0^{t/2} \frac{1}{(1+t-s)^{\alpha_1}(1+s)^{\alpha_2}} ds \le \frac{2^{\alpha_1}}{(1+t)^{\alpha_1}} M_1(t), \tag{2.41}$$

where

$$\frac{t}{C(1+t)}(1+t)^{-\min(0,\alpha_2-1)} \le M_1(t) = \int_0^{t/2} \frac{1}{(1+s)^{\alpha_2}} ds \le Ct^{-\min(0,\alpha_2-1)}.$$

Thus, the integral in (2.41) decays like $t^{-\alpha}$, where $\alpha = \min\{\alpha_1, \alpha_1+\alpha_2-1\}$.

Similarly, on the interval $s \in [t/2, t]$ we observe that $(1+t)/2 \le 1+s \le 1+t$ and

$$\frac{1}{(1+t)^{\alpha_2}} M_2(t) \le \int_{t/2}^{t} \frac{1}{(1+t-s)^{\alpha_1}(1+s)^{\alpha_2}} ds \le \frac{2^{\alpha_2}}{(1+t)^{\alpha_2}} M_2(t), \tag{2.42}$$

where

$$\frac{t}{C(1+t)}(1+t)^{-\min(0,\alpha_1-1)} \leq M_2(t) = \int_{t/2}^{t} \frac{1}{(1+t-s)^{\alpha_1}} ds \leq Ct^{-\min(0,\alpha_1-1)}.$$

Therefore, the decay of the upper and lower bounds in (2.42) is of the order $t^{-\alpha}$, where $\alpha = \min\{\alpha_2, \alpha_1 + \alpha_2 - 1\}$.

Combining the above results we obtain the assertion of the Lemma. □

The next theorem provides restrictions on operators L and $\mathbf{N}$ in (2.40) which guarantee decay of small solutions in nonlinear system (2.40).

Theorem 2.27. *Let U_0, V, and X be nested Banach spaces such that $U_0 \subset V \subset X$. Suppose that the operators L and $\mathbf{N}$ in (2.40) satisfy*

$$\left\|e^{tL}\mathbf{u}\right\|_X \leq \frac{C_L}{(1+t)^{\alpha}}\|\mathbf{u}\|_{U_0},$$
$$\|\mathbf{N}(\mathbf{u})\|_{U_0} \leq C_N\|\mathbf{u}\|_V^{\beta_1}\|\mathbf{u}\|_X^{\beta_2}, \qquad \beta_1+\beta_2=\beta, \qquad \beta_1,\beta_2 \geq 0.$$

Assume further that there exist positive δ and ν such that for all $\mathbf{u}_0$ with $\|\mathbf{u}_0\|_{U_0} \leq \epsilon$ the unique solution to (2.40) satisfies the estimate

$$\|\mathbf{u}(t)\|_V \leq \frac{C_V}{(1+t)^{\nu}}\|\mathbf{u}_0\|_{U_0}, \text{ for all } t \geq 0.$$

Let $\min\{\beta_1\nu+\beta_2\alpha, \beta_1\nu+\beta_2\alpha+\alpha-1\} \geq \alpha$ and $\alpha \neq 1 \neq \beta_1\nu+\beta_2\alpha$. Then for $\mathbf{u}_0$ with $\|\mathbf{u}_0\|_{U_0} \leq \epsilon$ the solutions to (2.40) satisfy

$$\|\mathbf{u}(t)\|_X \leq \frac{C_X}{(1+t)^{\alpha}}\|\mathbf{u}_0\|_{U_0}, \text{ for all } t \geq 0.$$

Proof. Using the variation-of-constants formula

$$\mathbf{u}(t) = e^{tL}\mathbf{u}_0 + \int_0^t e^{(t-s)L}\mathbf{N}(\mathbf{u}(s))ds,$$

and the assumptions of the theorem we obtain the following estimate:

$$\begin{aligned}
\|\mathbf{u}(t)\|_X &\leq \left\|e^{tL}\mathbf{u}_0\right\|_X + \int_0^t \left\|e^{(t-s)L}\mathbf{N}(\mathbf{u}(s))\right\|_X ds \\
&\leq \frac{C_L}{(1+t)^\alpha}\|\mathbf{u}_0\|_{U_0} + C_L\int_0^t \frac{\|\mathbf{N}(\mathbf{u}(s))\|_{U_0}}{(1+t-s)^\alpha} ds \\
&\leq \frac{C_L}{(1+t)^\alpha}\|\mathbf{u}_0\|_{U_0} + C_L\int_0^t \frac{\|\mathbf{u}(s)\|_V^{\beta_1}\|\mathbf{u}(s)\|_X^{\beta_2}}{(1+t-s)^\alpha} ds \\
&\leq \frac{C_L}{(1+t)^\alpha}\|\mathbf{u}_0\|_{U_0} + C_L C_V\|\mathbf{u}_0\|_{U_0}^{\beta_1}\int_0^t \frac{\|\mathbf{u}(s)\|_X^{\beta_2}}{(1+t-s)^\alpha(1+s)^{\beta_1\nu}} ds.
\end{aligned}$$

Let $R(t) = \max_{0\leq s\leq t}(1+s)^\alpha\|\mathbf{u}(s)\|_X$, $\zeta = \|\mathbf{u}_0\|_{U_0}$ and $\mu = \beta_1\nu+\beta_2\alpha$. Then by Lemma 2.26 we obtain the following bound:

$$\begin{aligned}
(1+t)^\alpha\|\mathbf{u}(t)\|_X &= C_L\zeta + C_L C_V(1+t)^\alpha\zeta^{\beta_1}\int_0^t \frac{((1+s)^\alpha\|\mathbf{u}(s)\|_X)^{\beta_2}}{(1+t-s)^\alpha(1+s)^{\beta_1\nu+\beta_2\alpha}} ds \\
&\leq C_L\zeta + C_L C_V(1+t)^\alpha\zeta^{\beta_1}R(t)^{\beta_2}\int_0^t \frac{1}{(1+t-s)^\alpha(1+s)^\mu} ds \\
&\leq C_L\zeta + C(1+t)^{\alpha-\rho}\zeta^{\beta_1}R(t)^{\beta_2},
\end{aligned}$$

where $\rho = \min\{\alpha, \mu, \alpha+\mu-1\}$. Since $\rho \geq \alpha$, we find that $R(t) \leq C_L\zeta + C\zeta^{\beta_1}R(t)^{\beta_2}$ for all $t \geq 0$. If ζ is small enough we can bound $|R(t)|$ for all $t \geq 0$ as follows:

$$\begin{aligned}
|R(t)| &\leq \left|R(t) - C\zeta^{\beta_1}R(t)^{\beta_2}\right| + \left|C\zeta^{\beta_1}R(t)^{\beta_2}\right| \\
&\leq C_L\zeta + \left|C\zeta^{\beta_1}R(t)^{\beta_2}\right| \leq 2C_L\zeta.
\end{aligned}$$

This concludes the proof of the theorem. □

Corollary 2.28. *Let U_0 and X be nested Banach spaces such that $U_0 \subset X$. Suppose the operators L and $\mathbf{N}$ satisfy*

$$\left\|e^{Lt}\mathbf{u}\right\|_X \leq \frac{C_L}{(1+t)^\alpha}\|\mathbf{u}\|_{U_0},$$
$$\|\mathbf{N}(\mathbf{u})\|_{U_0} \leq C_N\|\mathbf{u}\|_X^\beta,$$

where $\min\{\alpha\beta, \alpha\beta+\alpha-1\} \geq \alpha$ and $\alpha \neq 1 \neq \alpha\beta$. Then there is $\epsilon > 0$ such that the unique solution to (2.40) with $\|\mathbf{u}_0\|_{U_0} \leq \epsilon$ satisfies

$$\|\mathbf{u}(t)\|_X \leq \frac{C}{(1+t)^\alpha}\|\mathbf{u}_0\|_{U_0}, \textit{ for all } t \geq 0.$$

Proof. The proof is the same as that for Theorem 2.27 with $\beta_1 = 0$ and $\beta_2 = \beta$. □

Using Theorem 2.27 and Corollary 2.28 we can now develop the proof of Theorem 2.14, which establishes scattering of small solutions to the dNLS equation (2.15).

Proof of Theorem 2.14. Let $1 < s < p$ which implies that $U_0 = l^1 \subset V = l^s \subset X = l^p$. We notice that

$$\|\mathbf{N}(\mathbf{u})\|_{l^1} = \left\| \{|u_n|^{\beta-1} u_n\} \right\|_{l^1} = \|\mathbf{u}\|_{l^\beta}^\beta \le C \|\mathbf{u}\|_{l^p}^\beta$$

provided $p \le \beta$. Since the constant of linear decay α_p given in (2.19) belongs to $[0, \frac{1}{3}]$, Corollary 2.28 is applicable whenever $\alpha_p \beta > 1$. Using the explicit formula for α_p again, we conclude that the linear and nonlinear decay rates are the same provided $p \in \left(\frac{2\beta}{\beta-2}, 4\right) \cup (4, \beta)$. It remains to extend this result for $p \in \left[2, \frac{2\beta}{\beta-2}\right]$ and for $p \in [\beta, \infty]$.

Let us next consider the endpoints, $p = 2$ and $p = \infty$. For $p = 2$, the decay formula (2.20) is valid due to the conservation law $\|\mathbf{u}(t)\|_{l^2} = \|\mathbf{u}_0\|_{l^2}$. For the case $p = \infty$, we employ Theorem 2.27 with $V = l^s$ and $4 < s < \beta$ so that

$$\|\mathbf{N}(\mathbf{u})\|_{l^1} = \|\mathbf{u}\|_{l^\beta}^\beta = \sum_{n \in \mathbb{Z}} |u_n|^s |u_n|^{\beta - s} \le \|\mathbf{u}\|_{l^s}^s \|\mathbf{u}\|_{l^\infty}^{\beta - s}.$$

Using $\alpha_\infty = \frac{1}{3}$ and conditions in Theorem 2.27 we obtain the constraint

$$s\nu + \frac{\beta - s}{3} > 1.$$

As we have shown above $\|\mathbf{u}(t)\|_{l^s} \le C(1+t)^{-\alpha_s} \|\mathbf{u}_0\|_{l^1}$ with $\alpha_s = \frac{s-1}{3s}$ and $s \in (4, \beta)$. Then, we set $V = l^s$ and $\nu = \alpha_s$ for $s \in (4, \beta)$. As a result, the condition

$$s\alpha_s + \frac{\beta - s}{3} = \frac{\beta - 1}{3} > 1$$

is satisfied for all $\beta > 4$ and hence (2.20) is true for $p = \infty$ as well.

To show that the linear and nonlinear decay rates are also the same for $p \in \left(2, \frac{2\beta}{\beta-2}\right] \cup [\beta, \infty)$ we use the interpolation inequality

$$\|\mathbf{u}\|_{l^p} \le \|\mathbf{u}\|_{l^q}^{1-\theta} \|\mathbf{u}\|_{l^r}^{\theta},$$

where $\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{r}$ and $\theta \in (0,1)$. For the left subinterval, $p \in \left(2, \frac{2\beta}{\beta-2}\right]$, we interpolate between $q = 2$ and $r \in \left(\frac{2\beta}{\beta-2}, 4\right)$ using $\theta = \left(\frac{1}{2} - \frac{1}{p}\right) / \left(\frac{1}{2} - \frac{1}{r}\right)$. Since the restriction $0 < \theta < 1$ yields $p < r$, we can cover all the interval $p \in [2, 4)$. Similarly, for $p \in [\beta, \infty)$, we interpolate between $q = \infty$ and $r \in (4, \beta)$ using $\theta = \frac{r}{p} \in (0,1)$ and thus cover all the interval $p \in (4, \infty]$. □

Chapter 3

Existence of discrete breathers near the anti-continuum limit

In this chapter we study existence of discrete breathers in the dNLS and dKG equations near the anti-continuum limit using the method of MacKay & Aubry [56] which is based on the implicit function arguments.

3.1 Existence of discrete breathers in the dNLS equation

Consider the dNLS equation in the form

$$i\dot{u}_n + \epsilon(\Delta \mathbf{u})_n + |u_n|^{2p} u_n = 0, \quad n \in \mathbb{Z}, \tag{3.1}$$

where $u_n(t) : \mathbb{R} \to \mathbb{C}$ is the set of amplitude functions, and parameters $\epsilon \in \mathbb{R}$ and $p \in \mathbb{N}$ define the coupling constant and the power of nonlinearity. The anti-continuum limit corresponds to $\epsilon = 0$, in which case the dNLS equation (3.1) becomes an infinite system of uncoupled differential equations.

Let us consider time-periodic solutions to (3.1) in the form

$$u_n(t) = \phi_n e^{i\omega t}, \tag{3.2}$$

where ω is the frequency. In the context of the dNLS equation, localized solutions in the form (3.2) are often called *discrete solitons*. Let us note, however, that the dNLS equation is not integrable and it does not admit moving solitary waves which interact elastically.

Thanks to homogeneity of the nonlinear term, we normalize the solution frequency and set $\omega = 1$. The time-independent solution profile $\boldsymbol{\phi}$ satisfies

$$(1 - |\phi_n|^{2p})\phi_n = \epsilon(\Delta\boldsymbol{\phi})_n, \quad n \in \mathbb{Z}. \tag{3.3}$$

The following lemma by Panayotaros & Pelinovsky [68] shows that it would suffice to establish existence of a localized stationary solution $\boldsymbol{\phi}$ in the space of real sequences.

Lemma 3.1. *The solution $\boldsymbol{\phi}$ to (3.3) satisfying the asymptotic decay condition $|\phi_n| \to 0$ as $|n| \to \infty$ is real-valued modulo a factor of $e^{i\theta}$ with $\theta \in [0, 2\pi)$.*

Proof. Multiplying (3.3) by $\bar{\phi}_n$ we obtain

$$\begin{aligned}(1 + 2\epsilon - |\phi_n|^{2p})|\phi_n|^2 &= \epsilon(\phi_{n-1} + \phi_{n+1})\bar{\phi}_n, \\ (1 + 2\epsilon - |\phi_n|^{2p})|\phi_n|^2 &= \epsilon(\bar{\phi}_{n-1} + \bar{\phi}_{n+1})\phi_n,\end{aligned}$$

where the second equation is just a conjugate of the first one. Equating the left sides of these equations yields

$$\bar{\phi}_n\phi_{n+1} - \phi_n\bar{\phi}_{n+1} = \text{const}, \quad n \in \mathbb{Z}, \quad \epsilon \neq 0.$$

Due to the decay requirement $|\phi_n| \to 0$ as $|n| \to \infty$ the constant on the right is zero which implies that $\bar{\phi}_n\phi_{n+1} = \phi_n\bar{\phi}_{n+1}$ for all $n \in \mathbb{Z}$. If $\phi_n\phi_{n+1} \neq 0$, we see that $\arg\phi_n = \arg\phi_{n+1} \mathrm{mod}\pi$. If, however, there is $n \in \mathbb{Z}$ such that $\phi_n = 0$ then $\phi_{n-1} = -\phi_{n+1}$ and we can draw a conclusion that $\arg\phi_{n-1} = \arg\phi_{n+1}\mathrm{mod}\pi$. Since the phase of $\boldsymbol{\phi}$ at different lattice sites vary only by π, we can always make $\boldsymbol{\phi}$ real by the phase transformation $\boldsymbol{\phi} \mapsto e^{i\theta}\boldsymbol{\phi}$ where $\theta \in [0, 2\pi)$. □

Let us now consider existence of l^2 solutions to the stationary dNLS equation (3.3). Thanks to Lemma 3.1, these solutions are real-valued modulo the phase transformation. Hence, it is enough to consider

$$(1 - \phi_n^{2p})\phi_n = \epsilon(\Delta\boldsymbol{\phi})_n, \quad n \in \mathbb{Z}, \quad p \in \mathbb{N} \tag{3.4}$$

for real $\boldsymbol{\phi} \in l^2$.

Definition 3.2. At $\epsilon = 0$ the *limiting configuration* of the stationary solution to (3.4) is given by the compact solution

$$\epsilon = 0: \quad \boldsymbol{\phi}^{(0)} = \sum_{n \in S_+} \boldsymbol{e}_n - \sum_{n \in S_-} \boldsymbol{e}_n, \tag{3.5}$$

where $S_\pm$ are compact disjoint subsets of $\mathbb{Z}$ and $\boldsymbol{e}_n$ is the standard unit vector in l^2

expressed via the Kronecker symbol by

$$(\boldsymbol{e}_n)_m = \delta_{n,m}, \quad m \in \mathbb{Z}. \tag{3.6}$$

The next proposition gives a unique analytic continuation of the compact limiting solution (3.5) to a particular family of discrete solitons. The idea of the proof comes from [68] and [56].

Proposition 3.3. *Fix disjoint compact subsets S_+ and S_- on $\mathbb{Z}$. There exists $\epsilon_0 > 0$ such that the stationary dNLS equation (3.4) with $\epsilon \in (-\epsilon_0, \epsilon_0)$ admits a unique solution $\boldsymbol{\phi} \in l^2$ near the limiting configuration $\boldsymbol{\phi}^{(0)}$ given by (3.5). Moreover, the map $(-\epsilon_0, \epsilon_0) \ni \epsilon \mapsto \boldsymbol{\phi} \in l^2$ is analytic and*

$$\exists C > 0: \quad \|\boldsymbol{\phi} - \boldsymbol{\phi}^{(0)}\|_{l^2} \le C|\epsilon|. \tag{3.7}$$

Proof. Consider the vector field $\mathbf{F}$ induced by the stationary equation (3.4):

$$F_n(\boldsymbol{\phi}, \epsilon) = (1 - \phi_n^{2p})\phi_n - \epsilon(\Delta\boldsymbol{\phi})_n.$$

Since l^2 is a Banach algebra and the operator Δ is bounded in l^2, the map $\mathbf{F} : l^2 \times \mathbb{R} \mapsto l^2$ is also bounded. To prove the proposition, it is enough to show that the Implicit Function Theorem can be applied to uniquely solve for $\boldsymbol{\phi}$ near the anti-continuum limit.

We make the following observations:

i. The point $(\boldsymbol{\phi}^{(0)}, 0) \in l^2 \times \mathbb{R}$ is the zero of the operator $\mathbf{F}$

$$\mathbf{F}(\boldsymbol{\phi}^{(0)}, 0) = 0.$$

ii. The map $\mathbf{F}$ is analytic in ϵ and $\boldsymbol{\phi}$ ($p \in \mathbb{N}$).

iii. The linearization operator L_+ of $\mathbf{F}$ given by

$$(L_+\mathbf{u})_n = \left(1 - (2p+1)\phi_n^{2p}\right) u_n - \epsilon(\Delta\mathbf{u})_n$$

is a bijective map from a small open neighbourhood of $(\boldsymbol{\phi}^{(0)}, 0)$ in $l^2 \times \mathbb{R}$ to l^2 since $\sigma\left(L_+|_{(\boldsymbol{\phi}^{(0)},0)}\right) = \{-2p, 1\}$.

Thus, by Implicit Function Theorem there exists $\epsilon_0 > 0$ such that the map $(-\epsilon_0, \epsilon_0) \ni \epsilon \mapsto \boldsymbol{\phi} \in l^2$ is analytic. As a result, the bound in (3.7) holds for all $\epsilon \in (-\epsilon_0, \epsilon_0)$. □

Remark 3.4. The proof can be adapted to discrete solitons with non-compact limiting configuration (3.5). To apply the implicit function theorem one needs to look for the solution in the form $\boldsymbol{\phi} = \boldsymbol{\phi}^{(0)} + \boldsymbol{\varphi}$, where $\boldsymbol{\varphi} \in l^2$ [70].

On a finite lattice, continuation of the discrete soliton $\boldsymbol{\phi}$ from the anti-continuum limit can be done using the Newton's method (see e.g. papers of Panayotaros, [66, 67]). Let us note that an alternative method based on variational techniques was recently justified by Chong, Pelinovsky & Schneider [19].

While Proposition 3.3 shows that the stationary solution $\boldsymbol{\phi}$ stays close to the initial configuration $\boldsymbol{\phi}^{(0)}$, we can be even more specific on the decay rate of $\boldsymbol{\phi}$ at infinity. The following proposition implies exponential decay of the stationary solution.

Proposition 3.5. *Fix disjoint compact subsets S_+ and S_- on $\mathbb{Z}$. Let $\epsilon > 0$ be small enough to guarantee existence and uniqueness of the l^2 solution to (3.4) in Proposition 3.3. Let m_- and m_+ be the smallest and the largest numbers in the set $\{S_+ \cup S_-\}$ respectively. Then there are positive constants $A_\pm$ and A_0 such that*

$$|\phi_n| \leq \begin{cases} A_- \epsilon^{|n-m_-|}, & n < m_- \\ A_0, & m_- \leq n \leq m_+ \\ A_+ \epsilon^{|n-m_+|}, & n > m_+ \end{cases} . \tag{3.8}$$

Proof. Let us first give a proof for the case of "fundamental" solution $\boldsymbol{\phi}$ defined by the limiting configuration supported on one site, $\phi_n^{(0)} = \delta_{n,0}$. Thanks to analyticity in ϵ, the solution can be expanded into the series

$$\phi_n = \sum_{k=0}^{\infty} \epsilon^k \phi_n^{(k)}.$$

Since only the adjacent sites interact, we have excitation of orders ϵ on the sites with $n = \pm 1$, ϵ^2 on the sites with $n = \pm 2$ and so on. For $n \geq 1$ we have $\phi_n^{(0)} = \phi_n^{(1)} = \cdots = \phi_n^{(n-1)} = 0$ and $\phi_n^{(n)} = \phi_{n-1}^{(n-1)} = \cdots = \phi_0^{(0)} = 1$. Therefore $\phi_n = \epsilon^n + \mathcal{O}(\epsilon^{n+1})$ where $n \geq 0$. Due to the symmetry of the soliton about $n = 0$ we write $\phi_n = \epsilon^{|n|} + \mathcal{O}(\epsilon^{|n|+1})$ for any $n \in \mathbb{Z}$.

For the solution extended from an arbitrary limiting configuration $\boldsymbol{\phi}^{(0)}$, the fluxes ϕ_n at $n < m_-$ and $n > m_+$ are determined in the leading order by excitations coming from the sites $n = m_-$ and $n = m_+$ respectively. Hence, the first and the last estimates in (3.8) follow readily. As for $m_- \leq n \leq m_+$, according to (3.7) we get

$$\left|\phi_n - \phi_n^{(0)}\right| \leq \|\boldsymbol{\phi} - \boldsymbol{\phi}^{(0)}\|_{l^2} \leq C|\epsilon|,$$

so that

$$|\phi_n| \leq C|\epsilon| + |\phi_n^{(0)}| \leq C|\epsilon| + 1 =: A_0.$$

□

Remark 3.6. Thanks to the exponential decay (3.8), the l^2 solution in Proposition 3.3 also belongs to the weighted space l^2_s for any $s \geq 0$.

3.2 Existence of multi-site breathers in the dKG equation

In this section, we study existence of discrete breather solutions in the dKG equation

$$\ddot{u}_n + V'(u_n) = \epsilon(\Delta \mathbf{u})_n, \quad n \in \mathbb{Z}, \tag{3.9}$$

where $t \in \mathbb{R}$ is the evolution time, $u_n(t) \in \mathbb{R}$ is the displacement of the n-th particle, $V : \mathbb{R} \to \mathbb{R}$ is an on-site potential for the external forces, and $\epsilon \in \mathbb{R}$ is the coupling constant of the linear interaction between neighbouring particles. For simplicity of arguments, we require that the potential V is even and smooth. The components $\{u_n\}_{n\in\mathbb{Z}}$ of the T-periodic breather solution to (3.9) are considered in Hilbert–Sobolev spaces $H^s_{\text{per}}(0,T)$ equipped with the norm,

$$\|f\|_{H^s_{\text{per}}} := \left(\sum_{m\in\mathbb{Z}} (1+m^2)^s |\hat{f}_m|^2\right)^{1/2}, \quad s \geq 0,$$

where the coefficients $\{\hat{f}_m\}$ are defined by the Fourier series of the T-periodic function,

$$f(t) = \sum_{m\in\mathbb{Z}} \hat{f}_m \exp\left(\frac{2\pi i m t}{T}\right), \quad t \in [0,T].$$

Accounting for the temporal symmetry of the dKG equation, we shall work in the restriction of $H^s_{\text{per}}(0,T)$ to the space of even T-periodic functions,

$$H^s_e(0,T) = \left\{f \in H^s_{\text{per}}(0,T) : \ f(-t) = f(t), \ t \in \mathbb{R}\right\}, \quad s \geq 0.$$

We are going to study existence of breathers in KG lattice which belong to the $l^2(\mathbb{Z}, H^2_{\text{per}}(0,T))$ space defined by the norm

$$\|\mathbf{u}\|_{l^2(\mathbb{Z},H^2_{\text{per}}(0,T))} := \left\|\left\{\|u_n\|_{H^2_{\text{per}}(0,T)}\right\}_{n\in\mathbb{Z}}\right\|_{l^2} = \sqrt{\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} (1+m^2)^2 \, |\hat{u}_{n,m}|^2}. \tag{3.10}$$

At $\epsilon = 0$, we have an arbitrary family of multi-site breathers,

$$\mathbf{u}^{(0)}(t) = \sum_{k \in S} \sigma_k \varphi(t) \mathbf{e}_k, \tag{3.11}$$

where $\mathbf{e}_k$ is the unit vector in l^2 defined in (3.6), $S \subset \mathbb{Z}$ is a compact set of *excited sites,* and $\sigma_k \in \{+1, -1\}$ encodes the phase factor of the k-th oscillator, and $\varphi \in H^2_{\text{per}}(0, T)$ is an even solution of the nonlinear oscillator equation at the energy level E,

$$\ddot{\varphi} + V'(\varphi) = 0 \quad \Rightarrow \quad E = \frac{1}{2}\dot{\varphi}^2 + V(\varphi). \tag{3.12}$$

The unique even solution $\varphi(t) \in H^2_{\text{e}}(0, T)$ satisfies the initial conditions,

$$\varphi(0) = a, \quad \dot{\varphi}(0) = 0,$$

where a is the smallest positive root of $V(a) = E$. The period T is uniquely defined from the energy level E,

$$T = \sqrt{2} \int_{-a}^{a} \frac{d\varphi}{\sqrt{E - V(\varphi)}}. \tag{3.13}$$

Definition 3.7. Suppose oscillators at the excited sites $S \subset \mathbb{Z}$ in the limiting configuration (3.11) have the same period T. We say that two oscillators at the j-th and k-th sites are *in-phase* (*anti-phase)* if $\sigma_j \sigma_k = 1$ ($\sigma_j \sigma_k = -1$).

Remark 3.8. In this thesis, we study discrete breathers with in-phase or anti-phase adjacent sites. We do not consider the *phase-shift breathers* where neighbouring excited sites can have phase difference other than 0 or π. In fact, it was proven by Koukouloyannis [51] that phase-shift breathers without "holes" do not persist in the KG lattice (3.9) with a generic potential satisfying $V'(0) = 0$ and $V''(0) > 0$. This result, however, does not rule out existence of phase-shift breathers with "holes". Let us also mention that phase-shift breathers have been recently shown to exist in KG chains with interactions beyond nearest neighbours by Koukouloyannis *et al.* [53].

Extension of the limiting configuration (3.11) as a space-localized and time-periodic breather of the dKG equation (3.9) is established by MacKay & Aubry [56] for small values of ϵ. The following theorem gives the relevant details of the theory that are useful in our analysis.

Theorem 3.9. *Fix the period T and the solution $\varphi \in H^2_e(0, T)$ of the nonlinear oscillator equation (3.9) with an even $V \in C^\infty(\mathbb{R})$ and assume that $T \neq 2\pi n$, $n \in \mathbb{N}$ and $T'(E) \neq 0$. Define $\mathbf{u}^{(0)}$ by the representation (3.11) with fixed $S \subset \mathbb{Z}$ and $\{\sigma_k\}_{k \in S}$. There are $\epsilon_0 > 0$ and $C > 0$ such that for all $\epsilon \in (-\epsilon_0, \epsilon_0)$ there exists a unique solution*

$\mathbf{u}^{(\epsilon)} \in l^2(\mathbb{Z}, H_e^2(0,T))$ *of the dKG equation (3.9) satisfying*

$$\|\mathbf{u}^{(\epsilon)} - \mathbf{u}^{(0)}\|_{l^2(\mathbb{Z}, H^2_{\text{per}}(0,T))} \leq C|\epsilon|. \tag{3.14}$$

Moreover, the map $\epsilon \mapsto \mathbf{u}^{(\epsilon)} \in l^2(\mathbb{Z}, H_e^2(0,T))$ *is* C^∞ *for all* $\epsilon \in (-\epsilon_0, \epsilon_0)$.

Proof. To prove the existence result, it is enough to show that the vector field defined by the dKG equation,

$$F_n(\mathbf{u}, \epsilon) = \ddot{u}_n + V'(u_n) - \epsilon(\Delta \mathbf{u})_n, \quad n \in \mathbb{Z},$$

satisfies the conditions of the Implicit Function Theorem (Theorem 4E in [103]) near the solution to the decoupled dKG equation, $(\mathbf{u}^{(0)}, 0)$. It is clear that $\mathbf{F}(\mathbf{u}, \epsilon)$ is C^∞ in $\mathbf{u}$ and analytic in ϵ, and $\mathbf{F}(\mathbf{u}^{(0)}, 0) = 0$. Hence, it is left to show that at $(\mathbf{u}^{(0)}, 0)$ the linearization of the above vector field,

$$(\mathcal{N}(\mathbf{u}, \epsilon)\boldsymbol{\xi})_n = \ddot{\xi}_n + V''(u_n)\xi_n - \epsilon(\Delta\boldsymbol{\xi})_n, \quad n \in \mathbb{Z},$$

defines an invertible mapping from $H_e^2(0,T)$ to $L_e^2(0,T)$. This happens if the homogenous equation

$$\ddot{\xi} + V''(u_n^{(0)})\xi = 0, \quad n \in \mathbb{Z}$$

has no nontrivial solutions in $H_e^2(0,T)$.

On the excited sites, $n \in S$, we have $u_n^{(0)} = \sigma_n \varphi(t)$ so that $V''(u_n^{(0)}) = V''(\varphi(t))$, thanks to the symmetry of the potential. The equation $\ddot{\xi} + V''(\varphi)\xi = 0$ admits two linearly independent solutions $\dot{\varphi}$ and $\partial_E \varphi$. The former solution is T-periodic with $\dot{\varphi}(0) = \dot{\varphi}(T) = 0$ and $\ddot{\varphi}(0) = \ddot{\varphi}(T) = -V'(a)$. However, $\dot{\varphi}$ is an odd function and does not belong to $H_e^2(0,T)$. For the latter solution, $\partial_E \varphi$, the periodicity condition is not satisfied provided $T'(E) \neq 0$. Indeed, differentiating the identity $\dot{\varphi}(T) = 0$ with respect to E, we obtain

$$\partial_E \dot{\varphi}(T) + \ddot{\varphi}(T) T'(E) = 0 \qquad \Longrightarrow \qquad \partial_E \dot{\varphi}(T) = V'(a) T'(E) \neq 0 = \partial_E \dot{\varphi}(0).$$

For the passive lattice sites, $n \in \mathbb{Z} \backslash S$, we have $V''(0) = 1$ so that the governing equation $\ddot{\xi} + \xi = 0$ does not admit T-periodic solutions if $T \neq 2\pi k$, $k \in \mathbb{N}$. □

Remark 3.10. We derive expansions for multi-site breathers with "holes" in Chapter 5, where we also consider stability of such breathers.

Chapter 4

Linear and asymptotic stability of the dNLS breathers

In Section 3.1, we studied existence of discrete breathers in the dNLS equation

$$i\dot{u}_n + \epsilon(\Delta \mathbf{u})_n + |u_n|^{2p} u_n = 0, \qquad n \in \mathbb{Z}, \qquad p \in \mathbb{N}. \tag{4.1}$$

Such breathers are sought in the form $\mathbf{u}(t) = \boldsymbol{\phi}(\omega) e^{i\omega t}$, where $\boldsymbol{\phi}(\omega)$ is spatially localized, and are often called *discrete solitons*. By Lemma 3.1, the associated stationary solution $\boldsymbol{\phi}(\omega)$ is real-valued and satisfies the lattice equation

$$(\omega - \phi_n^{2p}(\omega))\phi_n(\omega) = \epsilon(\Delta \boldsymbol{\phi}(\omega))_n, \qquad n \in \mathbb{Z}, \qquad p \in \mathbb{N}.$$

We fix $\omega = 1$ by rescaling the solution and the coupling constant:

$$(1 - \phi_n^{2p})\phi_n = \epsilon(\Delta \boldsymbol{\phi})_n, \qquad n \in \mathbb{Z}, \qquad p \in \mathbb{N}. \tag{4.2}$$

At $\epsilon = 0$ we consider the *limiting configuration* for the discrete soliton,

$$\boldsymbol{\phi}^{(0)} = \sum_{n \in S_+} \boldsymbol{e}_n - \sum_{n \in S_-} \boldsymbol{e}_n, \tag{4.3}$$

where $S_\pm$ are compact disjoint subsets of $\mathbb{Z}$. To simplify notations, we denote the *set of excited sites* $S_+ \cup S_- \subset \mathbb{Z}$ by S. The number of elements in S is denoted by N.

In Proposition 3.3, we proved that for sufficiently small value of ϵ there is a unique analytic continuation of the limiting configuration $\boldsymbol{\phi}^{(0)}$ to the l^2 solution $\boldsymbol{\phi}$ of (4.2).

Thanks to the analyticity of the solution, we can expand $\boldsymbol{\phi}$ in the power series

$$\boldsymbol{\phi} = \boldsymbol{\phi}^{(0)} + \sum_{k=1}^{\infty} \epsilon^k \boldsymbol{\phi}^{(k)}, \tag{4.4}$$

where correction terms $\{\boldsymbol{\phi}^{(k)}\}_{k\in\mathbb{N}}$ are uniquely determined by recursion from (4.2).

To study the spectral stability of the breather $\boldsymbol{\phi}e^{it}$ let us introduce a generic complex-valued perturbation:

$$\mathbf{u}(t) = \boldsymbol{\phi}e^{it} + (\mathbf{A}(t;\lambda) + i\mathbf{B}(t;\lambda))\, e^{it},$$

where $\mathbf{A}(t;\lambda), \mathbf{B}(t;\lambda) : \mathbb{R}\times\mathbb{C} \to l^2$ and $\lambda \in \mathbb{C}$ is the spectral parameter that controls the growth of $\mathbf{u}(t)$. We separate the variables by setting

$$\begin{aligned} \mathbf{A}(t;\lambda) &= \mathbf{v}e^{\lambda t} + \bar{\mathbf{v}}e^{\bar{\lambda} t}, \\ \mathbf{B}(t;\lambda) &= \mathbf{w}e^{\lambda t} + \bar{\mathbf{w}}e^{\bar{\lambda} t}, \end{aligned}$$

where $\mathbf{v}, \mathbf{w} \in l^2$. Extracting the terms linear in $\mathbf{A}$ and $\mathbf{B}$ from (4.1) and setting to zero the factors at $e^{it+\lambda t}$ and $e^{it+\bar{\lambda} t}$ we obtain a non-self-adjoint eigenvalue problem

$$\mathcal{L}\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix} = \lambda \begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix}, \qquad \mathcal{L} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}, \tag{4.5}$$

where $L_\pm$ are discrete Schrödinger operators given by

$$\begin{aligned} (L_+\mathbf{v})_n &= -\epsilon(\Delta\mathbf{v})_n + (1-(2p+1)\phi_n^{2p})v_n, \\ (L_-\mathbf{v})_n &= -\epsilon(\Delta\mathbf{v})_n + (1-\phi_n^{2p})v_n, \end{aligned} \qquad n \in \mathbb{Z}.$$

The eigenvalues of this spectral problem come in quartets. Indeed, for each eigenvalue $\lambda \in \mathbb{C}$ with an eigenvector $(\mathbf{v},\mathbf{w})^T$ we also find eigenvalues $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$ with eigenvectors $(\mathbf{v},-\mathbf{w})^T$, $(\bar{\mathbf{v}},\bar{\mathbf{w}})^T$, and $(\bar{\mathbf{v}},-\bar{\mathbf{w}})^T$ respectively.

Definition 4.1. We say that a soliton is *spectrally unstable* if there is an eigenvalue λ with $\mathrm{Re}\lambda > 0$. If all eigenvalues are located on the imaginary axis, we say that the soliton is *spectrally stable*.

It is straightforward to compute the eigenvalues of the operator $\mathcal{L}$ in the anti-continuum limit. We notice that in that limit, the operators $L_\pm^{(0)} := L_\pm|_{\epsilon=0}$ are multiplicatory:

$$(L_\pm^{(0)}\mathbf{v})_n = (L_\pm^{(0)})_n v_n,$$

where

$$\left(L_+^{(0)}\right)_n = \begin{cases} -2p, & n \in S \\ 1, & n \in \mathbb{Z}\backslash S \end{cases}, \qquad \left(L_-^{(0)}\right)_n = \begin{cases} 0, & n \in S \\ 1, & n \in \mathbb{Z}\backslash S \end{cases}.$$

From these properties, spectra of operators $L_\pm^{(0)}$ and

$$\mathcal{L}^{(0)} := \begin{bmatrix} 0 & L_-^{(0)} \\ -L_+^{(0)} & 0 \end{bmatrix}$$

immediately follow.

Definition 4.2. We say that an eigenvalue is *semi-simple* if its geometric and algebraic multiplicities are equal.

Proposition 4.3. *In the anti-continuum limit, the operators $L_\pm$ and $\mathcal{L}$ posses the following properties:*

- *The spectrum of $L_+^{(0)}$ (resp. $L_-^{(0)}$) includes a semi-simple eigenvalue $-2p$ (resp. 0) of multiplicity N and a semi-simple eigenvalue 1 of infinite multiplicity.*
- *The spectrum of the operator $\mathcal{L}^{(0)}$ consists of a pair of eigenvalues $\lambda = \pm i$ of infinite multiplicity and the eigenvalue $\lambda = 0$ of geometric multiplicity N and algebraic multiplicity $2N$.*

This chapter is structured as follows. In Section 4.1, we study what happens to the zero eigenvalues of the spectral problem (4.5) as we deviate from the anti-continuum limit. The main objective is to find out which limiting configurations of discrete solitons of the dNLS equation are spectrally stable near the anti-continuum limit. Then, in Section 4.2, we prove that no discrete eigenvalues bifurcate from the edges of the continuous spectrum near the anti-continuum limit. In Section 4.3, we prove that a small dNLS soliton bifurcating from the eigenvalue of the operator $-\Delta + V$ is asymptotically stable.

4.1 Unstable and stable eigenvalues

In this section, we summarize the spectral properties of operator $\mathcal{L}$ in (4.5) at small non-vanishing coupling ϵ.

Let us first study the kernel of the operator $\mathcal{L}$ in (4.5). Let us recall that the stationary solution associated with the breather of frequency $\omega > 0$ satisfies the lattice equation

$$(\omega - \phi_n^{2p}(\omega))\phi_n(\omega) = \epsilon(\Delta\boldsymbol{\phi}(\omega))_n, \qquad n \in \mathbb{Z}, \qquad p \in \mathbb{N}. \tag{4.6}$$

In the anti-continuum limit, this equation decouples producing a solution

$$\boldsymbol{\phi}^{(0)}(\omega) = \sqrt[2p]{\omega}\left(\sum_{n\in S_+}\mathbf{e}_n - \sum_{n\in S_-}\mathbf{e}_n\right).$$

Thanks to Proposition 3.3, the extension of this limiting configuration for sufficiently small values of ϵ produces a unique solution $\boldsymbol{\phi}(\omega)$ to (4.6). To account for the frequency parameter ω, we have to replace the Schrödinger operators $L_\pm$ in (4.5) with $L_\pm(\omega)$ given by

$$\begin{aligned} (L_+(\omega)\mathbf{v})_n &= -\epsilon(\Delta\mathbf{v})_n + (\omega - (2p+1)\phi_n^{2p}(\omega))v_n, \\ (L_-(\omega)\mathbf{v})_n &= -\epsilon(\Delta\mathbf{v})_n + (\omega - \phi_n^{2p}(\omega))v_n, \end{aligned} \qquad n\in\mathbb{Z}.$$

Since these operators satisfy

$$L_-(\omega)\boldsymbol{\phi}(\omega) = 0, \qquad L_+(\omega)\boldsymbol{\phi}'(\omega) = -\boldsymbol{\phi}(\omega), \tag{4.7}$$

the operator

$$\mathcal{L}(\omega) = \begin{bmatrix} 0 & L_-(\omega) \\ -L_+(\omega) & 0 \end{bmatrix}$$

has at least two eigenvectors in its generalized kernel:

$$\mathrm{Ker}\mathcal{L}(\omega) = \mathrm{span}\left\{\begin{bmatrix} 0 \\ \boldsymbol{\phi}(\omega)\end{bmatrix}\right\}, \qquad \mathrm{Ker}(\mathcal{L}^2(\omega)) = \mathrm{span}\left\{\begin{bmatrix} 0 \\ \boldsymbol{\phi}(\omega)\end{bmatrix}, \begin{bmatrix} \boldsymbol{\phi}'(\omega) \\ 0\end{bmatrix}\right\}.$$

In fact, these two vectors exhaust the generalized kernel of the operator $\mathcal{L}(\omega)$. If it was not the case, then the equation

$$\mathcal{L}(\omega)\begin{bmatrix}\mathbf{v} \\ \mathbf{w}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\phi}'(\omega) \\ 0\end{bmatrix},$$

would have a solution. This is not possible since the spectrum of $L_+(\omega)$ is bounded away from zero for $\omega > 0$ (see Figure 4.1), and the kernel of $L_-(\omega)$ given by $\boldsymbol{\phi}(\omega)$, is not orthogonal to $\boldsymbol{\phi}'(\omega)$ in l^2. To clarify on the last statement we notice that $\boldsymbol{\phi}(\omega) = \boldsymbol{\phi}^{(0)}(\omega) + \mathcal{O}(\epsilon)$ which yields

$$\|\boldsymbol{\phi}(\omega)\|_{l^2}^2 = N\omega^{1/p} + \mathcal{O}(\epsilon).$$

Since for $\omega > 0$ we have

$$\left\langle\boldsymbol{\phi}(\omega), \boldsymbol{\phi}'(\omega)\right\rangle_{l^2} = \frac{1}{2}\frac{d}{d\omega}\|\boldsymbol{\phi}(\omega)\|_{l^2}^2 = \frac{N}{2p}\omega^{1/p-1} \neq 0, \quad \omega\in\mathbb{R}, \quad p\in\mathbb{N}, \tag{4.8}$$

then the generalized kernel of the operator $\mathcal{L}(\omega)$ is

$$\mathrm{Ng}(\mathcal{L}(\omega)) = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ \boldsymbol{\phi}(\omega) \end{bmatrix}, \begin{bmatrix} \boldsymbol{\phi}'(\omega) \\ 0 \end{bmatrix} \right\}. \tag{4.9}$$

It is important to note that the eigenvector $(\mathbf{v}, \mathbf{w})^T$ of the spectral problem

$$\mathcal{L}(\omega) \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}, \quad \lambda \neq 0,$$

are *symplectically orthogonal* to the generalized kernel of $\mathcal{L}(\omega)$ in (4.9) in the sense that

$$\langle \mathbf{v}, \boldsymbol{\phi}(\omega) \rangle_{l^2} = \left\langle \mathbf{w}, \boldsymbol{\phi}'(\omega) \right\rangle_{l^2} = 0. \tag{4.10}$$

This property comes directly from self-adjointness of the operators $L_\pm(\omega)$ and (4.7).

Let us now fix the frequency parameter by setting $\omega = 1$ and examine splitting of the spectra of $L_\pm$ and $\mathcal{L}$ near the anti-continuum limit (cf. Proposition 4.3). As ϵ deviates from zero, the eigenvalues $\pm i$ of the operator $\mathcal{L}$ (infinite multiplicity) produce bands of continuous spectrum. Indeed, setting the decaying potential $\boldsymbol{\phi} \in l^2$ to zero we get

$$L_+|_{\boldsymbol{\phi}=0} = L_-|_{\boldsymbol{\phi}=0} = 1 - \epsilon\Delta := L_0,$$

so that the spectral problem (4.5) simplifies to

$$L_0 \mathbf{a} = i\lambda \mathbf{a}, \qquad L_0 \mathbf{b} = -i\lambda \mathbf{b},$$

where $\mathbf{a} = \mathbf{v} + i\mathbf{w}$, $\mathbf{b} = \mathbf{u} - i\mathbf{w}$. Since the continuous spectrum of the operator $(-\Delta)$ is $\sigma_c(-\Delta) = [0, 4]$ we find that

$$\sigma_c(\mathcal{L}|_{\phi=0}) = i[-1 - 4\epsilon, -1] \cup i[1, 1 + 4\epsilon], \qquad \epsilon > 0.$$

In Section 4.2, we perform resolvent analysis to show that the continuous spectrum of the operator $\mathcal{L}$ is the same as that of $\mathcal{L}|_{\phi=0}$.

To explain the bifurcation of semi-simple eigenvalues from the anti-continuum limit, let us recall basic definitions and results from the stability analysis of the spectral problem (4.5).

Definition 4.4. The eigenvalues of the spectral problem (4.5) with $\mathrm{Re}\lambda > 0$ (resp. with $\mathrm{Re}\lambda = 0$) are called *unstable* (resp. *neutrally stable*). If $\lambda \in i\mathbb{R}$ is a simple isolated eigenvalue, then the eigenvalue λ is said to have a *positive energy* if $\langle L_+\mathbf{u}, \mathbf{u}\rangle_{l^2} > 0$ and a *negative energy* if $\langle L_+\mathbf{u}, \mathbf{u}\rangle_{l^2} < 0$.

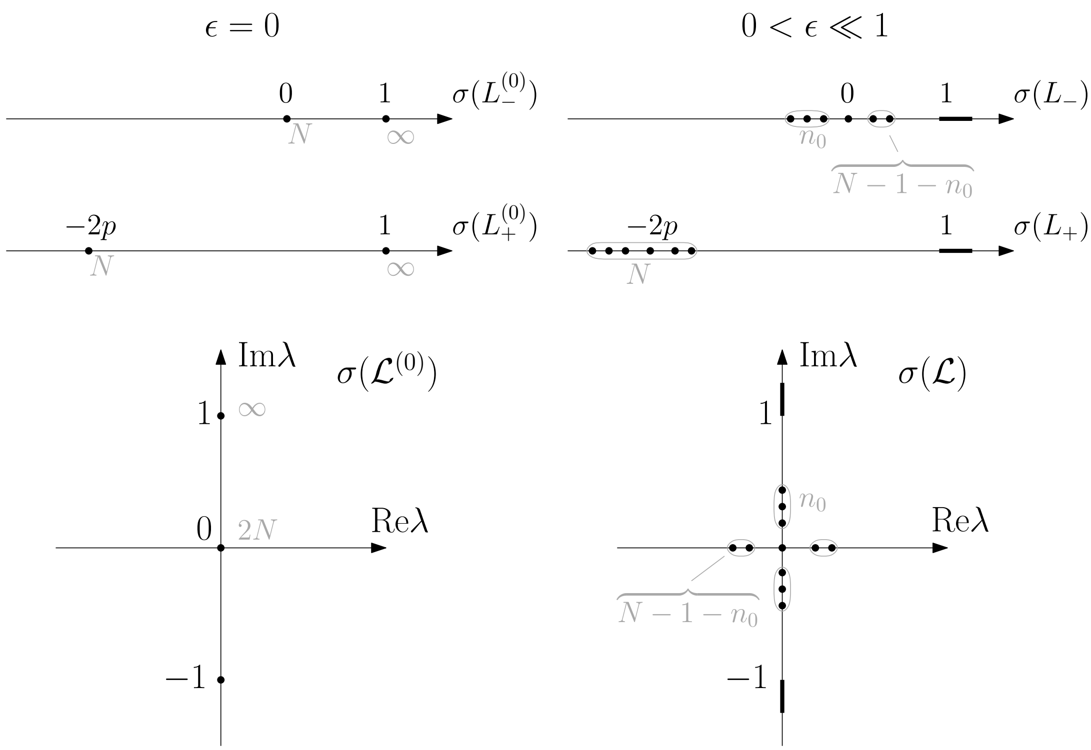

Figure 4.1: Spectra of operators $L_\pm$ and $\mathcal{L}$ at $\epsilon = 0$ (left) and at small $\epsilon > 0$ (right). The dots represent isolated eigenvalues, while the bold lines represent continuous spectra. The algebraic multiplicities of the isolated eigenvalues are shown in grey colour.

Remark 4.5. If $\lambda \in i\mathbb{R}$ is an isolated eigenvalue and $\langle L_+\mathbf{u}, \mathbf{u}\rangle_{l^2} = 0$, then λ is not a simple eigenvalue. In this case, the concept of eigenvalues of positive and negative energies is defined by the diagonalization of the quadratic form $\langle L_+\mathbf{u}, \mathbf{u}\rangle_{l^2}$, where $\mathbf{u}$ belongs to the subspace of l^2 associated with the eigenvalue λ of the spectral problem (4.5) and invariant under the action of the corresponding linearized operator (see [21] for the relevant theory).

The following proposition describes the splitting of the zero eigenvalue near the anti-continuum limit for $\epsilon > 0$ (see [72] for the proof).

Proposition 4.6. *Denote the number of sign differences in* $\{\phi_n^{(0)}\}_{n\in S}$ *by* n_0. *For sufficiently small* $\epsilon > 0$,

- *There are exactly* n_0 *negative and* $N-1-n_0$ *small positive eigenvalues of the operator* L_- *counting multiplicities and a simple zero eigenvalue.*
- *In addition to a double zero eigenvalue, the operator* $\mathcal{L}$ *has exactly* n_0 *pairs of small eigenvalues* $\lambda \in i\mathbb{R}$ *of negative energy and* $N-1-n_0$ *pairs of small eigenvalues* $\lambda \in \mathbb{R}$ *counting multiplicities.*

Remark. Since the discrete spectrum of operator L_+ is bounded away from zero, its splitting has no effect on the kernel of operator $\mathcal{L}$. The splitting of the zero eigenvalue in the operator $\mathcal{L}$ is described in terms of that for the operator L_-.

Proposition 4.6 completes the characterization of unstable eigenvalues and neutrally stable eigenvalues of negative energy from negative eigenvalues of L_+ and L_-. In particular, we know from [21] that if $\mathrm{Ker}L_+ = \{0\}$, $\mathrm{Ker}L_- = \mathrm{span}\{\boldsymbol{\phi}\}$, and $\langle L_+^{-1}\boldsymbol{\phi}, \boldsymbol{\phi}\rangle_{l^2} \neq 0$, then

$$\left\{ \begin{array}{lcl} n(L_+) - p_0 & = & N_r^- + N_i^- + N_c, \\ n(L_-) & = & N_r^+ + N_i^- + N_c, \end{array} \right. \tag{4.11}$$

where $n(L_\pm)$ denotes the number of negative eigenvalues of $L_\pm$, N_i^- denotes the number of eigenvalues $\lambda \in i\mathbb{R}$ with negative energy, N_c denotes the number of eigenvalues with $\mathrm{Re}\lambda > 0$ and $\mathrm{Im}\lambda > 0$, N_r^+ (resp. N_r^-) denotes the number of eigenvalues $\lambda \in \mathbb{R}$ with $\langle L_+\mathbf{u}, \mathbf{u}\rangle_{l^2} \geq 0$ (resp. $\langle L_+\mathbf{u}, \mathbf{u}\rangle_{l^2} \leq 0$), and

$$p_0 = \left\{ \begin{array}{ll} 1 & \text{if} \quad \langle L_+^{-1}\boldsymbol{\phi}, \boldsymbol{\phi}\rangle_{l^2} < 0, \\ 0 & \text{if} \quad \langle L_+^{-1}\boldsymbol{\phi}, \boldsymbol{\phi}\rangle_{l^2} > 0. \end{array} \right.$$

To compute p_0, we refer to (4.8) and find that

$$\langle L_+^{-1}\boldsymbol{\phi}, \boldsymbol{\phi}\rangle_{l^2} = -\langle \boldsymbol{\phi}, \boldsymbol{\phi}'(1)\rangle_{l^2} = -\frac{N}{2p} + \mathcal{O}(\epsilon).$$

Therefore, $p_0 = 1$ for small values of ϵ.

By Proposition 4.6, we have $n(L_-) = n_0$, $N_c = 0$, and $N_i^- \geq n_0$. Also, $n(L_+) = N$. For small $\epsilon > 0$, we use the eigenvalue count (4.11) to find that

$$N_r^+ = 0, \quad N_r^- = N - 1 - n_0, \quad N_i^- = n_0, \quad N_c = 0. \tag{4.12}$$

Equality (4.12) shows that besides the small and zero eigenvalues described by Proposition 4.6, the operator $\mathcal{L}$ may only have the continuous spectrum and *internal modes,* pairs of isolated eigenvalues $\lambda \in i\mathbb{R}$ of positive energy. The internal modes are produced by the splitting of eigenvalues of infinite multiplicity. The phenomena has been demonstrated for kinks of the dNLS equation by Pelinovsky & Kevrekidis [70]. For the fundamental breather in the dNLS equation, numerical simulations of Johansson & Aubry (Figure 1 in [43]) and Kevrekidis (Figure 2.5 in [46]) suggest that internal modes bifurcate from the continuum spectrum only for sufficiently large coupling ϵ. With these facts, we summarize the results on spectral properties of operators $L_\pm$ and $\mathcal{L}$ on Figure 4.1.

It is important to know the details on existence of internal modes because of several reasons. First, these internal modes may collide with eigenvalues of negative energy to produce the Hamilton–Hopf instability bifurcations [72]. Second, analysis of asymptotic stability of discrete solitons depends on the number and location of the internal modes [25, 49]. Third, the presence of internal modes may result in long-term quasi-periodic oscillations of discrete solitons [24].

In Section 4.2, we prove that no internal modes bifurcate from the continuous spectrum near the anti-continuum limit provided the initial configuration of the discrete soliton is supported on a simply-connected sets and $p \geq 2$. We also briefly discuss existence of internal modes in the case of the cubic dNLS equation, $p = 1$.

4.2 Internal modes

In this section, we analyze the spectrum of the problem (4.5) using the resolvent operator. As we already know all the details on splitting of the zero eigenvalue of the operator $\mathcal{L}$, our main focus here is on the spectrum of $\mathcal{L}$ near the points $\pm i$. Without loss of generality, we restrict the analysis to positive values of the coupling ϵ.

In Sections 4.2.1–4.2.4, we study the properties of the resolvent of a truncated operator $\mathcal{L}|_{\phi^{(0)}}$. In Section 4.2.5, we run perturbative arguments to extend these results for the case of the full operator $\mathcal{L}$. We prove the main theorem (Theorem 4.21) which shows that simply-connected discrete solitons in the dNLS equation with quintic or higher nonlinearity ($p \geq 2$) have no internal modes bifurcating from the continuous spectrum if the coupling constant ϵ is sufficiently small. In Sections 4.2.6 and 4.2.7, we address issues beyond applicability of our analytic results, namely, non-simply-connected solitons and the cubic dNLS equation.

4.2.1 The resolvent operator for the limiting configuration

Let us consider the truncated spectral problem (4.5) after $\boldsymbol{\phi}$ is replaced by its limiting configuration $\boldsymbol{\phi}^{(0)}$ (4.3). The resolvent operator is defined from the spectral problem (4.5) modified with an inhomogeneous term:

$$\left(\mathcal{L}|_{\boldsymbol{\phi}^{(0)}} - I\lambda\right)\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix} = \begin{bmatrix}\mathbf{F}\\ \mathbf{G}\end{bmatrix},$$

where I is the identity operator and $\mathbf{F}, \mathbf{G} \in l^2$ are given. This can be rewritten as

$$\begin{cases} -\epsilon(\Delta\mathbf{v})_n + v_n - (2p+1)\sum_{m\in S}\delta_{n,m}v_m + \lambda w_n = F_n, \\ -\epsilon(\Delta\mathbf{w})_n + w_n - \sum_{m\in S}\delta_{n,m}w_m - \lambda v_n = G_n, \end{cases} \quad n \in \mathbb{Z}, \tag{4.13}$$

where $\mathbf{F}, \mathbf{G} \in l^2$ are given. Since we are interested in the continuous spectrum and eigenvalues on $i\mathbb{R}$, we set $\lambda = -i\Omega$ and use new coordinates

$$\begin{aligned} a_n &:= v_n + iw_n, \quad f_n := F_n + iG_n, \\ b_n &:= v_n - iw_n, \quad g_n := F_n - iG_n. \end{aligned}$$

The inhomogeneous system (4.13) transforms to the equivalent form

$$\begin{cases} -\epsilon(\Delta\mathbf{a})_n + a_n - \sum_{m\in S}\delta_{n,m}((1+p)a_m + pb_m) - \Omega a_n = f_n, \\ -\epsilon(\Delta\mathbf{b})_n + b_n - \sum_{m\in S}\delta_{n,m}(pa_m + (1+p)b_m) + \Omega b_n = g_n, \end{cases} \tag{4.14}$$

which can be rewritten in the operator form

$$(L - I\Omega)\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} = \begin{bmatrix}\mathbf{f}\\ -\mathbf{g}\end{bmatrix}, \tag{4.15}$$

where

$$L = \begin{bmatrix} -\epsilon\Delta + I - (1+p)V & -pV \\ pV & \epsilon\Delta - I + (1+p)V, \end{bmatrix},$$

with $V : l^2 \to l^2$ being a compact potential

$$(V\mathbf{u})_n = \sum_{m\in S} \delta_{n,m} u_m, \quad n \in \mathbb{Z}.$$

It turns out that the spectral properties of the operator L can be studied using the *free resolvent*, the resolvent operator of the discrete Schrödinger operator $-\Delta$, defined as $R_0(\lambda) = (-\Delta - \lambda)^{-1} : l^2 \to l^2$. The free resolvent was studied recently by Komech, Kopylova, & Kunze [50], who showed that it can be expressed in the Green's function form

$$\forall \mathbf{f} \in l^2 : \quad (R_0(\lambda)\mathbf{f})_n = \frac{1}{2i \sin z(\lambda)} \sum_{m\in\mathbb{Z}} e^{-iz(\lambda)|n-m|} f_m, \tag{4.16}$$

where $z(\lambda)$ is the unique solution of the transcendental equation for $\lambda \notin [0,4]$

$$2 - 2\cos z(\lambda) = \lambda, \quad \mathrm{Re}z(\lambda) \in [-\pi,\pi), \ \ \mathrm{Im}z(\lambda) < 0. \tag{4.17}$$

Note that the choice $\mathrm{Im}z(\lambda) < 0$ corresponds to the exponentially decaying kernel of Green's function. If one opts for the roots with $\mathrm{Im}z(\lambda) > 0$, then the sign in the exponent of (4.16) has to be switched to positive.

It is shown in the work of Pelinovsky & Stefanov [76] that the bounded operator $R_0(\lambda) : l^2 \to l^2$ for $\lambda \notin [0,4]$ admits the limits

$$R_0^{\pm}(\omega) = \lim_{\mu\downarrow 0} R_0(\omega \pm i\mu) : l^2_\sigma \to l^2_{-\sigma}, \quad \sigma > \frac{1}{2} \tag{4.18}$$

for any fixed $\omega \in (0,4)$. Since $l^2_\sigma \subset l^2 \subset l^2_{-\sigma}$, the free resolvent operator $R_0(\lambda)$ can be extended to $\lambda \in \mathbb{C}\backslash(\{0\} \cup \{4\})$ as a bounded operator mapping l^2_σ to $l^2_{-\sigma}$

To get an explicit expression for $R_0^{\pm}(\omega)$ we need to examine the roots of transcendental equation (4.17) for the case of spectral parameter λ with $\mathrm{Re}\lambda \in (0,4)$ fixed approaching real axis. Setting

$$\lambda = \omega + i\mu, \qquad \omega \in (0,4), \qquad \mu \in \mathbb{R},$$

and

$$z(\lambda) = \theta - iy, \qquad \theta \in [-\pi,\pi), \qquad y > 0,$$

we get from equation (4.17) that

$$(\cos\theta\cosh y - 1) + i\sin\theta\sinh y = -\tfrac{\omega}{2} - i\tfrac{\mu}{2}.$$

From the imaginary part of this equation we conclude that $\mathrm{sign}\theta = -\mathrm{sign}\mu$. Therefore,

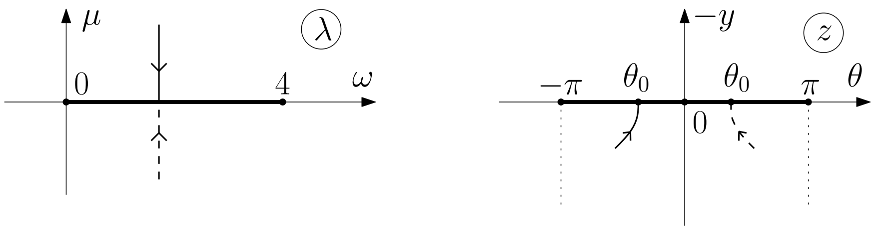

Figure 4.2: The path for approaching the continuous spectrum in λ-plane (left). The corresponding root $z(\lambda)$ to equation (4.17) (right).

if λ approaches real axis from above (if $\mu \geq 0$) then $z(\lambda)$ approaches $\theta \in (-\pi, 0)$, whereas if λ approaches real axis from below (if $\mu \leq 0$) then $z(\lambda)$ approaches $\theta \in (0, \pi)$. As a result, $\mathrm{Re}z(\mu)$ experiences a jump discontinuity when μ crosses zero. We can get even more information on the behaviour of $z(\mu)$ near $\mu = 0$ by constructing the power series of y and θ in μ:

$$\begin{cases} y &= -\frac{\mu}{2\sin\theta_0} + \mathcal{O}(\mu^3), \\ \theta &= \theta_0 + \frac{\mu^2}{8\sin^2\theta_0 \tan\theta_0} + \mathcal{O}(\mu^4), \end{cases} \qquad \text{where} \qquad 2 - 2\cos\theta_0 = \omega.$$

On Figure 4.2, we show the behaviour of $z(\mu)$ near $\mu = 0$. Note that the curves $z(\lambda)$ do not extend to the upper half-plane due to the restriction $\mathrm{Im}z(\lambda) < 0$ that guarantees exponential decay of the kernel in Green's function representation (4.16).

Using the above asymptotic expressions for $z(\lambda)$ near $\lambda \in (0, 4)$ we can express the limiting free resolvent operators $R_0^{\pm}(\omega)$ in the Green's function form

$$\forall \mathbf{f} \in l^1 : \quad (R_0^{\pm}(\omega)\mathbf{f})_n = \frac{1}{2i\sin\theta_{\pm}(\omega)} \sum_{m\in\mathbb{Z}} e^{-i\theta_{\pm}(\omega)|n-m|} f_m, \tag{4.19}$$

where $\theta_{\pm}(\omega) = \pm\theta(\omega)$ and $\theta(\omega)$ is a unique solution of the transcendental equation for $\omega \in (0, 4)$

$$2 - 2\cos\theta(\omega) = \omega, \quad \theta(\omega) \in (-\pi, 0). \tag{4.20}$$

The limiting operators $R_0^{\pm}(\omega) : l^1 \to l^\infty$ are bounded for any fixed $\omega \in (0, 4)$. Since $l^2_\sigma \subset l^1$ and $l^\infty \subset l^2_{-\sigma}$ for any $\sigma > 1/2$ this implies the limiting behaviour of the free resolvent in (4.18) with $\omega \in (0, 4)$.

In the limits $\omega \downarrow 0$ and $\omega \uparrow 4$ the limiting operators $R_0^{\pm}(\omega) : l^1 \to l^\infty$ are divergent. These divergences follow from the Puiseux expansion

$$\forall \mathbf{f} \in l^1_2 : \quad (R_0^{\pm}(\omega)\mathbf{f})_n = \frac{1}{2i\theta^{\pm}(\omega)} \sum_{m\in\mathbb{Z}} f_m - \frac{1}{2} \sum_{m\in\mathbb{Z}} |n-m| f_m + (\hat{R}_0^{\pm}(\omega)\mathbf{f})_n, \tag{4.21}$$

where

$$\exists C > 0 : \quad \|\hat{R}_0^{\pm}(\omega)\mathbf{f}\|_{l^\infty} \leq C|\theta^{\pm}(\omega)|\|\mathbf{f}\|_{l^1_2}.$$

Divergences of $R_0^{\pm}(\omega)$ at the end points $\omega = 0$ and $\omega = 4$ indicate *resonances*, which may result in the bifurcation of new eigenvalues from the continuous spectrum on $[0, 4]$ either for $\lambda < 0$ or $\lambda > 4$, when $-\Delta$ is perturbed by a small potential in l^2.

Let us denote the solution of the inhomogeneous system (4.15) by

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = R_L(\Omega) \begin{bmatrix} \mathbf{f} \\ -\mathbf{g} \end{bmatrix}. \tag{4.22}$$

The following theorem represents the main result of this subsection. This theorem is valid for simply-connected sets S, which are introduced by the following definition.

Definition 4.7. We say that the set $S \in \mathbb{Z}$ is *simply-connected* if no elements in $\mathbb{Z}\backslash S$ are located between elements in S.

Theorem 4.8. *Fix disjoint compact subsets S_+ and S_- on $\mathbb{Z}$ such that $S_+ \cup S_-$ is a simply-connected and set of N elements. Let $B_\delta(0) \subset \mathbb{C}$ denote a ball of radius δ centred at the origin. For any integer $p \geq 2$, there exist small $\epsilon_0 > 0$ and $\delta > 0$ such that for any fixed $\epsilon \in (0, \epsilon_0)$ the resolvent operator*

$$R_L(\Omega) : l^2 \times l^2 \to l^2 \times l^2$$

is bounded for any $\Omega \notin B_\delta(0) \cup [-1 - 4\epsilon, -1] \cup [1, 1 + 4\epsilon]$ and has exactly $2N$ poles (counting multiplicities) inside $B_\delta(0)$. Moreover, for any $\epsilon \in (0, \epsilon_0)$ there is $C > 0$ such that the limiting operators

$$R_L^{\pm}(\Omega) := \lim_{\mu \downarrow 0} R_L(\Omega \pm i\mu), \qquad \Omega \in [-1 - 4\epsilon, -1] \cup [1, 1 + 4\epsilon],$$

admit the uniform bounds

$$\|R_L^{\pm}(\Omega)\|_{l^1_1 \times l^1_1 \to l^\infty \times l^\infty} \leq C\epsilon^{-1}, \qquad \forall \Omega \in [-1 - 4\epsilon, -1] \cup [1, 1 + 4\epsilon].$$

Remark 4.9. The other way to formulate this theorem is to say that the end points of the continuous spectrum $\sigma_c(L) \equiv [-1 - 4\epsilon, -1] \cup [1, 1 + 4\epsilon]$ are not resonances and no eigenvalues of the linear operator L may exist outside a small disk $B_\delta(0) \subset \mathbb{C}$. The $2N$ eigenvalues inside the small disk $B_\delta(0)$ are characterized in Proposition 4.6.

To prove Theorem 4.8 we need to study solvability conditions for the linear system (4.14). Using the Green's function (4.16) we can rewrite (4.14) for any $n \in \mathbb{Z}$ in the equivalent form:

$$\begin{cases} a_n = \dfrac{1}{2i\epsilon \sin z(\lambda_+)} \left(\displaystyle\sum_{m\in\mathbb{Z}} e^{-iz(\lambda_+)|n-m|} f_m \right. \\ \qquad\qquad \left. + \displaystyle\sum_{m\in S} e^{-iz(\lambda_+)|n-m|}((1+p)a_m + pb_m) \right), \\ b_n = \dfrac{1}{2i\epsilon \sin z(\lambda_-)} \left(\displaystyle\sum_{m\in\mathbb{Z}} e^{-iz(\lambda_-)|n-m|} g_m \right. \\ \qquad\qquad \left. + \displaystyle\sum_{m\in S} e^{-iz(\lambda_-)|n-m|}(pa_m + (1+p)b_m) \right), \end{cases} \tag{4.23}$$

where the maps $\mathbb{C} \ni \lambda_\pm \mapsto z \in \mathbb{C}$ are defined by the transcendental equation (4.17) with

$$\lambda_\pm = \frac{\pm\Omega - 1}{\epsilon}.$$

The solution is closed if the set $\{(a_n, b_n)\}_{n\in S}$ is found from the linear system of finitely many equations:

$$\begin{cases} 2i\epsilon \sin z(\lambda_+) a_n - \displaystyle\sum_{m\in S} e^{-iz(\lambda_+)|n-m|}((1+p)a_m + pb_m) \\ \qquad\qquad = \displaystyle\sum_{m\in\mathbb{Z}} e^{-iz(\lambda_+)|n-m|} f_m, \\ 2i\epsilon \sin z(\lambda_-) b_n - \displaystyle\sum_{m\in S} e^{-iz(\lambda_-)|n-m|}(pa_m + (1+p)b_m) \\ \qquad\qquad = \displaystyle\sum_{m\in\mathbb{Z}} e^{-iz(\lambda_-)|n-m|} g_m, \end{cases} \quad n \in S. \tag{4.24}$$

Let us order lattice sites $n \in S$ such that the first site is placed at $n = 0$, the second site is placed at $m_1 \geq 1$, the third site is placed at $m_1 + m_2 \geq 2$, and so on, the last site is placed at $m_1 + m_2 + \cdots + m_{N-1} \geq N - 1$, where all $m_j > 0$. If S is a simply-connected set, then all $m_j = 1$.

Let $Q(q_1, q_2, \cdots, q_{N-1})$ be the matrix in $\mathbb{C}^{N\times N}$ defined by

$$Q(q_1, q_2, \cdots, q_{N-1}) := \begin{bmatrix} 1 & q_1 & q_1 q_2 & \cdots & q_1 q_2 \cdots q_{N-1} \\ q_1 & 1 & q_2 & \cdots & q_2 q_3 \cdots q_{N-1} \\ q_1 q_2 & q_2 & 1 & \cdots & q_3 \cdots q_{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_1 q_2 \cdots q_{N-1} & q_2 \cdots q_{N-1} & q_3 \cdots q_{N-1} & \cdots & 1 \end{bmatrix}.$$

Let $q_j^\pm = e^{-im_j z(\lambda_\pm)}$ and $Q^\pm(\Omega, \epsilon) := Q(q_1^\pm, q_2^\pm, \cdots, q_{N-1}^\pm)$. The coefficient matrix of the linear system (4.24) is given by

$$A(\Omega,\epsilon) := \begin{bmatrix} 2i\epsilon\sin z(\lambda_+)I-(1+p)Q^+(\Omega,\epsilon) & -pQ^+(\Omega,\epsilon) \\ -pQ^-(\Omega,\epsilon) & 2i\epsilon\sin z(\lambda_-)I-(1+p)Q^-(\Omega,\epsilon) \end{bmatrix}, \tag{4.25}$$

where I is an identity matrix in $\mathbb{C}^{N\times N}$.

We split the proof of Theorem 4.8 into three parts, described in Sections 4.2.2–4.2.4, where solutions of the system (4.24) are analyzed with respect to Ω in three different sets composing the complex plane.

4.2.2 Resolvent outside the continuous spectrum

We consider the resolvent operator $R_L(\Omega)$ for a fixed small $\epsilon \in (0,\epsilon_0)$. The following lemma shows that $R_L(\Omega)$ is a bounded operator from $l^2 \times l^2$ to $l^2 \times l^2$ for all $\Omega \in \mathbb{C}$ except three disks of small radii centred at $\{0, 1, -1\}$.

Lemma 4.10. *There are $\epsilon_0 > 0$ and $\delta, \delta_\pm > 0$ such that for any $\epsilon \in (0,\epsilon_0)$, the resolvent operator $R_L(\Omega) : l^2\times l^2 \to l^2\times l^2$ is bounded for all $\Omega \in \mathbb{C}\backslash\{B_\delta(0)\cup B_{\delta_+}(1)\cup B_{\delta_-}(-1)\}$. Moreover, $R_L(\Omega)$ has exactly $2N$ poles (counting multiplicities) inside $B_\delta(0)$.*

Proof. From the property of the free resolvent operator $R_0(\lambda)$, we know that the Green function in the representation (4.23) is bounded and exponentially decaying as $|n| \to \infty$ for any Ω such that $\lambda_\pm \notin [0,4]$. This gives $\Omega \notin \sigma_c(L) \equiv [1, 1+4\epsilon] \cup [-1-4\epsilon, -1]$. Therefore, $R_L(\Omega)$ is bounded map from $l^2\times l^2$ to $l^2\times l^2$ for any $\Omega \notin \sigma_c(L)$ if and only if the system of linear equations (4.24) is uniquely solvable. We shall study invertibility of the coefficient matrix $A(\Omega,\epsilon)$ of the linear system (4.24) for small $\epsilon > 0$ in various domains of the Ω-plane.

To find the expansion of $A(\Omega,\epsilon)$ for small $\epsilon > 0$, let us first study solutions of the transcendental equation (4.17),

$$2 - 2\cos z(\lambda) = \lambda, \quad \mathrm{Re} z(\lambda) \in [-\pi,\pi), \ \ \mathrm{Im} z(\lambda) < 0,$$

for real λ outside $[0,4]$. For $\lambda < 0$, we obtain $\cos z(\lambda) > 1$ and parametrize the solution as $z(\lambda) = -i\kappa$ with $\kappa > 0$. For $\lambda > 4$, we have $\cos z(\lambda) < -1$ so that the root can be parametrized by $z(\lambda) = -\pi - i\kappa$ with $\kappa > 0$. These parametrizations can be continuously extended for complex values $\lambda \notin [0,4]$, as shown on Figure 4.3, however, for the sake of asymptotic expansions that we perform below it would be simpler to treat the cases with $\mathrm{Re}\lambda < 0$ and $\mathrm{Re}\lambda > 4$ separately.

We are now going to construct the solutions $z(\lambda_\pm)$ to (4.17) for $\lambda_\pm = (\pm\Omega - 1)/\epsilon$ and $\Omega \in \mathbb{C}$ bounded away from $[-1-4\epsilon, -1] \cup [1, 1+4\epsilon]$, the continuous spectrum of the operator L. To do that, let us divide the complex plane Ω into domains where

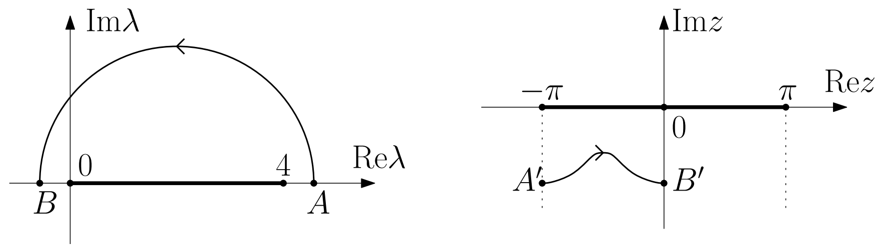

Figure 4.3: A continuous extension of $z(\lambda)$ for λ outside $[0, 4]$.

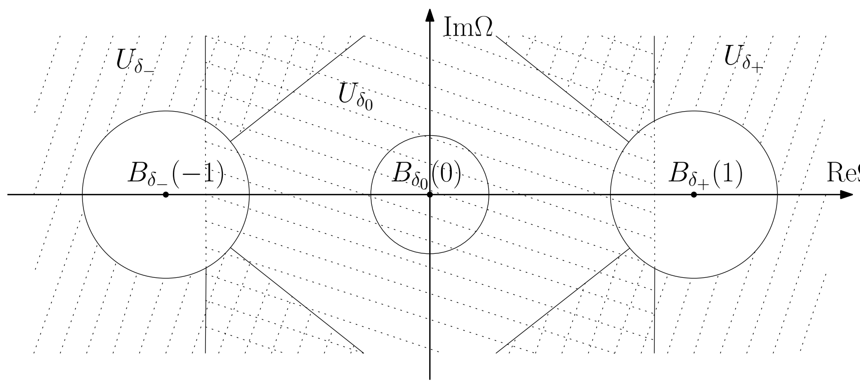

Figure 4.4: Schematic display of various domains in the Ω-plane.

$z(\lambda_\pm)$ admits continuous expansions in small $\epsilon > 0$. Figure 4.4 shows schematically the location of these domains on the Ω-plane.

Fix $\delta_0 \in (0, 1)$ and let Ω belong to the vertical strip

$$U_{\delta_0} := \{\Omega \in \mathbb{C} : \quad \mathrm{Re}(\Omega) \in [-\delta_0, \delta_0]\}.$$

If $\mathrm{Im}\Omega = 0$, then $\lambda_\pm < (\delta_0 - 1)/\epsilon < 0$. Hence, for $\Omega \in U_{\delta_0}$ the roots $z(\lambda_\pm) = -i\kappa_\pm$ are uniquely determined from the equation

$$e^{\kappa_\pm} + e^{-\kappa_\pm} - 2 = \frac{1 \mp \Omega}{\epsilon}, \quad \mathrm{Re}(\kappa_\pm) > 0, \quad \mathrm{Im}(\kappa_\pm) \in [-\pi, \pi),$$

which admits the asymptotic expansion

$$e^{\kappa_\pm} = \frac{1 \mp \Omega}{\epsilon} + 2 - \frac{\epsilon}{1 \mp \Omega} + \mathcal{O}(\epsilon^2) \quad \text{as} \quad \epsilon \to 0$$

and

$$e^{-\kappa_\pm} = \frac{\epsilon}{1 \mp \Omega} + \mathcal{O}(\epsilon^2) \quad \text{as} \quad \epsilon \to 0.$$

Therefore, both $\epsilon \sinh(\kappa_\pm)$ and $Q^\pm(\Omega,\epsilon)$ are analytic in ϵ near $\epsilon = 0$ and

$$2i\epsilon \sin z(\lambda_\pm) = 2\epsilon \sinh(\kappa_\pm) = 1 \mp \Omega + 2\epsilon + \mathcal{O}(\epsilon^2) \quad \text{as} \quad \epsilon \to 0,$$

and

$$Q^\pm(\Omega,\epsilon) = I + \mathcal{O}(\epsilon) \quad \text{as} \quad \epsilon \to 0.$$

It becomes now clear that $A(\Omega,\epsilon)$ is analytic in $\Omega \in U_{\delta_0}$ and $\epsilon \in [0,\epsilon_0)$ with the limit

$$A(\Omega,0) = \begin{bmatrix} -(p+\Omega)I & -pI \\ -pI & -(p-\Omega)I \end{bmatrix}. \tag{4.26}$$

Matrix $A(\Omega,0) \in \mathbb{C}^{2N\times 2N}$ is singular only for $\Omega = 0$. Thanks to analyticity of $A(\Omega,\epsilon)$, the determinant $D(\Omega,\epsilon) = \det A(\Omega,\epsilon)$ is also analytic in these variables and

$$D(\Omega,\epsilon) = (-\Omega^2)^N + \mathcal{O}(\epsilon) \quad \text{as} \quad \epsilon \to 0.$$

Therefore, there exist $2N$ zeros of $D(\Omega,\epsilon)$ for small $\epsilon \in (0,\epsilon_0)$ in a small disk $B_\delta(0)$ with $\delta = \mathcal{O}(\epsilon^{1/2N})$. By Cramer's rule, these zeros of $D(\Omega,\epsilon)$ give poles of $R_L(\Omega)$.

Fix $\delta_+ \in (4\epsilon,1)$ and $\theta_+ \in (\frac{\pi}{2},\pi)$. We now consider Ω in the domain (Figure 4.4)

$$U_{\delta_+} := \left\{\Omega = 1 + re^{i\theta}, \quad r > \delta_+, \;\; \theta \in (-\theta_+,\theta_+)\right\}.$$

Along the real axis in this domain, we have $\Omega > 1+\delta_+$ which gives $\lambda_- < (-2-\delta_+)/\epsilon < 0$ and $\lambda_+ > \delta_+/\epsilon > 4$. Hence, we have the same presentation for $z(\lambda_-) = -i\kappa_-$ but a different presentation for $z(\lambda_+) = -i\kappa_+ - \pi$. Now κ_+ is uniquely determined from the equation

$$e^{\kappa_+} + e^{-\kappa_+} + 2 = \frac{\Omega-1}{\epsilon} = \frac{r}{\epsilon}e^{i\theta}, \quad \text{Re}(\kappa_+) > 0, \quad \text{Im}(\kappa_+) \in [0,2\pi),$$

which admits the asymptotic expansions

$$e^{\kappa_+} = \frac{\Omega-1}{\epsilon} - 2 - \frac{\epsilon}{\Omega-1} + \mathcal{O}(\epsilon^2) \quad \text{as} \quad \epsilon \to 0$$

and

$$2i\epsilon \sin z(\lambda_+) = 1 - \Omega + 2\epsilon + \mathcal{O}(\epsilon^2) \quad \text{as} \quad \epsilon \to 0.$$

Since $\text{Re}(\kappa_+) \to \infty$ as $\epsilon \to 0$, $A(\Omega,0)$ is the same as matrix (4.26) and it is invertible for $\Omega \in S_{\delta_+}$. Similar arguments can be developed for

$$U_{\delta_-} := \left\{\Omega = -1 + re^{i\theta}, \quad r > \delta_-, \;\; \theta \in (\theta_-,2\pi-\theta_-)\right\},$$

where $\delta_- \in (4\epsilon, 1)$ and $\theta_- \in (0, \frac{\pi}{2})$. Since for sufficiently small ϵ there are ϵ-independent $\delta_0, \delta_\pm > 0$ such that

$$U_{\delta_0} \cup U_{\delta_+} \cup U_{\delta_-} = \mathbb{C}\backslash\{B_{\delta_+}(1) \cup B_{\delta_-}(-1)\},$$

we obtain the assertion of the lemma. □

Remark 4.11. The proof of Lemma 4.10 implies that poles of $R_L(\Omega)$ may have size $|\Omega| = \mathcal{O}(\epsilon^{1/2N})$. The results of the perturbation expansions (see [72] for details) imply that the eigenvalues bifurcating from 0 in the full spectral problem (4.5) have size $\mathcal{O}(\epsilon^{1/2})$. Moreover, the same perturbation expansion technique can be applied to show that eigenvalues of the truncated spectral problem (4.13) have the same size $\mathcal{O}(\epsilon^{1/2})$.

Remark 4.12. The parameter ϵ_0 governs the upper bound for δ_-, δ_0 and δ_+, the radii of balls centred at 0, -1 and 1. We need to keep ϵ_0 small enough to avoid collisions of eigenvalues bifurcating from zero with the rest of the spectrum.

4.2.3 Resolvent inside the continuous spectrum

We shall now consider the resolvent operator $R_L(\Omega)$ inside the continuous spectrum

$$\sigma_c(L) = [-1 - 4\epsilon, -1] \cup [1, 1 + 4\epsilon].$$

Thanks to the symmetry of system (4.23)–(4.24) in Ω, we can consider only one branch of the continuous spectrum $[1, 1+4\epsilon]$. Therefore, we set $\Omega = 1 + \epsilon\omega$ with $\omega \in [0, 4]$ and define

$$z(\lambda_+) = z(\omega) \equiv \theta \quad \text{and} \quad z(\lambda_-) = z(-2\epsilon^{-1} - \omega) \equiv -i\kappa.$$

It follows from (4.17) and (4.20) that $\theta \in [-\pi, 0]$ and $\kappa > 0$ are uniquely defined from equations

$$2 - 2\cos(\theta) = \omega, \quad 2\epsilon(\cosh(\kappa) - 1) = 2 + \epsilon\omega, \quad \omega \in [0, 4]. \tag{4.27}$$

The choice of $\theta \in [-\pi, 0]$ corresponds to the limiting operator $R_0^+(\omega)$ of the free resolvent. Since $R_0^+(\omega) : l^2_\sigma \to l^2_{-\sigma}$ is well defined for $\omega \in (0, 4)$ and $\sigma > \frac{1}{2}$, $R_L^+(1 + \epsilon\omega)$ is a bounded map from $l^2_\sigma \times l^2_\sigma$ to $l^2_{-\sigma} \times l^2_{-\sigma}$ for any $\omega \in (0, 4)$ and $\sigma > \frac{1}{2}$ if and only if there exists a unique solution of the linear system (4.24). On the other hand, the free resolvent is singular in the limits $\omega \downarrow 0$ and $\omega \uparrow 4$ and, therefore, we need to be careful in solving system (4.23)–(4.24) in this limit.

The following theorem describes the behaviour of the resolvent operator $R_L^+(\Omega)$ on the continuous spectrum of the operator L.

Theorem 4.13. *Let $m_1 = m_2 = \cdots = m_{N-1} = 1$ and let $p \geq 2$ be an integer. There exists $\epsilon_0 > 0$ such that for any $\omega \in [0,4]$ and any $\epsilon \in (0,\epsilon_0)$, there exist $C > 0$ such that*

$$\|R_L^+(1+\epsilon\omega)\|_{l_1^1 \times l_1^1 \to l^\infty \times l^\infty} \leq C\epsilon^{-1}, \tag{4.28}$$

where the upper sign indicates that ω is parametrized by $\omega = 2-2\cos(\theta)$ for $\theta \in [-\pi, 0]$.

To prove Theorem 4.13, we analyze solutions of system (4.24) for $\omega \in [0,4]$. Let us rewrite explicitly

$$q_j^+ = e^{-im_j\theta} \quad \text{and} \quad q_j^- = e^{-m_j\kappa}, \quad j \in \{1,2,...,N-1\}.$$

The coefficient matrix (4.25) for $\Omega = 1+\epsilon\omega$ with $\omega \in [0,4]$ is rewritten in the form

$$A(\theta,\epsilon) \equiv \begin{bmatrix} 2i\epsilon\sin(\theta)I - (1+p)M(\theta) & -pM(\theta) \\ -pN(\kappa) & 2\epsilon\sinh(\kappa)I - (1+p)N(\kappa) \end{bmatrix}, \tag{4.29}$$

where $M(\theta) \equiv Q(q_1^+, q_2^+, \cdots, q_{N-1}^+)$ and $N(\kappa) \equiv Q(q_1^-, q_2^-, \cdots, q_{N-1}^-)$. Note that θ and $M(\theta)$ are ϵ-independent, whereas $N(\kappa)$ depends on ϵ via κ. The linear system (4.24) is now expressed in the matrix form

$$A(\theta,\epsilon)c = h(\theta,\epsilon), \tag{4.30}$$

where components of $c \in \mathbb{C}^{2N}$ and $h \in \mathbb{C}^{2N}$ are given by

$$c = \left\{ \begin{array}{c} a_n \\ b_n \end{array} \right\}_{n\in S} \quad \text{and} \quad h(\theta,\epsilon) = \left\{ \begin{array}{c} \sum_{m\in\mathbb{Z}} e^{-i\theta|n-m|} f_m \\ \sum_{m\in\mathbb{Z}} e^{-\kappa|n-m|} g_m \end{array} \right\}_{n\in S}. \tag{4.31}$$

Thanks to the asymptotic expansion

$$e^\kappa = \frac{2}{\epsilon} + 2 + \omega - \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2) \quad \text{as} \quad \epsilon \to 0,$$

we have

$$2\epsilon\sinh(\kappa) = 2 + (2+\omega)\epsilon + \mathcal{O}(\epsilon^2) \quad \text{as} \quad \epsilon \to 0.$$

Both $A(\theta,\epsilon)$ and $h(\theta,\epsilon)$ are analytic in $\theta \in [-\pi, 0]$ and sufficiently small ϵ. The following lemma establishes the invertibility condition for matrix $A(\theta,\epsilon)$ with $\theta \in (-\pi, 0)$.

Lemma 4.14. *There exists $\epsilon_0 > 0$ such that the matrix $A(\theta,\epsilon)$ with $\epsilon \in [0,\epsilon_0)$ is invertible for any $\theta \in (-\pi, 0)$ provided $m_1 = m_2 = \cdots = m_{N-1} = 1$.*

Proof. We use the fact that matrix $A(\theta,\epsilon)$ is analytic in ϵ for sufficiently small value of this parameter. Therefore, it remains invertible if $A(\theta, 0)$ is invertible. To consider the limit $\epsilon \to 0$, we note that $\kappa \to \infty$ and $N(\kappa) \to I$ as $\epsilon \to 0$, so we have

$$A(\theta, 0) = \begin{bmatrix} -(1+p)M(\theta) & -pM(\theta) \\ -pI & (1-p)I \end{bmatrix}.$$

For any $p \in \mathbb{N}$, matrix $A(\theta, 0)$ is invertible if and only if matrix $M(\theta)$ is invertible. Let us then compute

$$D_N(q_1, q_2, \cdots, q_{N-1}) := \det Q(q_1, q_2, \cdots, q_{N-1}).$$

We note that $D_N(\pm 1, q_2, \cdots, q_{N-1}) = 0$ and $D_N(q_1, q_2, \cdots, q_{N-1})$ is a quadratic polynomial of q_1. Therefore,

$$D_N(q_1, q_2, \cdots, q_{N-1}) = (1 - q_1^2) D_N(0, q_2, \cdots, q_{N-1}) = (1 - q_1^2) D_{N-1}(q_2, \cdots, q_{N-1}).$$

Continuing the expansion recursively, we obtain the exact formula

$$D_N(q_1, q_2, \cdots, q_{N-1}) = (1 - q_1^2)(1 - q_2^2) \cdots (1 - q_{N-1}^2), \tag{4.32}$$

from which we conclude that $Q(q_1, q_2, \cdots, q_{N-1})$ is invertible if and only if all $q_j \neq \pm 1$. This implies that $M(\theta)$ is invertible if and only if all $e^{-im_j\theta} \neq \pm 1$, which is satisfied if all $m_j = 1$ and $\theta \in (-\pi, 0)$. Hence, there is $\epsilon_0 > 0$ such that for any $\epsilon \in [0, \epsilon_0)$, matrix $A(\theta, \epsilon)$ is invertible for $\theta \in (-\pi, 0)$ if all $m_j = 1$. □

In addition to the invertibility of matrix $A(\theta, \epsilon)$ for $\theta \in (-\pi, 0)$ we need to describe the null space of this matrix at $\theta = -\pi$ and $\theta = 0$.

Lemma 4.15. *For any integer $p \geq 2$ there exists $\epsilon_0 > 0$ such that at $\theta = -\pi$ and $\theta = 0$ the matrix $A(\theta, \epsilon)$ with $\epsilon \in [0, \epsilon_0)$ has a zero eigenvalue of geometric and algebraic multiplicities $N - 1$.*

Proof. The matrices $A_+(\epsilon) := A(0, \epsilon)$ and $A_-(\epsilon) := A(-\pi, \epsilon)$ can be written explicitly in the form

$$A_\pm(\epsilon) = \begin{bmatrix} -(1+p)M_\pm & -pM_\pm \\ -pN(\kappa_\pm) & 2\epsilon \sinh(\kappa_\pm) I - (1+p)N(\kappa_\pm) \end{bmatrix}, \tag{4.33}$$

where $\kappa_\pm > 0$ are uniquely defined by

$$2\epsilon(\cosh(\kappa_+) - 1) = 2, \quad 2\epsilon(\cosh(\kappa_-) - 1) = 2 + 4\epsilon,$$

whereas matrices $M_\pm$ are given by

$$M_+ = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

and

$$M_- = \begin{bmatrix} 1 & (-1)^{m_1} & (-1)^{m_1+m_2} & \dots & (-1)^{m_1+\dots+m_{N-1}} \\ (-1)^{m_1} & 1 & (-1)^{m_2} & \dots & (-1)^{m_2+\dots+m_{N-1}} \\ (-1)^{m_1+m_2} & (-1)^{m_2} & 1 & \dots & (-1)^{m_3+\dots+m_{N-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{m_1+\dots+m_{N-1}} & (-1)^{m_2+\dots+m_{N-1}} & (-1)^{m_3+\dots+m_{N-1}} & \dots & 1 \end{bmatrix}.$$

It is clear that $\text{Null}(M_+)$ and $\text{Null}(M_-)$ are $(N-1)$-dimensional.

The first N rows in $A_\pm(\epsilon)$ are multiples of the first row. In the limit $\epsilon \to 0$ we have

$$A_\pm(0) = \begin{bmatrix} -(1+p)M_\pm & -pM_\pm \\ -pI & (1-p)I \end{bmatrix}, \tag{4.34}$$

so that the last N rows in $A_\pm(0)$ are linearly independent. By continuity there exists $\epsilon_0 > 0$ such that the last N rows of $A_\pm(\epsilon)$ are linearly independent for all $\epsilon \in [0, \epsilon_0)$. Therefore, $\text{Null}(A_\pm(\epsilon))$ is $(N-1)$-dimensional for any $\epsilon \in [0, \epsilon_0)$.

It remains to prove that for $\epsilon \in (0, \epsilon_0)$ the zero eigenvalue of $A_\pm(\epsilon)$ is semi-simple. It is clear from the explicit form of $A_\pm(0)$ and $M_\pm$ that

$$u \in \text{Null}(A_\pm(0)) \quad \Leftrightarrow \quad u = \begin{bmatrix} (1-p)w \\ pw \end{bmatrix}, \quad w \in \text{Null}(M_\pm). \tag{4.35}$$

To construct a generalized kernel, we consider the inhomogeneous equation

$$A_\pm(0)\tilde{u} = u, \quad u \in \text{Null}(A_\pm(0)).$$

Then, we obtain for $w \in \text{Null}(M_\pm)$,

$$\tilde{u} = \begin{bmatrix} (1-p)\tilde{w} - w \\ p\tilde{w} \end{bmatrix}, \quad M_\pm \tilde{w} = (p-1)w.$$

Since $p \geq 2$, no non-trivial $\tilde{w} \in \mathbb{C}^N$ exists because $M_\pm$ is symmetric and the contradiction arises:

$$0 \equiv \langle w, M_\pm \tilde{w} \rangle_{\mathbb{C}^N} = (p-1)\|w\|^2_{\mathbb{C}^N}.$$

Therefore, the zero eigenvalue of matrices $A_\pm(0)$ has equal geometric and algebraic multiplicity. By continuity, this is also the case for the matrix $A_{\pm}(\epsilon)$ with $\epsilon \in [0, \epsilon_0)$. $\square$

Because the coefficient matrix $A(\theta, \epsilon)$ is singular at $\theta = 0$ and $\theta = -\pi$, we shall consider the limiting behaviour of solutions to linear system (4.30) near these points. Let us introduce a lemma that gives the sufficient condition that the unique solution c of the linear system (4.30) for small $\theta \neq 0$ and fixed $\epsilon \in (0, \epsilon_0)$ remains bounded in the limit $\theta \to 0$. Because ϵ is fixed, we can drop this parameter from the notations of the lemma.

Definition 4.16. Let A and B be square matrices of the same size, and let

$$\text{Null}B = \text{span}\{u_1, \dots, u_n\}, \qquad \text{Null}B^* = \text{span}\{v_1, \dots, v_n\}.$$

We define the $n \times n$ matrix $A|_{\text{Null}B}$ by its components:

$$(A|_{\text{Null}B})_{i,j} = \langle v_i, A u_j \rangle, \quad i, j = 1, \dots n.$$

Lemma 4.17. *Assume that $A(\theta) \in \mathbb{C}^{M \times M}$ and $h(\theta) \in \mathbb{C}^M$ are analytic in $\theta \in (-\theta_0, \theta_0)$ for $\theta_0 > 0$ and consider solutions of*

$$A(\theta)c = h(\theta), \quad c \in \mathbb{C}^M.$$

Assume that $A(\theta)$ is invertible for $\theta \neq 0$ and singular for $\theta = 0$ and that the zero eigenvalue of $A(0)$ has equal geometric and algebraic multiplicity $n \leq M$. A unique solution c for $\theta \neq 0$ is bounded as $\theta \to 0$ if

$$h(0) \perp \text{Null}(A^*(0)) \quad \text{and} \quad \text{Null}(A'(0)|_{\text{Null}(A(0))}) = \{0\}. \tag{4.36}$$

We denote the Hermite conjugate of a matrix $A_0 = A(0) \in \mathbb{C}^{M \times M}$ by $A_0^* = \overline{A_0^T}$. Let $J_0 = S^{-1} A_0 S$ be the Jordan normal form of matrix A_0. The null space of J_0 is spanned by a set of mutually orthogonal eigenvectors $\{u_i'\}_{i=1}^n$ satisfying $\langle u_i', u_j' \rangle_{\mathbb{C}^M} = \delta_{i,j}$. For matrices A_0 and A_0^* we have

$$\text{Null}(A_0) = \text{span}\{u_1, ..., u_n\} \quad \text{and} \quad \text{Null}(A_0^*) = \text{span}\{v_1, ..., v_n\}, \tag{4.37}$$

where $u_i = S u_i'$ and $v_i = (S^*)^{-1} u_i'$ for $1 \leq i \leq n$. The eigenvectors $\{u_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ also form mutually orthogonal bases:

$$\langle u_i, v_j \rangle_{\mathbb{C}^M} = \langle S u_i', (S^*)^{-1} u_i' \rangle_{\mathbb{C}^M} = \delta_{i,j} \quad \text{for all} \quad 1 \leq i, j \leq n. \tag{4.38}$$

The restriction of matrix $A_1 = A'(0) \in \mathbb{C}^{M \times M}$ on $\text{Null}(A_0)$ denoted by $A_1|_{\text{Null}(A_0)}$ can be expressed by the matrix $P \in \mathbb{C}^{n\times n}$ with elements

$$P_{ij} = \langle v_i, A_1 u_j \rangle_{\mathbb{C}^M} \quad \text{for all} \quad 1 \le i,j \le n. \tag{4.39}$$

Proof. The proof of the lemma is achieved with the method of Lyapunov–Schmidt reductions. Using analyticity of $A(\theta)$ and $h(\theta)$, let us expand

$$A(\theta) = A_0 + \theta A_1 + \theta^2 \tilde{A}(\theta), \quad h(\theta) = h_0 + \theta h_1 + \theta^2 \tilde{h}(\theta),$$

where $A_0 = A(0)$, $A_1 = A'(0)$, $h_0 = h(0)$, $h_1 = h'(0)$, and $\tilde{A}(\theta)$ and $\tilde{h}(\theta)$ are bounded as $\theta \to 0$. Given the basis for $\mathrm{Null}(A_0)$ in (4.37), we consider the orthogonal decomposition of the solution

$$c = \sum_{j=1}^{n} a_j u_j + b, \quad b \perp \mathrm{Null}(A_0). \tag{4.40}$$

The linear system becomes

$$\theta \sum_{j=1}^{n} a_j (A_1 + \theta \tilde{A}(\theta)) u_j + (A_0 + \theta A_1 + \theta^2 \tilde{A}(\theta)) b + = h_0 + \theta h_1 + \theta^2 \tilde{h}(\theta). \tag{4.41}$$

Projections of system (4.41) to the basis for $\mathrm{Null}(A_0^*)$ in (4.37) give n equations

$$\sum_{j=1}^{n} \left(P_{ij} + \theta \tilde{P}_{ij}(\theta) \right) a_j + \langle v_i, (A_1 + \theta \tilde{A}(\theta)) b \rangle_{\mathbb{C}^M} = \langle v_i, h_1 + \theta \tilde{h}(\theta) \rangle_{\mathbb{C}^M}, \quad 1 \le i \le n, \tag{4.42}$$

where P_{ij} is given in (4.39), $\tilde{P}_{ij}(\theta) = \langle v_i, \tilde{A}(\theta) u_j \rangle_{\mathbb{C}^M}$ is bounded as $\theta \to 0$, and we have used the condition $h_0 \perp \mathrm{Null}(A_0^*)$.

Let $Q : \mathbb{C}^M \to \mathrm{Ran}(A_0) \subset \mathbb{C}^M$ and $Q^* : \mathbb{C}^M \to \mathrm{Ran}(A_0^*) \subset \mathbb{C}^M$ be the projection operators. Recall that $\mathrm{Ran}(A_0) \perp \mathrm{Null}(A_0^*)$ and $\mathrm{Ran}(A_0^*) \perp \mathrm{Null}(A_0)$. Projection of system (4.41) to $\mathrm{Ran}(A_0)$ gives an equation for b

$$Q(A_0 + \theta A_1 + \theta^2 \tilde{A}(\theta)) Q^* b = h_0 + Q(\theta h_1 + \theta^2 \tilde{h}(\theta)) - \sum_{j=1}^{n} a_j Q (A_1 + \theta \tilde{A}(\theta)) u_j. \tag{4.43}$$

Because QA_0Q^* is invertible, there is a unique map $\mathbb{C}^n \ni (a_1, ..., a_n) \mapsto b \in \mathrm{Ran}(A_0^*)$ for any $\theta \in (-\theta_0, \theta_0)$ such that b is a solution of system (4.43) and for any $\theta \in (-\theta_0, \theta_0)$, there is $C > 0$ such that

$$\| b - (QA_0Q^*)^{-1} h_0 \|_{\mathbb{C}^M} \le C\theta. \tag{4.44}$$

Since $\mathrm{Null}(A_1|_{\mathrm{Null}(A_0)}) = \{0\}$, matrix P is invertible. For any b from solution of system (4.43) satisfying bound (4.44), there exists a unique solution of system (4.42) for $(a_1, ..., a_n)$ for any $\theta \in (-\theta_0, \theta_0)$ such that

$$\exists C > 0 : \quad \|a - P^{-1}(I - Q)(h_1 - A_1(QA_0Q^*)^{-1}h_0)\|_{\mathbb{C}^n} \leq C\theta.$$

For any $\theta \neq 0$, the solution of system $A(\theta)c = h(\theta)$ is unique. Therefore, the unique solution obtained from the decomposition (4.40) for any $\theta \in (-\theta_0, \theta_0)$ is equivalent to the unique solution of system $A(\theta)c = h(\theta)$ for $\theta \neq 0$. □

We shall check that the conditions (4.36) of Lemma 4.17 are satisfied for the matrix $A(\theta, \epsilon)$ (4.29) and the right-hand-side vector $h(\theta, \epsilon)$ (4.31) for both end points $\theta = 0$ and $\theta = -\pi$.

Lemma 4.18. *Let $h_+(\epsilon) := h(0, \epsilon)$, $h_-(\epsilon) := h(-\pi, \epsilon)$, and $\partial_\theta A_+(\epsilon) := \partial_\theta A(\theta, \epsilon)|_{\theta=0}$, $\partial_\theta A_-(\epsilon) := \partial_\theta A(\theta, \epsilon)|_{\theta=-\pi}$. There exists $\epsilon_0 > 0$ such that for any $\epsilon \in [0, \epsilon_0)$, it is true that*

$$h_\pm(\epsilon) \perp \mathrm{Null}(A^*_\pm(\epsilon)) \quad \text{and} \quad \mathrm{Null}(\partial_\theta A_\pm(\epsilon)|_{\mathrm{Null}(A_\pm(\epsilon))}) = \{0\}. \tag{4.45}$$

Proof. It is sufficient to develop the proof for $\theta = 0$. The proof for $\theta = -\pi$ is similar.

Recall that the first N rows of $A_+(\epsilon)$ are identical to the first row. Since components of the $N \times 1$ column vector $h(0, \epsilon)$ are given by

$$\left\{ \begin{array}{l} \sum_{m \in \mathbb{Z}} f_m \\ \sum_{m \in \mathbb{Z}} e^{-\kappa|n-m|} g_m \end{array} \right\}_{n \in S_+ \cup S_-},$$

the first N entries of $h(0, \epsilon)$ are also identical so that $h(0, \epsilon) \in \mathrm{Ran}(A_+(\epsilon)) \perp \mathrm{Null}(A^*_+(\epsilon))$ for any $\epsilon \in \mathbb{R}$. Therefore, the first condition (4.45) is satisfied.

Next, we compute $A_1(\epsilon) := \partial_\theta A(\theta, \epsilon)|_{\theta=0}$. We know that

$$2\epsilon(\cosh(\kappa) - 1) = 2 + \epsilon(2 - 2\cos(\theta)) \quad \Rightarrow \quad \frac{d\kappa}{d\theta} = \frac{\sin(\theta)}{\sinh(\kappa)},$$

therefore,

$$A_1(\epsilon) \equiv i \begin{bmatrix} 2\epsilon I + (1 + p)R & pR \\ 0 & 0 \end{bmatrix}, \tag{4.46}$$

where

$$R = \begin{bmatrix} 0 & m_1 & m_1+m_2 & \dots & m_1+\dots+m_{N-1} \\ m_1 & 0 & m_2 & \dots & m_2+\dots+m_{N-1} \\ m_1+m_2 & m_2 & 0 & \dots & m_3+\dots+m_{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1+\dots+m_{N-1} & m_2+\dots+m_{N-1} & m_3+\dots+m_{N-1} & \dots & 0 \end{bmatrix}.$$

Let $P(\epsilon)$ be the matrix in $\mathbb{C}^{(N-1)\times(N-1)}$ which represents the restriction $A_1(\epsilon)|_{\mathrm{Null}(A_0(\epsilon))}$:

$$P_{ij}(\epsilon) = \langle v_i, A_1(\epsilon)u_j\rangle_{\mathbb{C}^{2N}},$$

where $\{u_i\}_{i=1}^{N-1}$ and $\{v_i\}_{i=1}^{N-1}$ are bases for $\mathrm{Null}A_0(\epsilon)$ and $\mathrm{Null}A_0^*(\epsilon)$ respectively. If there is a non-zero eigenvector $a = [a^{(1)}, \dots, a^{(N-1)}]^T$ in the null space of $P(\epsilon)$, then for all $i \in \{1, \dots, N-1\}$ we have

$$0 = (P(\epsilon)a)_i = \sum_{j=1}^{N-1} \langle v_i, A_1(\epsilon)u_j\rangle a^{(j)} = \langle v_i, A_1(\epsilon)\sum_{j=1}^{N-1} a^{(j)}u_j\rangle.$$

This identity implies that $A_1(\epsilon)u \in \mathrm{Ran}A_0(\epsilon)$ for $u = \sum_{j=1}^{N-1} a_j u_j$. Hence, existence of $a \in \mathrm{Null}(P(\epsilon)) \subset \mathbb{C}^{N-1}$ is equivalent to existence of $u \in \mathrm{Null}(A_0(\epsilon)) \subset \mathbb{C}^{2N}$ such that $A_1(\epsilon)u \in \mathrm{Ran}(A_0(\epsilon)) \perp \mathrm{Null}(A_0^*(\epsilon))$. In other words, we need to find $u \in \mathrm{Null}(A_0(\epsilon))$ such that the first N entries of $A_1(\epsilon)u$ are identical (the other N entries of $A_1(\epsilon)u$ are zeros).

By continuity in ϵ, the second condition (4.45) is satisfied if it is satisfied for $\epsilon = 0$. Therefore, it is sufficient to check the existence of $u \in \mathrm{Null}(A_0(0))$ such that the first N entries of $A_1(0)u$ are identical.

It follows from relations (4.35) and (4.46) that existence of $u \in \mathrm{Null}(A_0(0))$ such that the first N entries of $A_1(0)u$ are identical is equivalent to the existence of $w \in \mathrm{Null}(M_+) \subset \mathbb{C}^N$ such that all entries of Rw are identical.

If $w = [w_1, w_2, ..., w_N]^T \in \mathrm{Null}(M_+)$, then

$$w_1 + w_2 + ... + w_N = 0. \tag{4.47}$$

Condition $(Rw)_1 = (Rw)_2$ gives

$$m_1(w_2 + ... + w_N) = m_1 w_1.$$

Constraint (4.47) implies that if $m_1 \neq 0$, then $w_1 = 0$ and $w_2 + \cdots + w_N = 0$. Continuing by induction for condition $(Rw)_j = (Rw)_{j+1}$, where $j \in \{1, 2, ..., N-1\}$, we obtain that if $m_j \neq 0$, then $w_j = 0$ for all $j \in \{1, 2, ..., N-1\}$. In view of constraint (4.47), we have $w_N = 0$ that is $w = 0 \in \mathbb{C}^N$. As a result, we have proved that $\mathrm{Null}(A_1(0)|_{\mathrm{Null}(A_0(0))}) = \{0\}$. By continuity in ϵ, $\mathrm{Null}(A_1(\epsilon)|_{\mathrm{Null}(A_0(\epsilon))}) = \{0\}$ for small $\epsilon > 0$, which gives the second condition (4.45) for $\theta = 0$. □

Remark 4.19. Lemma 4.18 is proved without assuming that all $m_j = 1$.

We now prove the main result of this section.

Proof of Theorem 4.13. By Lemmas 4.15 and 4.18, assumptions of Lemma 4.17 are satisfied and the unique solution of system (4.30) for $\theta \in (-\pi, 0)$ is continued to the unique bounded limit $c_0 = \lim_{\theta\to 0} c$. From the first N equations of system (4.24), we infer that

$$\theta = 0 : \quad \sum_{m\in S} ((1+p)a_m + pb_m) = -\sum_{m\in\mathbb{Z}} f_m.$$

As a result, the simple pole singularity at $\theta = 0$ ($z(\lambda_+) = 0$) in the Green's function representation (4.23) with the Puiseux expansion (4.21) is cancelled. Similarly, the simple pole singularity at $\theta = -\pi$ is cancelled.

On the other hand, the representation (4.23) contains ϵ in the denominator, which does not cancel out generally. As a result, Lemma 4.14 for all $m_j = 1$ and Lemma 4.18 give that for any $\omega \in [0, 4]$ and any $\epsilon \in (0, \epsilon_0)$, there exists $C > 0$ such that

$$\|\mathbf{a}\|_{l^\infty} \leq C\epsilon^{-1}.$$

This gives bound (4.28) and hence Theorem 4.13. □

4.2.4 Matching conditions for the resolvent operator

To complete the proof of Theorem 4.8, we need to prove that no singularities of linear system (4.24) are located inside the disks $B_{\delta_+}(1)$ and $B_{\delta_-}(-1)$ for ϵ-independent $\delta_\pm > 0$. It is again sufficient to consider the disk $B_{\delta_+}(1)$ because of the symmetry in the Ω-plane.

The free resolvent operator $R_0^+(\lambda) : l^2_\sigma \to l^2_{-\sigma}$ with $\sigma > \frac{1}{2}$ is extended meromorphically in variable $\theta(\lambda)$ for $\lambda \in \mathbb{C}^+\backslash(\{0\}\cup\{4\})$ with simple poles at $\theta = 0$ ($\lambda = 0$) and $\theta = -\pi$ ($\lambda = 4$). Since $R_0^+(\lambda) : l^1 \to l^\infty$ for $\lambda \in (0, 4)$ is a bounded operator, it follows that $R_L^+(1+\epsilon\omega) : l^1 \times l^1 \to l^\infty \times l^\infty$ with $\omega \in (0, 4)$ is also bounded. In addition, we know from Theorem 4.13 that resolvent operator $R_L^+(1+\epsilon\omega) : l^1_1 \times l^1_1 \to l^\infty \times l^\infty$ is bounded for $\omega \in [0, 4]$ and the pole singularities are cancelled. As a result, the resolvent operator $R_L^+(1+\epsilon\lambda)$ can be extended as a bounded operator from $l^2_\sigma \times l^2_\sigma$ to $l^2_{-\sigma} \times l^2_{-\sigma}$ with $\sigma > \frac{1}{2}$ for any $\lambda \in \mathbb{C}^+\backslash(\{0\}\cup\{4\})$. We need to show that no singularities of the resolvent operator $R_L(1+\epsilon\lambda)$ exist in the upper semi-annulus

$$D_{\delta_+} = \left\{\lambda \in \mathbb{C}^+ : \quad \gamma_+ < |\lambda| < \delta_+\epsilon^{-1}\right\} \subset B_{\delta_+}(1),$$

where $\gamma_+ > 4$ and $\delta_+ \in (0, 1)$. A similar analysis can also be used to show that the resolvent operator $R_L^-(1+\epsilon\lambda)$ can be extended as a bounded operator in the lower semi-disk in $B_{\delta_+}(1)$.

Lemma 4.20. *For any $\epsilon \in (0, \epsilon_0)$ and all $\lambda \in D_{\delta_+}$, the resolvent operator $R_L(1+\epsilon\lambda)$ is a bounded operator from $l^2 \times l^2$ to $l^2 \times l^2$.*

Proof. Since the continuous spectrum does not touch boundaries of D_{δ_+}, the statement is true if and only if there exists a unique solution of linear system (4.24).

Let us denote $z(\lambda_+) = z(\lambda)$ and $z(\lambda_-) = -i\kappa(\lambda)$, where $z(\lambda)$ is found from the transcendental equation (4.17) and $\kappa(\lambda)$ with $\mathrm{Re}(\kappa(\lambda)) > 0$ admits the asymptotic expansion for $\lambda \in D_{\delta_+}$

$$e^{\kappa(\lambda)} = \frac{2+\epsilon\lambda}{\epsilon} + 2 - \frac{\epsilon}{2+\epsilon\lambda} + \mathcal{O}(\epsilon^2) \quad \text{as} \quad \epsilon \to 0.$$

As earlier, we denote $q_j^+ = e^{-im_j z(\lambda)}$ and $q_j^- = e^{-m_j\kappa(\lambda)}$ for $j \in \{1, 2, ..., N-1\}$.

We write the coefficient matrix (4.25) for $\Omega = 1 + \epsilon\lambda$ in the form

$$A(\lambda,\epsilon) = \begin{bmatrix} -\epsilon\sqrt{\lambda(\lambda-4)}I-(1+p)M(\lambda) & -pM(\lambda) \\ -pN(\kappa) & \sqrt{(2+\epsilon\lambda)(2+\epsilon\lambda+4\epsilon)}I-(1+p)N(\kappa) \end{bmatrix}, \tag{4.48}$$

where $M(\lambda) \equiv Q(q_1^+, q_2^+, \cdots, q_{N-1}^+)$, $N(\kappa(\lambda)) \equiv Q(q_1^-, q_2^-, \cdots, q_{N-1}^-)$, and the appropriate branches of $\sin z(\lambda)$ and $\sinh(\kappa(\lambda))$ are chosen in the domain D_{δ_+}.

Let $|\lambda| = \mathcal{O}(\epsilon^{-r})$ as $\epsilon \to 0$ for $r \in [0, 1)$. Then, we have

$$A(\lambda,\epsilon) \to \begin{bmatrix} -(1+p)M(\lambda) & -pM(\lambda) \\ -pI & (1-p)I \end{bmatrix} \quad \text{as} \quad \epsilon \to 0, \tag{4.49}$$

where $M(\lambda) \to I$ as $\epsilon \to 0$ if $r \in (0, 1)$ and $M(\lambda) \nrightarrow I$ as $\epsilon \to 0$ if $r = 0$. The limiting matrix (4.49) is not singular if $\gamma_+ > 4$. Hence $A(\lambda, \epsilon)$ is not singular for small $\epsilon \geq 0$ if $|\lambda| = \mathcal{O}(\epsilon^{-r})$ with $r \in [0, 1)$.

Let $|\lambda| = \mathcal{O}(\epsilon^{-r})$ as $\epsilon \to 0$ for $r \in (0, 1]$. Then, we have

$$A(\lambda,\epsilon) \to \begin{bmatrix} -(1+\epsilon\lambda+p)I & -pI \\ -pI & (1+\epsilon\lambda-p)I \end{bmatrix} \quad \text{as} \quad \epsilon \to 0.$$

Again, the limiting matrix is not singular if $\epsilon\lambda \neq -1$ (that is $\delta_+ < 1$) and hence $A(\lambda, \epsilon)$ is not singular for small $\epsilon \geq 0$ if $|\lambda| = \mathcal{O}(\epsilon^{-r})$ with $r \in (0, 1]$.

Since the above asymptotic scaling overlap at any $r \in (0, 1)$, the matrix $A(\lambda, \epsilon)$ is not singular in the domain D_{δ_+} for small $\epsilon > 0$. ☐

Theorem 4.8 is now proven with Lemma 4.10, Theorem 4.13, and Lemma 4.20.

4.2.5 Perturbation arguments for the full resolvent

Let us now consider the full spectral problem (4.5). Thanks to Proposition 3.3 and expansion (4.4), we can represent ϕ_n^{2p} by

$$\phi_n^{2p} = \sum_{m \in S} \delta_{n,m}(1 + \epsilon\chi_m) + \epsilon^2 W_n, \tag{4.50}$$

where $\{\chi_m\}_{m \in S}$ is a set of numerical coefficients and $\{W_n\}_{n \in \mathbb{Z}} \in l^2$ is a new potential such that $\|\mathbf{W}\|_{l^2} = \mathcal{O}(1)$ as $\epsilon \to 0$.

In variables $\{(a_n, b_n)\}_{n \in \mathbb{Z}}$, the resolvent problem can be rewritten in the operator form

$$(\tilde{L} + \epsilon^2 \tilde{W}) \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} - \Omega \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ -\mathbf{g} \end{bmatrix}, \tag{4.51}$$

where

$$\tilde{L} = \begin{bmatrix} -\epsilon\Delta + I - (1+p)\tilde{V} & -p\tilde{V} \\ p\tilde{V} & \epsilon\Delta - I + (1+p)\tilde{V}, \end{bmatrix},$$
$$\tilde{W} = \begin{bmatrix} -(1+p)W & -pW \\ pW & (1+p)W, \end{bmatrix},$$

and $\tilde{V}$ is the associated compact potential such that

$$(\tilde{V}u)_n = \sum_{m \in S} \delta_{n,m}(1 + \epsilon\chi_m)u_m, \quad n \in \mathbb{Z}.$$

Let us denote the solution of the inhomogeneous system (4.51) by

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = R(\Omega) \begin{bmatrix} \mathbf{f} \\ -\mathbf{g} \end{bmatrix}, \tag{4.52}$$

where $R(\Omega)$ is the resolvent operator of the full spectral problem (4.5). The following theorem represents the main result of this section.

Theorem 4.21. *Fix disjoint compact subsets S_+ and S_- on $\mathbb{Z}$ such that $S_+ \cup S_-$ is simply-connected with N elements. Let $B_\delta(0) \subset \mathbb{C}$ denote a ball of radius δ centred at the origin. For any integer $p \geq 2$, there are $\epsilon_0 > 0$ and $\delta > 0$ such that for any fixed $\epsilon \in (0, \epsilon_0)$ the resolvent operator*

$$R(\Omega) : l^2 \times l^2 \to l^2 \times l^2$$

is bounded for any $\Omega \notin B_\delta(0) \cup [-1-4\epsilon, -1] \cup [1, 1+4\epsilon]$ and has exactly $2N$ poles *(counting multiplicities) inside $B_\delta(0)$. Moreover, for any $\epsilon \in (0, \epsilon_0)$ there is $C > 0$ such that the limiting operators*

$$R^{\pm}(\Omega) := \lim_{\mu \downarrow 0} R(\Omega \pm i\mu), \qquad \Omega \in [-1-4\epsilon, -1] \cup [1, 1+4\epsilon],$$

admit the uniform bounds

$$\|R^{\pm}(\Omega)\|_{l^1_1 \times l^1_1 \to l^\infty \times l^\infty} \leq C\epsilon^{-1}, \qquad \forall \Omega \in [-1-4\epsilon, -1] \cup [1, 1+4\epsilon].$$

Proof. Let $R_{\tilde{L}}(\Omega)$ be the resolvent operator for the inverse operator $(\tilde{L} - \Omega I)^{-1}$ associated with the compactly supported potential $\tilde{V}$. We shall prove that Theorem 4.8 remains valid for the resolvent operator $R_{\tilde{L}}(\Omega)$. Assuming it, the rest of the proof relies on the perturbation arguments and the resolvent identities

$$R(\Omega) = R_{\tilde{L}}(\Omega)(I + \epsilon^2 \tilde{W} R_{\tilde{L}}(\Omega))^{-1} = (I + \epsilon^2 R_{\tilde{L}}(\Omega)\tilde{W})^{-1} R_{\tilde{L}}(\Omega).$$

Indeed, outside the continuous spectrum located at

$$\sigma_c(\tilde{L} + \epsilon^2 \tilde{W}) = \sigma_c(\tilde{L}) = \sigma_c(L) \equiv [-1-4\epsilon, -1] \cup [1, 1+4\epsilon],$$

the resolvent operator $R_{\tilde{L}}(\Omega)$ is only singular inside the disk $B_{\delta_0}(0)$, where perturbation theory of isolated eigenvalues apply. Inside the continuous spectrum, $R_{\tilde{L}}(\Omega)$ is extended as a bounded operator from $l^1_1 \times l^1_1$ to $l^\infty \times l^\infty$ such that for any $\Omega \in [1, 1+4\epsilon]$ and any $\epsilon \in (0, \epsilon_0)$, there is $C > 0$ such that

$$\exists C > 0 : \quad \|R^{\pm}_{\tilde{L}}(\Omega)\|_{l^1_1 \times l^1_1 \to l^\infty \times l^\infty} \leq C\epsilon^{-1}. \tag{4.53}$$

Since $\tilde{W}$ is a bounded (Ω,ϵ)-independent operator from $l^\infty \times l^\infty$ to $l^1_1 \times l^1_1$ (note here that $\boldsymbol{\phi} \in l^2_{1/2}$, see Remark 3.6), bound (4.53) implies that

$$\exists C > 0 : \quad \|\epsilon^2 \tilde{W} R_{\tilde{L}}(\Omega)\|_{l^1_1 \times l^1_1 \to l^1_1 \times l^1_1} \leq C\epsilon,$$

so that $(I + \epsilon^2 \tilde{W} R_{\tilde{L}}(\Omega))$ is an invertible bounded operator with a bounded inverse from $l^1_1 \times l^1_1$ to $l^1_1 \times l^1_1$ for small $\epsilon > 0$.

We only need to extend Theorem 4.8 to the resolvent operator $R_{\tilde{L}}(\Omega)$. The Green's function representation (4.23) and the linear system (4.24) are now written with the factor $(1+\epsilon\chi_m)$ in the sum over $m \in S$. This implies that the coefficient matrix $A(\Omega, \epsilon)$

is now written as

$$\tilde{A}(\Omega,\epsilon) := \begin{bmatrix} 2i\epsilon\sin z(\lambda_+)I-(1+p)Q^+(\Omega,\epsilon)(I+\epsilon D) & -pQ^+(\Omega,\epsilon)(I+\epsilon D) \\ -pQ^-(\Omega,\epsilon)(I+\epsilon D) & 2i\epsilon\sin z(\lambda_-)I-(1+p)Q^-(\Omega,\epsilon)(I+\epsilon D) \end{bmatrix}, \quad (4.54)$$

where D is a diagonal matrix of elements $\{\chi_m\}_{m\in S}$. Lemmas 4.10, 4.14, 4.15, 4.18, and 4.20 remain valid as these lemmas were proved from the limit $\epsilon = 0$, where $\tilde{A}(\Omega,0) = A(\Omega,0)$. Therefore, Theorem 4.8 also holds for the resolvent operator $R_{\tilde{L}}(\Omega)$. □

4.2.6 Case study for a non-simply-connected two-site soliton

We explain now why the resolvent operator associated with non-simply-connected multi-site discrete solitons have singularities near the anti-continuum limit. Lemma 4.14 suggests that the determinant $D_N(q_1,q_2,\cdots,q_{N-1})$ given by (4.32) has zeros for $\theta\in(-\pi,0)$ if $m_j\geq 2$ for some $1\leq j\leq N-1$.

Let us consider a case study of a two-site soliton with $n_1=0$ and $n_2=m\geq 2$. For clarity of presentation, we only consider $p\geq 2$. The power series expansions (4.4) give

$$m\geq 3: \quad \phi_n^{2p} = (\delta_{n,0}+\delta_{n,m})\left(1+2\epsilon-2\epsilon^2\right)+\epsilon^3 W_n, \quad n\in\mathbb{Z}, \quad (4.55)$$

and

$$m=2: \quad \phi_n^{2p} = (\delta_{n,0}+\delta_{n,m})\left(1+2\epsilon-3\epsilon^2\right)+\epsilon^3 W_n, \quad n\in\mathbb{Z}, \quad (4.56)$$

where $\{W_n\}_{n\in\mathbb{Z}}\in l^2$ is a new potential such that $\|\mathbf{W}\|_{l^2}=\mathcal{O}(1)$ as $\epsilon\to 0$.

Let us consider the coefficient matrix $A(\theta,\epsilon)$ (4.29) at the continuous spectrum which corresponds to $\theta\in[-\pi,0]$. We have explicitly

$$M(\theta) = \begin{bmatrix} 1 & e^{-im\theta} \\ e^{-im\theta} & 1 \end{bmatrix}, \quad N(\kappa) = \begin{bmatrix} 1 & e^{-2\kappa} \\ e^{-2\kappa} & 1 \end{bmatrix}.$$

Note that $\det M(\theta) = 1-e^{-2im\theta}$. Besides the end points $\theta=-\pi$ and $\theta=0$, the matrix $M(\theta)$ (and, therefore, the limiting matrix $A(\theta,0)$) is singular at the intermediate points $\theta_j=-\frac{\pi j}{m}$ for $j=1,2,...,m-1$.

If $m=2$, there is only one intermediate-point singularity of $A(\theta,0)$ at $\theta=-\frac{\pi}{2}$. We have $\dim\mathrm{Null}A(-\frac{\pi}{2},0)=1$ and

$$\mathrm{Null}A^*\left(-\frac{\pi}{2},0\right) = \mathrm{span}\{e_1\}, \quad e_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The first two entries of the right-hand-side vector $h(\theta,\epsilon)$ in the linear system (4.30) are given explicitly by

$$h_1(\theta,\epsilon) = \sum_{n\in\mathbb{Z}} e^{-i\theta|n|} f_n, \quad h_2(\theta,\epsilon) = \sum_{n\in\mathbb{Z}} e^{-i\theta|n-2|} f_n.$$

The constraint $\langle e_1, h(-\frac{\pi}{2},0)\rangle_{\mathbb{C}^4} = 0$ of Lemma 4.17 gives $h_1(-\frac{\pi}{2},0) = -h_2(-\frac{\pi}{2},0)$ and it is equivalent to the constraint $f_1 = 0$. If $f \in l^1$ with $f_1 \neq 0$, then the solution of the linear system (4.24) and hence the resolvent operator (4.23) has a singularity at $\Omega = 1+2\epsilon$ ($\theta = -\frac{\pi}{2}$) as $\epsilon \to 0$. This singularity indicates a resonance at the mid-point of the continuous spectrum in the anti-continuum limit.

We would like to show that the resonance does not actually occur at the continuous spectrum if $\epsilon > 0$ and does not lead to (unstable) eigenvalues off the continuous spectrum. To do so, we use the perturbation theory up to the quadratic order in ϵ.

Expanding solutions of the transcendental equation

$$2\epsilon(\cosh\kappa - 1) = 2 + \epsilon\omega, \quad \omega = 2 - 2\cos\theta,$$

we obtain

$$e^{-\kappa} = \frac{1}{2}\epsilon - \frac{2+\omega}{4}\epsilon^2 + \mathcal{O}(\epsilon^3) \quad \text{as} \quad \epsilon \to 0$$

and

$$2\epsilon\sinh\kappa = 2 + (2+\omega)\epsilon - \epsilon^2 + \mathcal{O}(\epsilon^3) \quad \text{as} \quad \epsilon \to 0.$$

Using expansion (4.56) for $m = 2$, we obtain the extended coefficient matrix $\tilde{A}(\theta,\epsilon)$ in the form

$$\tilde{A}(\theta,\epsilon) := \begin{bmatrix} 2i\epsilon\sin\theta I - (1+p)\nu(\epsilon)M(\theta) & -p\nu(\epsilon)M(\theta) \\ -p\nu(\epsilon)N(\kappa) & 2\epsilon\sinh\kappa I - (1+p)\nu(\epsilon)N(\kappa) \end{bmatrix},$$

where $\nu(\epsilon) = 1+2\epsilon-3\epsilon^2+\mathcal{O}(\epsilon^3)$. Using `Mathematica`, we expand roots of $\det\tilde{A}(\theta,\epsilon) = 0$ near $\theta = -\frac{\pi}{2}$ and $\epsilon = 0$ to obtain

$$\theta = -\frac{\pi}{2} + (p-1)\epsilon - 2(p-1)\epsilon^2 + i(p-1)^2\epsilon^2 + \mathcal{O}(\epsilon^3) \quad \text{as} \quad \epsilon \to 0. \tag{4.57}$$

Since $\text{Im}\theta > 0$ for small $\epsilon > 0$ and $z(\lambda_+) = \theta$, the solution of the linear system (4.30) is singular at the point $z(\lambda_+)$, which does not belong to the domain $\text{Im}z(\lambda_+) < 0$ and hence violates the condition (4.17).

The singularity of the solution of the linear system (4.30) is still located near the continuous spectrum for small $\epsilon > 0$ and, therefore, the resolvent operator $R(\Omega)$ becomes large near the points $\Omega = \pm(1+2\epsilon)$ (although, it is always a bounded operator from $l^2_\sigma \times l^2_\sigma$ to $l^2_{-\sigma} \times l^2_{-\sigma}$ for small $\epsilon > 0$ and fixed $\sigma > \frac{1}{2}$). Since $\sin\theta$ is nonzero for

$\theta = -\frac{\pi}{2}$, the norm of $R(\Omega)$ is proportional to the norm of inverse matrix $\tilde{A}^{-1}(\theta, \epsilon)$.

Figure 4.5 illustrates the singularities of the resolvent operator $R(\Omega)$ by plotting the 2-norm pseudospectra of the coefficient matrix $A(\Omega, \epsilon)$ in the complex Ω-plane for $p = 2$ and $\epsilon = 0.05$. The matrix 2-norm $\|\cdot\|_2$ is defined by

$$\|A\|_2 = \max_{\|\mathbf{u}\|_{l^2}=1} \|A\mathbf{u}\|_{l^2}.$$

The subplots (a) and (b) for $m = 1$ show that the matrix is singular at the edges of the continuous spectrum $\Omega = \pm 1$ and $\Omega = \pm(1 + 2\epsilon)$, and at four points on the imaginary axis, the latter being attributed to the splitting of zero eigenvalue in the anti-continuum limit. The subplots (c) and (d) for $m = 2$ and $m = 3$ respectively show that in addition to singularities at the edges of continuous spectrum there are also $m - 1$ local maxima at its intermediate points. This local maxima correspond to the minima of $\det A(\Omega, \epsilon)$. We also notice the wedges on the level sets as they cross the continuous spectrum. These features occur due to the jump discontinuities in $z(\lambda_+)$ across the continuous spectrum.

Figure 4.6 further illustrates what exactly happens at the continuous spectrum. On the left, we plot $\left\|A(\Omega, \epsilon)^{-1}\right\|_2$ versus $\theta \in (-\pi, 0)$ for the case $m = 2$. On the right, we show that the height of the local maxima near $\theta = -\pi/2$ is proportional to ϵ^{-2} as prescribed by the imaginary part of formula (4.57).

Let us give an illustration for pseudospectra of the resolvent operator $R(\Omega)$. Recall that on the continuous spectrum $\Omega \in [1, 1 + 4\epsilon]$, $R(\Omega)$ is a bounded operator from $l^2_\sigma \times l^2_\sigma$ to $l^2_{-\sigma} \times l^2_{-\sigma}$) for fixed $\sigma > \frac{1}{2}$. To incorporate the weighted l^2 spaces, we consider the renormalized resolvent operator

$$\tilde{R}_L(\Omega) = (\tilde{L} - \Omega \tilde{I}_2)^{-1} : l^2 \times l^2 \to l^2 \times l^2,$$

where $\tilde{L}$ is derived from L by replacing operators I, Δ and V with $\tilde{I}$, $\tilde{\Delta}$ and $\tilde{V}$, and $\tilde{I}_2 = \mathrm{diag}\{\tilde{I}, \tilde{I}\}$. Here

$$\tilde{I}_{n,m} = \kappa_n^2 \delta_{n,m}, \quad \tilde{V}_{n,m} = \tilde{I}_{n,m} \sum_{j \in S} \delta_{n,j},$$

$$\tilde{\Delta}_{n,n} = -2\kappa_n^2, \quad \tilde{\Delta}_{n,n+1} = \tilde{\Delta}_{n+1,n} = \kappa_n \kappa_{n+1},$$

and $\kappa_n = (1 + n^2)^{\sigma/2}$. The lattice problem is considered for $2K + 1$ grid points and the corresponding matrix representation of operators $\tilde{L}$ and $\tilde{I}_2$ is constructed subject to the Dirichlet boundary conditions.

The level sets for the $(2K + 1) \times (2K + 1)$ matrix approximation of the resolvent

$\tilde{R}(\Omega)$ are plotted on Figure 4.7. The subplots of Figure 4.7 correspond to the subplots of Figure 4.5. We observe that the norm of $\tilde{R}(\Omega)$ has the same global behaviour as the norm of $A(\Omega,\epsilon)^{-1}$ has. However, the resolvent operator $\tilde{R}(\Omega)$ has no singularities at the edges $\Omega = \pm 1$ and $\Omega = \pm(1+4\epsilon)$ because these singularities are cancelled according to Lemma 4.18 (which remains true for any $m \geq 1$, see Remark 4.19). As the operator L is not self-adjoint, convergence of the level sets of $\tilde{R}_L(\Omega)$ is not an obvious result. Nevertheless, we explored numerically that the results do converge as K gets large.

Although our analytical results do not exclude resonances at the intermediate points of the continuous spectrum for the linearized dNLS equation (4.5), the case study of a two-site discrete soliton suggests that the resonances do not happen at the continuous spectrum for small but finite values of $\epsilon > 0$. Moreover, the resonances do not bifurcate to the isolated eigenvalues off the continuous spectrum because isolated eigenvalues near the continuous spectrum would violate the count of unstable eigenvalues (4.12) provided by Proposition 4.6. Therefore, the only scenario for these resonances is to move to the resonant poles on the wrong sheets $\text{Im} z(\lambda_\pm) > 0$ of the definition of $z(\lambda_\pm)$ in (4.17).

4.2.7 Resolvent for the cubic dNLS case

For the cubic dNLS, $p = 1$, the proof of Theorem 4.21 cannot be achieved in the general case. Indeed, if $p = 1$ matrices $A_\pm(0)$ given by (4.34) have a zero eigenvalue of algebraic multiplicity $2N - 2$ and geometric multiplicity $N - 1$. This clearly violates a non-degeneracy condition of Lemma 4.17. Let us consider the cases of $N = 1$ and $N \geq 2$ separately.

Since for $N = 1$ we have $\text{Null}A_\pm(0) = \{0\}$, it is easy to show that Theorem 4.21 applies for the fundamental soliton of the cubic dNLS equation:

Corollary 4.22. *The result of Theorem 4.21 holds for $p = 1$ if $N = 1$.*

Proof. If $N = 1$ and $p = 1$, the coefficient matrix (4.54) reduces to a 2×2 matrix

$$\tilde{A}(\Omega,\epsilon) = \begin{bmatrix} 2i\epsilon \sin z(\lambda_+) - 2(1+\epsilon\chi_0) & -(1+\epsilon\chi_0) \\ -(1+\epsilon\chi_0) & 2i\epsilon \sin z(\lambda_-) - 2(1+\epsilon\chi_0) \end{bmatrix}.$$

For small $\epsilon > 0$, this matrix is only singular in a small ball centred at zero, where a double pole of $R_{\tilde{L}}(\Omega)$ and $R(\Omega)$ resides. ☐

For $N \geq 2$, when discrete solitons are supported on several sites in the anti-continuum limit, the null space of $A_\pm(0)$ is degenerate, but the degeneracy disappears for $\epsilon > 0$:

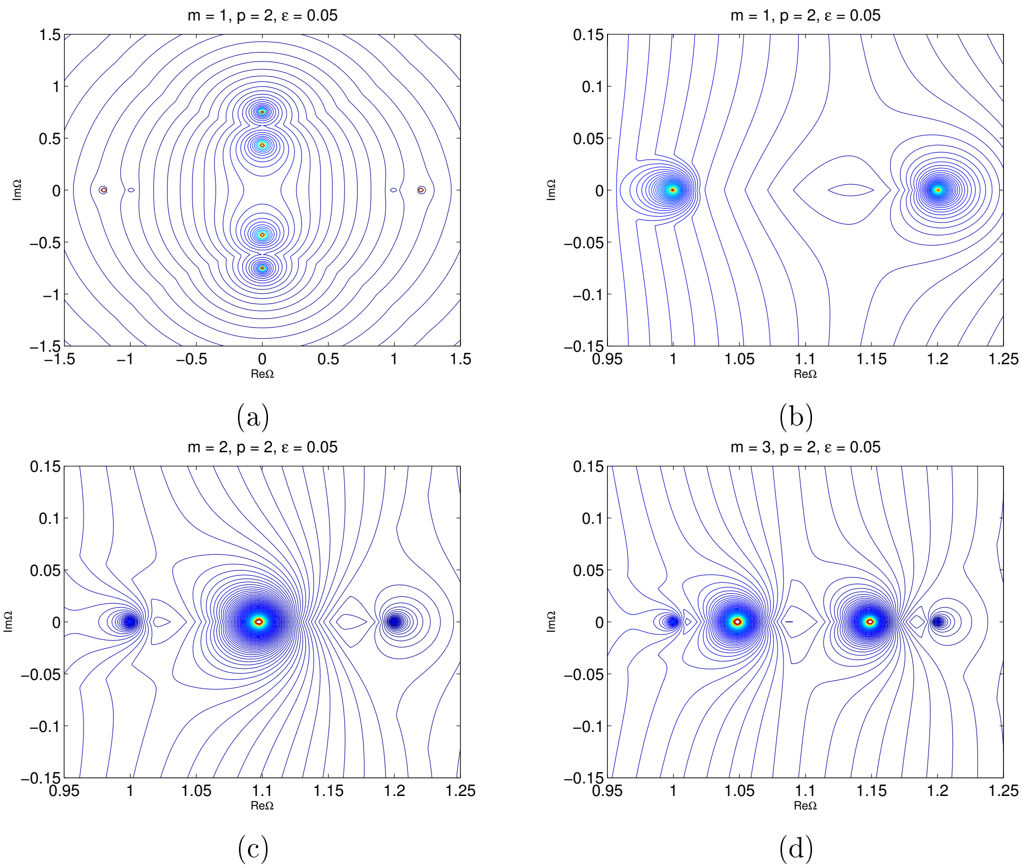

Figure 4.5: Level sets for $\left\|A(\Omega,\epsilon)^{-1}\right\|_2$ in the Ω-plane. The levels are equidistant on a logarithmic scale.

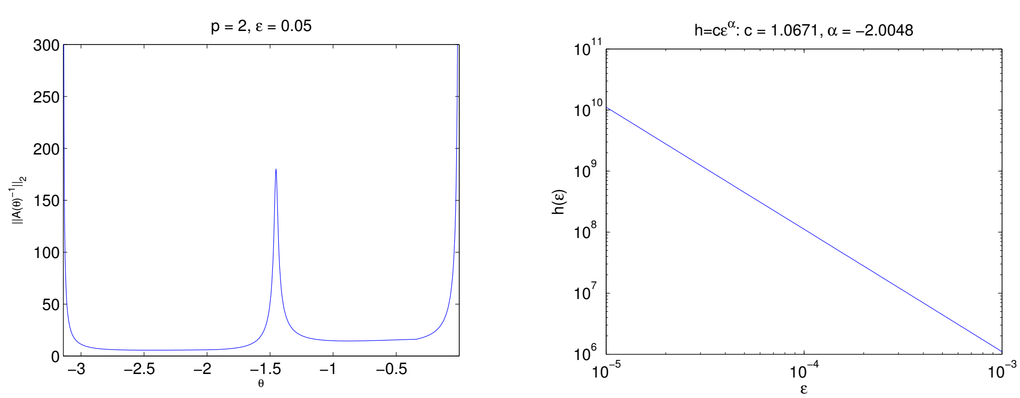

Figure 4.6: Left: Norm $\left\|A(\Omega,\epsilon)^{-1}\right\|_2$ versus $\theta \in (-\pi, 0)$ for $m = 2$. Right: The value of local maxima of $\left\|A(\Omega,\epsilon)^{-1}\right\|_2$ in the neighbourhood of $\theta = -\pi/2$ as a function of ϵ.

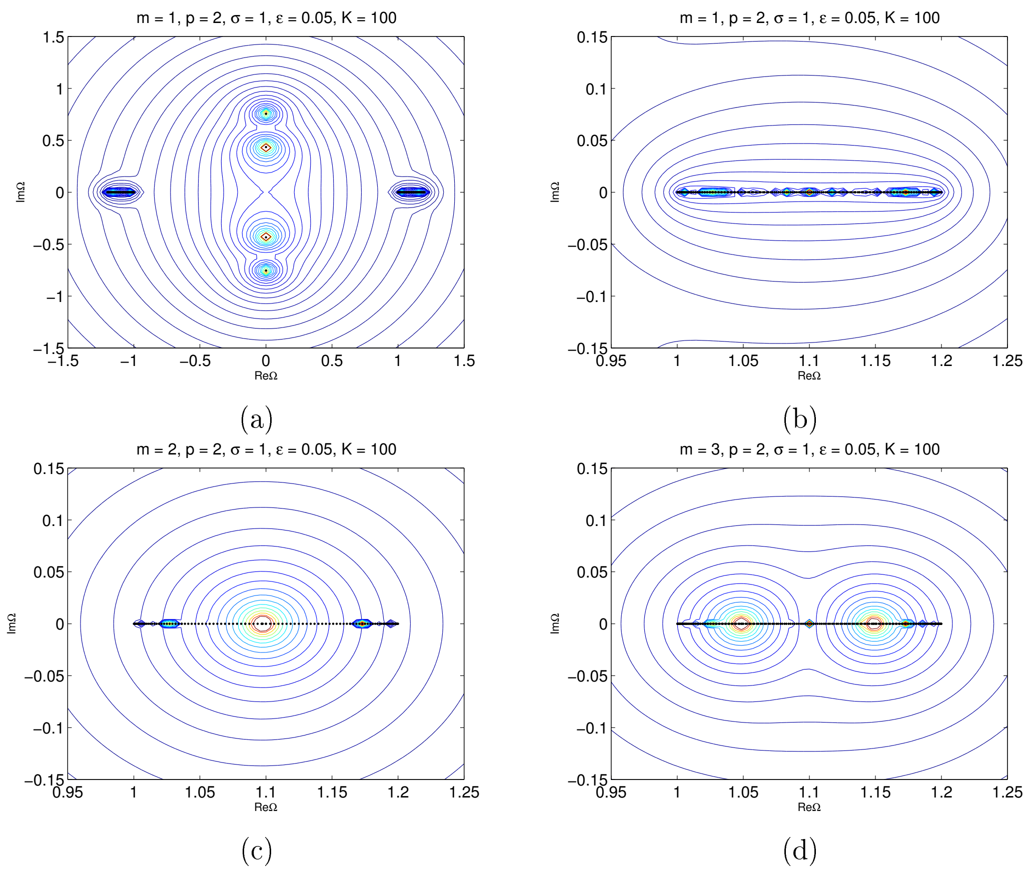

Figure 4.7: The level sets of $\left\|(\tilde{L} - \Omega\tilde{I}_2)^{-1}\right\|_2$ in the Ω-plane. The black dots represent eigenvalues of the matrix representation of operator $\tilde{L}$. The levels are equidistant on a logarithmic scale.

Lemma 4.23. *For $p = 1$ and any integer $N \geq 2$ there exist $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ the zero eigenvalue of $A_{\pm}(\epsilon)$ is semi-simple.*

Proof. We recall the coefficient matrices $A_{\pm}(\epsilon)$ from the proof of Lemma 4.15. In the case $p = 1$ (the cubic dNLS equation), these matrices are rewritten in the form

$$A_{\pm}(\epsilon) = \begin{bmatrix} -2M_{\pm} & -M_{\pm} \\ -N(\kappa_{\pm}) & 2\epsilon\sinh(\kappa_{\pm})I - 2N(\kappa_{\pm}) \end{bmatrix},$$

where $\kappa_{\pm} > 0$ are uniquely defined by

$$2\epsilon(\cosh(\kappa_+) - 1) = 2, \quad 2\epsilon(\cosh(\kappa_-) - 1) = 2 + 4\epsilon.$$

We recall that $\mathrm{Null}(A_{\pm}(\epsilon))$ and $\mathrm{Null}(M_{\pm})$ are $(N-1)$-dimensional for any $\epsilon \in [0, \epsilon_0)$. It is clear from the explicit form of $A^*_{\pm}(\epsilon)$ that

$$u \in \mathrm{Null}(A^*_{\pm}(\epsilon)) \quad \Leftrightarrow \quad u = \begin{bmatrix} w \\ 0 \end{bmatrix}, \quad w \in \mathrm{Null}(M_{\pm}).$$

At $\epsilon = 0$, we also recall that $\mathrm{Null}(A_{\pm}(0))^2$ is $(2N-2)$-dimensional because of $(N-1)$ eigenvectors and $(N-1)$ generalized eigenvectors,

$$A_{\pm}(0)\begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_{\pm}(0)\begin{bmatrix} -w \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ w \end{bmatrix}, \quad w \in \mathrm{Null}(M_{\pm}).$$

We would like to show that $\mathrm{Null}(A_{\pm}(\epsilon))^2 = \mathrm{Null}(A_{\pm}(\epsilon))$ is $(N-1)$-dimensional for any $\epsilon \in (0, \epsilon_0)$. In other words, we would like to show that no solution $\tilde{u} \in \mathbb{C}^{2N}$ of the inhomogeneous equation $A_{\pm}(\epsilon)\tilde{u} = u \in \mathrm{Null}(A_{\pm}(\epsilon))$ exists for $\epsilon \in (0, \epsilon_0)$. This task is achieved by the perturbation theory. We will only consider the case $A_+(\epsilon)$, which corresponds to $\theta = 0$. The case $A_-(\epsilon)$ which corresponds to $\theta = -\pi$ can be considered similarly.

We shall only consider the case of the simply-connected set $S_+ \cup S_-$ with $m_1 = m_2 = ... = m_{N-1} = 1$. The general case holds without any changes.

Thanks to the asymptotic expansions

$$e^{-\kappa_+} = \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2), \quad 2\epsilon\sinh(\kappa_+) = 2 + 2\epsilon + \mathcal{O}(\epsilon^2), \quad \text{as} \quad \epsilon \to 0,$$

we obtain the asymptotic expansion

$$A_+(\epsilon) = \begin{bmatrix} -2M_+ & -M_+ \\ -I & O \end{bmatrix} + \epsilon \begin{bmatrix} O & O \\ -\frac{1}{2}J & 2I - J \end{bmatrix} + \mathcal{O}(\epsilon^2),$$

where I and O are identity and zero matrices in $\mathbb{R}^N$ and J is the three-diagonal matrix in $\mathbb{R}^N$

$$J = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Note that $(2I-J)$ is a strictly positive matrix because it appears in the finite-difference approximation of the differential operator $-\partial_x^2$ subject to the Dirichlet boundary conditions.

Perturbative computations show that if $u \in \text{Null}(A_+(\epsilon))$, then u is represented asymptotically as

$$u = \begin{bmatrix} \epsilon(2I-J)v \\ v \end{bmatrix} + \mathcal{O}(\epsilon^2),$$

where $v + 2\epsilon(2I-J)v + \mathcal{O}(\epsilon^2) = w \in \text{Null}(M_+)$.

Now, there exists a solution $\tilde{u} \in \mathbb{C}^{2N}$ of the inhomogeneous equation $A_+(\epsilon)\tilde{u} = u \in \text{Null}(A_+(\epsilon))$ if and only if $u \perp \text{Null}(A_+^*(\epsilon))$. For small $\epsilon \in (0, \epsilon_0)$, this condition implies that

$$\epsilon(2I-J)v + \mathcal{O}(\epsilon^2) = \epsilon(2I-J)w + \mathcal{O}(\epsilon^2) \perp w \in \text{Null}(M_+),$$

which is not possible since $(2I-J)$ is a strictly positive matrix. □

Despite of this fact, we can not generally extend the result of Theorem 4.21 to multi-site discrete solitons because the perturbation theory for $\tilde{A}(\Omega,\epsilon)$ in (4.54) near the end points of the continuous spectrum $\Omega = \pm 1$ and $\Omega = \pm(1+4\epsilon)$ draws no conclusion in a general case. Letting

$$\tilde{A}_+(\epsilon) := \lim_{\Omega \to 1^+} \tilde{A}(\Omega,\epsilon), \quad \tilde{A}_-(\epsilon) := \lim_{\Omega \to (1+4\epsilon)^-} \tilde{A}(\Omega,\epsilon)$$

and reworking the perturbative arguments in the proof of Lemma 4.23, we obtain the necessary condition for $\text{Null}(\tilde{A}_\pm(\epsilon))^2 > \text{Null}(\tilde{A}_\pm(\epsilon))$ in the form

$$\epsilon(2I-J-2D)w + \mathcal{O}(\epsilon^2) \perp \text{Null}(M_+), \tag{4.58}$$

where I is the identity matrix in $\mathbb{R}^N$, and D is a diagonal matrix of $\{\chi_m\}_{m\in S}$. Because $(2I-J-2D)$ is no longer positive definite, the degenerate cases with $\text{Null}(\tilde{A}_\pm(\epsilon))^2 > \text{Null}(\tilde{A}_\pm(\epsilon))$ are possible. To illustrate this possibility let us notice that for

$$\text{Null}(M_+) = \text{span}\{w_1, w_2, \ldots, w_{N-1}\} \quad \text{and} \quad C := 2I-J-2D$$

the degeneracy condition (4.58) is equivalent to

$$\det P = 0, \quad \text{where} \quad P_{ij} = \langle C w_i, w_j \rangle_{\mathbb{C}^{N-1}}.$$

Let us set $N = 3$ and consider three distinct simply-connected discrete solitons associated with the sets

$$\text{(a) } S_+ = \{0,1,2\}; \quad \text{(b) } S_+ = \{0,1\},\ S_-\{2\}; \quad \text{(c) } S_+ = \{0,2\},\ S_-\{1\}.$$

Computations of the power expansions (4.50) give

$$\text{(a) } \chi_m = \begin{cases} 1, & m = 0, \\ 0, & m = 1, \\ 1, & m = 2, \end{cases} \quad \text{(b) } \chi_m = \begin{cases} 1, & m = 0, \\ 2, & m = 1, \\ 3, & m = 2, \end{cases} \quad \text{(c) } \chi_m = \begin{cases} 3, & m = 0, \\ 4, & m = 1, \\ 3, & m = 2. \end{cases}$$

As a result, matrix C is obtained in the form

$$\text{(a) } C = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \text{(b) } C = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -4 \end{bmatrix}, \quad \text{(c) } C = \begin{bmatrix} -4 & -1 & 0 \\ -1 & -6 & -1 \\ 0 & -1 & -4 \end{bmatrix}.$$

We have

$$\mathrm{Null}(M_+) = \mathrm{span}\{w_1, w_2\}, \quad w_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad w_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

from which we compute the matrix of projections $P_{ij} = \langle C w_i, w_j \rangle_{\mathbb{C}^3}$ in the form

$$\text{(a) } P = \begin{bmatrix} 0 & 0 \\ 0 & \frac{8}{3} \end{bmatrix}, \quad \text{(b) } P = \begin{bmatrix} -2 & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & -\frac{2}{3} \end{bmatrix}, \quad \text{(c) } P = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}.$$

The projection matrices in cases (a) and (b) are singular. In order to show that $\mathrm{Null}(A_\pm(\epsilon))^2 = \mathrm{Null}(A_\pm(\epsilon))$ for $\epsilon \in (0, \epsilon_0)$, we need to extend the arguments of Lemma 4.23 to the order $\mathcal{O}(\epsilon^2)$. Although it is quite possible that the non-degeneracy condition $\mathrm{Null}(A_\pm(\epsilon))^2 = \mathrm{Null}(A_\pm(\epsilon))$ is still satisfied for simply-connected multi-site discrete solitons for $p = 1$, we do not include computations of the higher-order perturbation theory in this thesis.

4.3 Scattering near solitons

In this section we discuss recent advances in the area of asymptotic stability of localized solutions to the dNLS equation. In what follows, we consider the dNLS equation in the form

$$i\dot{u}_n = Hu_n + |u_n|^{2p}u_n, \qquad n \in \mathbb{Z}, \tag{4.59}$$

where $p > 0$, $H = -\Delta + V$ and $\{V_n\}_{n\in\mathbb{Z}} \in l^\infty$ is non-zero.

The role of the potential V in asymptotic stability analysis can be understood by comparing the works of Cuccagna [23] and Mizumachi [60] in the context of continuous NLS equation. In the case of $V = 0$, Cuccagna followed the pioneering works of Buslaev & Perelman [15, 16], Buslaev and Sulem [17], and Gang & Sigal [35, 36]. He had to work with analysis of non-self-adjoint operators arising in the linearization of the NLS equation about the space-symmetric ground states. In contrast, to study the asymptotic stability of a small soliton bifurcating from the ground state of the Schrödinger operator $H = -\partial_x^2 + V$, Mizumachi could get by with the theory of self-adjoint operators. The fundamentals of analysis in this direction have been laid in the works of Soffer & Weinstein [86, 87, 88], Pillet & Wayne [78], and Yau & Tsai [98, 99, 100].

In the context of dNLS equation (4.59) dispersive decay estimates for the operator $e^{-iHt}P_{\text{a.c.}}(H)$, where $P_{\text{a.c.}}(H)$ denotes the projection onto the absolutely continuous spectrum of H, were established during the past decade. Some pointwise estimates as well as Strichartz estimates were developed by Stefanov & Kevrekidis [89] for the zero V, Komech, Kopylova & Kunze [50] for compact V, and Pelinovsky & Stefanov [76] for exponentially decaying V. Based on these papers, asymptotic stability of small bound states bifurcating from an isolated eigenvalue of the Schrödinger operator H has been studied for the case of septic or higher nonlinearities ($p \geq 3$) by Kevrekidis, Pelinovsky & Stefanov [49] as well as by Cuccagna & Tarulli [25].

Improved pointwise dispersive decay estimates were recently established for the case of zero potential by Mielke & Patz [58] (see Section 2.3 for the discussion of this result). These estimates allowed Mizumachi & Pelinovsky [61] to extend the main result of [49] to the case of $p \geq 2.75$. Along the same lines, very recently Bambusi [9] proved asymptotic stability of breathers in KG lattices using normal form theory for discrete Hamiltonian systems and dispersive decay estimates.

We are now going to discuss the techniques developed by Mizumachi & Pelinovsky in [61]. Let us impose the following assumptions on the potential:

Assumption 4.24. *We assume that non-zero potential* $\{V_n\}_{n\in\mathbb{Z}} \in l^\infty$ *satisfies the following requirements:*

1. *V_n decays to zero as $|n| \to \infty$ exponentially fast.*
2. *The discrete Schrödinger operator $H = -\Delta + V$ has no resonances at the endpoints of its continuous spectrum.*
3. *The operator H supports only one eigenvalue $\omega_0 < 0$.*

The first assumption implies that the continuous spectrum of H is the same as $\sigma(-\Delta) = [0, 4]$, while the second and third assumptions simplify the spectral formalism and allow us to stay within the framework of self-adjoint operators. It is important to note that the presence of two or more isolated eigenvalues may lead to non-vanishing oscillations as shown by Cuccagna in [24]. This is the reason for requiring that the point spectrum of H consists of a unique eigenvalue. We are going to denote the eigenvector corresponding to the eigenvalue ω_0 by $\boldsymbol{\psi}_0$:

$$H\boldsymbol{\psi}_0 = \omega_0 \boldsymbol{\psi}_0.$$

Example 4.25. As shown in [49, 50], Assumption 4.24 is satisfied for the single-site potential

$$V_n = -\delta_{n,0}, \qquad n \in \mathbb{Z}.$$

For this potential, H has a unique eigenvalue $\omega_0 = 2 - \sqrt{5} < 0$ and no resonances at the endpoints of the continuous spectrum i.e. the set $\{0, 4\}$. Explicit computations show that the eigenvector associated with ω_0 is given by

$$\psi_{0,n} = e^{-\kappa|n|}, \quad n \in \mathbb{Z}, \quad \kappa = \arcsin(2^{-1}).$$

4.3.1 Preliminary estimates

Let us set $\mathbf{u}(t) = e^{-i\omega t}\boldsymbol{\phi}(\omega)$ where $\boldsymbol{\phi}$ is the stationary solution satisfying

$$-(\Delta\boldsymbol{\phi})_n + V_n\phi_n + \phi_n^{2p+1} = \omega\phi_n, \qquad n \in \mathbb{Z}. \tag{4.60}$$

Thanks to Assumption 4.24, the linear version of this equation has a solution only in the case of $\omega = \omega_0$. The following lemma describes bifurcation of small solution to (4.60) from the eigenvalue $\omega_0 < 0$ of the linear operator H. This result is proved according to the standard method of Lyapunov–Schmidt reductions, but we omit the proof.

Lemma 4.26. *Assume that $\{V_n\}_{n\in\mathbb{Z}} \in l^\infty$ and that H has a simple eigenvalue $\omega_0 < 0$ with a normalized eigenfunction $\boldsymbol{\psi}_0 \in l^2$ such that $\|\boldsymbol{\psi}_0\|_{l^2} = 1$. For any $p > 0$, there exist positive constants ϵ_0, κ, and C, such that for any $\omega \in [\omega_0, \omega_0 + \epsilon_0)$ there exist* *a unique real-valued solution* $\boldsymbol{\phi}(\omega) \in C([\omega_0, \omega_0 + \epsilon_0), l^2) \cap C^2((\omega_0, \omega_0 + \epsilon_0), l^2)$ *to the stationary dNLS equation (4.60) satisfying*

$$\left\| e^{\kappa|\cdot|} \partial_\omega^i \left(\boldsymbol{\phi}(\omega) - c_0 (\omega - \omega_0)^{\frac{1}{2p}} \boldsymbol{\psi}_0 \right) \right\|_{l^2} \leq C (\omega - \omega_0)^{1 - i + \frac{1}{2p}}, \tag{4.61}$$

where $c_0 = \|\boldsymbol{\psi}_0\|_{l^{2p+2}}^{-1-\frac{1}{p}}$ *and* $i \in \{0, 1\}$.

Linearizing the right-hand-side of the dNLS equation (4.59) about its stationary solution $\mathbf{u}(t) = e^{-it\omega} \boldsymbol{\phi}(\omega)$ we obtain a linear operator $\mathcal{L}(\omega)$ such that

$$\mathcal{L}(\omega)\mathbf{z} = (H - \omega)\mathbf{z} + W(\omega)\mathbf{z} + pW(\omega)(\mathbf{z} + \bar{\mathbf{z}}), \tag{4.62}$$

where $W(\omega)$ is a diagonal operator with $W_n(\omega) = \phi_n^{2p}(\omega)$. Clearly, the generalized kernel of the operator $\mathcal{L}(\omega)$ is spanned by $\boldsymbol{\phi}(\omega)$ and $\boldsymbol{\phi}'(\omega)$. In the case of the zero potential, spectral problem (4.62) reduces to the one discussed in Section 4.1.

It follows from Lemma 4.26 that $\boldsymbol{\psi}_0$, the eigenvector of H, stays close to

$$\boldsymbol{\psi}_1(\omega) := \frac{\boldsymbol{\phi}(\omega)}{\|\boldsymbol{\phi}(\omega)\|_{l^2}}, \quad \boldsymbol{\psi}_2(\omega) := \frac{\boldsymbol{\phi}'(\omega)}{\|\boldsymbol{\phi}'(\omega)\|_{l^2}},$$

i.e. the generalized kernel of $\mathcal{L}(\omega)$, provided $\omega - \omega_0$ is small enough. In fact, one can easily show that given $s \geq 1$ and $\alpha \geq 0$ there exists a constant $C_{\alpha,s}$ such that

$$\|\boldsymbol{\psi}_1(\omega) - \psi_0\|_{l^s_\alpha} + \|\boldsymbol{\psi}_2(\omega) - \psi_0\|_{l^s_\alpha} \leq C_{\alpha,s}(\omega - \omega_0) \tag{4.63}$$

for all $\omega \in [\omega_0, \omega_0 + \epsilon_0)$.

Let us now work on stability of stationary solutions described in Lemma 4.26. We decompose a solution to the dNLS equation (4.59) into a family of stationary solutions with time-dependent parameters and a radiation part using the ansatz

$$\mathbf{u}(t) = e^{-i\theta(t)} \left[\boldsymbol{\phi}(\omega(t)) + \mathbf{z}(t) \right], \tag{4.64}$$

where $(\omega, \theta) \in \mathbb{R}^2$ represents a two-dimensional orbit of stationary solutions. Recalling that $\boldsymbol{\phi}(\omega)$ solves stationary dNLS equation (4.60), we obtain the following evolution equation for the radiation part $\mathbf{z}(t)$:

$$i\dot{\mathbf{z}} = (H - \omega)\mathbf{z} - (\dot{\theta} - \omega)(\boldsymbol{\phi}(\omega) + \mathbf{z}) - i\dot{\omega}\boldsymbol{\phi}'(\omega) + \mathbf{N}\left(\boldsymbol{\phi}(\omega) + \mathbf{z}\right) - \mathbf{N}\left(\boldsymbol{\phi}(\omega)\right), \tag{4.65}$$

where $H = -\Delta + V$ and $[\mathbf{N}(\boldsymbol{\phi})]_n = |\phi_n|^{2p}\phi_n$. To uniquely identify $\theta(t)$ and $\omega(t)$ in (4.64) we require that $\mathbf{z}(t)$ is *symplectically orthogonal* to the generalized kernel of the linear operator $\mathcal{L}(\omega)$ in (4.62):

$$\langle \operatorname{Re} \mathbf{z}(t), \boldsymbol{\psi}_1(\omega(t)) \rangle = \langle \operatorname{Im} \mathbf{z}(t), \boldsymbol{\psi}_2(\omega(t)) \rangle = 0. \tag{4.66}$$

This condition is the analogue of formula (4.10) that arises in the context of spectral stability of breathers, and it implies that $\mathbf{z}(t)$ belongs to the subspace associated with the continuous spectrum of $\mathcal{L}(\omega(t))$ for all $t \in \mathbb{R}$. In addition, according to [49], the symplectic orthogonality condition (4.66) guarantees uniqueness of decomposition (4.64):

Lemma 4.27. *Fix $\omega_* \in (\omega_0, \omega_0 + \epsilon_0)$. There exists δ_0, $C > 0$ such that for any $\delta \in (0, \delta_0)$ and any $\mathbf{u} \in l^2$ satisfying*

$$\|\mathbf{u} - \boldsymbol{\phi}(\omega_*)\|_{l^2} \leq \delta(\omega_* - \omega_0)^{\frac{1}{2p}},$$

there exist unique $(\omega, \theta) \in \mathbb{R}^2$ and $\mathbf{z} \in l^2$ in the decomposition

$$\mathbf{u} = e^{i\theta}(\boldsymbol{\phi}(\omega) + \mathbf{z})$$

subject to the symplectic orthogonality conditions

$$\langle \operatorname{Re} \mathbf{z}, \boldsymbol{\phi}(\omega) \rangle = \langle \operatorname{Im} \mathbf{z}, \boldsymbol{\phi}'(\omega) \rangle = 0,$$

and the bound

$$|\omega - \omega_*| \leq C\delta(\omega_* - \omega_0), \qquad |\theta| \leq C\delta, \qquad \|\mathbf{z}\|_{l^2} \leq C\delta(\omega_* - \omega_0)^{\frac{1}{2p}}.$$

The mapping $l^2 \ni \mathbf{u} \mapsto (\omega, \theta, \mathbf{z}) \in \mathbb{R}^2 \times l^2$ is a C^1 diffeomorphism.

Lemma 4.26 and the symplectic orthogonality condition (4.66) allow us to obtain some useful bounds on $\dot{\omega}$ and $\dot{\theta}$. Substitution of evolution equation (4.65) into the time derivative of orthogonality conditions (4.66) yields

$$A(\omega, \mathbf{z}) \begin{bmatrix} \dot{\omega} \\ \dot{\theta} - \omega \end{bmatrix} = \mathbf{f}(\omega, \mathbf{z}), \tag{4.67}$$

where

$$A(\omega, \mathbf{z}) = \begin{bmatrix} \langle \boldsymbol{\phi}'(\omega), \boldsymbol{\psi}_1(\omega) \rangle - \langle \operatorname{Re} \mathbf{z}, \boldsymbol{\psi}_1'(\omega) \rangle & \langle \operatorname{Im} \mathbf{z}, \boldsymbol{\psi}_1(\omega) \rangle \\ \langle \operatorname{Im} \mathbf{z}, \boldsymbol{\psi}_2'(\omega) \rangle & \langle \boldsymbol{\phi}(\omega), \boldsymbol{\psi}_2(\omega) \rangle + \langle \operatorname{Re} \mathbf{z}, \boldsymbol{\psi}_2(\omega) \rangle \end{bmatrix}$$

and

$$\mathbf{f}(\omega,\mathbf{z}) = \begin{bmatrix} \langle \mathrm{Im}\left\{\mathbf{N}(\boldsymbol{\phi}(\omega)+\mathbf{z}) - \mathbf{N}(\boldsymbol{\phi}(\omega)) - W(\omega)\mathbf{z}\right\}, \boldsymbol{\psi}_1(\omega)\rangle \\ \langle \mathrm{Re}\left\{\mathbf{N}(\boldsymbol{\phi}(\omega)+\mathbf{z}) - \mathbf{N}(\boldsymbol{\phi}(\omega)) - (2p+1)W(\omega)\mathbf{z}\right\}, \boldsymbol{\psi}_2(\omega)\rangle \end{bmatrix}.$$

Since according to Lemma 4.26 $\|\boldsymbol{\phi}(\omega)\|_{l^\infty} = \mathcal{O}\left((\omega-\omega_0)^{\frac{1}{2p}}\right)$ for $(\omega-\omega_0)\in(0,\epsilon_0)$ we can estimate the matrix of linear system (4.67) as

$$A(\omega,z) = \begin{bmatrix} \mathcal{O}\left((\omega-\omega_0)^{\frac{1}{2p}-1}\right) & 0 \\ 0 & \mathcal{O}\left((\omega-\omega_0)^{\frac{1}{2p}}\right) \end{bmatrix} + \mathcal{O}\left(\|\mathbf{z}\|_{l^2}\right)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Clearly, this matrix is invertible provided $\|\mathbf{z}\|_{l^2} \le C(\omega-\omega_0)^{\frac{1}{2p}}$ where the constant C is sufficiently small.

To obtain bounds on the solution to (4.67), let us recall that given $s \ge 1$, there exists a constant $C_s > 0$ such that

$$\left||a+b|^{2s}(a+b) - |a|^{2s}a\right| \le C_s\left(|a|^{2s}|b| + |b|^{2s+1}\right), \qquad \forall a,b\in\mathbb{C}.$$

The above inequality yields a pointwise estimate

$$\left|\mathbf{N}(\boldsymbol{\phi}(\omega)+\mathbf{z}) - \mathbf{N}(\boldsymbol{\phi}(\omega))\right| \le C_p\left(|\boldsymbol{\phi}(\omega)|^{2p}|\mathbf{z}| + |\mathbf{z}|^{2p+1}\right), \tag{4.68}$$

and results in a bound

$$\|\mathbf{f}(\omega,\mathbf{z})\| \le C\sum_{i=1}^{2}\left(\|\boldsymbol{\phi}(\omega)^{2p-1}\boldsymbol{\psi}_i\mathbf{z}^2\|_{l^1} + \|\boldsymbol{\psi}_i(\omega)\mathbf{z}^{2p+1}\|_{l^1}\right).$$

Now, since $\|\mathbf{f}(\omega,\mathbf{z})\| = \mathcal{O}\left(\|\boldsymbol{\phi}(\omega)^{2p-1}\|_{l^\infty}\right)$ it follows from Lemma 4.26 that the components of solution to (4.67) are bounded as follows:

$$|\dot{\omega}| \le C(\omega-\omega_0)^{2-\frac{1}{p}}\|e^{-\kappa|n|}\mathbf{z}^2\|_{l^1}, \tag{4.69}$$

$$|\dot{\theta}-\omega| \le C(\omega-\omega_0)^{1-\frac{1}{p}}\|e^{-\kappa|n|}\mathbf{z}^2\|_{l^1}. \tag{4.70}$$

4.3.2 Asymptotic stability of discrete solitons

In this section we prove a theorem on asymptotic stability of the discrete soliton of the dNLS equation (4.59) that bifurcate from the isolated eigenvalue of the linear operator $H = -\Delta + V$. Our main result is as follows:

Theorem 4.28. *Suppose that the linear operator $V : l^1 \to l^1$ satisfies Assumption 4.24. For any $p > 2.75$, there exist $\epsilon_0 > 0$ and $\delta > 0$ such that if $\epsilon = \omega_* - \omega_0 \in (0, \epsilon_0)$ and*

$$\|\mathbf{u}_0 - \boldsymbol{\phi}(\omega_*)\|_{l^1_1} \leq \delta \epsilon^{\frac{1}{2p}}, \tag{4.71}$$

then there exist $C > 0$, $\theta_\infty \in \mathbb{R}$, $\omega_\infty \in (\omega_0, \omega_0 + \epsilon_0)$, $(\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$ and a solution to dNLS equation (4.59)

$$\mathbf{u}(t) = e^{-i\theta(t)}\boldsymbol{\phi}(\omega(t)) + \mathbf{y}(t) \quad \in \quad C^1(\mathbb{R}_+, l^2)$$

such that

$$\lim_{t\to\infty} \left(\theta(t) - \int_0^t \omega(s) ds\right) = \theta_\infty, \qquad \lim_{t\to\infty} \omega(t) = \omega_\infty, \tag{4.72}$$

and

$$\sup_{t\geq 0} |\omega(t) - \omega_*| \leq C\delta\epsilon. \tag{4.73}$$

Moreover, for any $s \in (2, 4) \cup (4, \infty]$ and $t \geq 0$, there exists $C_s > 0$ such that

$$\|\mathbf{y}(t)\|_{l^s} \leq C_s \delta \epsilon^{\frac{1}{2p}} (1+t)^{-\alpha_s}, \quad \alpha_s = \begin{cases} \dfrac{s-2}{2s}, & \text{for } s \in [2, 4), \\ \dfrac{s-1}{3s}, & \text{for } s \in (4, \infty]. \end{cases} \tag{4.74}$$

Introducing the radiation part of the solution $\mathbf{y}(t) = e^{-i\theta(t)}\mathbf{z}(t)$, we rewrite the evolution equation (4.65) as

$$i\dot{\mathbf{y}} = H\mathbf{y} + \mathbf{g}, \qquad \mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3, \tag{4.75}$$

where

$$\begin{aligned} \mathbf{g}_1 &= \left[\mathbf{N}(\boldsymbol{\phi}(\omega) + \mathbf{y}e^{i\theta}) - \mathbf{N}(\boldsymbol{\phi}(\omega))\right]e^{-i\theta}, \\ \mathbf{g}_2 &= -(\dot{\theta} - \omega)\boldsymbol{\phi}(\omega)e^{-i\theta}, \\ \mathbf{g}_3 &= -i\dot{\omega}\boldsymbol{\phi}'(\omega)e^{-i\theta}. \end{aligned}$$

We now split the solution $\mathbf{y}(t)$ into the part parallel to the point spectrum of H and its orthogonal complement:

$$\mathbf{y}(t) = a(t)\boldsymbol{\psi}_0 + \boldsymbol{\eta}(t), \tag{4.76}$$

where $a(t) = \langle \mathbf{y}(t), \boldsymbol{\psi}_0 \rangle$ and $\langle \boldsymbol{\eta}(t), \boldsymbol{\psi}_0 \rangle = 0$. We also set the projection operators

$$P_0 = \langle \cdot, \boldsymbol{\psi}_0 \rangle \boldsymbol{\psi}_0, \qquad Q_0 = I - P_0,$$

and rewrite the evolution equation (4.75) as

$$\begin{cases} i\dot{a} = \omega_0 a + \langle \mathbf{g}, \boldsymbol{\psi}_0 \rangle, \\ i\dot{\boldsymbol{\eta}} = H\boldsymbol{\eta} + Q_0\mathbf{g}. \end{cases} \tag{4.77}$$

To analyze solutions to this system, we need a couple of auxiliary results.

Firstly, Lemma 2.24 for the semigroup of the operator $-\Delta$ can be extended to that for the semigroup of $H = -\Delta + V$:

Lemma 4.29. *Suppose that the linear operator $V : l^1 \to l^1$ satisfies Assumption 4.24. For any $s \geq 2$, there is $C_s > 0$ such that for all $t \in \mathbb{R}$,*

$$\left\| e^{-itH} Q_0 \mathbf{f} \right\|_{l^s} \leq C_s \big(1 + |t|\big)^{-\alpha_s} \|\mathbf{f}\|_{l^1}, \quad \alpha_s = \begin{cases} \dfrac{s-2}{2s}, & \text{for } s \in [2,4), \\ \dfrac{s-1}{3s}, & \text{for } s \in (4,\infty]. \end{cases}$$

The proof of this Lemma is given in [61] using the Jost function for the discrete Schrödinger operator.

Secondly, we need to extend the estimate from Section 2.3,

$$\|e^{it\Delta}\mathbf{u}_0\|_{l^\infty} \leq C(1+t)^{-1/3}\|\mathbf{u}_0\|_{l^1},$$

to non-zero potentials and weighted spaces.

Lemma 4.30. *Suppose that the linear operator $V : l^1 \to l^1$ satisfies Assumption 4.24. For any $\alpha \in [0,1]$, there is $C_\alpha > 0$ such that for all $t \in \mathbb{R}$,*

$$\left\| e^{-itH} Q_0 \mathbf{f} \right\|_{l^\infty_{-\alpha}} \leq C_\alpha \big(1 + |t|\big)^{-\frac{1}{3}-\alpha} \|\mathbf{f}\|_{l^1_\alpha}.$$

The proof of this lemma via inverse Laplace transform can be found in [61].

Lastly, we know that the l^2 norm of the solution to (4.59) is conserved. Let us now establish the upper bound on the weighted l^2 norm.

Lemma 4.31. *Let $\mathbf{u}(t) \in C(\mathbb{R}, l^2)$ be a solution to the initial value problem (4.59) with initial data $\mathbf{u}_0 \in l^2_1$. For any $\alpha \in [0,1]$, there is $C_\alpha > 0$ such that for all $t \in \mathbb{R}$*

$$\|\mathbf{u}(t)\|_{l^2_\alpha} \leq C_\alpha \big(1 + |t|\big)^{\alpha} \|\mathbf{u}_0\|_{l^2_1}. \tag{4.78}$$

Proof. It readily follows from the dNLS equation (4.59) that

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}\|_{l_1^2}^2 &= \frac{d}{dt}\sum_{n\in\mathbb{Z}}(1+n^2)|u_n|^2 \\ &= i\sum_{n\in\mathbb{Z}}(1+n^2)(\bar{u}_n u_{n-1} + \bar{u}_n u_{n+1} - u_n\bar{u}_{n-1} - u_n\bar{u}_{n+1}) \\ &= i\sum_{n\in\mathbb{Z}}(1+2n)(u_n\bar{u}_{n+1} - \bar{u}_n u_{n+1}).\end{aligned}$$

Applying the Cauchy–Schwarz inequality and using the l^2 norm conservation, we find

$$\|\mathbf{u}\|_{l_1^2}\left|\frac{d}{dt}\|\mathbf{u}\|_{l_1^2}\right| \leq \sum_{n\in\mathbb{Z}}(1+2n)|u_n u_{n+1}| \leq C\|\mathbf{u}\|_{l_1^2}\|\mathbf{u}_0\|_{l^2}.$$

After cancellation of $\|\mathbf{u}\|_{l_1^2}$ and integration in t this yields

$$\|\mathbf{u}(t)\|_{l_1^2} \leq C(1+|t|)\|\mathbf{u}_0\|_{l_1^2}. \tag{4.79}$$

Clearly, this is inequality (4.78) at $\alpha = 1$. Applying the Hölder inequality to

$$\|\mathbf{u}\|_{l_\alpha^2}^2 \equiv \sum_{n\in\mathbb{Z}}(1+n^2)^\alpha|u_n|^{2\alpha}|u_n|^{2(1-\alpha)},$$

we obtain

$$\|\mathbf{u}\|_{l_\alpha^2}^2 \leq \left\|\left\{(1+n^2)^\alpha|u_n|^{2\alpha}\right\}\right\|_{l^p}\left\|\left\{u_n^{2(1-\alpha)}\right\}\right\|_{l^q} = \|\mathbf{u}\|_{l_1^{2\alpha p}}^{2\alpha}\|\mathbf{u}\|_{l^{2(1-\alpha)q}}^{2(1-\alpha)}, \qquad \frac{1}{p}+\frac{1}{q}=1.$$

Finally, setting $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$ and using (4.79) together with l^2 norm conservation we get

$$\|\mathbf{u}\|_{l_\alpha^2} \leq \|\mathbf{u}\|_{l_1^2}^\alpha\|\mathbf{u}\|_{l^2}^{1-\alpha} \leq C_\alpha(1+|t|)^\alpha\|\mathbf{u}_0\|_{l_1^2}^\alpha\|\mathbf{u}_0\|_{l^2}^{1-\alpha} \leq C_\alpha\big(1+|t|\big)^\alpha\|\mathbf{u}_0\|_{l_1^2}.$$

which completes the proof of the lemma. □

Having the above lemmas, we are now ready to prove the main result, Theorem 4.28.

Proof of Theorem 4.28. The proof is based on establishing uniform bounds on the following quantities:

$$\begin{aligned}
M_1(t) &= \sup_{0\le\tau\le t}(1+\tau)^{\nu_\alpha}\|\mathbf{y}(\tau)\|_{l^\infty_{-\alpha}},\\
M_2(t) &= \sup_{0\le\tau\le t}(1+\tau)^{\alpha_{2p+1}}\|\mathbf{y}(\tau)\|_{l^{2p+1}} + \sup_{0\le\tau\le t}(1+\tau)^{\alpha_{4p}}\|\mathbf{y}(\tau)\|_{l^{4p}},\\
M_3(t) &= \sup_{0\le\tau\le t}|\omega(\tau)-\omega_*|,
\end{aligned}$$

where $\alpha_s = \frac{s-1}{3s}$ for $s > 4$ and the parameters α and ν_α will be determined later (see (4.90) and (4.93)).

To establish the second limit in (4.72) we use the bound (4.69) which leads to

$$\begin{aligned}
|\dot\omega(t)| &\le C(\omega-\omega_0)^{2-\frac{1}{p}}\|e^{-\kappa|\cdot|}\mathbf{y}^2(t)\|_{l^1}\\
&\le C(\omega-\omega_0)^{2-\frac{1}{p}}\|e^{-\kappa|\cdot|}\|_{l^1_{2\alpha}}\|\mathbf{y}(t)\|_{l^\infty_{-2\alpha}}\\
&\le C(\omega-\omega_0)^{2-\frac{1}{p}}(1+t)^{-2\nu_\alpha}M_1^2(t).
\end{aligned}\tag{4.80}$$

Hence, in addition to a uniform bound on $M_1(t)$ we need to require that $\boxed{\nu_\alpha>\frac{1}{2}}$. Since according to (4.70) we have

$$|\dot\theta(t)-\omega(t)| \le C(\omega-\omega_0)^{1-\frac{1}{p}}(1+t)^{-2\nu_\alpha}M_1^2(t),\tag{4.81}$$

the same conditions establish the first limit in (4.72).

In order to prove (4.73) we need to show that the uniform upper bound on $M_3(t)$ scales like $\delta\epsilon$ (see (4.97) below). Similarly, to prove the asymptotic stability result (4.74) it is enough to invoke Lemma 4.29 and show that the uniform upper bounds on $M_1(t)$ and $M_2(t)$ scale like $\delta\epsilon^{\frac{1}{2p}}$ (see (4.96) below).

Let us first focus on the estimates for $M_1(t)$. Using decomposition of $\mathbf{y}(t)$ in (4.76) we find that

$$\|\mathbf{y}(t)\|_{l^\infty_{-\alpha}} \le |a(t)|\|\boldsymbol{\psi}_0(t)\|_{l^\infty_{-\alpha}} + \|\boldsymbol{\eta}(t)\|_{l^\infty_{-\alpha}}.\tag{4.82}$$

To obtain the bound on the coefficient $|a(t)|$ we recall orthogonality conditions (4.66) and the fact that $\boldsymbol{\psi}_0$ is close to both $\boldsymbol{\psi}_1(\omega)$ and $\boldsymbol{\psi}_2(\omega)$ for small $\omega-\omega_0$ in (4.63) so that

$$\begin{aligned}
|a(t)| &= \langle\mathbf{z}(t),\boldsymbol{\psi}_0\rangle\\
&\le \left|\langle\operatorname{Re}\mathbf{z}(t),\boldsymbol{\psi}_0-\boldsymbol{\psi}_1\rangle\right| + \left|\langle\operatorname{Im}\mathbf{z}(t),\boldsymbol{\psi}_0-\boldsymbol{\psi}_2\rangle\right|\\
&\le \left(\|\boldsymbol{\psi}_0-\boldsymbol{\psi}_1\|_{l^1_\alpha} + \|\boldsymbol{\psi}_0-\boldsymbol{\psi}_2\|_{l^1_\alpha}\right)\|\mathbf{z}(t)\|_{l^\infty_{-\alpha}}\\
&\le C_\alpha(\omega-\omega_0)\|\mathbf{y}(t)\|_{l^\infty_{-\alpha}}.
\end{aligned}$$

Now since $\|\boldsymbol{\psi}_0\|_{l^\infty_{-\alpha}} \le C\|\boldsymbol{\psi}_0\|_{l^2} < \infty$ we know that for small $\omega-\omega_0$ formula (4.82) yields

$$\|\mathbf{y}(t)\|_{l^\infty_{-\alpha}} \leq 2\|\boldsymbol{\eta}(t)\|_{l^\infty_{-\alpha}}. \tag{4.83}$$

By Duhamel's principle the equation on η in (4.77) can be solved as,

$$\boldsymbol{\eta}(t) = e^{-itH}Q_0\boldsymbol{\eta}(0) - i\int_0^t e^{-i(t-s)H}Q_0\mathbf{g}(s)ds. \tag{4.84}$$

Using the result in Lemma 4.30 we get

$$\begin{aligned}\|\boldsymbol{\eta}(t)\|_{l^\infty_{-\alpha}} &\leq \left\|e^{-itH}Q_0\boldsymbol{\eta}_0\right\|_{l^\infty_{-\alpha}} + \int_0^t \left\|e^{-i(t-s)H}Q_0\mathbf{g}(s)\right\|_{l^\infty_{-\alpha}} ds \\ &\leq C_\alpha(1+t)^{-\frac{1}{3}-\alpha}\|\boldsymbol{\eta}_0\|_{l^1_\alpha} + C_\alpha\int_0^t (1+t-s)^{-\frac{1}{3}-\alpha}\|\mathbf{g}(s)\|_{l^1_\alpha}ds.\end{aligned} \tag{4.85}$$

To simplify estimates on $\|\mathbf{g}(s)\|_{l^1_\alpha}$ we introduce

$$\begin{aligned} I_1(s) &= (\omega(s)-\omega_0)(1+s)^{-\nu_\alpha}M_1(s), \\ I_2(s) &= (\omega(s)-\omega_0)^{1-\frac{1}{2p}}(1+s)^{-2\nu_\alpha}M_1^2(s), \\ I_3(s) &= \|\mathbf{y}(s)\|^{2p}_{l^{4p}}\|\mathbf{y}(s)\|_{l^2_\alpha}. \end{aligned}$$

Employing the asymptotics for $\boldsymbol{\phi}(\omega)$ in (4.61), bound (4.68) and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\|\mathbf{g}_1(s)\|_{l^1_\alpha} &\leq C\left\|\boldsymbol{\phi}^{2p}\big(\omega(s)\big)\mathbf{y}(s)\right\|_{l^1_\alpha} + C\left\|\mathbf{y}^{2p+1}(s)\right\|_{l^1_\alpha} \\ &\leq C\left\|\boldsymbol{\phi}^{2p}\big(\omega(s)\big)\right\|_{l^1_{2\alpha}}\|\mathbf{y}(s)\|_{l^\infty_{-\alpha}} + C\|\mathbf{y}(s)\|^{2p}_{l^{4p}}\|\mathbf{y}(s)\|_{l^2_\alpha} \\ &\leq C(I_1(s)+I_3(s)).\end{aligned}$$

Similarly, to obtain bounds on $\mathbf{g}_2(s)$ and $\mathbf{g}_3(s)$ we use asymptotics in (4.61), and recent estimates (4.80), (4.81):

$$\begin{aligned}\|\mathbf{g}_2(s)+\mathbf{g}_3(s)\|_{l^1_\alpha} &\leq |\dot\theta(s)-\omega(s)|\|\boldsymbol{\phi}(\omega(s))\|_{l^1_\alpha} + |\dot\omega(s)|\|\boldsymbol{\phi}'(\omega(s))\|_{l^1_\alpha} \\ &\leq C\left[(\omega(s)-\omega_0)^{1-\frac{1}{p}}\|\boldsymbol{\phi}(\omega(s))\|_{l^1_\alpha} + (\omega(s)-\omega_0)^{2-\frac{1}{p}}\|\boldsymbol{\phi}'(\omega(s))\|_{l^1_\alpha}\right] \\ &\quad \times(1+s)^{-2\nu_\alpha}M_2^2(s) \quad \leq \quad CI_2(s).\end{aligned} \tag{4.86}$$

To proceed with estimate (4.85) we need to establish the decay of integrals

$$\int_0^t (1+t-s)^{-\frac{1}{3}-\alpha}I_j(s)ds, \qquad j=1,2,3.$$

According to Lemma 2.26, if $\beta_1, \beta_2 \in (0,\infty)\backslash\{1\}$ then

$$\int_0^t (1+t-s)^{-\beta_1}(1+s)^{-\beta_2} ds \le C(1+t)^{-\gamma}, \tag{4.87}$$

where $\gamma = \min(\beta_1, \beta_2, \beta_1+\beta_2-1)$. As a result, we have

$$\int_0^t (1+t-s)^{-\frac{1}{3}-\alpha}(1+s)^{-\nu_\alpha} ds \le C(1+t)^{-\nu_\alpha}$$

provided $\boxed{\alpha > \frac{2}{3}}$ and $\boxed{\frac{1}{3}+\alpha \le \nu_\alpha}$, so we get the estimate

$$\begin{aligned} &\int_0^t (1+t-s)^{-\frac{1}{3}-\alpha}(I_1(s)+I_2(s))ds \\ &\qquad\qquad \le C(1+t)^{-\nu_\alpha}\left[(\omega-\omega_0)M_1(t) + (\omega-\omega_0)^{1-\frac{1}{2p}}M_1^2(t)\right]. \end{aligned} \tag{4.88}$$

We also need to establish the decay rate for $I_3(t)$. Using decomposition (4.64), Lemmas 4.26 and 4.31, we obtain the following bound:

$$\begin{aligned} I_3(t) &\le \|\mathbf{y}(s)\|_{l^{4p}}^{2p}\left(\|\mathbf{u}(t)\|_{l^2_\alpha} + \|\boldsymbol{\phi}(\omega(t))\|_{l^2_\alpha}\right) \\ &\le C(1+t)^{-2p\alpha_{4p}}M_2^{2p}(t)\left(C_\alpha(1+t)^\alpha\|\mathbf{u}(0)\|_{l^2_1} + \|\boldsymbol{\phi}(\omega(t))\|_{l^2_\alpha}\right) \\ &\le C(1+t)^{-2p\alpha_{4p}+\alpha}M_2^{2p}(t)\left(\epsilon^{\frac{1}{2p}} + (\omega-\omega_0)^{\frac{1}{2p}}\right). \end{aligned}$$

This allows us to apply (4.87) with $\beta_1 = \alpha + \frac{1}{3}$, $\beta_2 = \frac{2}{3}p - \frac{1}{6} - \alpha$ and find the bound

$$\begin{aligned} \int_0^t (1+t-s)^{-\alpha-\frac{1}{3}} I_3(s) ds &\le CM_2^{2p}(t)\int_0^t (1+t-s)^{-\alpha-\frac{1}{3}}(1+s)^{-2p\alpha_{4p}+\alpha} \\ &\qquad\qquad \times\left(\epsilon^{\frac{1}{2p}} + (\omega(s)-\omega_0)^{\frac{1}{2p}}\right)ds \\ &\le C(1+t)^{-\nu_\alpha}M_2^{2p}(t)\left(\epsilon^{\frac{1}{2p}} + (\omega(t)-\omega_0)^{\frac{1}{2p}}\right), \end{aligned} \tag{4.89}$$

where thanks to $\beta_1 > 1$ ($\alpha > \frac{2}{3}$) the parameter ν_α is defined as

$$\boxed{\nu_\alpha = \min\left(\frac{1}{3}+\alpha, \frac{2}{3}p - \frac{1}{6} - \alpha\right)}. \tag{4.90}$$

The constraint $\nu_\alpha > \frac{1}{2}$ that we have established above is equivalent to $\frac{2}{3}p - \frac{1}{6} - \alpha > \frac{1}{2}$, so we are going to require $\boxed{2 < \frac{3}{2}\alpha + 1 < p}$. Plugging estimates (4.88) and (4.89) into

(4.85) we close the bound on $M_1(t)$ and $M_2(t)$:

$$M_1(t) \leq C\Big\{\|\boldsymbol{\eta}(0)\|_{l^1_\alpha} + (\omega(t)-\omega_0)M_1(t) \\ + (\omega(t)-\omega_0)^{1-\frac{1}{2p}}M_1^2(t) + M_2^{2p}(t)\left(\epsilon^{\frac{1}{2p}} + (\omega(t)-\omega_0)^{\frac{1}{2p}}\right)\Big\}. \quad (4.91)$$

Similar to (4.91), we would also like to obtain an estimate on $M_2(t)$ in terms of $M_1(t)$ and $M_2(t)$. Using the same approach we took in deriving (4.83) we obtain

$$\|\mathbf{y}(t)\|_{l^s} \leq 2\|\boldsymbol{\eta}(t)\|_{l^s}.$$

The bound on the right hand side of this inequality comes from Lemma 4.29 applied to $\boldsymbol{\eta}(t)$ in the form (4.84):

$$\|\boldsymbol{\eta}(t)\|_{l^s} \leq C_s(1+t)^{-\alpha_s}\|\boldsymbol{\eta}(0)\|_{l^1} + C_s\int_0^t (1+t-s)^{-\alpha_s}\|\mathbf{g}(s)\|_{l^1}ds.$$

It follows from Lemma 4.26 and bound (4.68) that

$$\begin{aligned}\|\mathbf{g}_1(s)\|_{l^1} &\leq C\big\|\boldsymbol{\phi}^{2p}\big(\omega(s)\big)\mathbf{y}(s)\big\|_{l^1} + C\left\|\mathbf{y}^{2p+1}(s)\right\|_{l^1} \\ &\leq C\big\|\boldsymbol{\phi}^{2p}\big(\omega(s)\big)\big\|_{l^1_\alpha}\|\mathbf{y}(s)\|_{l^\infty_{-\alpha}} + C\|\mathbf{y}(s)\|_{l^{2p+1}}^{2p+1} \\ &\leq C\left(I_1(s) + (1+s)^{-(2p+1)\alpha_{2p+1}}M_2^{2p+1}(s)\right).\end{aligned}$$

Rewriting (4.86) in l^1 norm gives

$$\|\mathbf{g}_2(s) + \mathbf{g}_3(s)\|_{l^1} \leq CI_2(s).$$

To allow for (β_1, β_2) in (4.87) to be equal to either (α_s, ν_α) or $(\alpha_s, (2p+1)\alpha_{2p+1})$ we demand $\beta_1 = \min(\beta_1, \beta_2, \beta_1+\beta_2-1)$ which implies $\beta_1 \leq \beta_2$ and $\beta_2 \geq 1$. Since $\alpha_s \in [0, \frac{1}{3}]$, we need to add an extra condition of $\boxed{\nu_\alpha \geq 1}$ to add to the framed ones above. Thus,

$$\int_0^t (1+t-s)^{-\alpha_s}\|\mathbf{g}(s)\|_{l^1}ds \\ \leq C(1+t)^{-\alpha_s}\left\{(\omega-\omega_0)M_1(t) + (\omega-\omega_0)^{1-\frac{1}{2p}}M_1^2(t) + M_2^{2p+1}(t)\right\}. \quad (4.92)$$

Combining (4.90) with conditions $\nu_\alpha \geq 1$ and $\alpha > \frac{2}{3}$ we find that

$$\frac{2}{3}p - \frac{1}{6} - \alpha \geq 1 \qquad \Longrightarrow \qquad \boxed{\text{p}>\tfrac{11}{4}=2.75}.$$

Since we rely on Lemma 4.30 where $\alpha \leq 1$, the above inequality provides the following non-empty range for α:

$$\frac{2}{3} < \alpha < \min\left(1, \frac{2}{3}p - \frac{7}{6}\right). \tag{4.93}$$

Thanks to estimate (4.92) we obtain another closed form bound on $M_1(t)$ and $M_2(t)$:

$$M_2(t) \leq C\left\{\|\boldsymbol{\eta}(0)\|_{l^1} + (\omega-\omega_0)M_1(t) + (\omega-\omega_0)^{1-\frac{1}{2p}}M_1^2(t) + M_2^{2p+1}(t)\right\}. \tag{4.94}$$

To obtain the bound on $M_3(t)$ we recall the bound on $|\dot{\omega}|$ in (4.80) and the requirement $|\omega(0)-\omega_*| \leq C\delta\epsilon$, so that

$$\begin{aligned} M_3(t) &= \sup_{0\leq\tau\leq t}|\omega(0)-\omega_* + \omega(\tau)-\omega(0)| \\ &\leq C\delta\epsilon + C\sup_{0\leq\tau\leq t}\int_0^\tau (\omega(s)-\omega_0)^{2-\frac{1}{p}}(1+s)^{-2\nu_\alpha}M_1^2(s)ds \\ &\leq C\delta\epsilon + C\sup_{0\leq\tau\leq t}(\omega(\tau)-\omega_0)^{2-\frac{1}{p}}M_1^2(t). \end{aligned} \tag{4.95}$$

Finally, to establish the appropriate bounds on $M_1(t)$, $M_2(t)$, and $M_3(t)$ in (4.91), (4.94), and (4.95) we apply the triangle inequality

$$\begin{aligned} |\omega(t)-\omega_0| &= |\omega(t)-\omega_*| + |\omega_*-\omega_0| \\ &\leq M_3(t) + |\omega_*-\omega_0|, \end{aligned}$$

so that $|\omega(t)-\omega_0| = \mathcal{O}(\epsilon)$ if $M_1(t)$ bounded. Also, as

$$M_1(t)+M_2(t) \leq C\left(\|\boldsymbol{\eta}(0)\|_{l^1_\alpha} + M_1^2(t) + M_2^{2p}(t)\left[\epsilon^{\frac{1}{2p}} + M_2(t)\right]\right) \quad ,$$

it follows from continuity of $M_1(t)$ and $M_2(t)$, and the bound on initial data (4.71) that

$$\sup_{t\geq 0}(M_1(t)+M_2(t)) \leq 2C\|\mathbf{y}(0)\|_{l^1_\alpha} \leq 2C\delta\epsilon^{\frac{1}{2p}}. \tag{4.96}$$

Applying this bound to (4.95) we get

$$\sup_{t\geq 0} M_3(t) \leq 2C\delta\epsilon. \tag{4.97}$$

□

Chapter 5

Linear stability of the dKG breathers

Existence theory for multi-site breathers in KG lattices near the anti-continuum limit was described in Section 3.2. We considered the dKG equation

$$\ddot{u}_n + V'(u_n) = \epsilon(\Delta \mathbf{u})_n, \quad n \in \mathbb{Z}, \tag{5.1}$$

where $t \in \mathbb{R}$ is the evolution time, $u_n(t) \in \mathbb{R}$ is the displacement of the n-th particle, $V : \mathbb{R} \to \mathbb{R}$ is a smooth even on-site potential, and $\epsilon \in \mathbb{R}$ is the coupling constant of the linear interaction between neighbouring particles. Multi-site breathers were extended from the limiting solution in the anti-continuum limit which consists of excited oscillations at different lattice sites separated by a number of "holes" (sites at rest). In this chapter, we consider linear stability of such discrete breathers near the anti-continuum limit. We assume that the on-site potential is even and admits a Taylor expansion

$$V'(u) = u \pm u^3 + \mathcal{O}(u^5) \quad \text{as} \quad |u| \to \infty, \tag{5.2}$$

where the plus and minus signs correspond to *hard* and *soft* potentials respectively, and the coefficients at the first and second terms are normalized by rescaling the variables in (5.1).

The first analytical work on stability of discrete breathers near the anti-continuum limit is due to Aubry [4] who proposed a method based on analysis of the band structure of the perturbed Newton's operator in the linearized problem. This method is now known as *Aubry's band theory.* Spectral stability of multi-site breathers continued from the anti-continuum limit, was also considered by Morgante *et al.* [62] with the help of numerical computations. These computations suggested that spectral stability of small-amplitude multi-site breathers in the dKG equation (5.1) is the same as that for the dNLS equation arising in the small-amplitude approximation. The numerical results in [62] can be summarized as follows: *in the case of the focusing nonlinearity, the only stable multi-site breathers of the dNLS equation (5.3) near the anti-continuum limit correspond to the anti-phase oscillations on the excited sites of the lattice.* This conclusion does not depend on the number of “holes" between the excited sites in the anti-continuum limit.

Let us recall that the dNLS approximation for small-amplitude slowly varying oscillations in the KG lattice (5.1) with potential (5.2) relies on the asymptotic solution,

$$u_n(t) = \sqrt{\frac{\epsilon}{3}}\left[a_n(\epsilon t)e^{it} + \bar{a}_n(\epsilon t)e^{-it}\right] + \mathcal{O}_{l^\infty}(\epsilon^{3/2}),$$

where $\epsilon > 0$ is assumed to be small, $\tau = \epsilon t$ is the slow time, and $a_n(\tau) \in \mathbb{C}$ is an envelope amplitude of nearly harmonic oscillations with the unit frequency. This approximation yields the dNLS equation to the leading order in ϵ,

$$2i\dot{a}_n = (\Delta \mathbf{a})_n \mp |a_n|^2 a_n, \quad n \in \mathbb{Z}. \tag{5.3}$$

The hard and soft potentials (5.2) result in the *focusing* and *defocusing* cubic nonlinearities of the dNLS equation (5.3), respectively. Let us note that existence and continuous approximations of small-amplitude breathers in the dKG and dNLS equations were recently justified by Bambusi *et al.* [10, 11]. The problem of bifurcation of small-amplitude breathers in KG lattices, in connection to homoclinic bifurcations in the dNLS equations, was also studied by James *et al.* [42].

We recall that multi-site solitons of the dNLS equation (5.3) can be constructed similarly to the multi-site breathers in the dKG equation (5.1). The time-periodic solutions are given by $a_n(\tau) = A_n e^{-i\omega\tau}$, where $\omega \in \mathbb{R}$ is a frequency of oscillations and $\{A_n\}_{n\in\mathbb{Z}}$ is a real-valued sequence of amplitudes decaying to zero as $|n| \to \infty$. In the anti-continuum limit (which corresponds here to the limit $|\omega| \to \infty$ [69]), the multi-site solitons are supported on a finite number of lattice sites. The oscillations are in-phase or anti-phase, depending on the sign difference between the amplitudes $\{A_n\}_{n\in\mathbb{Z}}$ on the excited sites of the lattice.

The stable oscillations in the case of the defocusing nonlinearity can be recovered from the stable anti-phase oscillations in the focusing case using the staggering transformation,

$$a_n(\tau) = (-1)^n \bar{b}_n(\tau) e^{2i\tau},$$

which changes the dNLS equation (5.3) to the form,

$$2i\dot{b}_n = (\Delta \mathbf{b})_n \pm |b_n|^2 b_n, \quad n \in \mathbb{Z}.$$

Consequently, the results in [62] also imply: *in the case of the defocusing nonlinearity, the only stable multi-site solitons of the dNLS equation (5.3) with adjacent excited sites near the anti-continuum limit correspond to the in-phase oscillations on the excited sites of the lattice.* The numerical observations of [62] were rigorously proved for the dNLS equation (5.3) by Pelinovsky, Kevrekidis & Frantzeskakis [72].

Similar to [62] conclusions on spectral stability of breathers in the dKG equation (5.1) were reported in the literature under some simplifying assumptions. Archilla *et al.* [3] used the Aubry's band theory to consider two-site, three-site, and generally multi-site breathers. Theorem 6 in [3] states that *in-phase multi-site breathers are stable for hard potentials and anti-phase breathers are stable for soft potentials for* $\epsilon > 0$. The statement of this result misses, however, that the corresponding computations are only justified for multi-site breathers with adjacent excited sites: no "holes" in the limiting configuration at $\epsilon = 0$ are allowed. More recently, Koukouloyannis & Kevrekidis [52] recovered exactly the same conclusion using the averaging theory for Hamiltonian systems in action–angle variables developed earlier by Ahn, MacKay & Sepulchre [2] and MacKay [55]. To justify the use of the first-order perturbation theory, the multi-site breathers were considered to have adjacent excited sites and no "holes". The equivalence between the method of Hamiltonian averaging and the Aubry's band theory for stability of multi-site breathers was addressed by Cuevas *et al.* [26].

In this chapter, we follow our recent work [75] to prove the linear stability criterion for all multi-site breathers, including breathers with "holes" between excited sites in the anti-continuum limit. We use perturbative arguments for characteristic exponents of the Floquet monodromy matrices. In order to work with higher-order perturbation theory, we combine these perturbative arguments with the theory of tail-to-tail interactions of individual breathers in lattices. Although the tail-to-tail interaction theory is well-known for continuous partial differential equations [80, 81], this theory was not previously developed in the context of nonlinear lattices.

Nonlinear stability of discrete breathers in the KG chain (5.1) with potential (5.2) has not been studied analytically yet. Some results on nonlinear stability of discrete breathers in Hamiltonian networks were established by Bambusi [8, 9]. In [9], asymptotic stability of discrete breathers in the KG lattice (5.1) near the anti-continuum limit is established provided the on-site potential admits the asymptotic expansion $V'(u) = u + \mathcal{O}(u^7)$ as $|u| \to 0$.

Multi-site breathers with "holes" have been recently considered by Yoshimura [101] in diatomic FPU lattice near the anti-continuum limit. In order to separate variables n and t and to perform computations using the discrete Sturm theorem (similar to the one used in the context of NLS lattices in [72]), the interaction potential was assumed to be nearly homogeneous of degree four and higher. Similar work was also performed by Yoshimura for KG lattices with a purely anharmonic interaction potential [102].

We discover new important details on the spectral stability of multi-site breathers, which were missed in the previous works [3, 52, 62]. In the case of soft potentials, breathers of the dKG equation (5.1) cannot be continued far away from the small-amplitude limit described by the dNLS equation (5.3) because of the resonances between the nonlinear oscillators at the excited sites and the linear oscillators at the sites at rest. Branches of breather solutions continued from the anti-continuum limit above and below the resonance are disconnected. In addition, these resonances change the stability conclusion. In particular, the anti-phase oscillations may become unstable in soft nonlinear potentials even if the coupling constant is sufficiently small.

Another interesting feature of soft potentials is the symmetry-breaking (pitchfork) bifurcation of one-site and multi-site breathers that occurs near the point of resonances. In symmetric potentials, the first non-trivial resonance occurs near $\omega = \frac{1}{3}$, that is, at 1:3 resonance. We analyze this bifurcation by using asymptotic expansions and reduction of the dKG equation (5.1) to a normal form, which coincides with the nonlinear Duffing oscillator perturbed by a small harmonic forcing. It is interesting that the normal form equation for 1:3 resonance which we analyze here is different from the normal form equations considered in the previous studies of 1:3 resonance [14, 84, 85]. While the standard normal form equations for 1:3 resonance are derived in a neighbourhood of equilibrium points, in this chapter we are looking at bifurcations of periodic solutions far from the equilibrium points. Note that an analytical study of bifurcations of small breather solutions close to a point of 1:3 resonance for a diatomic FPU lattice was performed by James & Kastner [39].

This chapter is organized as follows. The tail-to-tail interaction theory is developed in Section 5.1. The main result on spectral stability of multi-site breathers for small coupling constants is formulated and proved in Section 5.2. Section 5.3 illustrates the existence and spectral stability of multi-site breathers in soft potentials numerically. Section 5.4 is devoted to studies of the symmetry-breaking (pitchfork) bifurcation using asymptotic expansions and normal forms for the 1:3 resonance.

5.1 Tail-to-tail interactions

In this section, we study leading-order interactions between excited sites in the weakly-coupled KG lattice (5.1). As we will see in Section 5.2, these interactions are key in finding linearly stable configurations of multi-site discrete breathers.

Let us recall some existence theory from Section 3.2, where we extended limiting breather configurations on the KG lattice (5.1) with smooth even potential (5.2) away from the anti-continuum limit. We consider limiting configurations consisting of *in-phase* or *anti-phase* adjacent sites [51] in the form

$$\mathbf{u}^{(0)}(t) = \sum_{k\in S} \sigma_k \varphi(t)\mathbf{e}_k, \tag{5.4}$$

where $\mathbf{e}_k$ is a discrete delta function centred on the kth site (3.6), $S \subset \mathbb{Z}$ is a finite set of *excited sites*, $\sigma_k \in \{+1,-1\}$ encodes the phase factor of the k-th oscillator, and $\varphi \in H^2_{\rm e}(0,T)$ is an even solution of the nonlinear oscillator equation at the energy level E,

$$\ddot{\varphi} + V'(\varphi) = 0 \quad \Rightarrow \quad E = \frac{1}{2}\dot{\varphi}^2 + V(\varphi). \tag{5.5}$$

The unique even solution $\varphi(t)$ satisfies the initial condition,

$$\varphi(0) = a, \quad \dot{\varphi}(0) = 0, \tag{5.6}$$

where a is the smallest positive root of $V(a) = E$, and has a period T which is uniquely defined from the energy level E,

$$T = \sqrt{2}\int_{-a}^{a} \frac{d\varphi}{\sqrt{E - V(\varphi)}}. \tag{5.7}$$

Since $\varphi(t)$ is T-periodic, we also have

$$\begin{aligned} \partial_E \varphi(T) &= a'(E) = \frac{1}{V'(a)}, &(5.8)\\ \partial_E \dot{\varphi}(T) &= -\ddot{\varphi}(T)T'(E) = V'(a)T'(E). &(5.9) \end{aligned}$$

Example 5.1. Let us consider the truncation of the expansion (5.2) for the nonlinear potential at the first two terms:

$$V'(u) = u \pm u^3. \tag{5.10}$$

The dependence of the period T of the anharmonic oscillator on its energy E is computed numerically from (5.2) and is shown on Figure 5.1. For the *hard potential* with the plus sign, the period $T \in (0, 2\pi)$ is a decreasing function of E, whereas for the *soft potential* with the minus sign, the period $T > 2\pi$ is an increasing function of E, which diverges to infinity as E approaches 0.25.

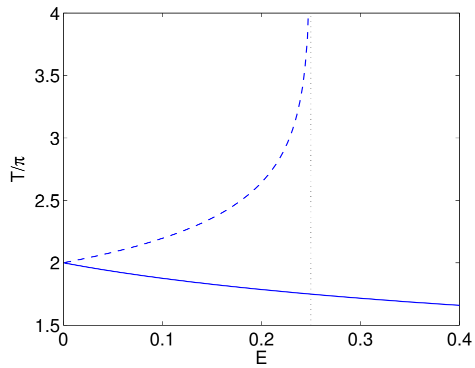

Figure 5.1: The period T versus energy E for the hard (solid) and soft (dashed) potentials.

We recall that according to Theorem 3.9, the limiting breather $\mathbf{u}^{(0)}(t)$ (5.4) satisfying non-resonance ($T \neq 2\pi n$, $n \in \mathbb{N}$) and non-degeneracy ($T'(E) \neq 0$) conditions can be uniquely extended to a solution $\mathbf{u}^{(\epsilon)}(t)$ of the dKG equation (5.1) in $l^2(\mathbb{Z}, H^2_e(0,T))$ space with norm in (3.10) provided the coupling constant ϵ is sufficiently small.

We are now going to look into specifics of leading-order interactions between the adjacent excited sites. Let us first introduce the concept of the fundamental breather that is constructed for the particular case of one excited site in the anti-continuum limit. For small $\epsilon > 0$, multi-site breathers can be approximated by superposition of fundamental breathers up to and including the order at which the tail-to-tail interactions of these breathers occur.

Definition 5.2. Let $\epsilon > 0$ be sufficiently small and $\mathbf{u}^{(\epsilon)} \in l^2(\mathbb{Z}, H^2_e(0,T))$ be a solution of the dKG equation (3.9) that is uniquely extended from the one-site limiting configuration $\mathbf{u}^{(0)}(t) = \varphi(t)\mathbf{e}_0$. This solution is called the *fundamental breather* and we denote it by $\boldsymbol{\phi}^{(\epsilon)}$.

By Theorem 3.9, we can use the Taylor approximation,

$$\boldsymbol{\phi}^{(\epsilon)} = \boldsymbol{\phi}^{(\epsilon,N)} + \mathcal{O}_{l^2(\mathbb{Z},H^2_{\text{per}}(0,T))}(\epsilon^{N+1}), \quad \boldsymbol{\phi}^{(\epsilon,N)} = \sum_{k=0}^{N} \frac{\epsilon^k}{k!} \frac{d^k}{d\epsilon^k} \boldsymbol{\phi}^{(\epsilon)} \bigg|_{\epsilon=0}, \tag{5.11}$$

up to any integer $N \geq 0$. Thanks to the discrete translational invariance of the lattice, the fundamental breather can be centred at any site $j \in \mathbb{Z}$. Let $\tau_j : l^2 \to l^2$ be the shift operator defined by

$$(\tau_j \mathbf{u})_n = u_{n-j}, \quad n \in \mathbb{Z}.$$

If $\boldsymbol{\phi}^{(\epsilon)}$ is centred at site 0, then $\tau_j \boldsymbol{\phi}^{(\epsilon)}$ is centred at site $j \in \mathbb{Z}$. The simplest multi-site breather is given by the two excited nodes at $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$ with $j \neq k$. The following lemma determines the leading-order interaction term for such discrete breather.

Lemma 5.3. *Let* $\mathbf{u}^{(0)}(t) = \sigma_j \varphi(t)\mathbf{e}_j + \sigma_k \varphi(t)\mathbf{e}_k$ *with* $j \neq k$ *and* $N = |j-k| \geq 1$. *Let* $\mathbf{u}^{(\epsilon)} \in l^2(\mathbb{Z}, H_e^2(0,T))$ *be the corresponding solution of the dKG equation (3.9) for small* $\epsilon > 0$ *defined by Theorem 3.9. Let* $\{\varphi_m\}_{m=1}^N \in H_e^2(0,T)$ *be defined recursively by*

$$\mathcal{L}_0 \varphi_m := (\partial_t^2 + 1)\varphi_m = \varphi_{m-1}, \quad m = 1, 2, ..., N, \tag{5.12}$$

starting with $\varphi_0 = \varphi$, *and let* $\psi_N \in H_e^2(0,T)$ *be defined by*

$$\mathcal{L}_e \psi_N := (\partial_t^2 + V''(\varphi(t)))\psi_N = \varphi_{N-1}. \tag{5.13}$$

Then, we have

$$\mathbf{u}^{(\epsilon)} = \sigma_j \tau_j \boldsymbol{\phi}^{(\epsilon,N)} + \sigma_k \tau_k \boldsymbol{\phi}^{(\epsilon,N)} + \epsilon^N (\sigma_j \mathbf{e}_k + \sigma_k \mathbf{e}_j)(\psi_N - \varphi_N) + \mathcal{O}_{l^2(\mathbb{Z}, H^2_{\rm per}(0,T))}(\epsilon^{N+1}). \tag{5.14}$$

Proof. By Theorem 3.9, the limiting configuration $\mathbf{u}^{(0)}(t) = \sigma_j \varphi(t)\mathbf{e}_j + \sigma_k \varphi(t)\mathbf{e}_k$ with two excited sites generates a C^∞ map, which can be expanded up to the $N+1$-order,

$$\mathbf{u}^{(\epsilon)} = \sum_{k=0}^{N} \frac{\epsilon^k}{k!} \frac{d^k}{d\epsilon^k} \mathbf{u}^{(\epsilon)} \Big|_{\epsilon=0} + \mathcal{O}_{l^2(\mathbb{Z}, H^2_{\rm per}(0,T))}(\epsilon^{N+1}). \tag{5.15}$$

Substituting (5.15) into (3.9) generates a sequence of equations at each order of ϵ, which we consider up to and including the terms of order N.

The central excited site at $n = 0$ in the fundamental breather $\boldsymbol{\phi}^{(\epsilon)}$ generates fluxes, which reach sites $n = \pm m$ at the m-th order. Because $\boldsymbol{\phi}^{(\epsilon,m)}$ is compactly supported on $\{-m, -m+1, ..., m\}$ and all sites with $n \neq 0$ contain no oscillations at the 0-th order, we have

$$\phi_{\pm m}^{(\epsilon,m)} = \epsilon^m \varphi_m, \tag{5.16}$$

where $\{\varphi_m\}_{m=1}^N \in H_e^2(0,T)$ are computed from the linear inhomogeneous equations (5.12) starting with $\varphi_0 = \varphi$. Note that equations (5.12) are uniquely solvable because $T \neq 2\pi n$, $n \in \mathbb{N}$.

For definiteness, let us assume that $j = 0$ and $k = N \geq 1$. The fluxes from the excited sites $n = 0$ and $n = N$ meet at the $N/2$-th order at the middle site $n = N/2$ if N is even or they overlap at the $(N+1)/2$-th order at the two sites $n = (N-1)/2$ and $n = (N+1)/2$ if N is odd. In either case, because of the expansion (5.2), the nonlinear superposition of these fluxes affects terms at the order $3N/2$-th or

$3(N+1)/2$-th orders, that is, beyond the N-th order of the expansion (5.14). Therefore, the nonlinear superposition of fluxes in higher orders of ϵ will definitely be beyond the N-th order of the expansion (5.14).

Up to the N-th order, all correction terms are combined together as a sum of correction terms from the decomposition (5.11) centred at the j-th and k-th sites, that is, we have

$$\mathbf{u}(\epsilon) = \sigma_j \tau_j \boldsymbol{\phi}^{(\epsilon,N-1)}(\epsilon) + \sigma_k \tau_k \boldsymbol{\phi}^{(\epsilon,N-1)}(\epsilon) + \mathcal{O}_{l^2(\mathbb{Z},H^2_{\text{per}}(0,T))}(\epsilon^N).$$

At the N-th order, the flux from j-th site arrives to the k-th site and vice versa. Therefore, besides the N-th order correction terms from the decomposition (5.11), we have additional terms $\epsilon^N (\sigma_j \mathbf{e}_k + \sigma_k \mathbf{e}_j) \psi_N$ at the sites $n = j$ and $n = k$. Thanks to the linear superposition principle, these additional terms are given by solutions of the inhomogeneous equations (5.13), which are uniquely solvable in $H^2_e(0,T)$ because $T'(E) \neq 0$. We also have to subtract $\epsilon^N (\sigma_j \mathbf{e}_k + \sigma_k \mathbf{e}_j) \varphi_N$ from the N-th order of $\sigma_j \tau_j \boldsymbol{\phi}^{(\epsilon,N)} + \sigma_k \tau_k \boldsymbol{\phi}^{(\epsilon,N)}$, because these terms were computed under the assumption that the k-th site contained no oscillations at the order 0 for $\sigma_j \tau_j \boldsymbol{\phi}^{(\epsilon,N)}$ and vice versa. Combined all together, the expansion (5.14) is justified up to terms of the N-th order. □

5.2 Stability of multi-site breathers

Let $\mathbf{u} \in l^2(\mathbb{Z}, H^2_e(0,T))$ be a multi-site breather in Theorem 3.9 and $\epsilon > 0$ be a small coupling parameter in the dKG equation (5.1). We introduce a small perturbation $\mathbf{w} \in l^2(\mathbb{Z}, H^2_e(0,T))$ to the multi-site breather, and substitute the decomposition $\mathbf{u}(t)+\mathbf{w}(t)$ into the dKG equation (5.1). Collecting the terms linear in $\mathbf{w}$, we obtain

$$\ddot{w}_n + V''(u_n) w_n = \epsilon (\Delta \mathbf{w})_n, \quad n \in \mathbb{Z}. \tag{5.17}$$

Because $\mathbf{u}(t+T) = \mathbf{u}(t)$, an infinite-dimensional analogue of the Floquet theorem applies and the Floquet monodromy matrix $\mathcal{M}$ is defined by

$$\left[\left\{ \begin{array}{c} w_n(T) \\ \dot{w}_n(T) \end{array} \right\}_{n\in\mathbb{Z}} \right] = \mathcal{M} \left[\left\{ \begin{array}{c} w_n(0) \\ \dot{w}_n(0) \end{array} \right\}_{n\in\mathbb{Z}} \right].$$

Definition 5.4. We say that the breather is *spectrally stable* if all eigenvalues of the monodromy matrix $\mathcal{M}$, called Floquet multipliers, are located on the unit circle and it is *spectrally unstable* if there is at least one Floquet multiplier outside the unit disk.

Because the linearized system (5.17) is Hamiltonian, Floquet multipliers come in pairs μ_1 and μ_2 with $\mu_1\mu_2 = 1$. Moreover, if a multiplier μ_1 is on a unit circle, so is its partner μ_2.

For $\epsilon = 0$, Floquet multipliers can be computed explicitly because $\mathcal{M}$ is decoupled into a diagonal combination of 2-by-2 matrices $\{M_n\}_{n\in\mathbb{Z}}$, which are computed from solutions of the linearized equations

$$\ddot{w}_n + w_n = 0, \quad n \in \mathbb{Z}\backslash S \tag{5.18}$$

and

$$\ddot{w}_n + V''(\varphi)w_n = 0, \quad n \in S. \tag{5.19}$$

The first problem (5.18) admits the exact solution,

$$w_n(t) = w_n(0)\cos(t) + \dot{w}_n(0)\sin(t) \quad \Rightarrow \quad M_n = \begin{bmatrix} \cos(T) & \sin(T) \\ -\sin(T) & \cos(T) \end{bmatrix}, \quad n \in \mathbb{Z}\backslash S.$$

Each M_n for $n \in \mathbb{Z}\backslash S$ has two Floquet multipliers at $\mu_1 = e^{iT}$ and $\mu_2 = e^{-iT}$. If $T \neq 2\pi n$, $n \in \mathbb{N}$, the Floquet multipliers μ_1 and μ_2 are located on the unit circle bounded away from the point $\mu = 1$.

The second problem (5.19) admits the exact solution,

$$w_n(t) = \frac{\dot{w}_n(0)}{\ddot{\varphi}(0)}\dot{\varphi}(t) + \frac{w_n(0)}{\partial_E\varphi(0)}\partial_E\varphi(t), \quad n \in S,$$

where $\varphi(t)$ is a solution of the nonlinear oscillator equation (5.5) with the initial condition (5.6). Using identities (5.8)–(5.9), we obtain,

$$M_n = \begin{bmatrix} 1 & 0 \\ T'(E)[V'(a)]^2 & 1 \end{bmatrix}, \quad n \in S.$$

Note that $V'(a) \neq 0$ (or T is infinite). If $T'(E) \neq 0$, each M_n for $n \in S$ has the Floquet multiplier $\mu = 1$ of geometric multiplicity one and algebraic multiplicity two.

We conclude that if $T \neq 2\pi n$, $n \in \mathbb{N}$ and $T'(E) \neq 0$, the limiting multi-site breather (5.4) at the anti-continuum limit $\epsilon = 0$ has an infinite number of semi-simple Floquet multipliers at $\mu_1 = e^{iT}$ and $\mu_2 = e^{-iT}$ bounded away from the Floquet multiplier $\mu = 1$ of algebraic multiplicity $2|S|$ and geometric multiplicity $|S|$, where $|S|$ represents the number of excited sites in limiting breather configuration (5.4).

Semi-simple multipliers on the unit circle are structurally stable in Hamiltonian dynamical systems (Chapter III in [96]). Under perturbations in the Hamiltonian, Floquet multipliers of the same Krein signature do not move off the unit circle unless they coalesce with Floquet multipliers of the opposite Krein signature [13]. Therefore,

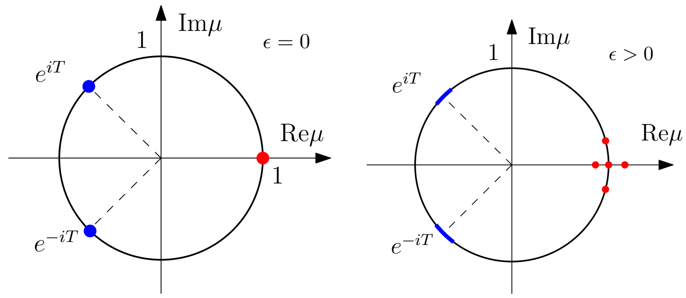

Figure 5.2: Spectrum of the monodromy matrix associated with a multi-site breather in the dKG equation (5.1) at $\epsilon = 0$ (left) and at small $\epsilon > 0$ (right).

the instability of the multi-site breather may only arise from the splitting of the Floquet multiplier $\mu = 1$ of algebraic multiplicity $2|S|$ for $\epsilon \neq 0$. We show the splitting of the Floquet multipliers near the anti-continuum limit on Figure 5.2. Details on the splitting of the unit Floquet multiplier will follow in Lemmas 5.5 and 5.6.

5.2.1 Perturbation analysis for the unit Floquet multiplier

To consider Floquet multipliers, we can introduce the *characteristic exponent* λ in the decomposition $\mathbf{w}(t) = \mathbf{W}(t)e^{\lambda t}$. If $\mu = e^{\lambda T}$ is the Floquet multiplier of the monodromy operator $\mathcal{M}$, then $\mathbf{W} \in l^2(\mathbb{Z}, H^2_{\text{per}}(0,T))$ is a solution of the eigenvalue problem,

$$\ddot{W}_n + V''(u_n)W_n + 2\lambda\dot{W}_n + \lambda^2 W_n = \epsilon(\Delta\mathbf{W})_n, \quad n \in \mathbb{Z}. \tag{5.20}$$

In particular, Floquet multiplier $\mu = 1$ corresponds to the characteristic exponent $\lambda = 0$. The generalized eigenvector $\mathbf{Z} \in l^2(\mathbb{Z}, H^2_{\text{per}}(0,T))$ of the eigenvalue problem (5.20) for $\lambda = 0$ solves the inhomogeneous problem,

$$\ddot{Z}_n + V''(u_n)Z_n = \epsilon(\Delta\mathbf{Z})_n - 2\dot{W}_n, \quad n \in \mathbb{Z}, \tag{5.21}$$

where $\mathbf{W}$ is the eigenvector of (5.20) for $\lambda = 0$. To normalize $\mathbf{Z}$ uniquely, we add a constraint that $\mathbf{Z}$ is orthogonal to $\mathbf{W}$ with respect to the inner product

$$\langle \mathbf{W}, \mathbf{Z} \rangle_{l^2(\mathbb{Z}, L^2_{\text{per}}(0,T))} := \sum_{n\in\mathbb{Z}} \int_0^T \bar{W}_n(t) Z_n(t) dt.$$

At $\epsilon = 0$, the eigenvector $\mathbf{W}$ of the eigenvalue problem (5.20) for $\lambda = 0$ is spanned by the linear combination of $|S|$ solutions, so that

$$\mathbf{W}^{(0)}(t) = \sum_{k\in S} c_k \dot{\varphi}(t)\mathbf{e}_k. \tag{5.22}$$

Recalling that the operator

$$\mathcal{L}_e = \partial_t^2 + V''(\varphi(t)) : H_e^2(0,T) \to L_e^2(0,T)$$

is invertible and $\ddot{\varphi} \in L_e^2(0,T)$ (see the proof of Theorem 3.9), we write the generalized eigenvector $\mathbf{Z}$ as a linear combination of $|S|$ solutions, so that

$$\mathbf{Z}^{(0)}(t) = -\sum_{k\in S} c_k v(t)\mathbf{e}_k, \qquad v = 2\mathcal{L}_e^{-1}\ddot{\varphi}. \tag{5.23}$$

We also notice that $\langle \dot{\varphi}, v\rangle_{L^2_{\rm per}(0,T)} = 0$ since $\dot{\varphi}$ is odd and v is even in t.

Let $\boldsymbol{\phi}^{(\epsilon)}$ be the fundamental breather in Definition 5.2. Because of the translational invariance in t, the time derivative of the fundamental breather,

$$\mathbf{W} = \partial_t \boldsymbol{\phi}^{(\epsilon)} \equiv \boldsymbol{\theta}^{(\epsilon)} \in l^2(\mathbb{Z}, H^2_{\rm per}(0,T)),$$

is the eigenvector of the eigenvalue problem (5.20) for $\lambda = 0$ and small $\epsilon > 0$. Moreover, since $\boldsymbol{\theta}^{(\epsilon)}$ and $\partial_t \boldsymbol{\theta}^{(\epsilon)}$ have the opposite parity in t, there exists a corresponding generalized eigenvector

$$\mathbf{Z} \equiv \boldsymbol{\mu}^{(\epsilon)} \in l^2(\mathbb{Z}, H^2_{\rm per}(0,T))$$

of the inhomogeneous problem (5.21).

By Taylor approximation (5.11), for any integer $N \geq 0$, we have

$$\begin{aligned} \boldsymbol{\theta}^{(\epsilon)} &= \boldsymbol{\theta}^{(\epsilon,N)} + \mathcal{O}_{l^2(\mathbb{Z},H^2_{\rm per}(0,T))}(\epsilon^{N+1}), \\ \boldsymbol{\mu}^{(\epsilon)} &= \boldsymbol{\mu}^{(\epsilon,N)} + \mathcal{O}_{l^2(\mathbb{Z},H^2_{\rm per}(0,T))}(\epsilon^{N+1}), \end{aligned}$$

where $\boldsymbol{\theta}^{(\epsilon,N)}$ and $\boldsymbol{\mu}^{(\epsilon,N)}$ are polynomials in ϵ of degree N. It follows from equations (5.22) and (5.23) that

$$\boldsymbol{\theta}^{(0)} = \dot{\varphi}(t)\mathbf{e}_0, \quad \boldsymbol{\mu}^{(0)} = -v(t)\mathbf{e}_0. \tag{5.24}$$

This formalism sets up the scene for the perturbation theory, which is used to prove the main result on spectral stability of multi-site breathers. We start with a simple multi-site breather configuration with equal distances between excited sites and then generalize this result to multi-site breathers with non-equal distances between excited sites.

Lemma 5.5. *Fix the period T and the solution $\varphi \in H_e^2(0,T)$ of the nonlinear oscillator equation (5.5) with an even $V \in C^\infty(\mathbb{R})$ and assume that $T \neq 2\pi n$, $n \in \mathbb{N}$ and*

$T'(E) \neq 0$. Let

$$\mathbf{u}^{(0)}(t) = \sum_{j=1}^{N} \sigma_j \varphi(t) \mathbf{e}_{jM}$$

with fixed $N, M \in \mathbb{N}$ and $\mathbf{u}^{(\epsilon)} \in l^2(\mathbb{Z}, H_e^2(0,T))$ be the corresponding solution of the dKG equation (5.1) for small $\epsilon > 0$ defined by Theorem 3.9. Let $\{\varphi_m\}_{m=0}^{M}$ be defined by inhomogeneous equation (5.12) starting with $\varphi_0 = \varphi$. Then the eigenvalue problem (5.20) for small $\epsilon > 0$ has $2N$ small eigenvalues,

$$\lambda = \epsilon^{M/2} \Lambda + \mathcal{O}(\epsilon^{(M+1)/2}),$$

where Λ is an eigenvalue of the matrix eigenvalue problem

$$-\frac{T^2(E)}{T'(E)} \Lambda^2 \mathbf{c} = K_M \mathcal{S} \mathbf{c}, \quad \mathbf{c} \in \mathbb{C}^N. \tag{5.25}$$

Here the numerical coefficient K_M is given by

$$K_M = \int_0^T \dot{\varphi} \dot{\varphi}_{M-1} dt$$

and the matrix $\mathcal{S} \in \mathbb{M}^{N \times N}$ is given by

$$\mathcal{S} = \begin{bmatrix} -\sigma_1\sigma_2 & 1 & 0 & \dots & 0 & 0 \\ 1 & -\sigma_2(\sigma_1+\sigma_3) & 1 & \dots & 0 & 0 \\ 0 & 1 & -\sigma_3(\sigma_2+\sigma_4) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\sigma_{N-1}(\sigma_{N-2}+\sigma_N) & 1 \\ 0 & 0 & 0 & \dots & 1 & -\sigma_N\sigma_{N-1} \end{bmatrix}.$$

Proof. At $\epsilon = 0$, the eigenvalue problem (5.20) admits eigenvalue $\lambda = 0$ of geometric multiplicity N and algebraic multiplicity $2N$, which is isolated from the rest of the spectrum. Perturbation theory in ϵ applies thanks to the smoothness of $\mathbf{u}^{(\epsilon)}$ in ϵ and V' in u. Perturbation expansions (so-called Puiseux series, see Chapter 2 in [45] and recent work [95]) are smooth in powers of $\epsilon^{1/2}$ thanks to the Jordan block decomposition at $\epsilon = 0$.

We need to find out how the eigenvalue $\lambda = 0$ of algebraic multiplicity $2N$ splits for small $\epsilon > 0$. Therefore, we are looking for the eigenvectors of the eigenvalue problem (5.20) in the subspace associated with the eigenvalue $\lambda = 0$ using the substitution

$\lambda = \epsilon^{M/2}\tilde{\lambda}$ and the decomposition

$$\mathbf{W} = \sum_{j=1}^{N} c_j \left(\tau_{jM}\boldsymbol{\theta}^{(\epsilon,M)} - \epsilon^M(\mathbf{e}_{(j-1)M} + \mathbf{e}_{(j+1)M})\dot{\varphi}_M\right) + \epsilon^{M/2}\tilde{\lambda}\sum_{j=1}^{N} c_j\tau_{jM}\boldsymbol{\mu}^{(\epsilon,M_*)} + \epsilon^M\tilde{\mathbf{W}}, \quad (5.26)$$

where $M_* = M/2$ if M is even and $M_* = (M-1)/2$ if M is odd, whereas $\tilde{\mathbf{W}}$ is the remainder term at the M-th order in ϵ. The decomposition formula (5.26) follows from the superposition (5.14) up to the M-th order in ϵ. The terms $\epsilon^M\sum_{j=1}^{N} c_j(\mathbf{e}_{(j-1)M} + \mathbf{e}_{(j+1)M})\dot{\psi}_N$ from the superposition (5.14) are to be accounted at the equation for $\tilde{\mathbf{W}}$. Note that our convention in writing (5.26) is to drop the boundary terms with $\mathbf{e}_{0M}$ and $\mathbf{e}_{(N+1)M}$.

Substituting (5.26) to (5.20), all equations are satisfied up to the M-th order. At the M-th order, we divide (5.20) by ϵ^M and collect equations at the excited sites $n = jM$ for $j \in \{1,2,...,N\}$,

$$\ddot{\tilde{W}}_{jM} + V''(\varphi)\tilde{W}_{jM} = (c_{j+1} + c_{j-1})\dot{\varphi}_{M-1} - \sigma_j(\sigma_{j+1} + \sigma_{j-1})c_j V'''(\varphi)\psi_M\dot{\varphi} + \tilde{\lambda}^2 c_j(2\dot{v} - \dot{\varphi}) + \mathcal{O}(\epsilon^{1/2}), \quad (5.27)$$

where we admit another convention that $\sigma_0 = \sigma_{N+1} = 0$ and $c_0 = c_{N+1} = 0$. In the derivation of equations (5.27), we have used the fact that the term $\dot{\varphi}_{M-1}$ comes from the fluxes from $n = (j+1)M$ and $n = (j-1)M$ sites generated by the derivatives of the linear inhomogeneous equations (5.12) and the term $\sigma_j(\sigma_{j+1} + \sigma_{j-1})c_j V'''(\varphi)\psi_M\dot{\varphi}$ comes from the expansion of the nonlinear potential $V''(u_{jM})$ by using the expansion (5.14).

Expanding $\tilde{\lambda} = \Lambda + \mathcal{O}(\epsilon^{1/2})$ and projecting the system of linear inhomogeneous equations (5.27) to $\dot{\varphi} \in H^2_{\text{per}}(0,T)$, the kernel of the linear operator

$$\mathcal{L} = \partial_t^2 + V''(\varphi) : H^2_{\text{per}}(0,T) \to L^2_{\text{per}}(0,T),$$

we obtain the system of difference equations,

$$\Lambda^2 c_j \int_0^T \left(\dot{\varphi}^2 + 2v\ddot{\varphi}\right) dt = (c_{j+1} + c_{j-1})\int_0^T \dot{\varphi}\dot{\varphi}_{M-1} dt - \sigma_j(\sigma_{j+1} + \sigma_{j-1})c_j \int_0^T V'''(\varphi)\psi_M\dot{\varphi}^2 dt,$$

where the integration by parts is used to simplify the left-hand side. Differentiating the linear inhomogeneous equation (5.13) and projecting it to $\dot{\varphi}$, we infer that

$$\int_0^T V'''(\varphi)\psi_M \dot{\varphi}^2 dt = \int_0^T \dot{\varphi}\dot{\varphi}_{M-1} dt \equiv K_M.$$

The system of difference equations yields the matrix eigenvalue problem (5.25), provided we can verify that

$$\int_0^T \left(\dot{\varphi}^2 + 2v\ddot{\varphi}\right) dt = -\frac{T^2(E)}{T'(E)}.$$

To do so, we note that $v = 2\mathcal{L}_e^{-1}\ddot{\varphi}$ in (5.23) is even in $t \in \mathbb{R}$, so that it is given by the exact solution,

$$v(t) = t\dot{\varphi}(t) + C\partial_E \varphi(t),$$

where $C \in \mathbb{R}$. From the condition of T-periodicity for $v(t)$ and $\dot{v}(t)$, we obtain

$$\begin{aligned} v(0) &= v(T) = Ca'(E), \\ \dot{v}(0) &= 0 = \dot{v}(T) = T\ddot{\varphi}(0) - CT'(E)\ddot{\varphi}(0), \end{aligned}$$

hence $C = T(E)/T'(E)$ and

$$\begin{aligned} \int_0^T \left(\dot{\varphi}^2 + 2v\ddot{\varphi}\right) dt &= 2C\int_0^T \ddot{\varphi}\partial_E\varphi dt = -C\int_0^T \left(\dot{\varphi}\partial_E\dot{\varphi} + V'(\varphi)\partial_E\varphi\right) dt \\ &= -C\int_0^T \frac{\partial}{\partial E}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) dt = -CT(E) = -\frac{T^2(E)}{T'(E)}, \end{aligned}$$

where equation (5.5) has been used. Finally, the matrix eigenvalue problem (5.25) defines $2N$ small eigenvalues that bifurcate from $\lambda = 0$ for small $\epsilon > 0$. The proof of the lemma is complete. ☐

To classify stable and unstable configurations of multi-site breathers near the anti-continuum limit we shall now count eigenvalues of the matrix eigenvalue problem (5.25).

Lemma 5.6. *Let n_0 be the numbers of negative elements in the sequence $\{\sigma_j\sigma_{j+1}\}_{j=1}^{N-1}$. If $T'(E)K_M > 0$, the matrix eigenvalue problem (5.25) has exactly n_0 pairs of purely imaginary eigenvalues Λ and $N-1-n_0$ pairs of real eigenvalues μ counting their multiplicities, in addition to the double zero eigenvalue. If $T'(E)K_M < 0$, the conclusion changes to the opposite.*

Proof. We shall prove that the matrix $\mathcal{S}$ has exactly n_0 positive and $N-1-n_0$ negative eigenvalues counting their multiplicities, in addition to the simple zero eigenvalue. If this is the case, the assertion of the lemma follows from the correspondence

$$\Lambda^2 = -\frac{T'(E)K_M}{T^2(E)}\gamma,$$

where γ is an eigenvalue of $\mathcal{S}$.

Setting $c_j = \sigma_j b_j$, we rewrite the eigenvalue problem $\mathcal{S}\mathbf{c} = \gamma\mathbf{c}$ as the difference equation,

$$\sigma_j\sigma_{j+1}(b_{j+1} - b_j) + \sigma_j\sigma_{j-1}(b_{j-1} - b_j) = \gamma b_j, \quad j \in \{1, 2, ..., N\}, \tag{5.28}$$

subject to the conditions $\sigma_0 = \sigma_{N+1} = 0$. Therefore, $\gamma = 0$ is always an eigenvalue with the eigenvector $\mathbf{b} = [1, 1, ..., 1] \in \mathbb{R}^N$. The coefficient matrix in (5.28) coincides with the one analyzed by Sandstede in Lemma 5.4 and Appendix C [80]. This correspondence yields the assertion on the number of eigenvalues of $\mathcal{S}$. □

5.2.2 General stability result

Before applying the results of Lemmas 5.5 and 5.6 to multi-site breathers, we consider two examples, which are related to the truncated potential (5.10). We shall use the Fourier cosine series for the solution $\varphi \in H_e^2(0, T)$,

$$\varphi(t) = \sum_{n\in\mathbb{N}} c_n(T)\cos\left(\frac{2\pi nt}{T}\right), \tag{5.29}$$

for some square summable set $\{c_n(T)\}_{n\in\mathbb{N}}$. Because of the symmetry of V, we have $\varphi(T/4) = 0$, which imply that $c_n(T) \equiv 0$ for all even $n \in \mathbb{N}$. Solving the linear inhomogeneous equations (5.12), we obtain

$$\varphi_k(t) = \sum_{n\in\mathbb{N}_{\text{odd}}} \frac{T^{2k}c_n(T)}{(T^2 - 4\pi^2n^2)^k}\cos\left(\frac{2\pi nt}{T}\right), \quad k \in \mathbb{N}. \tag{5.30}$$

Using Parseval's equality, we compute the constant K_M in Lemma 5.5,

$$K_M = \int_0^T \dot{\varphi}(t)\dot{\varphi}_{M-1}(t)dt = 4\pi^2 \sum_{n\in\mathbb{N}_{\text{odd}}} \frac{T^{2M-3}n^2|c_n(T)|^2}{(T^2 - 4\pi^2n^2)^{M-1}}. \tag{5.31}$$

For the hard potential with $V'(u) = u + u^3$, we know from Figure 5.1 that the period $T(E)$ is a decreasing function of E from $T(0) = 2\pi$ to $\lim_{E\to\infty} T(E) = 0$. Since $T'(E) < 0$ and $T(E) < 2\pi$, we conclude that $T'(E)K_M < 0$ if M is odd and $T'(E)K_M > 0$ if M is even. By Lemma 5.6, if M is odd, the only stable configuration of

	M odd	M even
Hard potential $V'(u) = u + u^3$ $0 < T < 2\pi$	In-phase	Anti-phase
Soft potential $V'(u) = u - u^3$ $2\pi < T < 6\pi$	Anti-phase	Anti-phase $2\pi < T < T_M$ In-phase $T_M < T < 6\pi$

Table 5.1: Stable multi-site breathers in hard and soft potentials. The stability threshold T_M corresponds to the zero of K_M for $T \in (2\pi, 6\pi)$.

the multi-site breathers is the one with all equal $\{\sigma_j\}_{j=1}^N$ (in-phase breathers), whereas if M is even, the only stable configuration of the multi-site breathers is the one with all alternating $\{\sigma_j\}_{j=1}^N$ (anti-phase breathers). This conclusion is shown in the first line of Table 5.1.

For the soft potential with $V'(u) = u - u^3$, we know from Figure 5.1 that the period $T(E)$ is an increasing function of E from $T(0) = 2\pi$ to $\lim_{E\to E_0} T(E) = \infty$, where $E_0 = \frac{1}{4}$. If $T(E)$ is close to 2π, then the first positive term in the series (5.31) dominates and $K_M > 0$ for all $M \in \mathbb{N}$. At the same time, $T'(E) > 0$ and Lemma 5.6 implies that the only stable configuration of the multi-site breathers is the one with all alternating $\{\sigma_j\}_{j=1}^N$ (anti-phase breathers). The conclusion holds for any $T > 2\pi$ if M is odd, because $K_M > 0$ in this case.

This precise conclusion is obtained in Theorem 3.6 of [72] in the framework of the dNLS equation (5.3) (see also Section 4.1). It is also in agreement with perturbative arguments in [3, 52], which are valid for $M = 1$ (all excited sites are adjacent on the lattice). To elaborate this point further, we prove in [75] the equivalence between the matrix eigenvalue problem (5.25) with $M = 1$ and the criteria used in [3, 52].

For even $M \in \mathbb{N}$, we observe a new phenomenon, which arises for the soft potentials with larger values of $T(E) > 2\pi$. To be specific, we restrict our consideration of multi-site breathers with the period T in the interval $(2\pi, 6\pi)$. Similar results can be obtained in the intervals $(6\pi, 10\pi)$, $(10\pi, 14\pi)$, and so on. For even $M \in \mathbb{N}$, there exists a period $T_M \in (2\pi, 6\pi)$ such that the constant K_M in (5.31) changes sign from $K_M > 0$ for $T \in (2\pi, T_M)$ to $K_M < 0$ for $T \in (T_M, 6\pi)$. When it happens, the conclusion on stability of the multi-site breather change directly to the opposite: the only stable configuration of the multi-site breathers is the one with all equal $\{\sigma_j\}_{j=1}^N$ (in-phase breathers). This observation is shown in the second line of Table 5.1.

We conclude this section with the stability theorem for general multi-site breathers. For the sake of clarity, we formulate the theorem in the case when $T'(E) > 0$ and all $K_M > 0$, which arises for the soft potential with odd M. Using Lemma 5.6, the count can be adjusted to the cases of $T'(E) < 0$ and/or $K_M < 0$.

Theorem 5.7. *Let $\{n_j\}_{j=1}^N \subset \mathbb{Z}$ be an increasing sequence with $N \in \mathbb{N}$. Let $\mathbf{u}^{(\epsilon)} \in l^2(\mathbb{Z}, H_e^2(0,T))$ be a solution of the dKG equation (5.1) in Theorem 3.9 with*

$$\mathbf{u}^{(0)}(t) = \sum_{j=1}^{N} \sigma_j \varphi(t) \mathbf{e}_{n_j} \tag{5.32}$$

for small $\epsilon > 0$. Let $\{\varphi_m\}_{m=0}^\infty$ be defined by the linear equations (5.12) starting with $\varphi_0 = \varphi$. Define $\{M_j\}_{j=1}^{N-1}$ and $\{K_{M_j}\}_{j=1}^{N-1}$ by

$$M_j = n_{j+1} - n_j \quad \text{and} \quad K_{M_j} = \int_0^T \dot{\varphi}\dot{\varphi}_{M_j-1} dt.$$

Assume $T'(E) > 0$ and $K_{M_j} > 0$ for all M_j. Let n_0 be the numbers of negative elements in the sequence $\{\sigma_j \sigma_{j+1}\}_{j=1}^{N-1}$. The eigenvalue problem (5.20) at the discrete breather $\mathbf{u}^{(\epsilon)}$ has exactly n_0 pairs of purely imaginary eigenvalues λ and $N - 1 - n_0$ pairs of purely real eigenvalues λ counting their multiplicities, in addition to the double zero eigenvalue.

Proof. The limiting configuration (5.32) defines clusters of excited sites with equal distances M_j between the adjacent excited sites. According to Lemma 5.5, splitting of N double Jordan blocks associated with the decompositions (5.22) and (5.23) occurs in different orders of the perturbation theory, which are determined by the set $\{M_j\}_{j=1}^{N-1}$. At each order of the perturbation theory, the splitting of eigenvalues associated with one cluster with equal distance between the adjacent excited sites obeys the matrix eigenvalue problem (5.25), which leaves exactly one double eigenvalue at the zero and yields symmetric pairs of purely real or purely imaginary eigenvalues, in accordance to the count of Lemma 5.6.

The double zero eigenvalue corresponds to the eigenvector $\mathbf{W}$ and the generalized eigenvector $\mathbf{Z}$ generated by the translational symmetry of the multi-site breather bifurcating from a particular cluster of excited sites in the limiting configuration $\mathbf{u}^{(0)}$. The splitting of the double zero eigenvalues associated with the cluster happens at the higher orders in ϵ, when the fluxes from adjacent clusters reach each others. Since the end-point fluxes from the multi-site breathers are equivalent to the fluxes (5.16) generated from the fundamental breathers, they still obey Lemma 5.3 and the splitting of the double zero eigenvalue associated with different clusters still obeys Lemma 5.5.

At the same time, the small pairs of real and imaginary eigenvalues arising at a particular order in ϵ remain at the real and imaginary axes in higher orders of the perturbation theory because their geometric and algebraic multiplicity coincides, thanks

	M odd	M even
Hard potential $V'(u) = u + u^3$	In-phase	Anti-phase
Soft potential $V'(u) = u - u^3$	Anti-phase In-phase	Anti-phase

Table 5.2: Stable two-site breathers in the KG lattice with anharmonic coupling (5.33).

to the fact that these eigenvalues are related to the eigenvalues of the symmetric matrix $\mathcal{S}$ in the matrix eigenvalue problem (5.25).

Avoiding lengthy algebraic proofs, these arguments yield the assertion of the theorem. □

5.2.3 Breathers in the dKG equation with anharmonic coupling

We have considered existence and stability of multi-site breathers in the KG lattices with linear couplings between neighbouring particles. In Table 5.1 and Theorem 5.7, we have described explicitly how the stability or instability of a multi-site breather depends on the phase difference and distance between the excited oscillators.

It is instructive to compare our results to those obtained by Yoshimura [102] for the lattices with purely anharmonic coupling:

$$\ddot{u}_n + u_n \pm u_n^k = \epsilon(u_{n+1} - u_n)^k - \epsilon(u_n - u_{n-1})^k, \tag{5.33}$$

where $k \geq 3$ is an odd integer. Table 5.2 summarizes the result of [102] for stable configurations of two-site breathers, which are continued from the limiting solution

$$\mathbf{u}^{(0)}(t) = \sigma_j \varphi(t)\mathbf{e}_j + \sigma_k \varphi(t)\mathbf{e}_k,$$

where $M = |j - k| \geq 1$.

Note that the original results of [102] were obtained for finite lattices with open boundary conditions but can be extrapolated to infinite lattices, which preserve the symmetry of the multi-site breathers.

Table 5.2 is to be compared with Table 5.1. Note that Table 5.1 actually covers N-site breathers with equal distance M between the excited sites, whereas Table 5.2 only gives the results in the case $N = 2$. We have identical results for hard potentials and different results for soft potentials. First, spectral stability of a two-site breather in the anharmonic potentials is independent of its period of oscillations and is solely determined by its limiting configuration (Table 5.2). This is different from the transition from stable anti-phase to stable in-phase breathers for even M in soft potentials (Table

5.1). Second, both anti-phase and in-phase two-site breathers with odd M are stable in the anharmonic lattice. The surprising stability of in-phase breathers is explained by additional symmetries of these two-site breathers in the anharmonic potentials. The symmetries trap the unstable Floquet multipliers μ associated with in-phase breathers for odd M at the point $\mu = 1$. Once the symmetries are broken (e.g., for even M), the Floquet multipliers $\mu = 1$ split along the real axis and the in-phase two-site breather becomes unstable in soft potentials.

5.3 Numerical results

We illustrate our analytical results on existence and stability of discrete breathers near the anti-continuum limit using numerical approximations. The dKG equation (5.1) can be truncated at a finite system of differential equations by applying the Dirichlet conditions at the ends.

5.3.1 Three-site model

The simplest model which allows gaps in the limiting configuration $\mathbf{u}^{(0)}$ is the one restricted to three lattice sites, e.g. $n \in \{-1, 0, 1\}$. We choose the soft potential $V'(u) = u - u^3$ and rewrite the truncated dKG equation (5.1) as a system of three Duffing oscillators with linear coupling terms,

$$\begin{cases} \ddot{u}_0 + u_0 - u_0^3 = \epsilon(u_1 - 2u_0 + u_{-1}), \\ \ddot{u}_{\pm 1} + u_{\pm 1} - u_{\pm 1}^3 = \epsilon(u_0 - 2u_{\pm 1}). \end{cases} \tag{5.34}$$

A fast and accurate approach to construct T-periodic solutions for this system is the shooting method. The idea is to find $\mathbf{a} \in \mathbb{R}^3$ such that the solution $\mathbf{u}(t) \in C^1(\mathbb{R}_+, \mathbb{R}^3)$ with initial conditions

$$\mathbf{u}(0) = \mathbf{a}, \qquad \dot{\mathbf{u}}(0) = 0,$$

satisfy the conditions of T-periodicity, $\mathbf{u}(T) = \mathbf{a}$, $\dot{\mathbf{u}}(T) = 0$. However, these constraints would generate an over-determined system of equations on $\mathbf{a}$. To set up the square system of equations, we can use the symmetry $t \mapsto -t$ of system (5.34). If we add the constraint $\dot{\mathbf{u}}(T/2) = 0$, then even solutions of system (5.34) satisfy $\mathbf{u}(-T/2) = \mathbf{u}(T/2)$ and $\dot{\mathbf{u}}(-T/2) = -\dot{\mathbf{u}}(T/2) = 0$, that is, these solutions are T-periodic. Hence, the values of $\mathbf{a} \in \mathbb{R}^3$ become the roots of the vector $\mathbf{F}(\mathbf{a}) = \dot{\mathbf{u}}(T/2) \in \mathbb{R}^3$.

We construct a T-periodic solution $\mathbf{u}$ to system (5.34) that corresponds to the limiting configuration $\mathbf{u}^{(0)}$ as follows. Using the initial data $\mathbf{u}^{(0)}(0)$ as an initial guess for the shooting method, we first continue the initial displacement $\mathbf{u}(0)$ with respect to

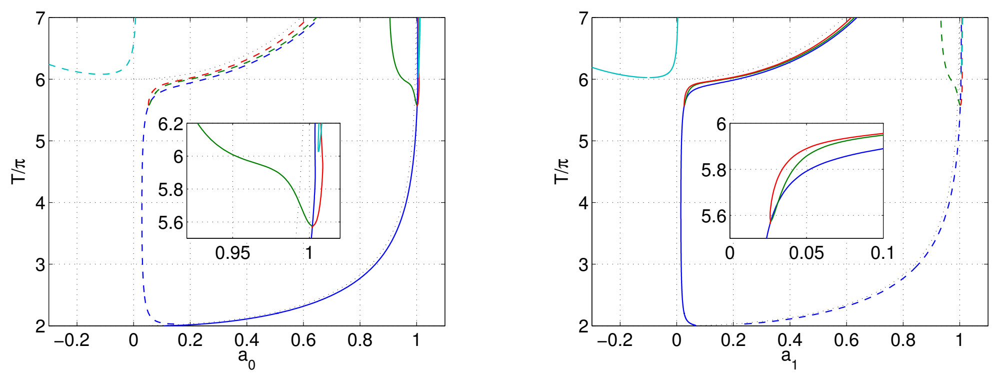

Figure 5.3: The initial displacements a_0 and a_1 for the T-periodic solutions to system (5.34) with $\epsilon = 0.01$. The solid and dashed lines correspond to the fundamental (5.35) and two-site (5.36) breathers respectively. The dotted lines correspond to the T-periodic solutions to equation (5.5). The insets show the pitchfork bifurcation of the fundamental breather.

the coupling constant $\epsilon > 0$. After that, we use the shooting method again to continue the initial displacement $\mathbf{u}(0)$ with respect to period T at the fixed value of ϵ.

Let us apply this method to determine initial conditions for the fundamental breather,

$$u_0^{(0)} = \varphi, \quad u_{\pm 1}^{(0)} = 0, \tag{5.35}$$

and for a two-site breather with a "hole",

$$u_0^{(0)} = 0, \quad u_{\pm 1}^{(0)} = \varphi. \tag{5.36}$$

In both cases, we can use the symmetry $u_{-1}(t) = u_1(t)$ to reduce the dimension of the shooting method to two unknowns a_0 and a_1.

Figure 5.3 shows solution branches for these two breathers on the period–amplitude plane by plotting T versus a_0 and a_1 for $\epsilon = 0.01$. For $2\pi < T < 6\pi$, solution branches are close to the limiting solutions (dotted line), in agreement with Theorem 3.9. However, a new phenomenon emerges near $T = 6\pi$: both breather solutions experience a pitchfork bifurcation and two more solution branches split off the main solution branch. The details of the pitchfork bifurcation for the fundamental solution branch are shown on the insets of Figure 5.3.

Let T_S be the period at the point of the pitchfork bifurcation. We may think intuitively that T_S should approach to the point of $1:3$ resonance for small ϵ, that is, $T_S \to 6\pi$ as $\epsilon \to 0$. We have checked numerically that this conjecture is in fact false and the value of T_S gets larger as ϵ gets smaller. This property of the pitchfork bifurcation is analyzed in Section 5.4 below (see Remark 5.12 and Figure 5.13).

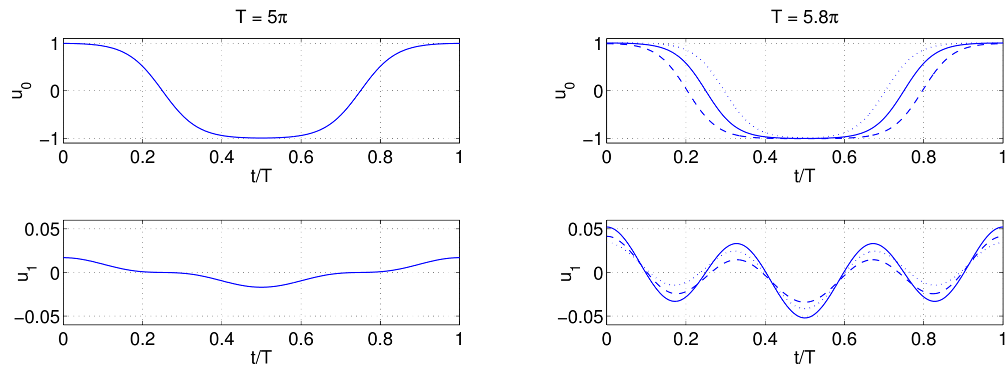

Figure 5.4: Fundamental breathers for system (5.34) before (left) and after (right) the symmetry-breaking bifurcation at $\epsilon = 0.01$.

Figure 5.3 also shows two branches of solutions for $T > 6\pi$ with negative values of a_1 for positive values of a_0 and vice versa. One of the two branches is close to the breathers at the anti-continuum limit, as prescribed by Theorem 3.9. We note that the breather solutions prescribed by Theorem 3.9 for $T < 6\pi$ and $T > 6\pi$ belong to different solution branches. This property is also analyzed in Section 5.4 below (see Remark 5.10 and Figure 5.10).

Figure 5.4 shows the fundamental breather before ($T = 5\pi$) and after ($T = 5.8\pi$) pitchfork bifurcation. The symmetry condition $\mathbf{u}(T/4) = 0$ for the solution at the main branch is violated for two new solutions that bifurcate from the main branch. Note that the two new solutions bifurcating for $T > T_S$ look different on the graphs of a_0 and a_1 versus T. Nevertheless, these two solutions are related to each other by the symmetry of the system (5.34). If $\mathbf{u}(t)$ is one solution of the system (5.34), then $-\mathbf{u}(t + T/2)$ is another solution of the same system. If $\mathbf{u}(T/4) \neq 0$, then these two solutions are different from each other.

Let us now illustrate the stability result of Theorem 5.7 using the fundamental breather (5.35) and the breather with a hole (5.36). We draw a conclusion on spectral stability of these breathers by testing whether their Floquet multipliers, found from the monodromy matrix associated with the linearization of system (5.34), stay on the unit circle.

Figure 5.5 shows the real part of Floquet multipliers versus the breather's period for the fundamental breather (left) and the new solution branches (right) bifurcating from the fundamental breather due to the pitchfork bifurcation. Because Floquet multipliers are on the unit circle for all periods below the bifurcation value T_S, the fundamental breather remains stable for these periods, in agreement with Theorem 5.7. Once the bifurcation occurs, one of the Floquet multiplier becomes real and unstable (outside

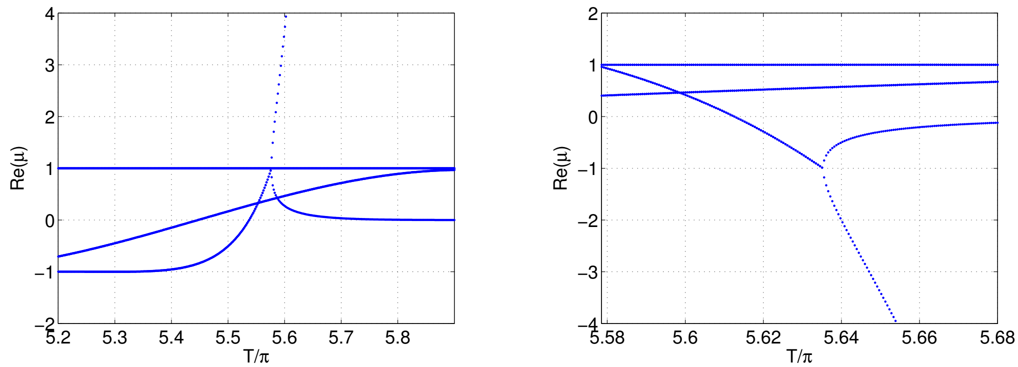

Figure 5.5: Real parts of Floquet multipliers μ for the fundamental breather at $\epsilon = 0.01$ near the bifurcation for the main branch (left) and side branches (right).

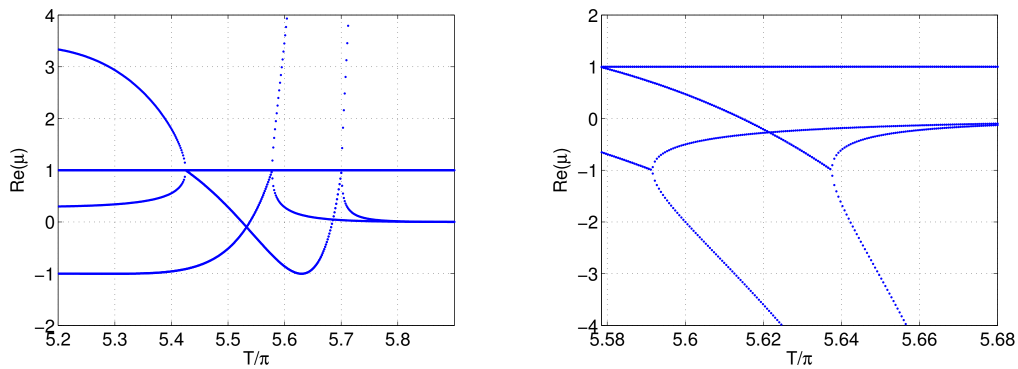

Figure 5.6: Real parts of Floquet multipliers μ for the two-site breather with a hole at $\epsilon = 0.01$ near the bifurcation for the main branch (left) and side branches (right).

the unit circle). Two new stable solutions appear during the bifurcation and have the identical Floquet multipliers because of the aforementioned symmetry between the new solutions. These solutions become unstable for periods slightly larger than the bifurcation value T_S, because of the period-doubling bifurcation associated with Floquet multipliers at -1.

We perform similar computations for the two-site breather with the central hole (5.36). Figure 5.6 (left) shows that at the coupling $\epsilon = 0.01$ the breather is unstable for periods $2\pi < T < T_*^{(\epsilon)}$ and stable for periods $T \gtrapprox T_*^{(\epsilon)}$ with $T_*^{(\epsilon)} \approx 5.425\pi$ for $\epsilon = 0.01$. This can be compared using the change of stability predicted by Theorem 5.7. According to equation (5.31), K_2 changes sign from positive to negative at $T_{M=2} \approx 5.476\pi$. Since $T'(E)$ is positive for the soft potential, Theorem 5.7 predicts that in the anti-continuum limit the two-site breather is unstable for $2\pi < T < T_{M=2}$ and stable for $T_{M=2} < T < 6\pi$. This change of stability agrees with Figure 5.6 where we note

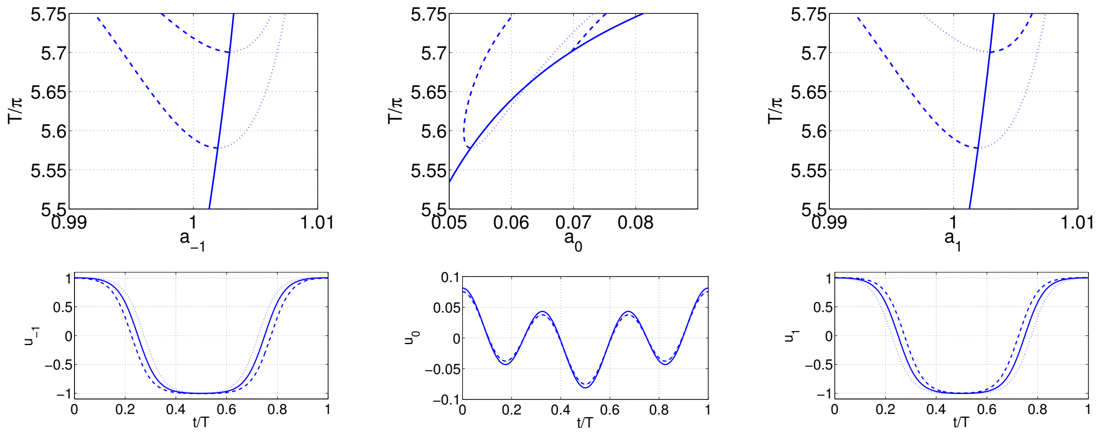

Figure 5.7: Top: The initial displacements a_{-1}, a_0, and a_1 for the T-periodic breather with a hole on the three-site lattice with $\epsilon = 0.01$. Bottom: Asymmetric breathers with period $T = 5.75\pi$ on the three-site lattice with $\epsilon = 0.01$.

that $|T_*^{(\epsilon)} - T_{M=2}| \approx 0.05\pi$ at $\epsilon = 0.01$.

At $T \approx 5.6\pi$ and $T \approx 5.7\pi$, two bifurcations occur for the two-site breather with the central hole and unstable multipliers bifurcate from the unit multiplier for larger values of T. The behaviour of Floquet multipliers is similar to the one on Figure 5.5 (left) and it marks two consequent pitchfork bifurcations for the two-site breather with the hole. The first bifurcation is visible on Figure 5.3 in the space of symmetric two-site breathers with $u_{-1}(t) = u_1(t)$. The Floquet multipliers for the side branches of these symmetric two-site breathers is shown on Figure 5.6 (right), where we can see two consequent period-doubling bifurcations in comparison with one such bifurcation on Figure 5.5 (right). The second bifurcation is observed in the space of asymmetric two-site breathers with $u_{-1}(t) \neq u_1(t)$.

We display the two pitchfork bifurcations on the top panel of Figure 5.7. One can see for the second bifurcation that the value of a_0 is the same for both breathers splitting of the main solution branch. Although the values of a_{-1} and a_1 look the same for the second bifurcation, dashed and dotted lines indicate that a_1 is greater than a_{-1} at one asymmetric branch and vice versa for the other one. The bottom panels of Figure 5.7 show the asymmetric breathers with period $T = 5.75\pi$ that appear as a result of the second pitchfork bifurcation.

It is important to note that a similar behaviour is observed near points of $1 : k$ resonance, with k being an odd natural number. For the non-resonant periods, a breather has large amplitudes on excited sites and small amplitudes on the other sites. As we increase the breather's period approaching a resonant point $T = 2\pi k$ for odd k, the amplitudes at all sites become large, a cascade of pitchfork bifurcations occurs for

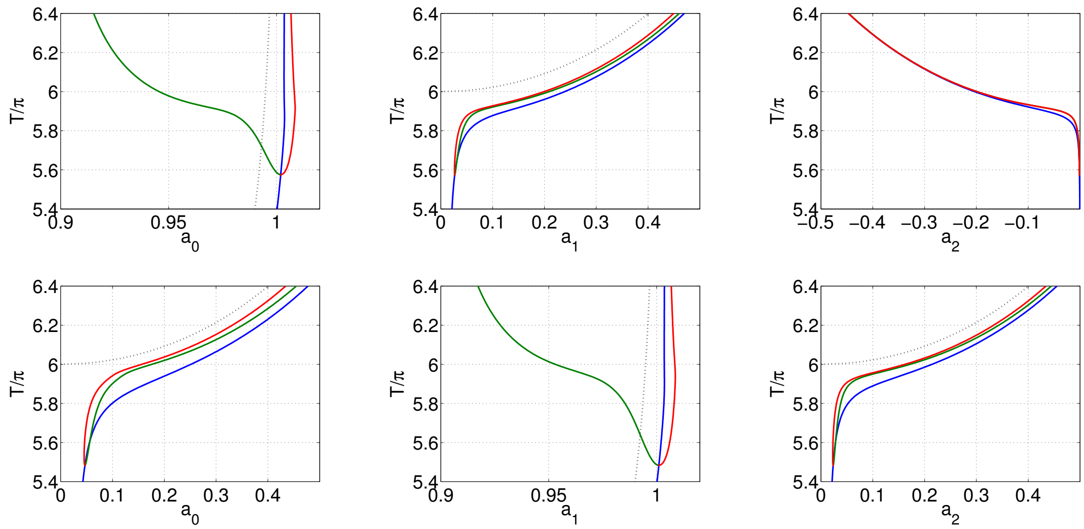

Figure 5.8: Top: The initial displacements a_0, a_1, and a_2 for the T-periodic fundamental breather of the five-site lattice with $\epsilon = 0.01$. Bottom: The same for the two-site breather with a hole. The dotted lines correspond to the T-periodic solutions to equation (5.5).

these breathers, and families of these breathers deviate from the one prescribed by the anti-continuum limit. However, due to the saddle-node bifurcation, another family of breathers satisfying Theorem 3.9 emerges for periods just above the resonance value. The period–amplitude curves, similar to those on Figure 5.3, start to look like trees with branches at all resonant points $T = 2\pi k$ for odd k. In the anti-continuum limit, the gaps at the period–amplitude curves vanish while the points of the pitchfork bifurcations go to infinity. The period–amplitude curves turn into those for the set of uncoupled anharmonic oscillators.

5.3.2 Five-site model

We can now truncate the dKG equation (5.1) at five lattice sites, e.g. at $n \in \{-2, -1, 0, 1, 2\}$. The fundamental breather (5.35) and the breather with a central hole (5.36) are continued in the five-site lattice subject to the symmetry conditions $u_n(t) = u_{-n}(t)$ for $n = 1, 2$. We would like to illustrate that increasing the size of the lattice does not qualitatively change the previous existence and stability results, in particular, the properties of the pitchfork bifurcations.

Figure 5.8 gives analogues of Figure 5.3 for the fundamental breather and the breather with a hole. The associated Floquet multipliers are shown on Figure 5.9, in full analogy with Figures 5.5 and 5.6. We can see that both existence and stability results are analogous between the three-site and five-site lattices.

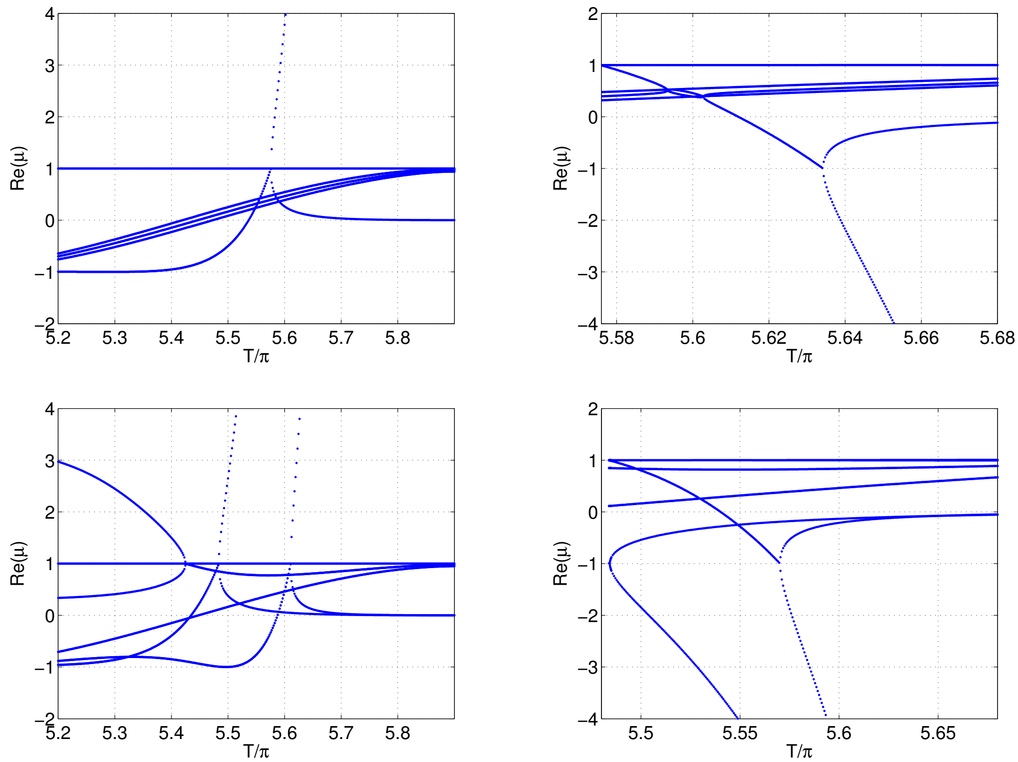

Figure 5.9: Top: Real parts of Floquet multipliers μ for the fundamental breather near the bifurcation for the main branch (left) and side branches (right). Bottom: The same for the two-site breather with a hole.

5.4 Pitchfork bifurcation near 1:3 resonance

We study here the symmetry-breaking (pitchfork) bifurcation of the fundamental breather. This bifurcation, illustrated on Figure 5.4, occurs for soft potentials near the point of 1:3 resonance, which corresponds to $T = 6\pi$. We point out that the period T_S of the pitchfork bifurcation is close to 6π for small but finite values of ϵ. As we have discovered numerically, T_S gets larger as ϵ gets smaller. This property indicates that the asymptotic analysis of this bifurcation is not uniform with respect to two small parameters ϵ and $T - 6\pi$, which we explain below in more details.

When $\mathbf{u} = \boldsymbol{\phi}^{(\epsilon)}$ is the fundamental breather and $T \neq 2\pi n$ is fixed, Theorem 3.9 and Lemma 5.3 imply that

$$\left\{ \begin{array}{rcrcll} u_0(t) & = & \varphi(t) - 2\epsilon\psi_1(t) & + & \mathcal{O}_{H^2_{\rm per}(0,T)}(\epsilon^2), & \\ u_{\pm 1}(t) & = & \epsilon\varphi_1(t) & + & \mathcal{O}_{H^2_{\rm per}(0,T)}(\epsilon^2), & \\ u_{\pm n}(t) & = & & + & \mathcal{O}_{H^2_{\rm per}(0,T)}(\epsilon^2), & n \geq 2, \end{array} \right. \tag{5.37}$$

where φ can be expanded in the Fourier series,

$$\varphi(t) = \sum_{n \in \mathbb{N}_{\rm odd}} c_n(T) \cos\left(\frac{2\pi n t}{T}\right), \tag{5.38}$$

and the Fourier coefficients $\{c_n(T)\}_{n \in \mathbb{N}_{\rm odd}}$ are uniquely determined by the period T. The correction terms φ_1 and ψ_1 are determined by the solution of the linear inhomogeneous equations (5.12) and (5.13), in particular, we have

$$\varphi_1(t) = \sum_{n \in \mathbb{N}_{\rm odd}} \frac{T^2 c_n(T)}{T^2 - 4\pi^2 n^2} \cos\left(\frac{2\pi n t}{T}\right). \tag{5.39}$$

In what follows, we restrict our consideration of soft potentials to the case of the quartic potential $V'(u) = u - u^3$. In agreement with a numerical approximation for the quartic potential, we shall assume that $c_3(6\pi) < 0$.

Expansion (5.37) and solution (5.39) imply that if T is fixed in $(2\pi, 6\pi)$, then $\|u_{\pm 1}\|_{H^2_{\rm per}(0,T)} = \mathcal{O}(\epsilon)$ and the cubic term $u_{\pm 1}^3$ is neglected at the order $\mathcal{O}(\epsilon)$, where the linear inhomogeneous equation (5.12) is valid. Near the resonant period $T = 6\pi$, the norm $\|u_{\pm 1}\|_{H^2_{\rm per}(0,T)}$ is much larger than $\mathcal{O}(\epsilon)$ if $c_3(6\pi) \neq 0$. As a result, the cubic term $u_{\pm 1}^3$ must be incorporated at the leading order of the asymptotic approximation.

We shall reduce the dKG equation (5.1) for the fundamental breather near 1 : 3 resonance to a normal form equation, which coincides with the nonlinear Duffing oscillator perturbed by a small harmonic forcing (equation (5.56) below). The normal form equation features the same properties of the pitchfork bifurcation of T-periodic solutions as the dKG equation (5.1). To prepare for the reduction to the normal form equation, we introduce the scaling transformation,

$$T = \frac{6\pi}{1+\delta\epsilon^{2/3}}, \quad \tau = (1+\delta\epsilon^{2/3})t, \quad u_n(t) = (1+\delta\epsilon^{2/3})U_n(\tau), \tag{5.40}$$

where δ is a new parameter, which is assumed to be ϵ-independent. The dKG equation (5.1) with $V'(u) = u - u^3$ can be rewritten in new variables (5.40) as follows,

$$\ddot{U}_n + U_n - U_n^3 = \beta U_n + \gamma(U_{n+1} + U_{n-1}), \quad n \in \mathbb{Z}, \tag{5.41}$$

where

$$\beta = 1 - \frac{1+2\epsilon}{(1+\delta\epsilon^{2/3})^2}, \quad \gamma = \frac{\epsilon}{(1+\delta\epsilon^{2/3})^2}. \tag{5.42}$$

T-periodic solutions of the dKG equation (5.1) in variables $\{u_n(t)\}_{n\in\mathbb{Z}}$ become now 6π-periodic solutions of the rescaled dKG equation (5.41) in variables $\{U_n(\tau)\}_{n\in\mathbb{Z}}$.

5.4.1 Deriving the normal form

To reduce the system (5.41) to the Duffing oscillator perturbed by a small harmonic forcing near 1:3 resonance, we consider the fundamental breather, for which $U_n = U_{-n}$ for all $n \in \mathbb{N}$. Using this reduction, we write equations (5.41) separately at $n = 0$, $n = 1$, and $n \geq 2$:

$$\ddot{U}_0 + U_0 - U_0^3 = \beta U_0 + 2\gamma U_1, \tag{5.43}$$

$$\ddot{U}_1 + U_1 - U_1^3 = \beta U_1 + \gamma U_2 + \gamma U_0, \tag{5.44}$$

$$\ddot{U}_n + U_n - U_n^3 = \beta U_n + \gamma(U_{n+1} + U_{n-1}), \quad n \geq 2. \tag{5.45}$$

Let us represent a 6π-periodic function U_0 with the symmetries

$$U_0(-\tau) = U_0(\tau) = -U_0(3\pi - \tau), \quad \tau \in \mathbb{R}, \tag{5.46}$$

by the Fourier series,

$$U_0(\tau) = \sum_{n\in\mathbb{N}_{\text{odd}}} b_n \cos\left(\frac{n\tau}{3}\right), \tag{5.47}$$

where $\{b_n\}_{n\in\mathbb{N}_{\text{odd}}}$ are some Fourier coefficients. If U_0 converges to φ in H^2 norm as $\epsilon \to 0$ (when $\beta, \gamma \to 0$), then $b_n \to c_n(6\pi)$ as $\epsilon \to 0$ for all $n \in \mathbb{N}_{\text{odd}}$, where the Fourier coefficients $\{c_n(6\pi)\}_{n\in\mathbb{N}_{\text{odd}}}$ are uniquely defined by the Fourier series (5.38) for $T = 6\pi$. We assume again that $c_3(6\pi) \neq 0$ and δ is fixed independently of small $\epsilon > 0$.

We shall now use a Lyapunov–Schmidt reduction method to show that the components $\{U_n\}_{n\in\mathbb{N}}$ are uniquely determined from the system (5.44)–(5.45) for small $\epsilon > 0$

if U_0 is represented by the Fourier series (5.47). To do so, we decompose the solution into two parts:

$$U_n(\tau) = A_n \cos(\tau) + V_n(\tau), \quad n \in \mathbb{N},$$

where $V_n(\tau)$ is orthogonal to $\cos(\tau)$ in the sense

$$\langle V_n, \cos(\cdot)\rangle_{L^2_{\rm per}(0,6\pi)} = 0, \quad n \in \mathbb{N}.$$

Projecting the system (5.44)–(5.45) to $\cos(\tau)$, we obtain a difference equation for $\{A_n\}_{n\in\mathbb{N}}$:

$$\beta A_1 + \gamma A_2 + \gamma b_3 = -\frac{1}{3\pi}\int_0^{6\pi} \cos(\tau)(A_1\cos(\tau) + V_1(\tau))^3 d\tau, \quad (5.48)$$

$$\beta A_n + \gamma(A_{n+1} + A_{n-1}) = -\frac{1}{3\pi}\int_0^{6\pi} \cos(\tau)(A_n\cos(\tau) + V_n(\tau))^3 d\tau, \quad (5.49)$$

where $n \geq 2$ in the second equation. Projecting the system (5.44)–(5.45) to the orthogonal complement of $\cos(\tau)$, we obtain a lattice differential equation for $\{V_n(\tau)\}_{n\in\mathbb{N}}$:

$$\begin{aligned}\ddot{V}_1 + V_1 = \beta V_1 + \gamma V_2 + \gamma \sum_{k\in\mathbb{N}_{\rm odd}\backslash\{3\}} b_k \cos\left(\frac{k\tau}{3}\right)\\ + (A_1\cos(\tau) + V_1)^3 - \cos(\tau)\frac{\langle\cos(\cdot), (A_1\cos(\cdot) + V_1)^3\rangle_{L^2_{\rm per}(0,6\pi)}}{\langle\cos(\cdot), \cos(\cdot)\rangle_{L^2_{\rm per}(0,6\pi)}},\end{aligned} \quad (5.50)$$

$$\begin{aligned}\ddot{V}_n + V_n = \beta V_n + \gamma(V_{n+1} + V_{n-1})\\ + (A_n\cos(\tau) + V_n)^3 - \cos(\tau)\frac{\langle\cos(\cdot), (A_n\cos(\cdot) + V_n)^3\rangle_{L^2_{\rm per}(0,6\pi)}}{\langle\cos(\cdot), \cos(\cdot)\rangle_{L^2_{\rm per}(0,6\pi)}},\end{aligned} \quad (5.51)$$

where $n \geq 2$ in the second equation. Recall that $\beta = \mathcal{O}(\epsilon^{2/3})$ and $\gamma = \mathcal{O}(\epsilon)$ as $\epsilon \to 0$ if δ is fixed independently of small $\epsilon > 0$. Provided that the sequence $\{A_n\}_{n\in\mathbb{N}}$ is bounded and $\|\mathbf{A}\|_{l^\infty(\mathbb{N})}$ is small as $\epsilon \to 0$, the Implicit Function Theorem applied to the system (5.50)–(5.51) yields a unique even solution for $\mathbf{V} \in l^2(\mathbb{N}, H^2_e(0,6\pi))$ such that $\langle\mathbf{V}, \cos(\cdot)\rangle_{L^2_{\rm per}(0,6\pi)} = \mathbf{0}$ in the neighbourhood of zero solution for small $\epsilon > 0$ and $\mathbf{A} \in l^\infty(\mathbb{N})$. Moreover, for all small $\epsilon > 0$ and $\mathbf{A} \in l^\infty(\mathbb{N})$, there is a positive constant $C > 0$ such that

$$\|\mathbf{V}\|_{l^2(\mathbb{N},H^2_{\rm per}(0,6\pi))} \leq C(\epsilon + \|\mathbf{A}\|^3_{l^\infty(\mathbb{N})}). \quad (5.52)$$

The balance occurs if $\|\mathbf{A}\|_{l^\infty(\mathbb{N})} = \mathcal{O}(\epsilon^{1/3})$ as $\epsilon \to 0$.

Recall now that $\beta = 2\delta\epsilon^{2/3} - 2\epsilon + \mathcal{O}(\epsilon^{4/3})$ and $\gamma = \epsilon + \mathcal{O}(\epsilon^{5/3})$ as $\epsilon \to 0$. Substituting the solution of the system (5.50)–(5.51) satisfying (5.52) to the system (5.48)–(5.49) and using the scaling transformation $A_n = \epsilon^{1/3} a_n$, $n \in \mathbb{N}$, we obtain the perturbed difference equation for $\{a_n\}_{n\in\mathbb{N}}$:

$$2\delta a_1 + \frac{3}{4}a_1^3 + b_3 = \epsilon^{1/3}(2a_1 - a_2) + \mathcal{O}(\epsilon^{2/3}), \quad (5.53)$$

$$2\delta a_n + \frac{3}{4}a_n^3 = \epsilon^{1/3}(2a_n - a_{n+1} - a_{n-1}) + \mathcal{O}(\epsilon^{2/3}), \quad n \geq 2. \quad (5.54)$$

At $\epsilon = 0$, the system (5.53) and (5.54) is decoupled. Let $a(\delta)$ be a root of the cubic equation:

$$2\delta a(\delta) + \frac{3}{4}a^3(\delta) + c_3(6\pi) = 0, \quad (5.55)$$

where $c_3(6\pi) \neq 0$ is given. Roots of the cubic equation (5.55) are shown on Figure 5.10 for $c_3(6\pi) < 0$. A positive root continues across $\delta = 0$ and the two negative roots bifurcate for $\delta < 0$ by means of a saddle-node bifurcation.

Let $a(\delta)$ denote any root of cubic equation (5.55) such that $8\delta + 9a^2(\delta) \neq 0$. Assuming that $b_3 = c_3(6\pi) + \mathcal{O}(\epsilon^{2/3})$ as $\epsilon \to 0$ (this assumption is proved later in Lemma 5.11), the Implicit Function Theorem yields a unique continuation of this root in the system (5.53)–(5.54) for small $\epsilon > 0$ and any fixed $\delta \neq 0$:

$$\begin{cases} a_1 = a(\delta) + \epsilon^{1/3}\frac{8a(\delta)}{8\delta+9a^2(\delta)} + \mathcal{O}(\epsilon^{2/3}), \\ a_2 = -\epsilon^{1/3}\frac{a(\delta)}{2\delta} + \mathcal{O}(\epsilon^{2/3}), \\ a_n = + \mathcal{O}(\epsilon^{2/3}), \quad n \geq 3. \end{cases}$$

Again, these expansions are valid for any fixed $\delta \neq 0$ such that $8\delta + 9a^2(\delta) \neq 0$.

Remark 5.8. The condition $8\delta + 9a^2(\delta) = 0$ implies bifurcations among the roots of the cubic equation (5.55), e.g., the fold bifurcation, when two roots coalesce and disappear after δ crosses a bifurcation value. The condition $\delta = 0$ does not lead to new bifurcations but implies that the values of a_n for $n \geq 2$ are no longer as small as $\mathcal{O}(\epsilon^{1/3})$. Refined scaling shows that if $\delta = 0$, then $a_1 = a(0) + \mathcal{O}(\epsilon^{1/3})$, $a_2 = \mathcal{O}(\epsilon^{1/9})$, and $a_n = \mathcal{O}(\epsilon^{4/27})$, $n \geq 3$, where $a(0)$ is a unique real root of the cubic equation (5.55) for $\delta = 0$.

We can now focus on the last remaining equation (5.43) of the rescaled dKG equation (5.41). Substituting $U_1 = \epsilon^{1/3}a(\delta)\cos(\tau) + \mathcal{O}_{H^2_{\text{per}}(0,6\pi)}(\epsilon^{2/3})$ into equation (5.43), we obtain the perturbed normal form for 1:3 resonance,

$$\ddot{U}_0 + U_0 - U_0^3 = \beta U_0 + \nu\cos(\tau) + \mathcal{O}_{H^2_{\text{per}}(0,6\pi)}(\epsilon^{5/3}), \quad (5.56)$$

where $\nu = 2\gamma\epsilon^{1/3}a(\delta) = \mathcal{O}(\epsilon^{4/3})$ as $\epsilon \to 0$. Because $a(\delta) \neq 0$, we know that $\nu \neq 0$ if $\epsilon \neq 0$. The perturbed normal form (5.56) coincides with the nonlinear Duffing oscillator perturbed by a small harmonic forcing. The following lemma summarizes the reduction of the dKG equation to the perturbed Duffing equation, which was proved above with

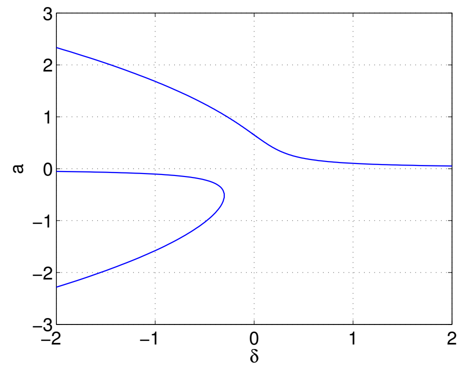

Figure 5.10: Roots of the cubic equation (5.55).

the Lyapunov–Schmidt reduction arguments.

Lemma 5.9. *Let $\delta \neq 0$ be fixed independently of small $\epsilon > 0$. Let $a(\delta)$ be a root of the cubic equation (5.55) such that $8\delta + 9a^2(\delta) \neq 0$. Assume that $c_3(6\pi) \neq 0$ among the Fourier coefficients (5.38). For any 6π-periodic solution U_0 of the perturbed Duffing equation (5.56) satisfying symmetries (5.46) such that*

$$U_0(\tau) = \varphi(\tau) + \mathcal{O}_{H^2_{\rm per}(0,6\pi)}(\epsilon^{2/3}) \quad \text{as} \quad \epsilon \to 0, \tag{5.57}$$

there exists a solution of the dKG equation (5.41) such that

$$\begin{cases} U_{\pm 1}(\tau) = \epsilon^{1/3} a(\delta)\cos(\tau) + \epsilon^{2/3} \dfrac{8a(\delta)}{8\delta + 9a^2(\delta)} \cos(\tau) + \mathcal{O}_{H^2_{\rm per}(0,6\pi)}(\epsilon), \\ U_{\pm 2}(\tau) = \qquad - \epsilon^{2/3} \dfrac{a(\delta)}{2\delta} \cos(\tau) \qquad + \mathcal{O}_{H^2_{\rm per}(0,6\pi)}(\epsilon), \\ U_{\pm n}(\tau) = \qquad + \mathcal{O}_{H^2_{\rm per}(0,6\pi)}(\epsilon), \quad n \geq 3 \end{cases}$$

Remark 5.10. Figure 5.10 shows that two negative roots of the cubic equation (5.55) bifurcate at $\delta_* < 0$ via the saddle-node bifurcation and exist for $\delta < \delta_*$. Negative values of δ correspond to $T > 6\pi$. As ϵ is small, this saddle-node bifurcation gives a birth of two periodic solutions with

$$u_1(0) = \epsilon^{1/3} a(\delta) + \mathcal{O}(\epsilon^{2/3}) < 0.$$

This bifurcation is observed on Figure 5.3 (right), one of the two new solutions still satisfies the asymptotic representation (5.37) as $\epsilon \to 0$ for fixed $T > 6\pi$.

5.4.2 Analysis of the normal form

In what follows, we shall consider the positive root of the cubic equation (5.55) that continues across $\delta = 0$. We are interested in 6π-periodic solutions of the perturbed normal form (5.56) in the limit of small $\epsilon > 0$ (when $\beta = \mathcal{O}(\epsilon^{2/3})$ and $\nu = \mathcal{O}(\epsilon^{4/3})$ are small). Since the remainder term is small as $\epsilon \to 0$ and the persistence analysis is rather straightforward, we obtain main results by studying the truncated Duffing equation with a small harmonic forcing:

$$\ddot{U} + U - U^3 = \beta U + \nu \cos(\tau). \tag{5.58}$$

The following lemma guarantees the persistence of 6π-periodic solutions with even symmetry in the Duffing equation (5.58) for all small values of β and ν. Note that this persistence is assumed in equation (5.57) of the statement of Lemma 5.9.

Lemma 5.11. *There are positive constants β_0, ν_0, and C such that for all $\beta \in (-\beta_0, \beta_0)$ and $\nu \in (-\nu_0, \nu_0)$, the normal form equation (5.58) admits a unique 6π-periodic solution $U_{\beta,\nu} \in H^2_e(0, 6\pi)$ satisfying symmetries*

$$U_{\beta,\nu}(-\tau) = U_{\beta,\nu}(\tau) = -U_{\beta,\nu}(3\pi - \tau), \quad \tau \in \mathbb{R}, \tag{5.59}$$

and bound

$$\|U_{\beta,\nu} - \varphi\|_{H^2_{\text{per}}} \leq C(|\beta| + |\nu|). \tag{5.60}$$

Moreover, the map $\mathbb{R} \times \mathbb{R} \ni (\beta, \nu) \mapsto U_{\beta,\nu} \in H^2_e(0, 6\pi)$ is C^∞ for all $\beta \in (-\beta_0, \beta_0)$ and $\nu \in (-\nu_0, \nu_0)$.

Proof. The proof follows by the Lyapunov–Schmidt reduction arguments. For $\nu = 0$ and small $\beta \in (-\beta_0, \beta_0)$, there exists a unique 6π-periodic solution $U_{\beta,0}$ satisfying the symmetry (5.59), which is $\mathcal{O}(\beta)$-close to φ in the $H^2_{\text{per}}(0, 6\pi)$ norm. Because the Duffing oscillator is non-degenerate, the Jacobian operator $L_{\beta,0}$ has a one-dimensional kernel spanned by the odd function $\dot{U}_{\beta,0}$, where

$$L_{\beta,\nu} = \partial_t^2 + 1 - \beta - 3U^2_{\beta,\nu}(t). \tag{5.61}$$

Therefore, $\langle \dot{U}_{\beta,0}, \cos(\cdot)\rangle_{L^2_{\text{per}}(0,6\pi)} = 0$, and the unique even solution persists for small $\nu \in (-\nu_0, \nu_0)$. The symmetry (5.59) persists for all $\nu \in (-\nu_0, \nu_0)$ because both the Duffing oscillator and the forcing term $\cos(\tau)$ satisfy this symmetry. □

Remark 5.12. Lemma 5.11 excludes the pitchfork bifurcation in the limit $\epsilon \to 0$ for fixed $\delta \neq 0$. This result implies that the period of the pitchfork bifurcation T_S does

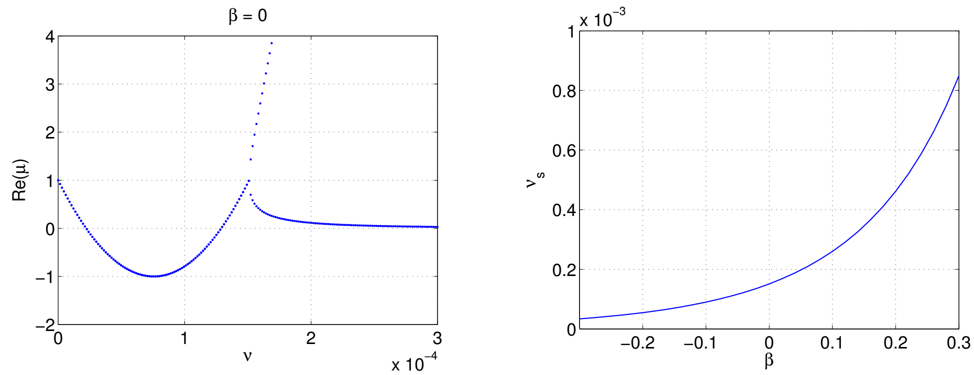

Figure 5.11: Left: Floquet multipliers μ of equation $L_{\beta,\nu}W = 0$. Right: Parameter ν versus β at the symmetry-breaking bifurcation.

not converge to 6π as $\epsilon \to 0$. Indeed, we mentioned in the context of Figure 5.3 that T_S gets larger as ϵ gets smaller.

5.4.3 Numerical results on the normal form

By the perturbation theory arguments, the kernel of the Jacobian operator $L_{\beta,\nu}$ is empty for small β and ν provided that $\nu \neq 0$. Indeed, expanding the solution of Lemma 5.11 in power series in β and ν, we obtain

$$U_{\beta,\nu} = \varphi + \beta\mathcal{L}_e^{-1}\varphi + \nu\mathcal{L}_e^{-1}\cos(\cdot) + \mathcal{O}_{H^2_{\text{per}}(0,6\pi)}(\beta^2,\nu^2), \tag{5.62}$$

where $\mathcal{L}_e$ is the operator in (5.13). Although $\mathcal{L}_e$ has a one-dimensional kernel spanned by $\dot{\varphi}$, this eigenfunction is odd in τ, whereas φ and $\cos(\cdot)$ are defined in the space of even functions. Expanding potentials of the operator $L_{\beta,\nu}$, we obtain

$$L_{\beta,\nu}\dot{U}_{\beta,\nu} = -\nu\sin(\cdot) + \mathcal{O}_{H^2_{\text{per}}(0,6\pi)}(\beta^2,\nu^2). \tag{5.63}$$

We note that

$$\langle\dot{\varphi}, \sin(\cdot)\rangle_{L^2_{\text{per}}(0,6\pi)} = \langle\varphi, \cos(\cdot)\rangle_{L^2_{\text{per}}(0,6\pi)} \neq 0$$

if $c_3(6\pi) \neq 0$, where $c_3(T)$ is defined by the Fourier series (5.38). By the perturbation theory, the kernel of $L_{\beta,\nu}$ is empty for small $\nu \in (-\nu_0, \nu_0)\backslash\{0\}$.

If the linearization operator $L_{\beta,\nu}$ becomes non-invertible along the curve $\nu = \nu_S(\beta)$ of the codimension one bifurcation, the symmetry-breaking (pitchfork) bifurcation occurs at $\nu = \nu_S(\beta)$. This property gives us a criterion to find the pitchfork bifurcation numerically, in the context of the Duffing equation (5.58). Figure 5.11 (left) shows the behaviour of Floquet multipliers of equation $L_{\beta,\nu}W = 0$ with respect to parameter ν at $\beta = 0$. We can see from this picture that the pitchfork bifurcation occurs at

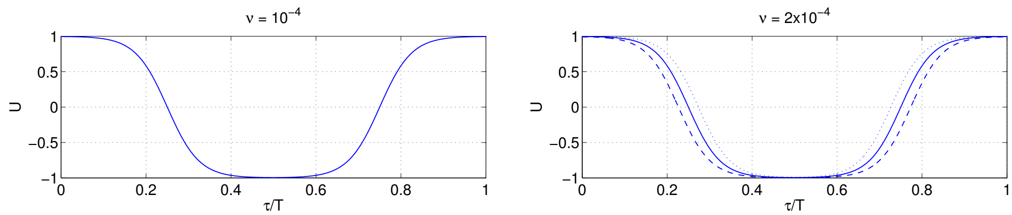

Figure 5.12: Solutions with period $T = 6\pi$ to equation (5.58) at $\beta = 0$ before (left) and after (right) the symmetry-breaking bifurcation.

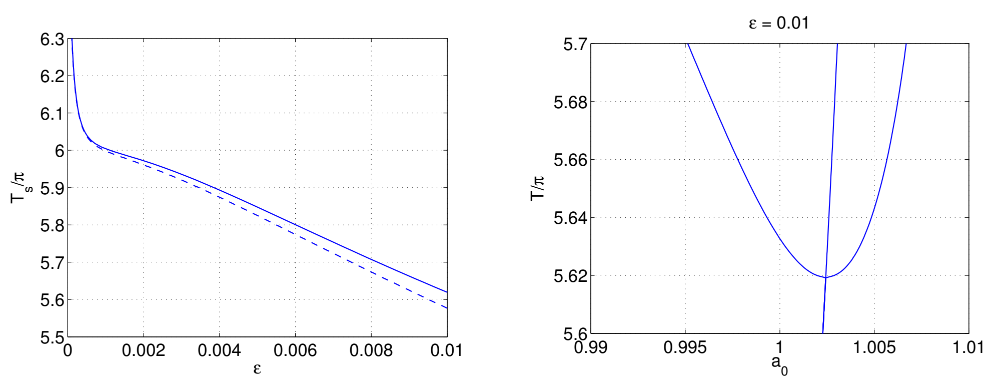

Figure 5.13: Left: Period T_S versus ϵ at the symmetry–breaking bifurcation of the fundamental breather modelled by equation (5.58) (solid line) and equation (5.34) (dashed line). Right: Bifurcation diagram for the initial displacement $u_0(0) = a_0$ and period T in variables (5.40) computed from the 6π-periodic solution to equation (5.58).

$\nu \approx 0.00015$.

The right panel of Figure 5.11 gives the dependence of the bifurcation value ν_S on β, for which the operator $L_{\beta,\nu_S(\beta)}$ in not invertible on $L^2_e(0, 6\pi)$. Using the formula for β in (5.42), we obtain

$$T = 6\pi \frac{\sqrt{1-\beta}}{\sqrt{1+2\epsilon}}.$$

As the coupling constant ϵ goes to zero, so does parameter ν. As shown on Figure 5.11 (right), parameter β at the bifurcation curve goes to negative infinity as $\nu \to 0$. This means that the closer we get to the anti-continuum limit, the further away from 6π moves the pitchfork bifurcation period T_S. This confirms the early observation that T_S gets larger as ϵ gets smaller (see Remark 5.12).

Figure 5.12 shows one solution of Lemma 5.11 for $0 \leq \nu \leq \nu_S(\beta)$ and three solutions for $\nu > \nu_S(\beta)$, where $\beta = 0$. The new solution branches are still given by even functions but the symmetry $U(\tau) = -U(3\pi - \tau)$ is now broken. This behaviour resembles the pitchfork bifurcation shown on Figure 5.4.

Figure 5.13 transfers the behaviour of Figures 5.11 and 5.12 to parameters T, ϵ, and $a_0 = u_0(0)$. The dashed line on the left panel shows the dependence of period T_S at the pitchfork bifurcation versus ϵ for the full system (5.34). The right panel of Figure 5.13 can be compared with the inset on the left panel of Figure 5.3.

Remark 5.13. Numerical results on Figures 5.12 and 5.13 indicate that the Duffing equation with a small harmonic forcing (5.58) allows us to capture the main features of the symmetry-breaking bifurcations in the dKG equation (5.34). Nevertheless, we point out that the rigorous results of Lemmas 5.9 and 5.11 are obtained far from the pitchfork bifurcation, because parameter δ is assumed to be fixed independently of ϵ in these lemmas. To observe the pitchfork bifurcation on Figures 5.12 and 5.13, parameter δ must be sent to $-\infty$ as ϵ reduces to zero.

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