Numerical Modelling of Waveguide Interface

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Outline of Presentation

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1. Formalism of Electrodynamics

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Numerical Algorithms:

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Numerical Algorithms:

2. Periodic Boundary Conditions

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4. Summary

Collaborators

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APOLLO
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The Research Objectives

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- To develop a computational algorithm for solving the stationary *Maxwell* equation at the interface between two planar waveguides

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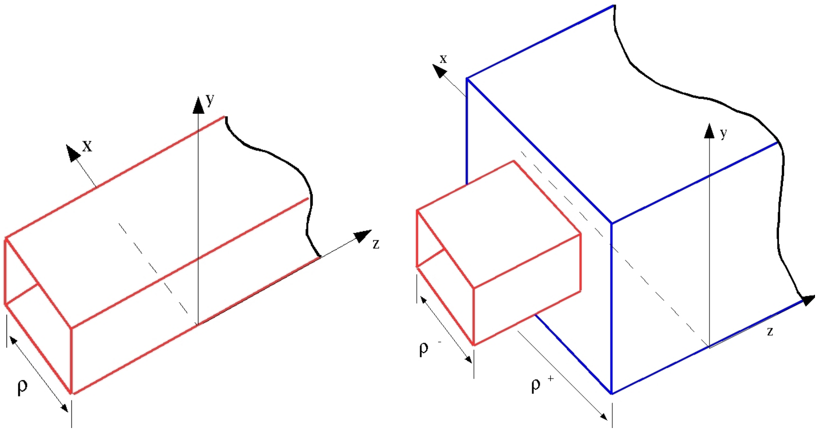

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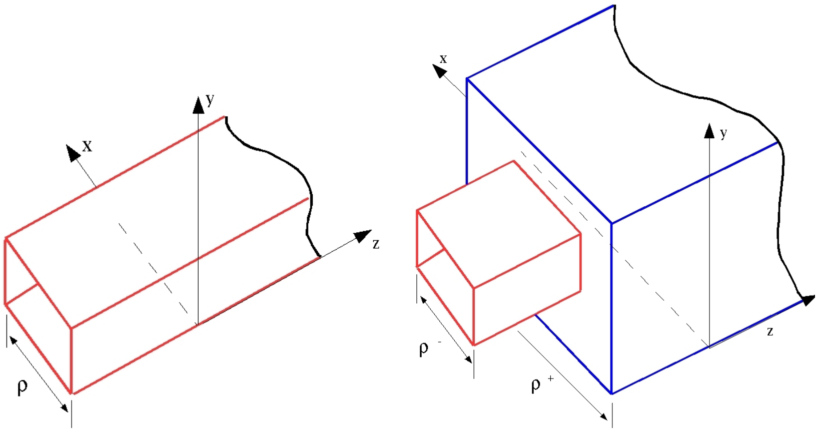

- Extent the computational algorithm to absorb outgoing waves from mirror-reflected waveguides

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Stationary Maxwell equation

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- c is the speed of light in vacuum

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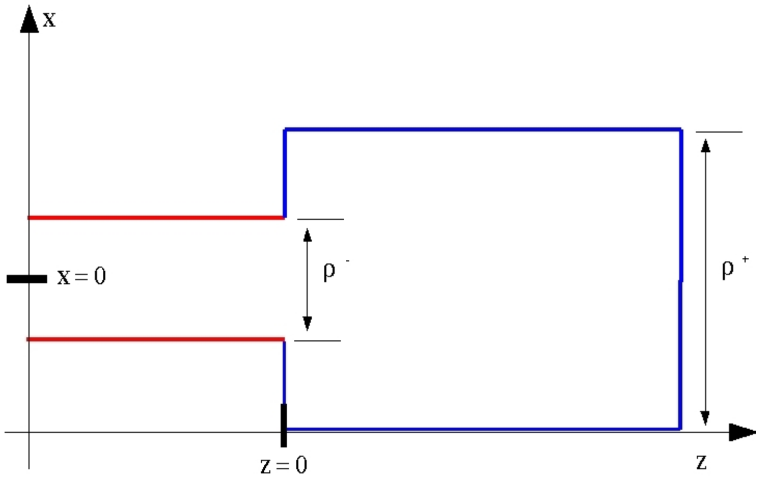

$$D^- = \{(x,z) \in \mathbb{R}^2 : z \leq 0\} \qquad D^+ = \{(x,z) \in \mathbb{R}^2 : z \geq 0\}.$$

TE Case: $\mathbf{E}_\omega(\mathbf{x}, \omega_0) = (0, E_{\omega,y}, 0)$, $\mathbf{x} = (x, z)$

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BC's: $\Psi^-, \Psi^+ \to 0$ as $|x| \to \infty$

MC's: $\Psi^-(x,0) = \Psi^+(x,0), \quad \frac{\partial\Psi^-}{\partial z}(x,0) = \frac{\partial\Psi^+}{\partial z}(x,0)$

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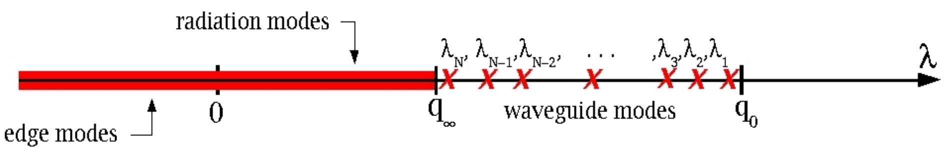

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$$\int_{-\infty}^{\infty} \left(S_{0,z}\right) dx = C$$

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The Propagation Problem

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$$\left[\frac{d^2}{dx^2} + q(x)\right]\Phi(x) = \lambda\Phi(x), \quad x \in \mathbb{R}$$

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- Φ_n denotes a numerical approximation for $\Phi(x_n)$
- $q_n = q(x_n)$

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- Φ_n denotes a numerical approximation for $\Phi(x_n)$
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$$A\mathbf{\Phi} = \lambda h^2 \mathbf{\Phi}$$

The Finite-Difference Frequency-Domain Method

- Divide $[-M, M]$ into $2N$ equal parts of width $h = \frac{M}{N}$
- Set $x_0 = -M$ and $x_{2N} = M$ as the boundary mesh points
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$$A = \begin{pmatrix} Q_1 & 1 & 0 & & \dots & 1 \\ 1 & Q_2 & 1 & 0 & \dots & 0 \\ 0 & 1 & Q_3 & 1 & 0\dots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & & \dots & 1 & Q_{2N-1} & 1 \\ 1 & 0 & \dots & 0 & 1 & Q_{2N} \end{pmatrix}, \boldsymbol{\Phi} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_{2N} \end{pmatrix}$$

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- $Q_n = h^2 q_n - 2, \quad n = 1, 2, ..., 2N$

Linear Systems for Two Waveguides

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Left of the Interface: $A^{-}\mathbf{\Phi}_{\mathbf{j}}^{-} = \lambda_j^{-} h^2 \mathbf{\Phi}_{\mathbf{j}}^{-}$

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$$\Downarrow$$

$$D^- = Q_-^{-1} A^- Q_- = Q_-^T A^- Q_- = \text{diag}\{\lambda_1^-, \lambda_2^-, ..., \lambda_{2N}^-\}$$

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$$D^{+} = Q_{+}^{-1} A^{+} Q_{+} = Q_{+}^{T} A^{+} Q_{+} = \text{diag}\{\lambda_1^{+}, \lambda_2^{+}, ..., \lambda_{2N}^{+}\}$$

- $Q_{\pm}$ is $2N$ x $2N$ matrix whose jth column is $\mathbf{\Phi}_{\mathbf{j}}^{\pm}$ respectively

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Left of the Interface: $z < 0$

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$$\mathbf{\Psi}^- = \sum_{j=1}^{2N} a_j \mathbf{\Phi}_{\mathbf{j}}^- e^{-i\beta_j^- z} + \sum_{j=1}^{2N} c_j \mathbf{\Phi}_{\mathbf{j}}^- e^{+i\beta_j^- z}, \quad \beta_j^- = \sqrt{\lambda_j^-}$$

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Right of the Interface: $z > 0$

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At the Interface: $z = 0$

$$\sum_{j=1}^{2N} c_j \mathbf{\Phi_j^-} + \sum_{j=1}^{2N} a_j \mathbf{\Phi_j^-} = \sum_{j=1}^{2N} b_j \mathbf{\Phi_j^+}$$

$$\sum_{j=1}^{2N} \beta_j^- c_j \mathbf{\Phi_j^-} - \sum_{j=1}^{2N} \beta_j^- a_j \mathbf{\Phi_j^-} = \sum_{j=1}^{2N} \beta_j^+ b_j \mathbf{\Phi_j^+}$$

Solutions of the Interface Equations

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Projection Operators P_k:

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$$P_k(\mathbf{f}) = \mathbf{\Phi}^+_{\mathbf{k}} \left\langle \mathbf{\Phi}^+_{\mathbf{k}}, \mathbf{f} \right\rangle, \quad \mathbf{f} \in \mathbb{R}^{2N}$$

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- a_j are the unknown reflected wave coefficients

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Recover numerical solutions for $\mathbf{\Psi}^-(z)$ and $\mathbf{\Psi}^+(z)$

The Conservation Law

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$$C = \int_{-\infty}^{\infty} (S_{0,z})\, dx \sim \int_{-\infty}^{\infty} \left(i\Psi \frac{\partial \overline{\Psi}}{\partial z} - i \frac{\partial \Psi}{\partial z} \overline{\Psi} \right) dx$$

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Right of the interface: $z > 0$

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- $R = \frac{\mathbf{I}_{ref}}{\mathbf{I}_{in}}$
- $T = \frac{\mathbf{I}_{trans}}{\mathbf{I}_{in}}$

Visualization of a Particular Solution

Visualization of a Particular Solution

Geometric Configuration:

Visualization of a Particular Solution

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- $D = \{(x, z) : -30 \leq x \leq 30, -30 \leq z \leq 30\}$
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- $q^{\pm}_{\infty} = 1$
- $q^-_0 = 2$ and $q^+_0 = 2 \rightarrow 10$

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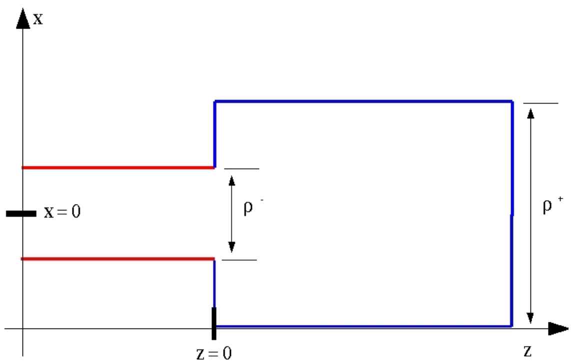
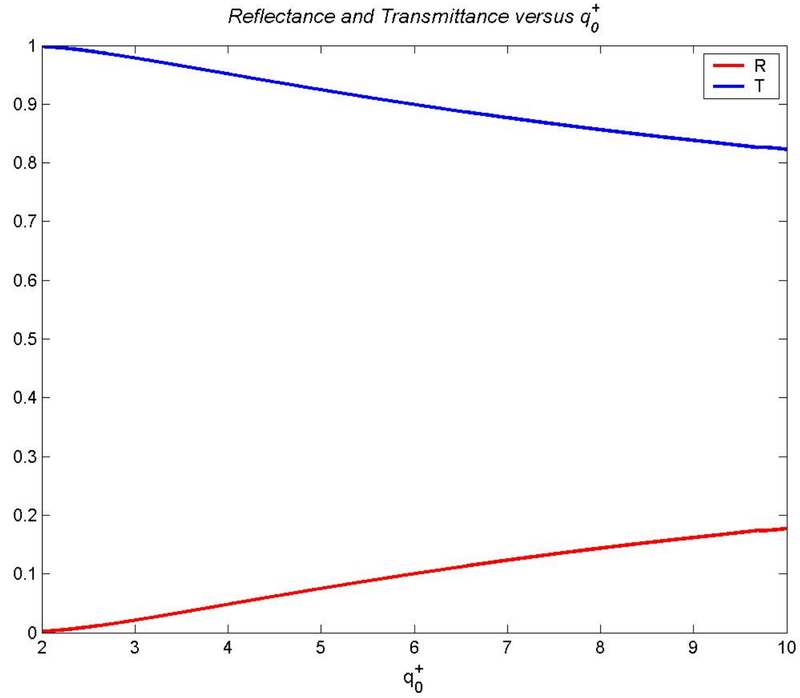
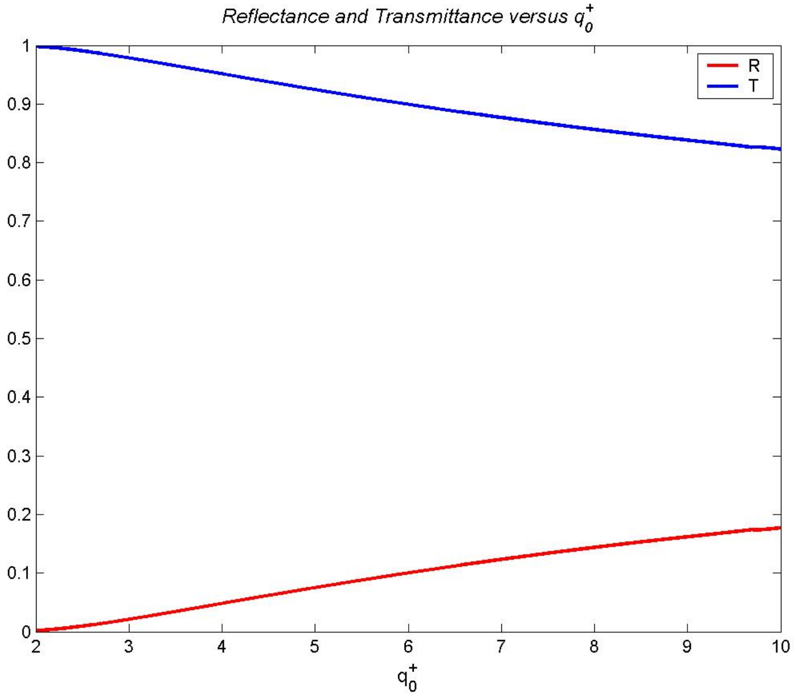



Reflectance and Transmittance versus q_0^+
R
T
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
2
3
4
5
6
7
8
9
10
q_0^+



- At each q_0^+: $R + T = 1$ (balance equation)

Fix $q_0^+ = 4$:

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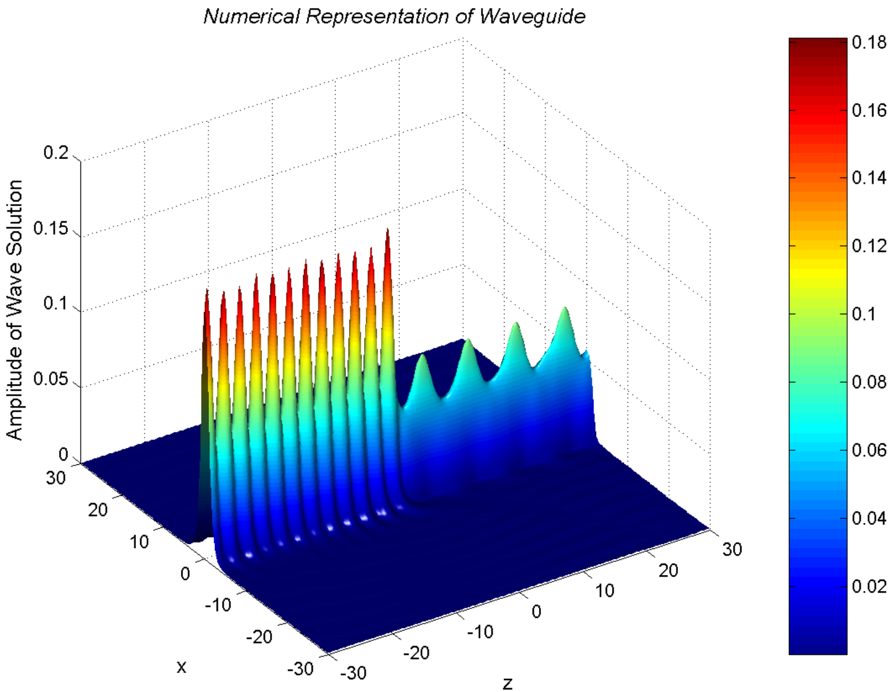
Numerical Representation of Waveguide
Amplitude of Wave Solution
0.2
0.15
0.1
0.05
0
x
30
20
10
0
-10
-20
-30
z
-30
-20
-10
0
10
20
30
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02

Energy Spectrum Left of the Interface

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Waveguide mode: $\lambda_1^- = 1.4794$
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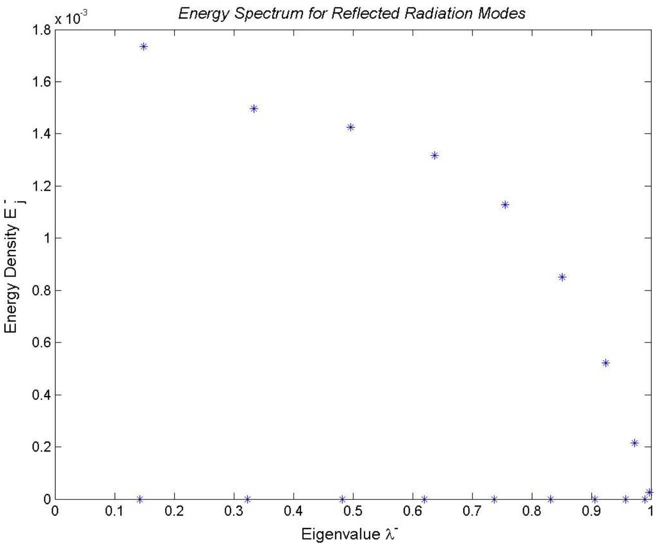
Radiation Modes:



Energy Spectrum Right of the Interface

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Waveguide modes: $\lambda_1^+ = 3.6238$, $\lambda_2^+ = 2.5544$, $\lambda_3^+ = 1.1270$
Transmitted Energies: 2.2698, 0, 0.0268

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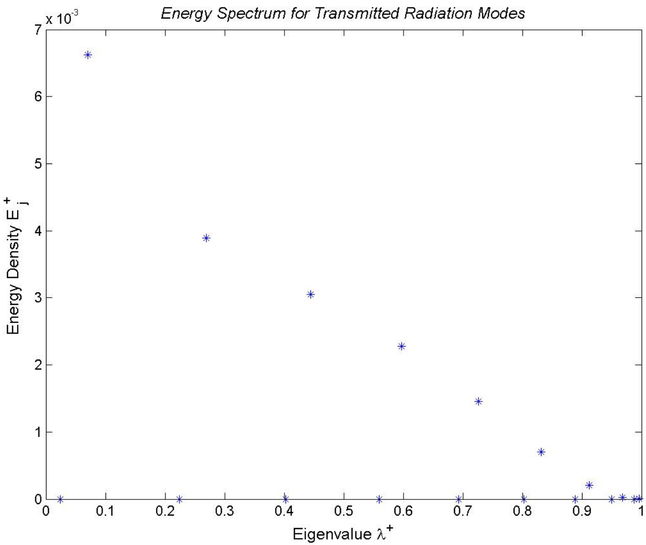

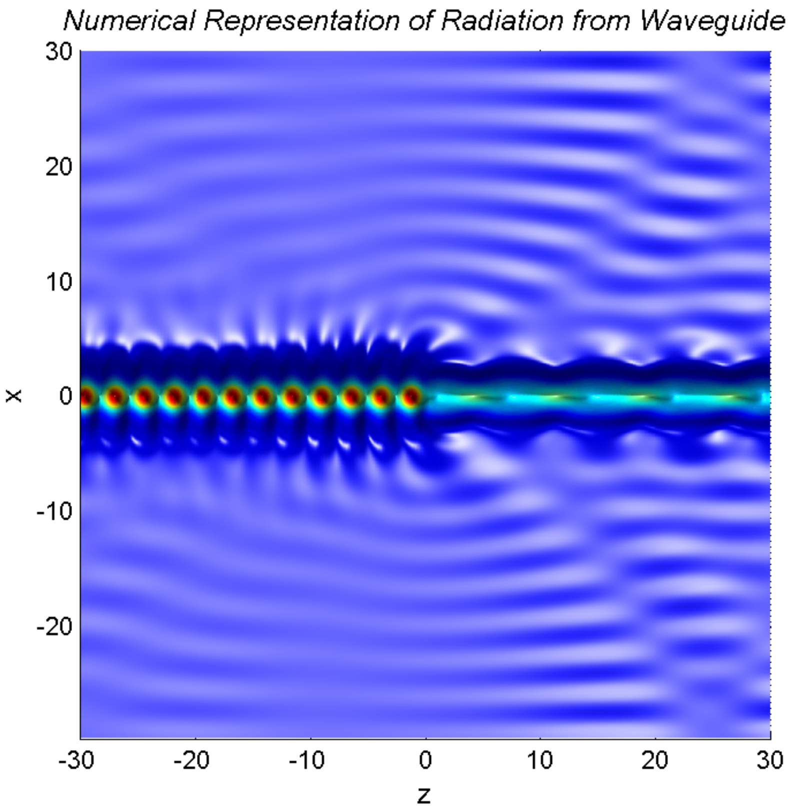
Numerical Representation of Radiation from Waveguide
30
20
10
0
-10
-20
x
-30
-20
-10
0
10
20
30
z

3. Absorbing Layers

The Method of Complexified Space

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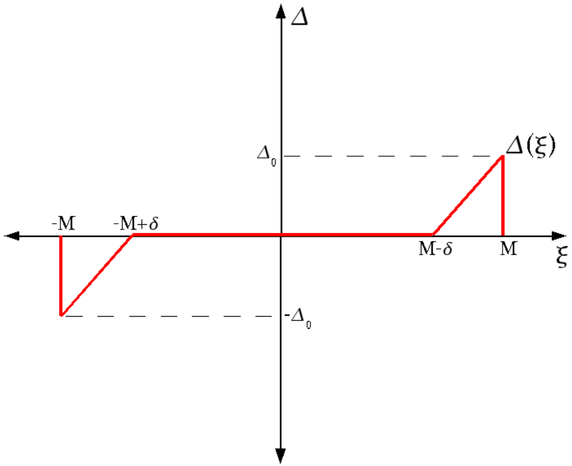

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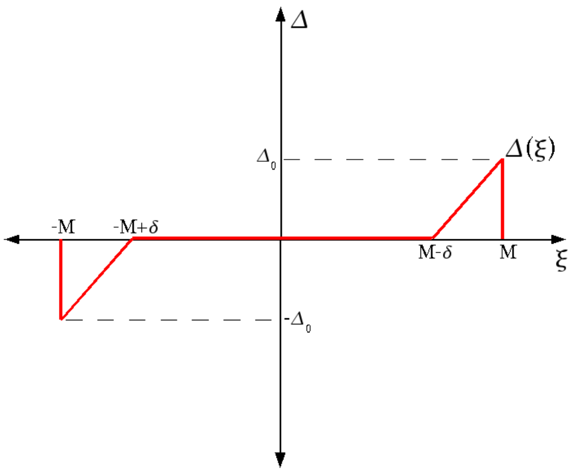

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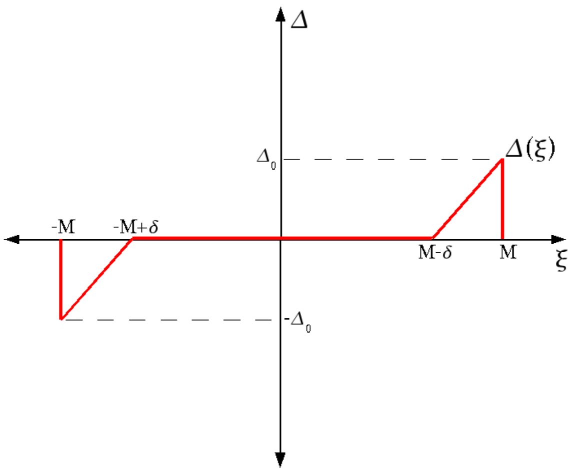

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$$\left[\frac{1}{C(\xi)}\frac{d^2}{d\xi^2} + q(\xi)\right]\Phi(\xi) = \lambda\Phi(\xi), \quad \lambda \in \mathbb{C}, \xi \in \mathbb{R}$$

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$\Downarrow$

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$\mathrm{Im}(\lambda_n) = \frac{\Delta_0}{\delta}(q_\infty - \mathrm{Re}(\lambda_n))$

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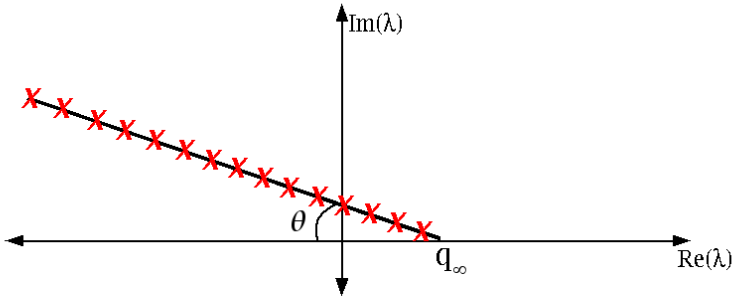

$\theta=-\frac{\Delta_0}{\delta}$

The Finite-Difference Frequency-Domain Method

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$$A = \begin{pmatrix} Q_1 & \frac{1}{C_1} & 0 & & \dots & 0 \\ \frac{1}{C_2} & Q_2 & \frac{1}{C_2} & 0 & \dots & 0 \\ 0 & \frac{1}{C_3} & Q_3 & \frac{1}{C_3} & 0\dots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & & \dots & \frac{1}{C_{2N-2}} & Q_{2N-2} & \frac{1}{C_{2N-2}} \\ 0 & 0 & \dots & 0 & \frac{1}{C_{2N-1}} & Q_{2N-1} \end{pmatrix}, \mathbf{\Phi} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_{2N-1} \end{pmatrix}$$

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Linear Systems for Two Wavguides

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Linear Systems for Two Wavguides

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$$\sum_{j=1}^{2N-1} \beta_j^{-} c_j \mathbf{\Phi}_{\mathbf{j}}^{-} - \sum_{j=1}^{2N-1} \beta_j^{-} a_j \mathbf{\Phi}_{\mathbf{j}}^{-} = \sum_{j=1}^{2N-1} \beta_j^{+} b_j \mathbf{\Phi}_{\mathbf{j}}^{+}$$

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$$\Downarrow$$

Recover numerical solutions for $\mathbf{\Psi}^{-}(z)$ and $\mathbf{\Psi}^{+}(z)$

Two Focal Issues

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- the ability to effectively absorb outgoing waves from artificial boundary-reflected waveguides, i.e. radiating waves

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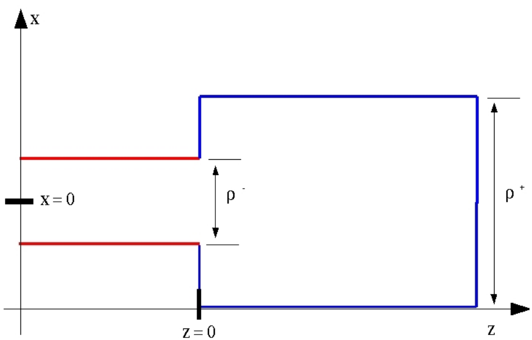

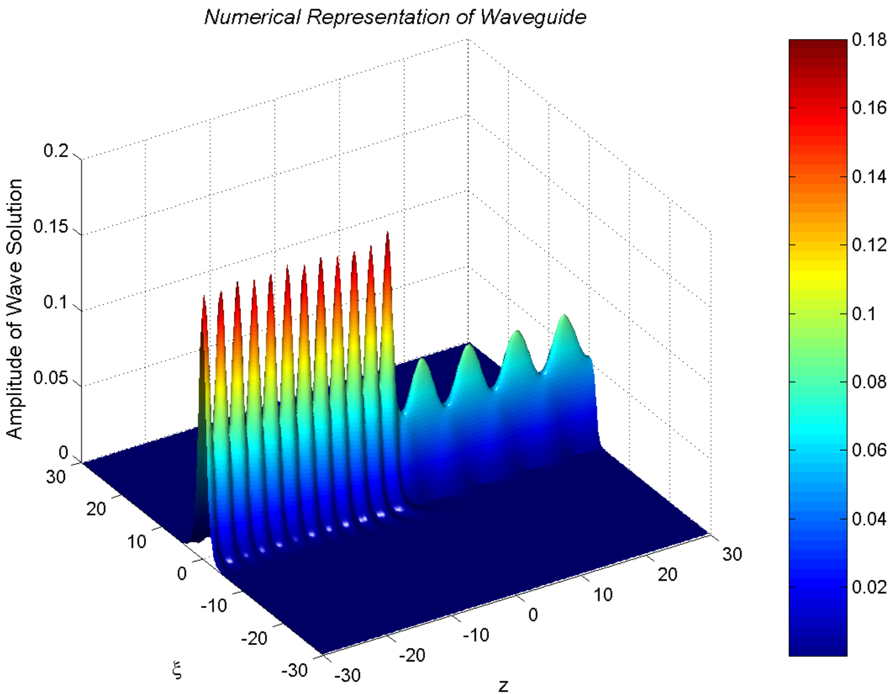
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0.2
0.15
0.1
0.05
0
30
20
10
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-20
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ξ
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10
20
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0.18
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Location of Eigenvalues Left of the Interface

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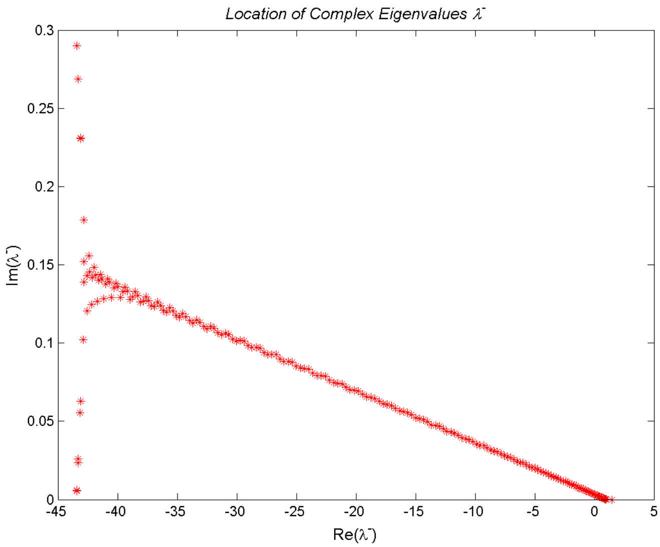

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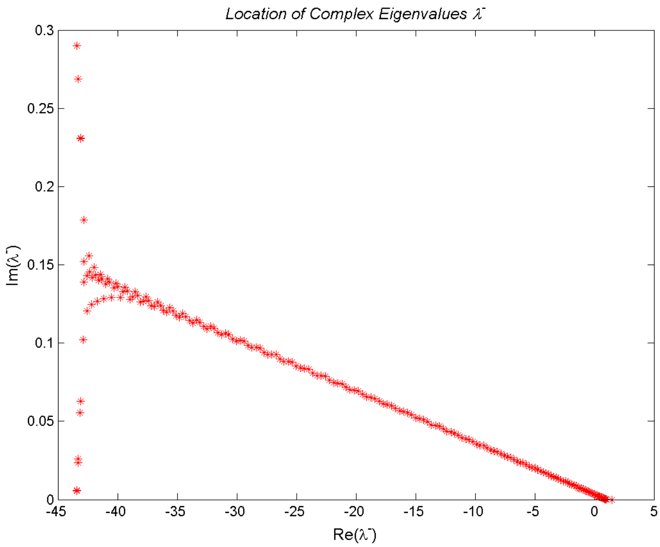

Waveguide mode:
$\lambda_1^- = 1.4794$

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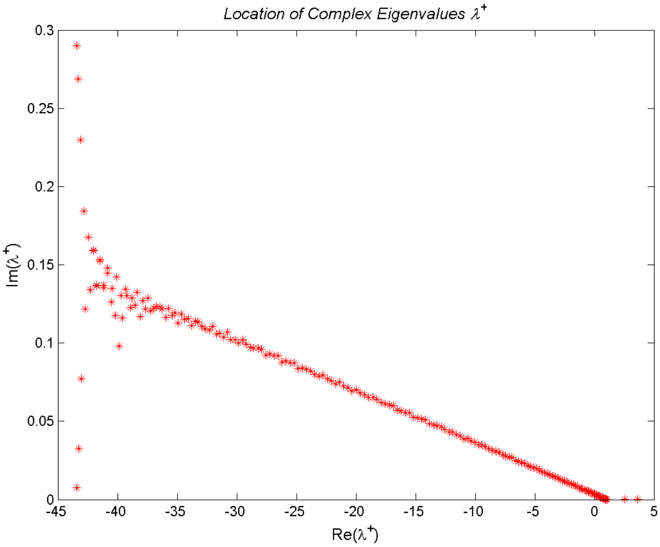

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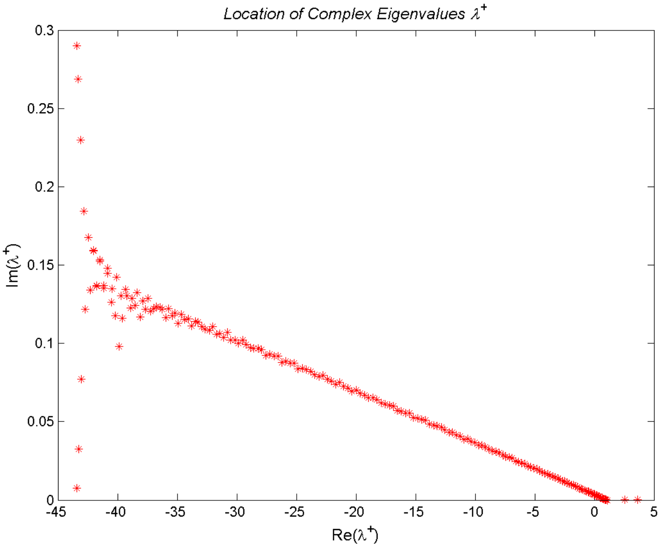


Waveguide modes:

$\lambda_1^+ = 3.6238$

$\lambda_2^+ = 2.5544$

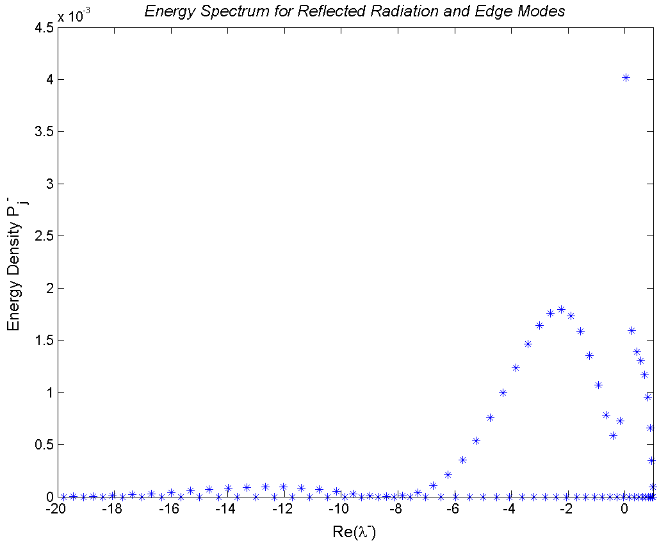
$\lambda_3^+ = 1.1270$

Energy Spectrum Left of the Interface

Energy Spectrum Left of the Interface

Waveguide mode: $\lambda_1^- = 1.4794$
Reflected Energy $= 0.1088$

Radiation Modes:



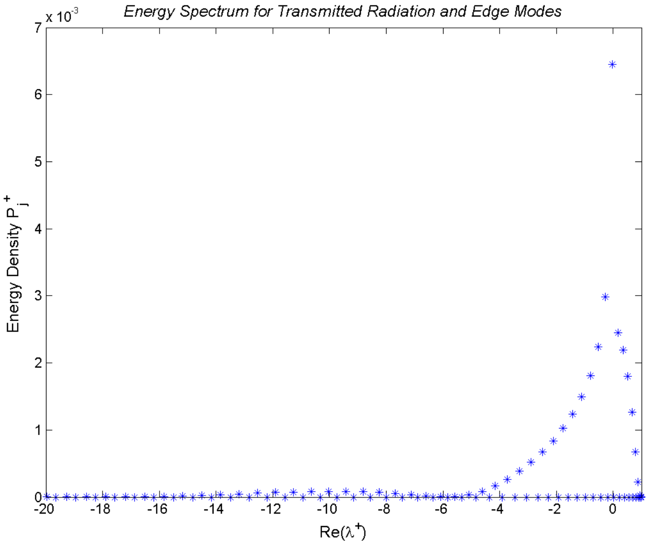
$$P_j^- = 2\sqrt{|\lambda^-|}|a_j|^2$$

Energy Spectrum Right of the Interface

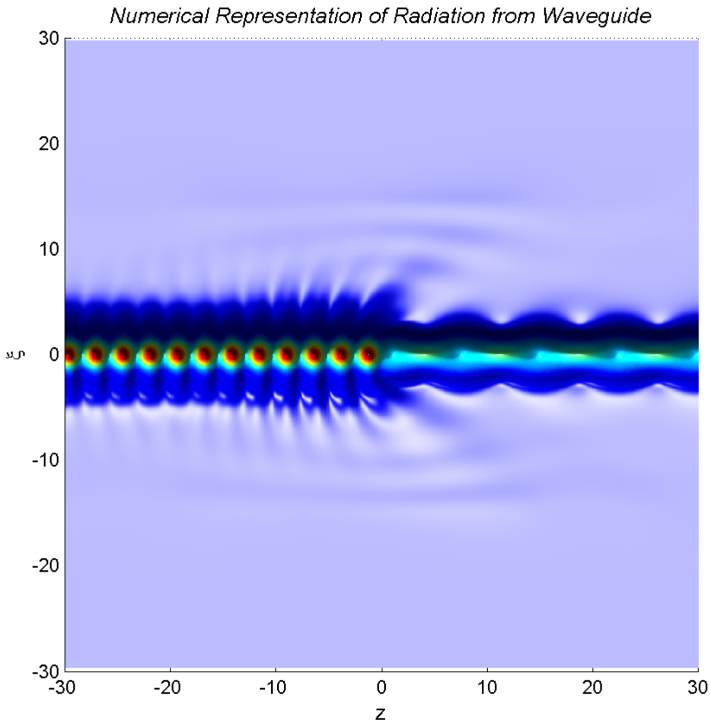
Energy Spectrum Right of the Interface

Waveguide modes: $\lambda_1^+ = 3.6238$, $\lambda_2^+ = 2.5544$, $\lambda_3^+ = 1.1270$
Transmitted Energies: 2.2690, 0, 0.0342

Radiation Modes:



$$P_j^+ = 2\sqrt{|\lambda^+|}|b_j|^2$$



Numerical Representation of Radiation from Waveguide
30
20
10
0
-10
-20
-30
ξ
-30
-20
-10
0
10
20
30
z

NUMERICAL CONCLUSIONS:

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- the absorbing layer does not change the location of eigenvalues of the discrete spectrum

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- the absorbing layer does not change the location of eigenvalues of the discrete spectrum
- the deformation of the continuous spectrum is a clockwise rotation about $q_\infty^\pm$ in the complex plane
- the absorbing layer induces splitting of the continuous spectrum into 2 branches

The Splitting Phenomena

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Simplifications:

- $q(\xi) = q_\infty \equiv \text{const}$ for $-M < \xi < M$ (no waveguide)

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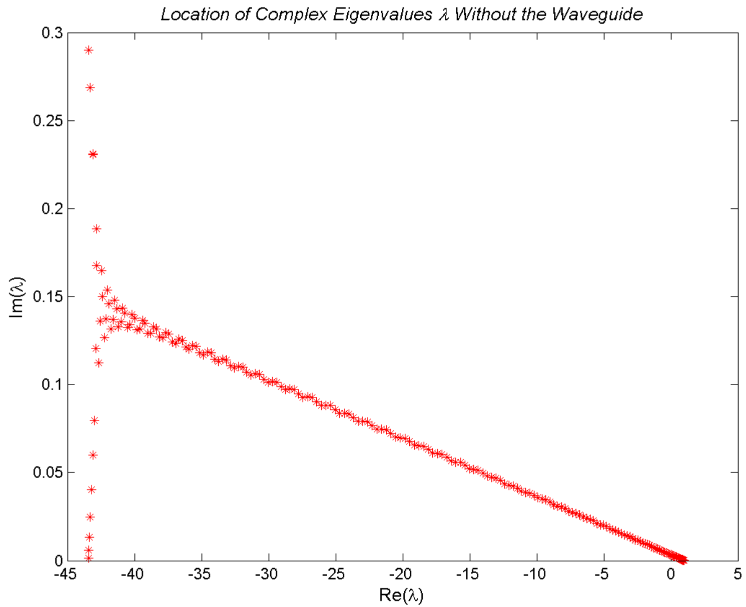

$$\Delta_0 = 0.1$$

Eigenvectors for Eigenvalues on the Lower Branch

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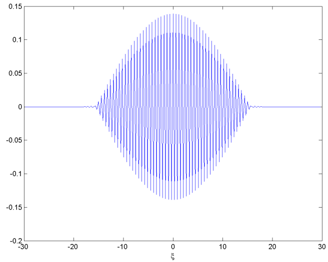

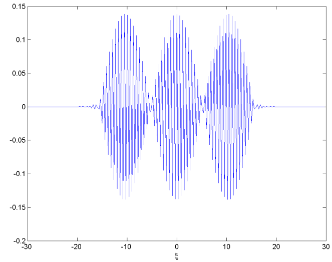

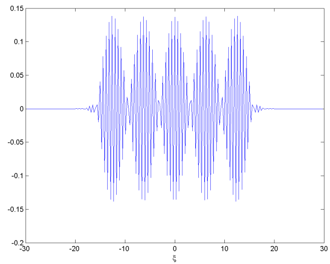

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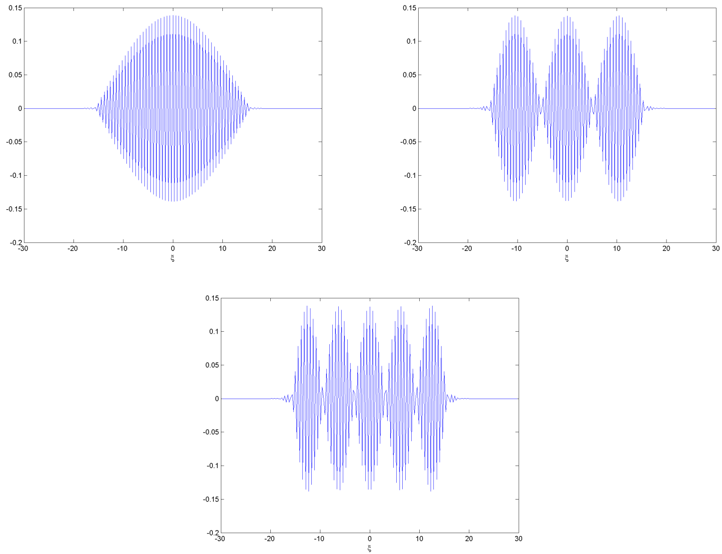
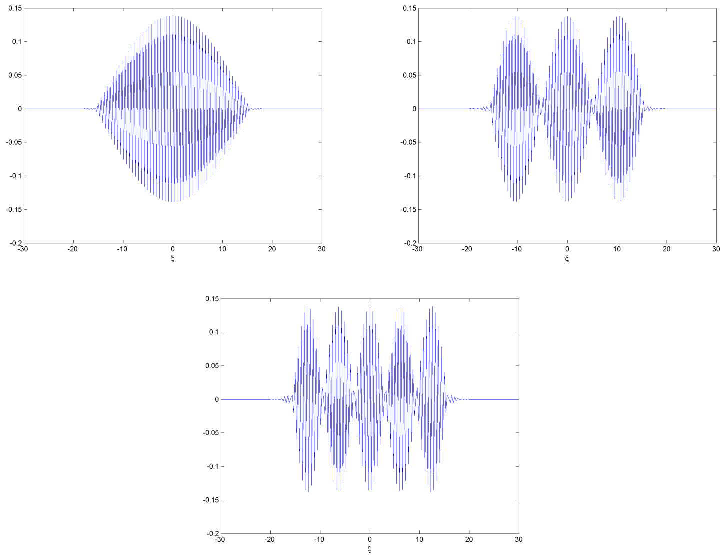
Eigenvectors for Eigenvalues on the Lower Branch



- $\mathbf{\Phi_j}$ are localized in the gap of the absorbing layer: $|\xi| < M - \delta = 15$

Eigenvectors for Eigenvalues on the Lower Branch



- $\mathbf{\Phi_j}$ are localized in the gap of the absorbing layer: $|\xi| < M - \delta = 15$
- modes on the lower branch represent waves trapped by the absorbing layer

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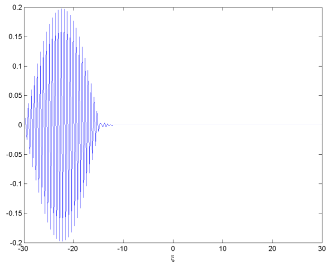
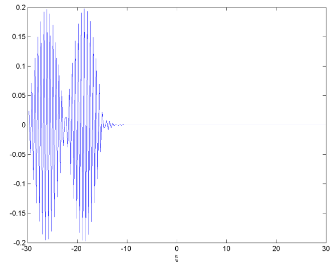
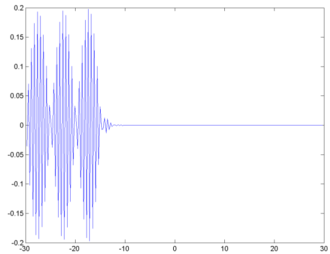
Eigenvectors for Eigenvalues on the Upper Branch

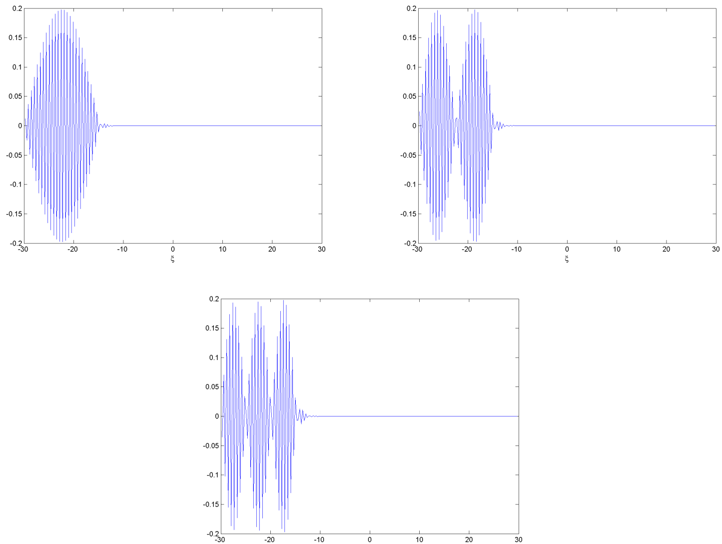
Eigenvectors for Eigenvalues on the Upper Branch







Eigenvectors for Eigenvalues on the Upper Branch



- $\mathbf{\Phi_j}$ are localized in the absorbing layer: $|\xi| > M - \delta = 15$

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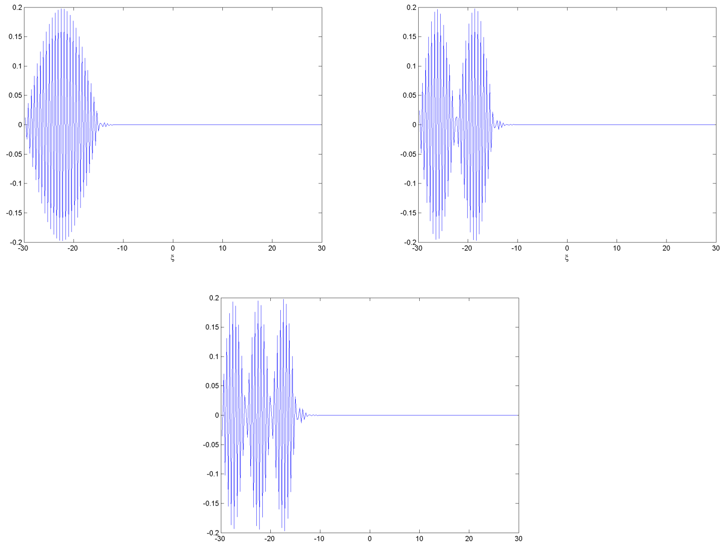
Eigenvectors for Eigenvalues on the Upper Branch



- $\mathbf{\Phi_j}$ are localized in the absorbing layer: $|\xi| > M - \delta = 15$
- modes on the upper branch represent waves escaping the absorbing layer

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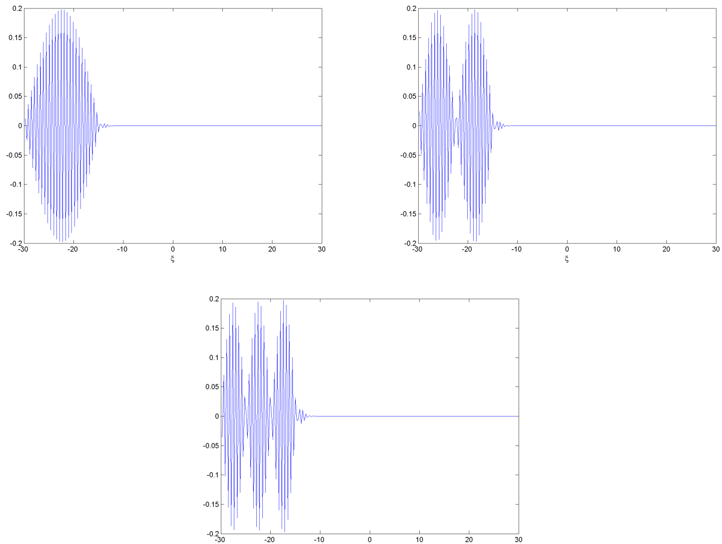
Eigenvectors for Eigenvalues on the Upper Branch



- $\mathbf{\Phi_j}$ are localized in the absorbing layer: $|\xi| > M - \delta = 15$
- modes on the upper branch represent waves escaping the absorbing layer
- modes are not excited since the incident wave is localized in the waveguide

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Damping coefficient $\Delta_0 = 1$

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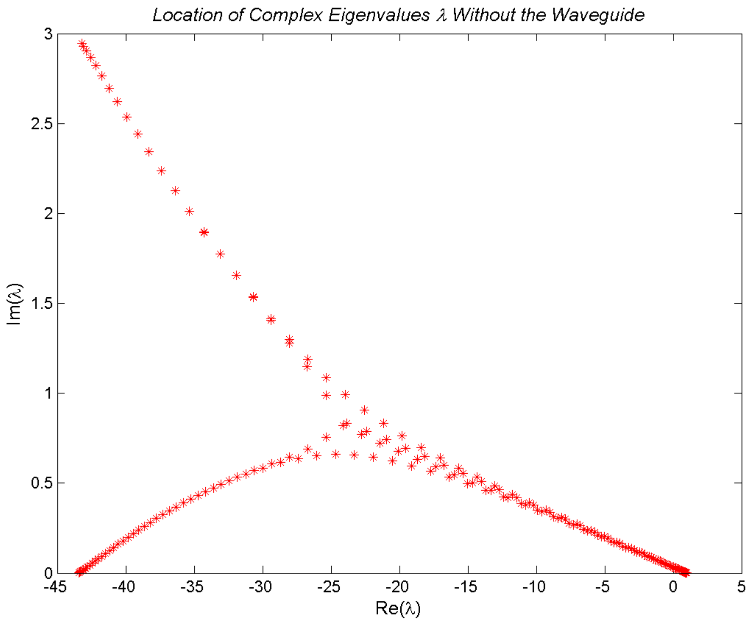

Damping coefficient $\Delta_0 = 10$

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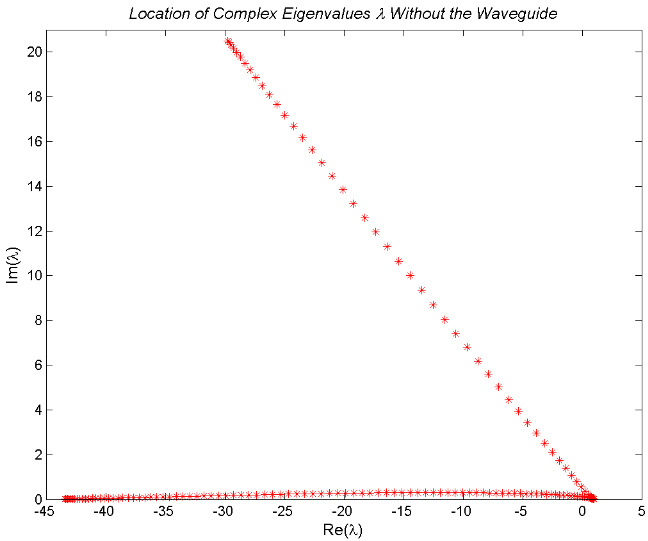

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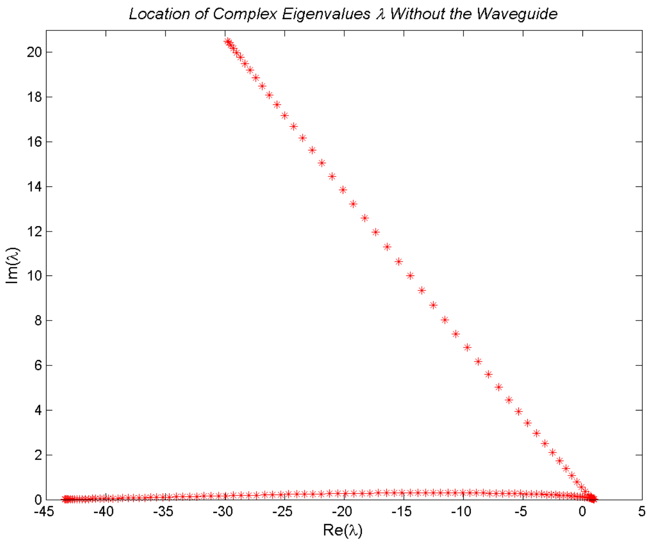

- eigenvalues on the upper branch increase as $\Delta_0 \nearrow$

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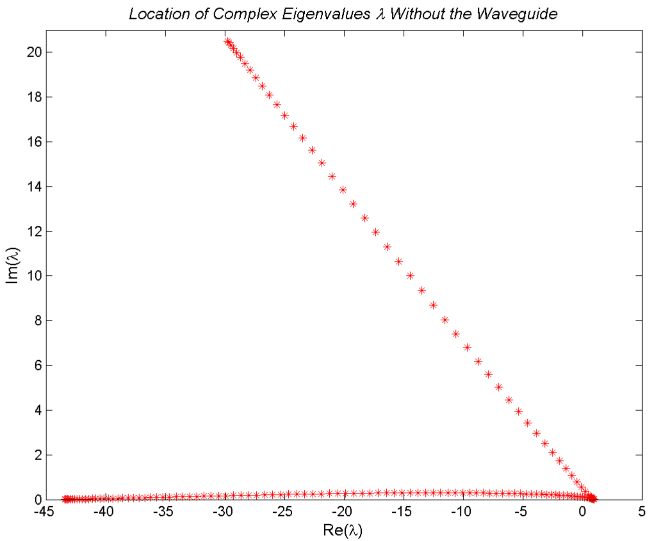
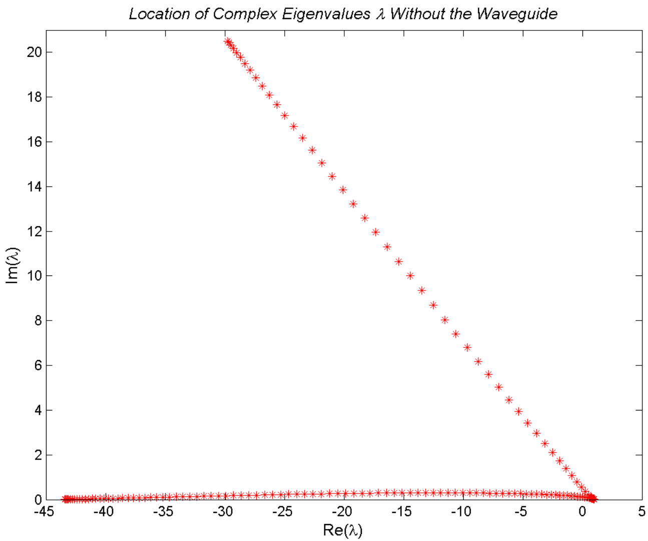


- eigenvalues on the upper branch increase as $\Delta_0 \nearrow$
- eigenvalues on the lower branch decrease as $\Delta_0 \nearrow$

Damping coefficient $\Delta_0 = 10$



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4. Summary

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- using the Finite-Difference Frequency-Domain method, we were able to describe a new technique for the simulation of electromagnetic wave propagation at the interface between two planar waveguides
- extended our algorithm due to the complexification of transverse space and were successful in absorbing radiating waves from artificial boundary-reflected waveguides