Advection-diffusion equations with forward-backward diffusion

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Reference: J. Math. Anal. Appl. **342** 970Ű-988 (2008)

The problem

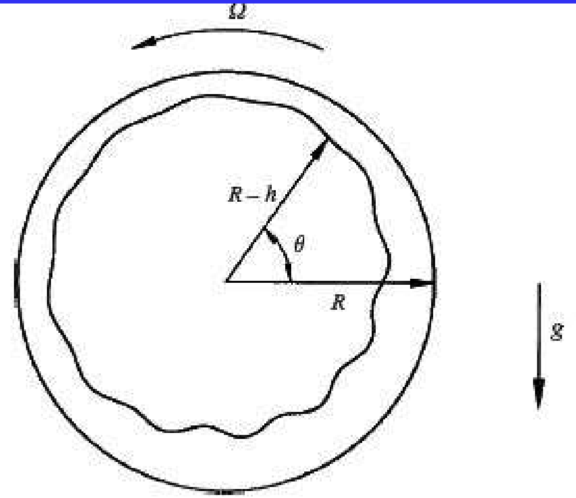


FIGURE 1. Formulation of the problem: film in a rotating cylinder with horizontal axis.

Reference: E. Benilov, S. O'Brien and I. Sazonov, J. Fluid Mech. 497, 201-224 (2003)

- A thin film of liquid on the inside surface of a cylinder rotating around its axis
- $h(\theta, t)$ is a thickness of the film in the limit $h \ll R$
- $\epsilon = \|h\|^4/R^4$ is a small parameter.

The Cauchy problem

Linear disturbances of a stationary flow satisfy

$$h_t + h_\theta + \epsilon \left(\sin\theta h_\theta\right)_\theta = 0.$$

The Cauchy problem for the advection–diffusion equation:

$$\begin{cases} \dot{h} = Lh, \quad L = -\partial_\theta - \epsilon\partial_\theta \sin\theta\partial_\theta, \\ h(0) = h_0, \end{cases}$$

subject to the periodic boundary conditions on $[-\pi, \pi]$.

We should expect heuristically that the Cauchy problem is ill-posed because of the backward heat equation on $(0, \pi)$ (for $\epsilon > 0$).

Previous claims on the spectrum of L

Let us consider the associated linear operator

$$L = -\epsilon \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) - \frac{\partial}{\partial\theta}$$

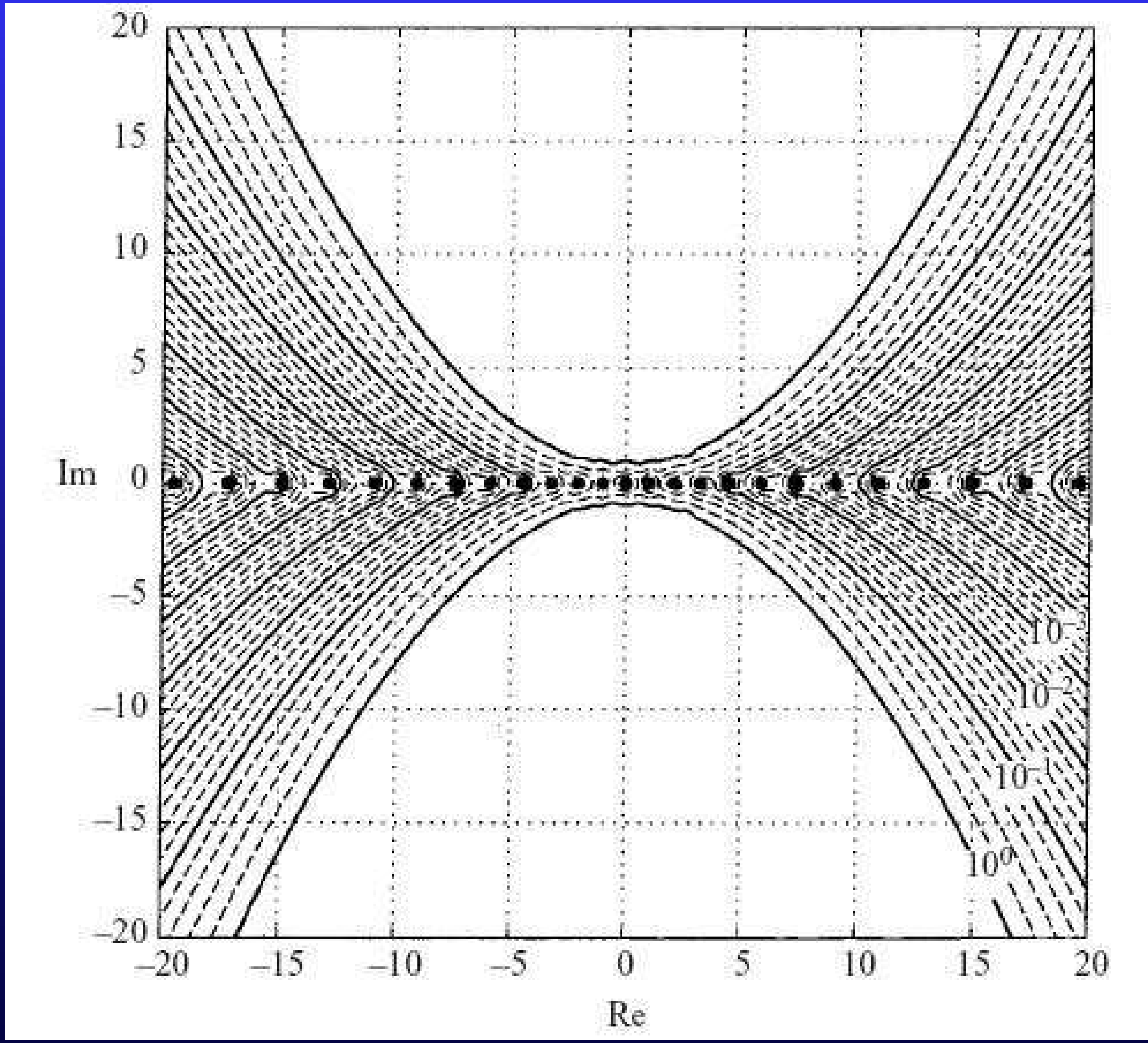
acting on smooth periodic functions $f(\theta)$ on $[-\pi, \pi]$.

1. All eigenvalues are simple and purely imaginary.
2. The series of eigenfunctions, even if it converges at $t = 0$, may diverge for some $t \geq t_0 > 0$.
3. The level set of $(\lambda - L)^{-1}$ form divergent curves to the left and right half-planes.

- E. Benilov (2004): an explicit example confirms (2).
- N. Trefethen (2005): the pseudospectral method confirms (3).

Level sets of the resolvent

From Benilov et al. (2003):



Main results

We study the relation between the spectral properties of the operator L and ill-posedness of the advection–diffusion equation.

- The operator L is closed in $L^2_{\text{per}}([-\pi, \pi])$ with a domain in $H^1_{\text{per}}([-\pi, \pi])$ for $0 < \epsilon < 2$.
- L has a compact inverse of the Hilbert–Shmidt type, so its spectrum consists of an infinite sequence of isolated eigenvalues accumulating to infinity. Moreover, all eigenvalues are simple and purely imaginary.
- The set of eigenfunctions is complete but does not form a basis in $L^2_{\text{per}}([-\pi, \pi])$.

Unexpected developments

- E.B. Davies (2007): same results from difference equations
- J. Weir (2008): transformation of iL to a self-adjoint operator
- E.B. Davies, J. Weir (2008): spectrum of iL in the asymptotic limit $\epsilon \to 0$
- L. Boulton, M. Levitin, M. Marletta (2008): generalization of the ODE approach for a class of operators L which admit a purely imaginary spectrum
- M. Chugunova, V. Strauss (2008): factorization of L in Krein spaces
- M. Chugunova, I. Karabash, S. Pyatkov (2008): characterization of the domain of L and proof of ill-posedness of $h_t = Lh$

Closure and domain of L

Claim: The operator L is closed in $L^2_{\text{per}}([-\pi,\pi])$ with a domain in $H^1_{\text{per}}([-\pi,\pi])$ for $0<\epsilon<2$.

$\lambda = 0$ is always an eigenvalue with eigenfunction $f=1$. We need to show that there exists at least one regular point $\lambda_0 \in \mathbb{C}$ with

$$\|(L-\lambda_0 I)f\|_{L^2} \geq k_0 \|f\|_{L^2}.$$

We use

$$(f',(L-\lambda_0 I)f) = -\int_{-\pi}^{\pi}(1+\epsilon\cos\theta)\,|f'|^2 d\theta - \int_{-\pi}^{\pi}\sin\theta\,\bar{f}'\,f'' d\theta,$$

from which the bound follows with $\lambda_0 = k_0 = \frac{1}{2\pi}\left(1-\frac{\epsilon}{2}\right)$.

Purely discrete spectrum of L

Claim: The spectrum of L consists of simple purely imaginary eigenvalues.

Eigenfunctions of L are represented by

$$f(\theta) = \sum_{n \geq 1} f_n e^{in\theta} = \sum_{n \geq 1} f_n z^n,$$

for $z = e^{i\theta}$. The interval $[-\pi, \pi]$ for θ transforms to a unit circle in $\mathbb{C}$ for z. Now $u(z) = \sum_{n \geq 1} f_n z^n$ satisfies the second-order ODE

$$z(1-z)(1+z)u''(z) - 2z(z + \frac{1}{\epsilon})u'(z) + \frac{2i\lambda}{\epsilon}u(z) = 0$$

and belong to the Hardy space of square-integrable functions on the unit circle which are analytically continued in the unit disk.

Proof of $\lambda \in i\mathbb{R}$

Consider solutions $u(z)$ on $\{\mathrm{Re}(z) \in [0,1], \mathrm{Im}(z) = 0\}$ and apply the singular point analysis:

$$u(x) \to \begin{cases} a + b(1-x)^{-1/\epsilon}, \quad \text{as} \quad x \to 1 \\ c + dx, \quad \text{as} \quad x \to 0 \end{cases}$$

For a proper eigenfunction, $b = 0$ and $c = 0$.

The second-order ODE is written in the self-adjoint form

$$-\left(p(x)u'(x)\right)' = \mu w(x) u(x), \qquad x \in [0,1],$$

where $\mu = 2i\lambda/\epsilon$, $p(x) = (1-x)^{1+1/\epsilon}(1+x)^{1-1/\epsilon}$, and $w(x) = (1+x)^{-1/\epsilon}(1-x)^{1/\epsilon}/x$. The solution belongs to $L^2_w([0,1])$, where $\mu \in \mathbb{R}$.

Eigenvalues of L

Lemma: Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a set of eigenvalues with $\mathrm{Im}\lambda_n > 0$, ordered in the ascending order of $|\lambda_n|$. There exists a $N \geq 1$, such that $\lambda_n \in i\mathbb{R}$ for all $n \geq N$ and

$$|\lambda_n| = Cn^2 + \mathrm{o}(n^2) \;\; \text{as} \;\; n \to \infty,$$

for some $C > 0$.

For $0 < \pm\theta < \pi$, let

$$\cos\theta = \tanh t, \quad \sin\theta = \pm\mathrm{sech} t, \qquad t \in \mathbb{R},$$

and find two uncoupled problems for $f_\pm(t) = f(\theta)$ on $0 < \pm\theta < \pi$:

$$-\epsilon f''_\pm(t) + f'_\pm(t) = \pm\lambda \mathrm{sech} t \; f_\pm(t),$$

allowing for the WKB solution $f_\pm(t) = e^{\int_\infty^t S_\pm(t')dt'}$.

Eigenvalues of L

The boundary conditions $f(\pi) = f(-\pi)$ or $\lim_{t\to-\infty} f_-(t) = \lim_{t\to-\infty} f_+(t)$ imply that λ is a root of

$$G_n(\lambda) = \frac{1}{4\pi i\epsilon}\int_{-\infty}^{\infty}\Big[\sqrt{1+4\epsilon\lambda\mathrm{sech}t - 4\epsilon^2 R_-(t)}$$
$$-\sqrt{1-4\epsilon\lambda\mathrm{sech}t - 4\epsilon^2 R_+(t)}\Big]\,dt - n, \quad n\in\mathbb{N}.$$

- $G_n(0) = -n$
- $G_n(i\omega)$ is real-valued for $\omega\in\mathbb{R}$.
- As $\omega\to\infty$

$$G_n(i\omega) = \frac{\sqrt{\omega}}{\sqrt{2\epsilon\pi}}\int_{-\infty}^{\infty}\frac{dt}{\sqrt{\cosh t}} + \mathrm{o}\left(\sqrt{\omega}\right) - n$$

Completeness of eigenfunctions

Definition: The set of functions $\{f_n\}_{n\in\mathbb{Z}}$ is said to be complete in Banach space X if any function $f \in X$ can be approximated by a finite linear combination $f_N(\theta) = \sum_{n=-N}^{N} c_n f_n(\theta)$ in the following sense: for any fixed $\varepsilon > 0$, there exists $N \geq 1$ and $\{c_n\}_{-N\leq n\leq N}$, such that $\|f - f_N\|_X < \epsilon$ holds.

Theorem: Let $\{f_n(\theta)\}_{n\in\mathbb{Z}}$ be the set of eigenfunctions of L corresponding to the set of eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$. The set of eigenfunctions is complete in $L^2_{\text{per}}([-\pi,\pi])$.

Completeness follows from Lidskii's Completeness Theorem since the two sufficient conditions are satisfied: (1) eigenvalues of L are purely imaginary and (2) $|\lambda_n| = \mathrm{O}(n^2)$ as $n \to \infty$.

Basis of eigenfunctions

Definition: The set of functions $\{f_n\}_{n\in\mathbb{Z}}$ is said to form a Schauder basis in Banach space X if, for every $f \in X$, there exists a unique representation $f(\theta) = \sum_{n\in\mathbb{Z}} c_n f_n(\theta)$ with some coefficients $\{c_n\}_{n\in\mathbb{Z}}$ such that $\lim_{N\to\infty} \|f - f_N\|_X = 0$.

Theorem: Let $\{f_n\}_{n\in\mathbb{Z}}$ be a complete set of eigenfunctions of L. It forms a basis in Hilbert space $L^2_{\text{per}}([-\pi,\pi])$ if and only if $\lim_{n\to\infty} \cos(\widehat{f_n, f_{n+1}}) < 1$ or $\lim_{n\to\infty} \|P_n\| < \infty$, where

$$\cos(\widehat{f_n, f_{n+1}}) = \frac{|(f_n, f_{n+1})|}{\|f_n\|\|f_{n+1}\|}, \quad \|P_n\| = \frac{\|f_n\|\|f_n^*\|}{|(f_n, f_n^*)|}.$$

Numerical shooting method

By the ODE theory near regular singular points, $f(\theta)$ is spanned by

$$f_1 = 1 + \sum_{n\in\mathbb{N}} c_n \theta^n, \quad f_2 = \theta^{-1/\epsilon} \left(1 + \sum_{n\in\mathbb{N}} d_n \theta^n \right)$$

near $\theta = 0$ and

$$f_1^{\pm} = 1 + \sum_{n\in\mathbb{N}} a_n^{\pm} (\pi \mp \theta)^n, \quad f_2^{\pm} = (\pi \mp \theta)^{1/\epsilon} \left(1 + \sum_{n\in\mathbb{N}} b_n^{\pm} (\pi \mp \theta)^n \right)$$

near $\theta = \pm\pi$. If $f \in H^1_{\text{per}}([-\pi, \pi])$, then

$$f = C f_1(\theta) = A_{\pm} f_1^{\pm}(\theta) + B_{\pm} f_2^{\pm}(\theta)$$

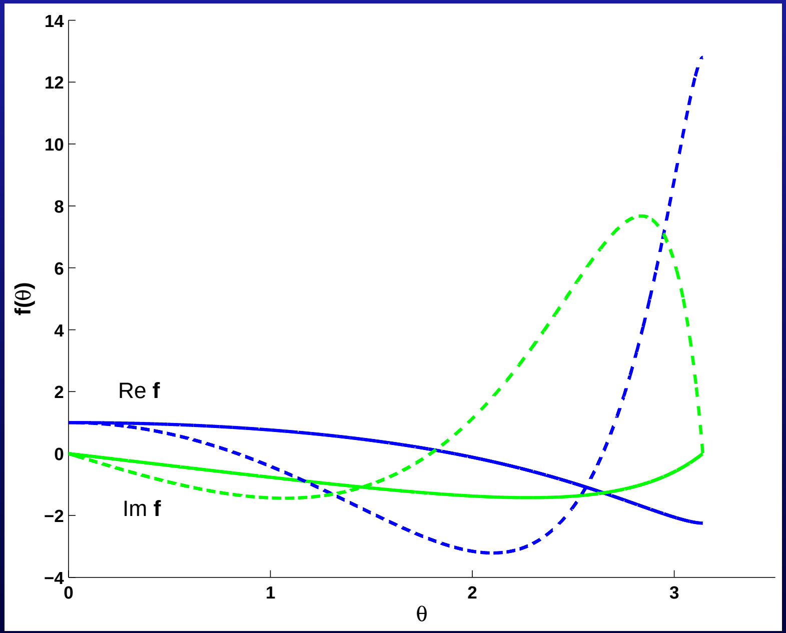
for some constants $C, A_{\pm}, B_{\pm}$ with $A_+ = A_-$.

Results of the shooting method

Purely imaginary eigenvalues:

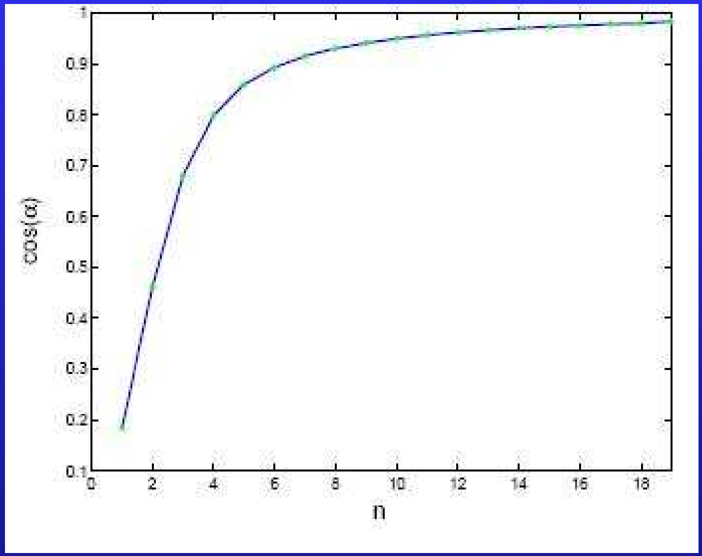
ϵ	ω_1	ω_2	ω_3	ω_4
0.5	1.167342	2.968852	5.483680	8.715534
1.0	1.449323	4.319645	8.631474	14.382886
1.5	1.757278	5.719671	11.846709	20.138824

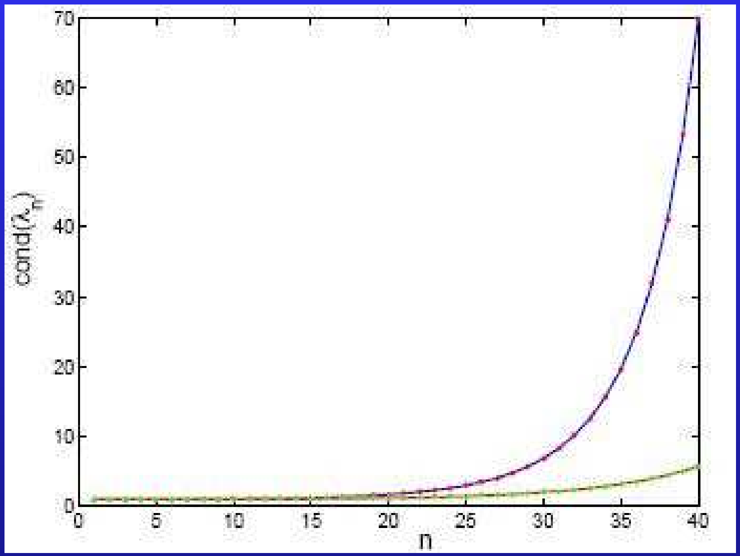
and their eigenfunctions:



Spectral projections

Criteria for eigenfunctions to form a basis:





Left - $\cos(\widehat{f_n, f_{n+1}})$, right - $\|P_n\|$, where

$$\cos(\widehat{f_n, f_{n+1}}) = \frac{|(f_n, f_{n+1})|}{\|f_n\|\|f_{n+1}\|}, \quad \|P_n\| = \frac{\|f_n\|\|f_n^*\|}{|(f_n, f_n^*)|}.$$

Numerical results indicate that the complete set of eigenfunctions does not form a basis in $L^2_{\text{per}}([-\pi, \pi])$.

Grande Finale

Summary: The spectrum of L is on the imaginary axis but the series of eigenfunctions can not be used for solutions of the advection-diffusion equation $\dot{h} = Lh$. Does it indicate ill-posedness of the advection equation?

Hille–Yosida theorem: A densely defined operator L forms a strongly continuous contraction semigroup in $L^2_{\text{per}}([-\pi, \pi])$ if and only if for any ray in $\text{Re}(\lambda) > 0$, the operator $\lambda I - L$ has an everywhere defined inverse such that

$$\|(\lambda I - L)^{-1}\|_{L^2 \to L^2} \leq \frac{1}{\lambda}.$$

From pseudo-spectrum, we know that this condition is not satisfied and, therefore, the Cauchy problem for the advection–diffusion equation is ill-posed.