Bifurcations of multi-vortex configurations in rotating Bose–Einstein condensates

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References:
D.P., P. Kevrekidis, AMRE **2013**, *127–164 (2013)*
C. Garcia–Azpeitia, D.P., arXiv:1701.01494 (2017)

Outline of the talk

1. Gross–Pitaevskii equation
2. Rotating vortices
3. Bifurcation of multi-vortex configurations
4. Examples of simplest bifurcations

Gross-Pitaevskii equation

Density waves in cigar–shaped Bose–Einstein condensates with repulsive inter-atomic interactions placed in a magnetic trap are modeled by the Gross-Pitaevskii equation with the harmonic potential and steady rotation:

$$i\,\varepsilon\,u_t = -\varepsilon^2(\partial_x^2 + \partial_y^2)u + (x^2 + y^2 + |u|^2)u + i\Omega(x\partial_y - y\partial_x)u,$$

where Ω is the rotation frequency and ε is related to the chemical potential.

The associated energy of the Gross–Pitaevskii equation:

$$E(u) = \int\int_{\mathbb{R}^2} \left[\varepsilon^2\,|\nabla u|^2 + |x|^2|u|^2 + \frac{1}{2}|u|^4 - i\Omega u(x\partial_y - y\partial_x)\bar{u}\right] dxdy.$$

Steadily rotating vortices are critical points of the energy $E(u)$ subject to the fixed mass

$$Q(u) = \int\int_{\mathbb{R}^2} |u|^2 dxdy.$$

Ground (vortex-free) state

If $\Omega = 0$ (no rotation), there exists ground state for every $\mathcal{Q} > 0$:

$$\mathcal{E} = \inf \left\{ u \in H^1(\mathbb{R}^2) \cap L^{2,1}(\mathbb{R}^2) : \quad Q(u) = \mathcal{Q} \right\}.$$

The ground state is given by the real, positive, radially symmetric solution η_ε of the stationary equation:

$$\mu \eta_\varepsilon = -\varepsilon^2 \nabla_x^2 \eta_\varepsilon + |x|^2 \eta_\varepsilon + \eta_\varepsilon^3, \quad x \in \mathbb{R}^2,$$

where $\mu > 0$ is the Lagrange multiplier.

In the semi-classical limit $\varepsilon \to 0$, the ground state decays to zero as $|x| \to \infty$ faster than any exponential function

$$0 < \eta_\varepsilon(x) \leq C \varepsilon^{1/3} \exp\left(\frac{\mu - |x|^2}{4\varepsilon^{2/3}} \right), \quad \text{for all } |x| \geq \mu^{1/2}$$

and is $\varepsilon^{1/3}$ close to the Thomas–Fermi approximation

$$\eta_0(x) := \begin{cases} (\mu - |x|^2)^{1/2}, & \text{for } |x| < \mu^{1/2}, \\ 0, & \text{for } |x| > \mu^{1/2}, \end{cases}$$

C.Gallo–D.P., Asymptotic Analysis **73**, *53-96 (2011)*

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C.Gallo–D.P., Asymptotic Analysis **73**, *53-96 (2011)*

Vortex states

If $\Omega = 0$ (no rotation), the vortex state u_ε is a complex-valued solution of the stationary equation,

$$\mu u_\varepsilon = -\varepsilon^2 \nabla_x^2 u_\varepsilon + |x|^2 u_\varepsilon + |u_\varepsilon|^2 u_\varepsilon, \quad x \in \mathbb{R}^2.$$

In the semi-classical limit $\varepsilon \to 0$, the vortex state u_ε is well approximated by the product representation

$$u_\varepsilon(x) = \eta_\varepsilon(x) v_\varepsilon(x),$$

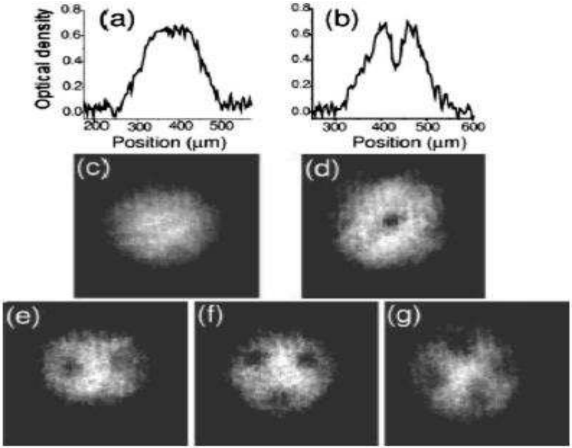
where v_ε satisfies the stationary equation

$$\varepsilon^2 \nabla_x \left(\eta_\varepsilon^2 \nabla_x v \right) + \eta_\varepsilon^4 (1 - |v|^2) v = 0,$$

subject to the boundary conditions $\lim_{|x|\to\infty} |v_\varepsilon(x)| = 1$.

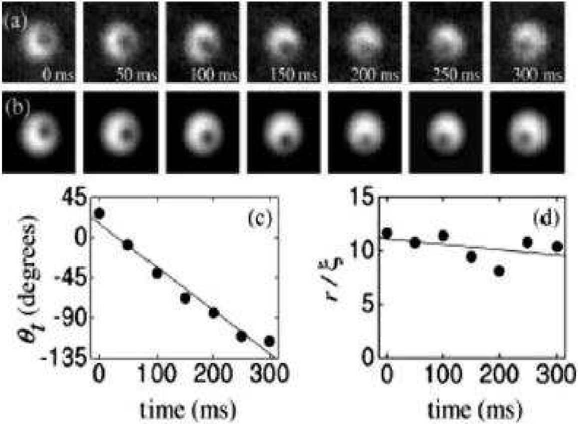
Symmetric vortex of charge $m \in \mathbb{N}$ corresponds to the choice $v_\varepsilon = \psi_\varepsilon(r) e^{im\theta}$, where (r, θ) are polar coordinates on $\mathbb{R}^2$ and $\psi_\varepsilon(r) \to 1$ as $r \to \infty$.

Experimental studies of vortices



Absorption images of a BEC stirred with a laser beam of increasing frequency. From Madison et al., 2000.

Experimental studies of vortex precession

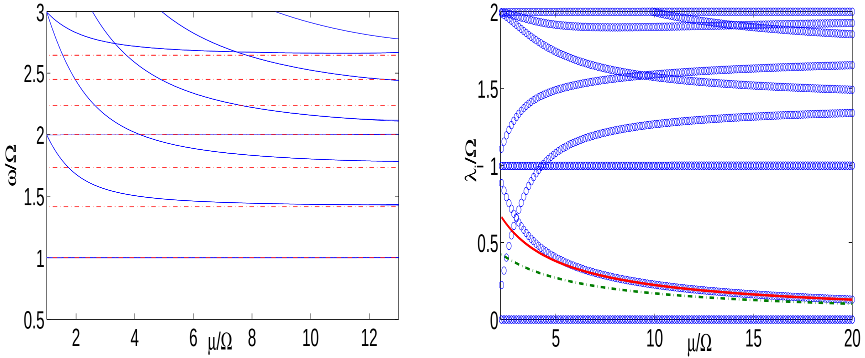


Vortex precession in a trapped two-component BEC.
From Anderson et al., 2000.

Theoretical studies of vortices

- Castin & Dum (1999) - rotating vortices can become local minimizers of energy for larger frequencies Ω
- Fetter & Svidzinsky (2001), Möttönen et al. (2005) - computations of effective energy for vortex configurations
- Aftalion & Du (2001), Ignat & Millot (2006) - variational proofs that a vortex of charge one is a global minimizer for larger frequencies
- Seiringer (2003) - proof that a multi-vortex configuration with charge m is energetically preferrable to that with charge $m-1$ for $\Omega > \Omega_m$, $m \in \mathbb{N}$
- Middlecamp et al. (2010), Kollar & Pego (2012) - numerical computations of eigenvalues for vortex configurations

Spectral stability of charge-one vortices



Left: ground state η_ε. Right: vortex of charge $m = 1$

D.P.–P.Kevrekidis, Nonlinearity **24**, 1271–1289 (2011)

Bifurcation of vortices

If $\Omega = 0$ (no rotation), spectrum of the vortex with charge $m = 1$ consists of neutrally stable eigenvalues with the lowest frequency ω_1. As $\varepsilon \to 0$, it is given asymptotically by

$$\omega_1 = 2\,\varepsilon\,|\log(\varepsilon)| + \mathcal{O}(\varepsilon).$$

The eigenmode corresponds to the periodic precession of the vortex around the origin $(0,0) \in \mathbb{R}^2$ with an infinitesimally small displacement from the origin.

At $\Omega = \omega_1$, a bifurcation occurs in the vortex solutions of the stationary equation:

$$\mu u = -\,\varepsilon^2(\partial_x^2 + \partial_y^2)u + (x^2 + y^2 + |u|^2)u + i\Omega(x\partial_y - y\partial_x)u.$$

In addition to the symmetric vortex u_ε of charge $m = 1$, which exists for every Ω and sufficiently large μ, there exists another asymmetric vortex of charge $m = 1$ displaced from the origin at a small but finite distance.

D.P.–P.Kevrekidis, AMRE **2013**, 127–164 (2013)

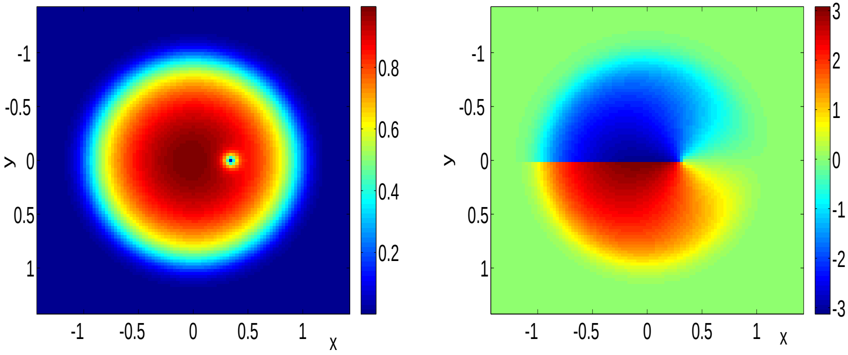
Main results from AMRE (2013)

Let $\varepsilon = 1$ and consider

$$\mu u = -(\partial_x^2 + \partial_y^2)u + (x^2 + y^2 + |u|^2)u + i\Omega(x\partial_y - y\partial_x)u.$$

- For every μ and Ω such that $\mu + \Omega > 4$, there exists a unique symmetric vortex $u = \psi(r)e^{i\theta}$:
  - It is a saddle point of the energy $E(u)$ for $\Omega < \omega_1$ with one double negative eigenvalue.
  - It is a local minimizer of energy for $\Omega > \omega_1$.
- For $\Omega > \omega_1$, there exists an additional asymmetric vortex with the center placed on the circle of radius $|a|$ centered at the origin $(0,0) \in \mathbb{R}^2$ at an arbitrary angle α such that $|a| \leq C\sqrt{\epsilon(\Omega - \omega_1)}$ for some $C > 0$.
  - The asymmetric vortex is a saddle point of the energy $E(u)$ with one simple negative eigenvalue.

Asymmetric vortex of charge one



Spatial contour plots of the amplitude (left) and phase (right) of the asymmetric charge-one vortex.

Stability theorem from AMRE (2013)

The energy space of the Gross–Pitaevskii equation

$$X = \left\{ u \in H^1(\mathbb{R}^2) : \; |x|u \in L^2(\mathbb{R}^2) \right\}.$$

Theorem

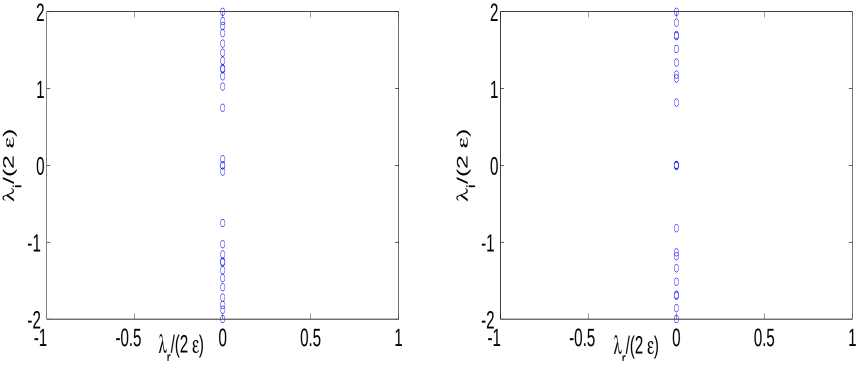
For $\Omega \gtrsim \omega_1$, the symmetric vortex of charge one is orbitally stable in the following sense: for any $\epsilon > 0$ there is a $\delta > 0$, such that if $\|u(x,0) - \psi(r)e^{i\theta}\|_X \leq \delta$, then

$$\inf_{\beta \in \mathbb{R}} \|e^{i\beta} u(x,t) - \psi(r) e^{i\theta}\|_X \leq \epsilon, \quad t \in \mathbb{R}_+,$$

For $\Omega \gtrsim \omega_1$, the asymmetric vortex is also orbitally stable in the following sense: for any $\epsilon > 0$ there is a $\delta > 0$, such that if $\|u(x,0) - u_\alpha(x)\|_X \leq \delta$, then

$$\inf_{(\alpha,\beta) \in \mathbb{R}^2} \|e^{i\beta} u(x,t) - u_\alpha(x)\|_X \leq \epsilon, \quad t \in \mathbb{R}_+.$$

Spectral stability of rotating charge-one vortices



Left: eigenvalues of the spectral stability problem for the symmetric vortex.
Right: eigenvalues of the spectral stability problem for the asymmetric vortex.

Spectrum of a harmonic oscillator

Consider the Schrödinger operator for the quantum harmonic oscillator

$$L := -\Delta_{(r,\theta)} + r^2 : \quad H^2(\mathbb{R}^2) \cap L^{2,2}(\mathbb{R}^2) \to L^2(\mathbb{R}^2),$$

The spectrum of L is purely discrete and equidistant. The eigenvectors are

$$f_{m,n}(r,\theta) = e_{m,n}(r) e^{im\theta}, \quad m \in \mathbb{Z}, \quad n \in \mathbb{N}_0$$

and the eigenvalues are

$$\lambda_{m,n} = 2(|m| + 2n + 1), \quad m \in \mathbb{Z}, \quad n \in \mathbb{N}_0.$$

Index m counts the angular momentum and index n counts zeros of $e_{m,n}$.

- Eigenvalue $\lambda = 2\ell$ with $\ell \in \mathbb{N}$ has multiplicity ℓ.
- For each $m \in \mathbb{Z}$, the spacing between eigenvalues is 4.

Primary branches of symmetric vortices

Consider the stationary Gross–Pitaevskii equation

$$\mu u = -\Delta_{(r,\theta)} u + r^2 u + |u|^2 u + i\Omega \partial_\theta u.$$

Radially symmetric vortex of charge $m_0 \in \mathbb{N}$ is

$$u(r,\theta) = e^{im_0\theta} \psi_{m_0}(r), \quad \omega = \mu + m_0 \Omega,$$

where (ψ_{m_0}, ω) is a root of the nonlinear operator

$$f(u,\omega) : \quad H_r^2(\mathbb{R}^+) \cap L_r^{2,2}(\mathbb{R}^+) \times \mathbb{R} \to L_r^2(\mathbb{R}^+), \tag{1}$$

given by $f(u,\omega) := -\Delta_{m_0} u + r^2 u + u^3 - \omega u$.

By local bifurcation theory (A. Contreras & C. Garcia–Azpeitia, 2016), there exists a unique smooth family:

$$\psi_{m_0}(r;a) = a e_{m_0,0}(r) + \mathcal{O}_{H_r^1}(a^3)$$

with

$$\omega_{m_0}(a) = 2(m_0+1) + a^2 \|e_{m_0,0}\|_{L_r^4}^4 + \mathcal{O}(a^4),$$

where a is the amplitude parameter.

Bifurcation results - analytical picture

C. Garcia–Azpeitia–D.P., arXiv:1701.01494 (2017):

(i) For $\Omega = 0$ and small a, the vortices are degenerate saddle points of the energy E with $2N(m_0)$ negative eigenvalues, a simple zero eigenvalue, and $2Z(m_0)$ small eigenvalues of order $\mathcal{O}(a^2)$, where

$$N(m_0) = \frac{1}{2} m_0 (m_0 + 1) \quad \text{and} \quad Z(m_0) = m_0.$$

(ii) $1 + B(m_0)$ global bifurcations occur when the parameter Ω is increased in the interval $[a^2 D_{m_0}, 2 - a^2 C_{m_0}]$ with some $C_{m_0}, D_{m_0} > 0$, where

$$B(m_0) = \frac{1}{2} m_0 (m_0 - 1).$$

The vortex family losses two negative eigenvalues past each bifurcation and has $2(m_0 - 1)$ negative eigenvalues for $\Omega \gtrsim 2 - a^2 C_{m_0}$.

(iii) A new smooth family of multi-vortex configurations is connected to the family of radially symmetric vortices on one side of each non-resonant bifurcation point (of the pitchfork type).

Bifurcation results - graphical picture

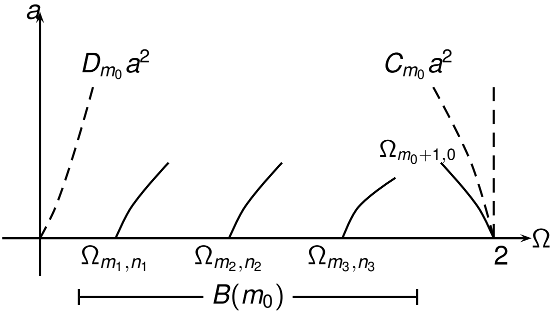


Figure: A schematic illustration of the bifurcation curves in the parameter plane (Ω, a), where a defines ω. The bifurcating solutions form surfaces parameterized by (Ω, a) close to the curves $\Omega_{m,n}$.

Energy of the symmetric vortex

The radially symmetric vortex $u = e^{im_0\theta}\psi_{m_0}(r)$ is a critical point of $E_\mu(u) := E(u) - \mu Q(u)$, where $Q(u) = \|u\|^2_{L^2}$. Then,

$$E_\mu(U + v) - E_\mu(U) = \langle \mathcal{H}\mathbf{v}, \mathbf{v}\rangle_{L^2} + \mathcal{O}(\|\mathbf{v}\|^3_{H^1\cap L^{2,1}}), \tag{2}$$

where $\mathbf{v} = [v, \bar{v}]^T$ and

$$\mathcal{H} = \begin{bmatrix} -\Delta_{(r,\theta)} + r^2 + i\Omega\partial_\theta - \mu + 2\psi^2_{m_0} & \psi^2_{m_0} e^{2im_0\theta} \\ \psi^2_{m_0} e^{-2im_0\theta} & -\Delta_{(r,\theta)} + r^2 - i\Omega\partial_\theta - \mu + 2\psi^2_{m_0} \end{bmatrix}.$$

By using the Fourier series

$$v = \sum_{m\in\mathbb{Z}} V_m e^{im\theta}, \quad \bar{v} = \sum_{m\in\mathbb{Z}} W_m e^{im\theta},$$

the operator $\mathcal{H}$ is diagonalized into blocks H_m, $m \in \mathbb{Z}$ given by

$$H_m(a, \Omega) = K_m(a) - \Omega(m - m_0)\sigma_3,$$

which act on $[V_m, W_{m-2m_0}]^T$ with $\omega = \mu + m_0\Omega = \omega_{m_0}(a)$.

Secondary branches of vortex configurations

In $H_m(a,\Omega) = K_m(a) - \Omega(m-m_0)\sigma_3$, we have

$$K_m(a) = \begin{bmatrix} -\Delta_m + r^2 - \omega_{m_0}(a) + 2\psi_{m_0}^2 & \psi_{m_0}^2 \\ \psi_{m_0}^2 & -\Delta_{m-2m_0} + r^2 - \omega_{m_0}(a) + 2\psi_{m_0}^2 \end{bmatrix}$$

For a given small a, we say that Ω is the *secondary bifurcation* point if $\mathrm{Ker}(K_m(a) - \Omega(m-m_0)\sigma_3)$ is nonempty.

The bifurcation problem in Ω coincides with the stability problem in the case of no rotation, where the eigenvalue of the stability problem is $\Omega(m-m_0)$.

The secondary branch of vortex configuration is given by $u = e^{im_0\theta}\psi_{m_0}(r) + v(r,\theta)$, where v is a root of the nonlinear operator

$$g(v;a,\Omega): \quad H^2(\mathbb{R}^2) \cap L^{2,2}(\mathbb{R}^2) \times \mathbb{R} \times \mathbb{R} \to L^2(\mathbb{R}^2),$$

given by

$$g(v;a,\Omega) = -\Delta_{(r,\theta)}v + r^2 v + i\Omega\left(\partial_\theta v - im_0 v\right) + 2\psi_{m_0}^2 v + e^{2im_0\theta}\psi_{m_0}^2 \bar{v} + e^{-im_0\theta}\psi_{m_0} v^2 + 2e^{im_0\theta}\psi_{m_0}|v|^2 + |v|^2 v - \omega_{m_0}(a)v.$$

Spectral information

In $H_m(a,\Omega) = K_m(a) - \Omega(m - m_0)\sigma_3$, we have

$$K_m(a) = \begin{bmatrix} -\Delta_m + r^2 - \omega_{m_0}(a) + 2\psi_{m_0}^2 & \psi_{m_0}^2 \\ \psi_{m_0}^2 & -\Delta_{m-2m_0} + r^2 - \omega_{m_0}(a) + 2\psi_{m_0}^2 \end{bmatrix}$$

- At $m = m_0$, $K_{m_0}(a) \geq 0$ with a simple zero eigenvalue due to gauge symmetry.
- Eigenvalues of $K_m(a)$ for $m > m_0$ are identical to those for $m < m_0$.
- For $a = 0$, there exist $N(m_0)$ negative and $Z(m_0)$ zero eigenvalues of $\{K_m(0)\}_{m>m_0}$ with

$$N(m_0) = \frac{m_0(m_0+1)}{2}, \quad Z(m_0) = m_0.$$

Example: $m_0 = 1$

$$\begin{cases} \sigma(K_2) = \{-2, 2, 2, 6, 6, \cdots\}, \\ \sigma(K_3) = \{0, 4, 4, 8, 8, \cdots\}, \\ \sigma(K_4) = \{2, 6, 6, 10, 10, \cdots\}, \\ \cdots \end{cases}$$

$m_0 = 2$

$$\begin{cases} \sigma(K_3) = \{-2, 2, 2, 6, 6, \cdots\}, \\ \sigma(K_4) = \{-4, 0, 4, 4, 8, 8, \cdots\}, \\ \sigma(K_5) = \{-2, 2, 6, 6, 10, 10, \cdots\}, \\ \sigma(K_6) = \{0, 4, 8, 8, 12, 12, \cdots\} \end{cases}$$

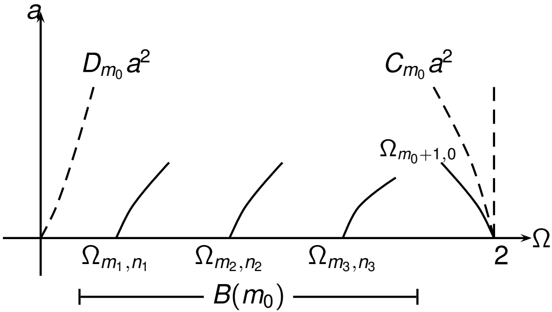
Count on the number of bifurcations

For $a = 0$, there exist $N(m_0)$ negative and $Z(m_0)$ zero eigenvalues of $\{K_m(0)\}_{m>m_0}$ with $N(m_0) = m_0(m_0 + 1)/2$ and $Z(m_0) = m_0$.

- The $Z(m_0)$ zero eigenvalues of $\{K_m(0)\}_{m>m_0}$ become positive eigenvalues of $\{K_m(a)\}_{m>m_0}$ for small $a \neq 0$.
- There exists $B(m_0)$ secondary bifurcations of $H_m(a, \Omega) = K_m(a) - \Omega(m - m_0)\sigma_3$ when Ω is increased from 0 to 2, where $B(m_0) = m_0(m_0 - 1)/2$.
- There is an additional secondary bifurcation near $\Omega = 2$.

After each bifurcation, the primary branch losses one pair of negative eigenvalues in $H_m(a, \Omega)$. After the last bifurcation, the primary branch has $N(m_0) - B(m_0) - 1 = m_0 - 1$ remaining pairs of negative eigenvalues in $H_m(a, \Omega)$.

Bifurcation results - graphical picture



Example: $m_0 = 1$

- The primary branch - the symmetric vortex of charge one:

$$u = e^{i\theta}\left[are^{-\frac{r^2}{2}} + \mathcal{O}(a^3)\right],$$

where $a > 0$ is a small parameter of the primary branch.

- $N(1) = 1$ and $B(1) = 0$ - no bifurcations for $\Omega \in (0,2)$

$$\begin{cases} \sigma(K_2) = \{-2, 2, 2, 6, 6, \cdots\}, \\ \sigma(K_3) = \{0, 4, 4, 8, 8, \cdots\}, \\ \sigma(K_4) = \{2, 6, 6, 10, 10, \cdots\}, \\ \cdots \end{cases}$$

- The secondary branch for K_2 - the asymmetric vortex of charge one near $\Omega = 2$:

$$u = are^{-\frac{r^2}{2}}e^{i\theta} + ab(1 + r^2e^{2i\theta})e^{-\frac{r^2}{2}} + \mathcal{O}(a^3, ab^2)$$

where $b = \mathcal{O}(a)$ is a small parameter of the secondary branch.

Example: $m_0 = 2$

- The primary branch - the symmetric vortex of charge two:

$$u = e^{2i\theta} \left[a r^2 e^{-\frac{r^2}{2}} + \mathcal{O}(a^3) \right],$$

where $a > 0$ is a small parameter of the primary branch.

- $N(1) = 3$ and $B(1) = 1$ - secondary bifurcation for $\Omega \approx 2/3$

$$\left\{ \begin{array}{l} \sigma(K_3) = \{-2, 2, 2, 6, 6, \cdots\}, \\ \sigma(K_4) = \{-4, 0, 4, 4, 8, 8, \cdots\}, \\ \sigma(K_5) = \{-2, 2, 6, 6, 10, 10, \cdots\}, \\ \sigma(K_6) = \{0, 4, 8, 8, 12, 12, \cdots\}, \\ \cdots \end{array} \right.$$

- The secondary branch for K_5 - the polygon of four vortices:

$$u = a r^2 e^{-\frac{r^2}{2}} e^{2i\theta} + b r e^{-i\theta} e^{-\frac{r^2}{2}} + \mathcal{O}(a^3, b^3)$$

where $b = \mathcal{O}(a)$ is a small parameter of the secondary branch.

Example: $m_0 = 2$

- $N(1) = 3$ and $B(1) = 1$ - additional secondary bifurcation for $\Omega \approx 2$

$$\left\{ \begin{array}{l} \sigma(K_3) = \{-2, 2, 2, 6, 6, \cdots\}, \\ \sigma(K_4) = \{-4, 0, 4, 4, 8, 8, \cdots\}, \\ \sigma(K_5) = \{-2, 2, 6, 6, 10, 10, \cdots\}, \\ \sigma(K_6) = \{0, 4, 8, 8, 12, 12, \cdots\}, \\ \cdots \end{array} \right.$$

- The secondary branch for K_3 - the pair of vortices:

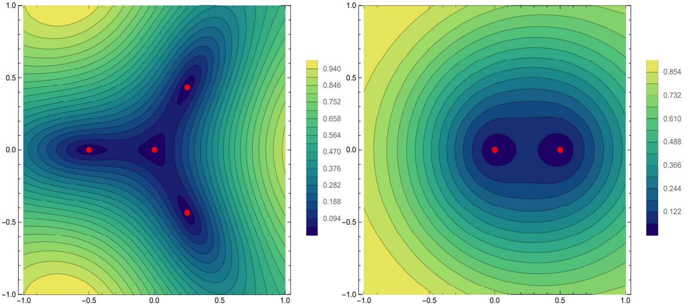
$$u = ar^2 e^{-\frac{r^2}{2}} e^{2i\theta} + ab(re^{i\theta} + r^3 e^{3i\theta})e^{-\frac{r^2}{2}} + \mathcal{O}(a^3, ab^2)$$

where $b = \mathcal{O}(a)$ is a small parameter of the secondary branch.

- The mode with $m = 4$ is unstable for the primary branch and hence does not bifurcate. The secondary branches inherit this instability for small a.

Example: $m_0 = 2$

Bifurcations near $\Omega = 2/3$ and near $\Omega = 2$



The method of analysis

Recall the bifurcation problem:

$$g(v;a,\Omega): \quad H^2(\mathbb{R}^2) \cap L^{2,2}(\mathbb{R}^2) \times \mathbb{R} \times \mathbb{R} \to L^2(\mathbb{R}^2),$$

given by

$$g(v;a,\Omega) = -\Delta_{(r,\theta)} v + r^2 v + i\Omega\left(\partial_\theta v - i m_0 v\right) + 2\psi_{m_0}^2 v + e^{2im_0\theta}\psi_{m_0}^2 \bar{v} \\ + e^{-im_0\theta}\psi_{m_0} v^2 + 2e^{im_0\theta}\psi_{m_0}|v|^2 + |v|^2 v - \omega_{m_0}(a) v.$$

The operator g is equivariant under the action of the group $O(2) = S^1 \cup \kappa S^1$ given by by

$$\rho(\varphi)v(r,\theta) = e^{-im_0\varphi} v(r,\theta+\varphi), \qquad \rho(\kappa)v(r,\theta) = \bar{v}(r,-\theta). \tag{3}$$

The subspace (V_m, W_{m-2m_0}) after the Fourier transform has as isotropy group D_{m-m_0} generated by the elements $\varphi = 2\pi/(m-m_0)$ and κ.

For a fixed value of $m \in \mathbb{Z}$, the action of $\rho(\varphi)$ is given by

$$\rho(\varphi)(V_j, W_{j-2m_0}) = \exp\left(2\pi i \frac{j - m_0}{m - m_0}\right)(V_j, W_{j-2m_0}), \quad j \in \mathbb{Z}.$$

The fixed point space

$$\mathrm{Fix}(D_{m-m_0}) = \{(v, w) \in L^2(\mathbb{R}^2) : \quad \rho(\gamma)(v, w) = (v, w) \text{ for } \gamma \in D_{m-m_0}\}$$

is composed of functions with real components (V_j, W_{j-2m_0}) such that $j - m_0$ is a multiple of $m - m_0$. If $(v, \bar{v}) \in \mathrm{Fix}(D_{m-m_0})$, then v can be characterized by

$$v(r, \theta) = \sum_{j \in m_0 + (m-m_0)\mathbb{Z}} V_j(r) e^{ij\theta} = e^{im_0\theta} \sum_{j \in (m-m_0)\mathbb{Z}} V_{m_0+j}(r) e^{ij\theta},$$

where all functions $\{V_j(r)\}_{j \in m_0 + (m-m_0)\mathbb{Z}}$ are real-valued.

In $\mathrm{Fix}(D_{m-m_0})$, the block H_{m_0} does not have zero eigenvalue and the double zero eigenvalues in the blocks H_{j-m_0} and H_{2m_0-j} become simple zero eigenvalues.

Schematic representation of the vortex bifurcations

Trivial solution	$O(2) \times O(2)$	$u = 0$
$\downarrow$	$\downarrow$	$\downarrow$
Vortex of charge m_0	$O(2)$	$u = e^{im_0\theta}\psi_{m_0}(r)$
$\downarrow$	$\downarrow$	$\downarrow$
$(m - m_0)$-polygons of vortices	D_{m-m_0}	$u = e^{im_0\theta}\psi_{m_0}(r) + v(r,\theta)$

More recent references

- Faou, Germain, & Hani (2016); Germain, Hani, & Thomann (2016) - resonant normal form for $\Omega = 0$
- Biasi, Bizon, Craps, & Evnin (2007) - Lowest Landau Level for $\Omega = 2$

The leading-order decomposition

$$\psi = \sum_{n=0}^{\infty} \alpha_n(t)\chi_n(z), \quad \chi_n(z) = z^n e^{-\frac{1}{2}|z|^2}, \quad z = x + iy$$

yields

$$i\dot{\psi} = \Pi(|\psi|^2\psi), \quad \Pi(\psi)(z') = e^{-\frac{1}{2}|z'|^2} \int_{\mathbb{C}} e^{\bar{z}z' - \frac{1}{2}|z|^2} \psi(z) dz.$$

The resonant normal form picks up the main bifurcation equations and truncates the unimportant terms.

Conclusion

- We have described bifurcation results for multi-vortex configurations in the small-amplitude limit.
- For $m_0 = 1$, symmetric vortices of charge one are local minimizers of energy and asymmetric vortices of charge one are saddle points of the energy. Nevertheless, both vortices are orbitally stable with respect to the time-dependent perturbations.
- For $m_0 = 2$, asymmetric pair of two vortices of charge one bifurcate but is unstable under the time-dependent perturbations.

Open questions:

- Can these results be extended in the semi-classical limit?
- Can we characterize minimizers of energy for two, three, and many vortices of charge one?

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