# Stability of breathers in nonlinear lattices

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# Stability of relative equilibria in Hamiltonian systems

Consider an abstract Hamiltonian dynamical system

$$\frac{du}{dt} = J\, H'(u), \quad u(t) \in X$$

where $X \subset L^2$ is a phase space, $J^+ = -J$ is a bounded invertible operator for the symplectic structure, and $H : X \to \mathbb{R}$ is the Hamilton function.

- Assume existence of the stationary state $u_0 \in X$ such that $H'(u_0) = 0$.
- Perform linearization $u(t) = u_0 + ve^{\lambda t}$, where $\lambda$ is the spectral parameter and $v \in X$ satisfies the spectral problem

  $$JH''(u_0)v = \lambda v,$$

  where $H''(u_0) : X \to L^2$ is a self-adjoint Hessian operator.

# Spectral stability

Consider the spectral problem:

$$JH''(u_0)v = \lambda v, \quad v \in X.$$

Assumptions:

- The spectrum of $H''(u_0)$ is positive except for finitely many negative and zero eigenvalues of finite multiplicity.
- The continuous wave spectrum of $JH''(u_0)$ is purely imaginary.
- Multiplicity of the zero eigenvalue of $JH''(u_0)$ is given by the number of parameters in $u_0$ (symmetries).

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

# Example: discrete NLS equation

Consider the discrete nonlinear Schrödinger equation in 1D,

$$i\frac{du_n}{dt} = (\Delta u)_n + 2|u_n|^2 u_n, \quad n \in \mathbb{Z}.$$

The stationary state (discrete soliton) is

$$u(t) = U_\omega e^{-i\omega t}, \quad \omega > 0, \quad U_\omega \in \ell^2(\mathbb{Z}).$$

- $U_\omega$ is a critical point of $H_\omega(u) = H(u) + \omega Q(u)$,

$$H(u) = \sum_{n\in\mathbb{Z}} |u_{n+1} - u_n|^2 - |u_n|^4, \quad Q(u) = \sum_{n\in\mathbb{Z}} |u_n|^2.$$

- The self-adjoint Hessian operator $H''_\omega(U_\omega)$ is given by

$$H''_\omega(U_\omega) = \left[\begin{array}{cc} -\Delta + \omega - 4|U_\omega|^2 & -2U_\omega^2 \\ -2\overline{U_\omega}^2 & -\Delta + \omega - 4|U_\omega|^2 \end{array}\right].$$

- $J = \mathrm{diag}(i, -i)$ is a bounded invertible symplectic operator.

# Main question

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

For a gradient system:

$$\frac{du}{dt} = -F'(u) \quad \Rightarrow \quad \lambda v = -F''(u_0)v,$$

**Theorem**

*The number of unstable eigenvalues of $-F''(u_0)$ equals the number of negative eigenvalues of $F''(u_0)$.*

The relation is less straightforward in a Hamiltonian system

$$\lambda v = JH''(u_0)v, \quad v \in X.$$

Quadruple Symmetry: If $\lambda$ is an eigenvalue, so is $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$.

# Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2)$$

The quadratic form for $H$ has four positive eigenvalues.

The two oscillators are stable:

$$\left\{ \begin{array}{l} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{y}_2 = -\omega_2^2 x_2, \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} \ddot{x}_1 + \omega_1^2 x_1 = 0, \\ \ddot{x}_2 + \omega_2^2 x_2 = 0. \end{array} \right.$$

# Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 - \lambda_2^2 x_2^2)$$

The quadratic form for $H$ has three positive and one negative eigenvalues.

One of the two oscillators is unstable:

$$\left\{ \begin{array}{l} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{y}_2 = \lambda_2^2 x_2, \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} \ddot{x}_1 + \omega_1^2 x_1 = 0, \\ \ddot{x}_2 - \lambda_2^2 x_2 = 0. \end{array} \right.$$

Negative index count:

$$N_{\mathrm{re}}(JH) = 1 = N_{\mathrm{neg}}(H)$$

# Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(-\lambda_1^2 x_1^2 - \lambda_2^2 x_2^2)$$

The quadratic form for $H$ has two positive and two negative eigenvalues.

Both oscillators are unstable:

$$\left\{ \begin{array}{l} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = \lambda_1^2 x_1, \\ \dot{y}_2 = \lambda_2^2 x_2, \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} \ddot{x}_1 - \lambda_1^2 x_1 = 0, \\ \ddot{x}_2 - \lambda_2^2 x_2 = 0. \end{array} \right.$$

Negative index count:

$$N_{\rm re}(JH) = 2 = N_{\rm neg}(H)$$

# Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 - y_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 - \omega_2^2 x_2^2)$$

The quadratic form for $H$ has two positive and two negative eigenvalues.

The two oscillators are nevertheless stable:

$$\left\{ \begin{array}{l} \dot{x}_1 = y_1, \\ \dot{x}_2 = -y_2, \\ \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{y}_2 = \omega_2^2 x_2, \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} \ddot{x}_1 + \omega_1^2 x_1 = 0, \\ \ddot{x}_2 + \omega_2^2 x_2 = 0. \end{array} \right.$$

Negative index count:

$$2N_{\rm im}^-(JH) = 2 = N_{\rm neg}(H)$$

# Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 - y_2^2) + \omega^2 x_1 x_2$$

The quadratic form for $H$ has two positive and two negative eigenvalues.

The two oscillators are unstable with a quartet of complex eigenvalues:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = -y_2, \\ \dot{y}_1 = -\omega^2 x_2, \\ \dot{y}_2 = -\omega^2 x_1, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 + \omega^2 x_2 = 0, \\ \ddot{x}_2 - \omega^2 x_1 = 0, \end{cases} \quad \Rightarrow \quad x_1^{(4)} + \omega^4 x_1 = 0.$$

Negative index count:

$$2N_{\mathrm{c}}(JH) = 2 = N_{\mathrm{neg}}(H)$$

# Spectral stability theorems

Consider the spectral stability problem:

$$JH''(u_0)v = \lambda v, \quad v \in X.$$

For simplicity, assume a zero-dimensional kernel of $H''(u_0)$.

- **Grillakis, Shatah, Strauss, 1990** Orbital Stability Theory:
  - If $H''(u_0)$ has no negative eigenvalues, then $JH''(u_0)$ has no unstable eigenvalues and $u_0$ is nonlinearly stable.
  - If $H''(u_0)$ has an odd number of negative eigenvalues, then $JH''(u_0)$ has at least one real unstable eigenvalue.

- **Kapitula, Kevrekidis, Sandstede, 2004**; **Pelinovsky, 2005** Negative Index Theory:

$$N_{\rm re}(JH''(u_0)) + 2N_{\rm c}(JH''(u_0)) + 2N_{\rm im}^{-}(JH''(u_0)) = N_{\rm neg}(H''(u_0)).$$

# What is Krein signature for eigenvalues?

- Suppose that $\lambda \in i\mathbb{R}$ is a simple isolated eigenvalue of $JH''(u_0)$ with the eigenvector $v$. Then, the sign of

$$E''(v) = \langle H''(u_0)v, v \rangle_{\ell^2}$$

  is called the Krein signature of the eigenvalue $\lambda$.

- If $\lambda$ is a multiple isolated eigenvalue of $JH''(u_0)$, then the number $N^-_{\rm im}(JH''(u_0))$ is introduced as the number of negative eigenvalues of the quadratic form $E''(v)$ restricted at the invariant subspace of $JH''(u_0)$ associated with the eigenvalue $\lambda$.

# What if $H''(u_0)$ has a kernel?

In the dNLS example

$$i\frac{du_n}{dt} = (\Delta u)_n + 2|u_n|^2 u_n, \quad n \in \mathbb{Z}.$$

with the discrete soliton

$$u(t) = U_\omega e^{-i\omega t}, \quad \omega > 0, \quad U_\omega \in \ell^2(\mathbb{Z}),$$

the kernel is one-dimensional:

$$H''_\omega(U_\omega) \left[ \begin{array}{c} U_\omega \\ -\overline{U_\omega} \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right],$$

where $H_\omega(u) = H(u) + \omega Q(u)$.

- Let $d(\omega) := H_\omega(U_\omega)$, then $d'(\omega) = Q(U_\omega) = \|U_\omega\|^2_{\ell^2}$.
  If $d''(\omega) = \frac{d}{d\omega}Q(U_\omega) > 0$, the negative index theory applies with

  $$N_{\rm neg}(H''_\omega(U_\omega)) \to N_{\rm neg}(H''_\omega(U_\omega)) - 1.$$

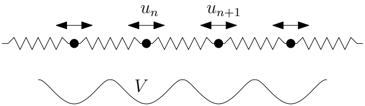
  The soliton is nonlinearly stable if $d''(\omega) > 0$ and $N_{\rm neg}(H''_\omega(U_\omega)) = 1$.

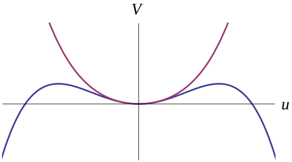
# Klein-Gordon lattice

Klein-Gordon (KG) lattice models a chain of coupled anharmonic oscillators with nearest-neighbour interactions

$$\frac{d^2 u_n}{dt^2} + V'(u_n) = \epsilon(u_{n+1} - 2u_n + u_{n-1}),$$

where $\{u_n(t)\}_{n\in\mathbb{Z}} : \mathbb{R} \to \mathbb{R}^{\mathbb{Z}}$, $\epsilon$ is the coupling constant, and $V : \mathbb{R} \to \mathbb{R}$ is an on-site potential, e.g.




Applications:

- dislocations in crystals (e.g. Frenkel & Kontorova '1938)
- oscillations in biological molecules (e.g. Peyrard & Bishop '1989)

# Relation to the discrete nonlinear Schrödinger equation

Discrete nonlinear Schrodinger equation (dNLS) corresponds to the small-amplitude weakly coupled limit of the KG lattice with $V'(u) = u \pm u^3$:

$$2i \frac{da_n}{d\tau} \pm 3|a_n|^2 a = a_{n+1} - 2a_n + a_{n-1},$$

where $\{a_n(\tau)\}_{n\in\mathbb{Z}} : \mathbb{R} \to \mathbb{C}^{\mathbb{Z}}$ and $\tau$ is new time variable.

By using the leading-order approximation

$$U_j(t) = \epsilon^{1/2} \left[ a_j(\epsilon t) e^{it} + \bar{a}_j(\epsilon t) e^{-it} \right],$$

in dKG, one can obtain dNLS and estimate the residual terms

$$\mathrm{Res}_j(t) := \pm\epsilon^{3/2} \left( a_j^3 e^{3it} + \bar{a}_j^3 e^{-3it} \right) + \epsilon^{5/2} \left( \ddot{a}_j e^{it} + \ddot{\bar{a}}_j e^{-it} \right),$$

For every $|t| \leq \tau_0 \epsilon^{-1}$, there is $C > 0$ such that

$$\|\mathbf{u}(t) - \mathbf{U}(t)\|_{l^2} + \|\dot{\mathbf{u}}(t) - \dot{\mathbf{U}}(t)\|_{l^2} \leq C\epsilon^{3/2}.$$

D.P., T. Penati, S. Paleari, Rev. Math. Phys. (2016), in press.

## Relation to the anti-continuum limit

In the anti-continuum limit ($\epsilon = 0$), each oscillator is governed by

$$\ddot{\varphi} + V'(\varphi) = 0, \quad \Rightarrow \quad \frac{1}{2}\dot{\varphi}^2 + V(\varphi) = E,$$

where $\varphi \in H^2_{per}(0, T)$.

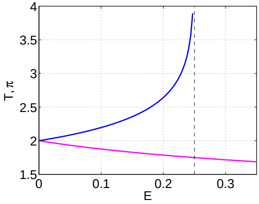



Figure: Period vs. energy in hard (magenta) and soft (blue) potential $V(u) = \frac{1}{2}u^2 \pm \frac{1}{4}u^4$.

The period of the oscillator is

$$T(E) = \sqrt{2}\int_{-a(E)}^{a(E)} \frac{dx}{\sqrt{E - V(x)}},$$

where $a(E)$, the amplitude, is the smallest root of $V(a) = E$.

# Multi-breathers near the anti-continuum limit

Breathers are spatially localized time-periodic solutions to the Klein-Gordon lattice. Multi-breathers are constructed by parameter continuation in $\epsilon$ from the limiting configuration:

$$\mathbf{u}^{(0)}(t) = \sum_{k \in S} \sigma_k \varphi(t) \mathbf{e}_k \quad \in \quad H^2_{per}((0,T); l^2(\mathbb{Z})),$$

where $S \subset \mathbb{Z}$ is a finite set of excited sites and $\mathbf{e}_k$ is the unit vector in $l^2(\mathbb{Z})$ at the node $k$. The oscillators are in-phase if $\sigma_k = +1$ and anti-phase if $\sigma_k = -1$.

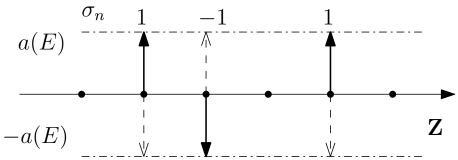



Figure: An example of a multi-site discrete breather at $\epsilon = 0$.

R. MacKay & S. Aubry, 1994; D. Bambusi, 2013.

# Spectral stability of breathers in the anti-continuum limit

- Archilla, Cuevas, Sánchez-Rey, Alvarez '2003
- Koukouloyannis, Kevrekidis '2009
- Pelinovsky, Sakovich '2012
- Youshimura '2012

Short summary of stability results near the anti-continuum limit:

- Single-site breather - spectrally stable
- Two-site breathers at two adjacent sites:
  - spectrally unstable if in-phase (soft) or anti-phase (hard)
  - spectrally stable if anti-phase (soft) or in-phase (hard)

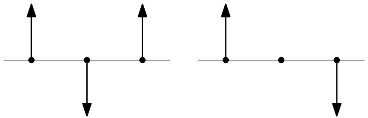
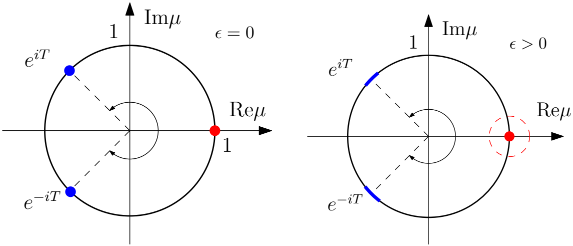


Figure: Stable configuration in soft potential: $T'(E) > 0$.

# Spectral stability via Floquet multipliers

For $\epsilon > 0$, Floquet multipliers split as follows:




One-site breathers have a double Floquet multiplier at $\mu = 1$.
Question: Do they remain stable far from the anti-continuum limit?

Two-site breathers have one split pair of multipliers:

- the pair is on the unit circle if the breathers are spectrally stable
- the pair is on the real line if the breathers are unstable

Question: Are spectrally stable two-site breathers also nonlinearly stable?

# Energy stability criterion for breathers

The KG lattice

$$\frac{d^2 u_n}{dt^2} + V'(u_n) = \epsilon(u_{n+1} - 2u_n + u_{n-1})$$

has the conserved energy

$$H = \sum_{n \in \mathbb{Z}} \frac{1}{2} \left( \frac{du_n}{dt} \right)^2 + V(u_n) + \frac{1}{2}\epsilon(u_{n+1} - u_n)^2.$$

Breathers (time-periodic solutions) are NOT relative equilibria of the energy function $H$. They can be written in the normalized form:

$$u(t) = U(\tau), \quad \tau = \omega t, \quad U(\tau + 2\pi) = U(\tau),$$

where $\omega = 2\pi/T$ is breather frequency and $U(\tau) \in H^2_{\rm per}((0, 2\pi), \ell^2(\mathbb{Z}))$.

Breathers with increasing (decreasing) energy-frequency dependence are generically unstable in soft (hard) nonlinear potentials.

P.G. Kevrekidis, J. Cuevas, D.P., Phys. Rev. Lett. **117**, 094101 (2016).

# A simple argument for energy stability criterion

Normalized breather profile $u(t) = U(\tau) \in H^2_{\rm per}((0,2\pi),\ell^2(\mathbb{Z}))$ satisfies

$$\omega^2 U_n''(\tau) + V'(U_n(\tau)) = \epsilon(\Delta U)_n(\tau), \quad n \in \mathbb{Z}.$$

Linearized equations for small perturbations are given by

$$\ddot{w}_n + V''(u_n) w_n = \epsilon(\Delta w)_n, \quad n \in \mathbb{Z}. \tag{1}$$

With Floquet theory,

$$w(t) = W(\tau)e^{\lambda t}, \quad \tau = \omega t, \quad W(\tau + 2\pi) = W(\tau),$$

we obtain the spectral stability problem

$$(LW)(\tau) = 2\lambda\omega W'(\tau) + \lambda^2 W(\tau),$$

where $L = \epsilon\Delta - V''(U(\tau)) - \omega^2 \partial_\tau^2$ acts on $H^2_{\rm per}((0,2\pi),\ell^2(\mathbb{Z}))$.

# A simple argument for energy stability criterion

Spectral stability problem is

$$(LW)(\tau) = 2\lambda\omega W'(\tau) + \lambda^2 W(\tau).$$

$\lambda = 0$ is at least a double eigenvalue because of the translational invariance:

$$LU'(\tau) = 0, \quad L\partial_\omega U(\tau) = 2\omega U''(\tau).$$

$\lambda = 0$ is at least a quadruple eigenvalue if $TH'(\omega) = 0$.

**Assumptions:**

- The spectral bands of the spectral stability problem are bounded away from $\lambda = 0$,
- The kernel of $L$ is exactly one-dimensional with the eigenvector $W(\tau) = U'(\tau)$.
- The energy $H$ of the breather $U$ is a $C^1$ function of frequency $\omega$.

# Energy stability criterion in the anti-continuum limit

In the anti-continuum limit ($\epsilon = 0$), each oscillator is governed by

$$\ddot{\varphi} + V'(\varphi) = 0, \quad \Rightarrow \quad \frac{1}{2}\dot{\varphi}^2 + V(\varphi) = E,$$

where $\varphi \in H^2_{per}(0, T)$.

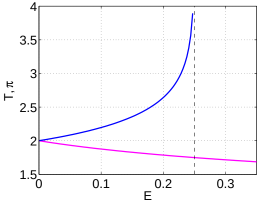



Figure: Period vs. energy in hard (magenta) and soft (blue) potential $V(u) = \frac{1}{2}u^2 \pm \frac{1}{4}u^4$.

Since $|T'(E)| < \infty$ in

$$H'(\omega) = -\frac{T}{\omega T'(E)},$$

the stability threshold $H'(\omega) = 0$ cannot be achieved.

Oscillators are always stable with $H'(\omega) > 0$ for hard potentials and $H'(\omega) < 0$ for soft potentials.

# Further arguments for energy stability criterion

Expanding in powers of $\lambda$:

$$W(\tau) = U'(\tau) + \lambda \partial_\omega U(\tau) + \lambda^2 Y(\tau) + \lambda^3 Z(\tau) + \mathcal{O}(\lambda^4)$$

and using Fredholm conditions yields the dispersion relation

$$0 = \lambda^2 T H'(\omega) + \lambda^4 M(\omega) + \mathcal{O}(\lambda^6),$$

where $M(\omega)$ is computed in terms of $U$ and $Y$.

The sign of $M(\omega)$ is not generally defined...

However, in the dNLS approximation limit, we can show that
$M(\omega) > 0$ for hard potentials [breathers are stable for $H'(\omega) > 0$];
$M(\omega) < 0$ for soft potentials [breathers are stable for $H'(\omega) < 0$].

# Energy stability criterion in the dNLS approximation

Consider the KG lattice

$$\frac{d^2 u_n}{dt^2} + u_n \pm \epsilon u_n^{1+2p} = \epsilon(u_{n+1} - 2u_n + u_{n-1}), \quad n \in \mathbb{Z}.$$

By using the leading-order approximation (P., Penati, Paleari, 2016),

$$U_n(\tau) = A_n e^{i\tau} + \bar{A}_n e^{-i\tau} + \mathcal{O}(\epsilon),$$

one can derive and justify the stationary dNLS equation

$$(\Delta A)_n = \epsilon^{-1}(1-\omega^2)A_n \pm \gamma |A_n|^{2p} A_n, \quad \gamma = \frac{(2p+1)!}{p!(p+1)!}.$$

# Energy stability criterion in the dNLS approximation

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Breathers exist for hard potentials if $\omega^2 > 1 + 4\epsilon$ and for soft potentials if $\omega^2 < 1$. Hence, we can introduce $\Omega > 0$ in either $\omega^2 = 1 + 4\epsilon + \epsilon\Omega$ or $\omega^2 = 1 - \epsilon\Omega$. Then, $A \in \ell^2(\mathbb{Z})$ depends on $\Omega$ and is independent of $\epsilon$.

$$H(\omega) = 2Q(\Omega) + \mathcal{O}(\epsilon).$$

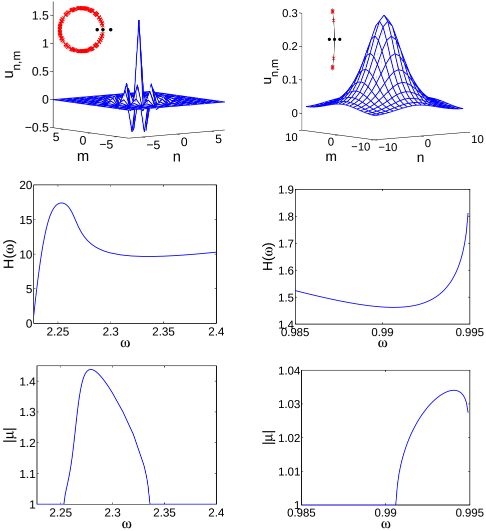
The energy stability criterion becomes the slope condition:

$$H'(\omega) = \pm 4\omega\epsilon^{-1} Q'(\Omega) + \mathcal{O}(1), \quad Q(\Omega) = \|A\|^2_{\ell^2}.$$

# Numerical illustration: 2D KG lattice.

Left - hard $\phi^4$ potential with $\epsilon = 0.5$.

Right - soft Morse potential with $\epsilon = 0.2$.



# Energy stability criterion for FPU lattices?

The FPU lattice

$$\frac{d^2 u_n}{dt^2} = W'(u_{n+1} - u_n) - W'(u_n - u_{n-1}), \quad n \in \mathbb{Z},$$

has the conserved energy

$$H = \sum_{n \in \mathbb{Z}} \frac{1}{2} \left( \frac{du_n}{dt} \right)^2 + W(u_{n+1} - u_n).$$

In the strain variables $r_n = u_{n+1} - u_n$, the FPU lattice can be rewritten as

$$\frac{d^2 r_n}{dt^2} = W'(r_{n+1}) - 2W'(r_n) + W'(r_{n-1}), \quad n \in \mathbb{Z},$$

and the normalized breather profile $r_n(t) = R_n(\tau) \in H^2_{\rm per}((0,2\pi), \ell^2(\mathbb{Z}))$.

The derivations and conclusions apply verbatim... In monoatomic chains, the dNLS approximation is valid at the maximal optical frequency and leads to breathers in hard potentials (G.James, 2003).

# Nonlinear instability of breathers

Consider the discrete KG equation

$$\frac{d^2 u_n}{dt^2} + V'(u_n) = \varepsilon(u_{n+1} - 2u_n + u_{n-1}), \quad n \in \mathbb{Z},$$

where $V$ is smooth and $V = \frac{1}{2}u^2 + \mathcal{O}(u^3)$.

Assumptions:

- The double eigenvalue $\lambda = 0$ is isolated from the spectral bands.
- There exists a pair of eigenvalues at $\lambda = \pm i\Omega$ isolated from the spectral bands.
- The double eigenvalue $\lambda = \pm 2i\Omega$ belongs to the spectral bands with nonzero Fermi golden rule.

If Krein signature of eigenvalues at $\lambda = \pm i\Omega$ is opposite to that of the spectral bands, the breather is spectrally stable but nonlinearly unstable.

P.G. Kevrekidis, D.P., A. Saxena, Phys. Rev. Lett. **114** (2015), 214101.
J. Cuevas, P.G. Kevrekidis, D.P., Stud. Appl. Math. **137** (2016), 214.

# Krein quantity

Linearized equations for small perturbations are given by

$$\ddot{w}_n + V''(u_n) w_n = \epsilon (\Delta w)_n, \quad n \in \mathbb{Z}. \tag{2}$$

With Floquet theory,

$$w(t) = W(t) e^{\lambda t}, \quad W(t+T) = W(\tau),$$

we obtain the spectral stability problem

$$\ddot{W}_n + 2\lambda \dot{W}_n + \lambda^2 W_n + V''(u_n) W_n = \epsilon (\Delta w)_n, \quad n \in \mathbb{Z}.$$

The symplectic structure is given by

$$\frac{dw_n}{dt} = \frac{\partial H}{\partial p_n}, \quad \frac{dp_n}{dt} = -\frac{\partial H}{\partial w_n}, \quad n \in \mathbb{Z}$$

The Krein quantity $K$ is real and constant in time $t$:

$$K = i \sum_{n \in \mathbb{Z}} (\bar{p}_n w_n - p_n \bar{w}_n) = 2\Omega \sum_{n \in \mathbb{Z}} |W_n|^2 + i \sum_{n \in \mathbb{Z}} \left( \dot{\bar{W}}_n W_n - \dot{W}_n \bar{W}_n \right).$$

# Krein quantity for two-site breathers

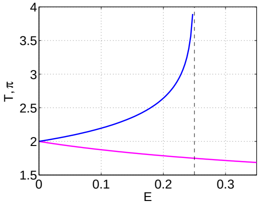
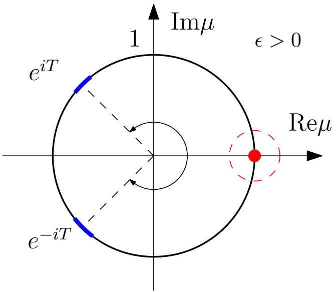


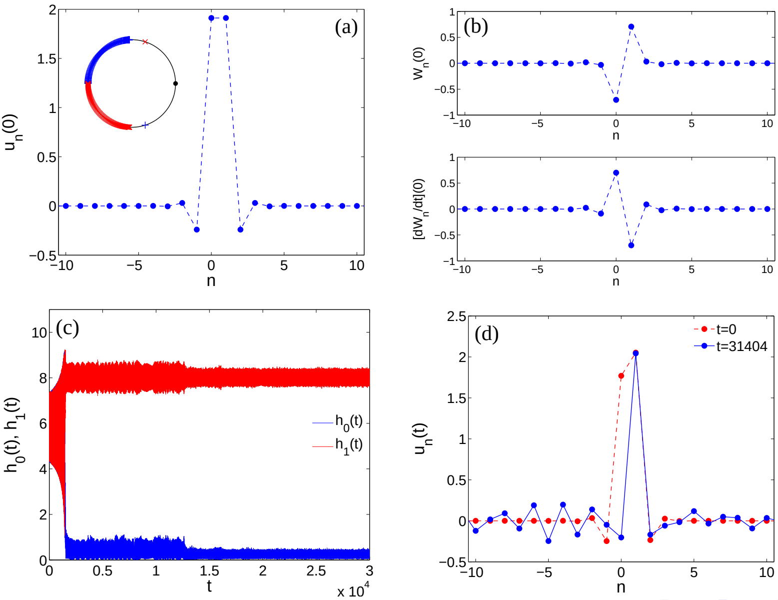

Figure: Period vs. energy in hard (magenta) and soft (blue) potential $V(u) = \frac{1}{2}u^2 \pm \frac{1}{4}u^4$.



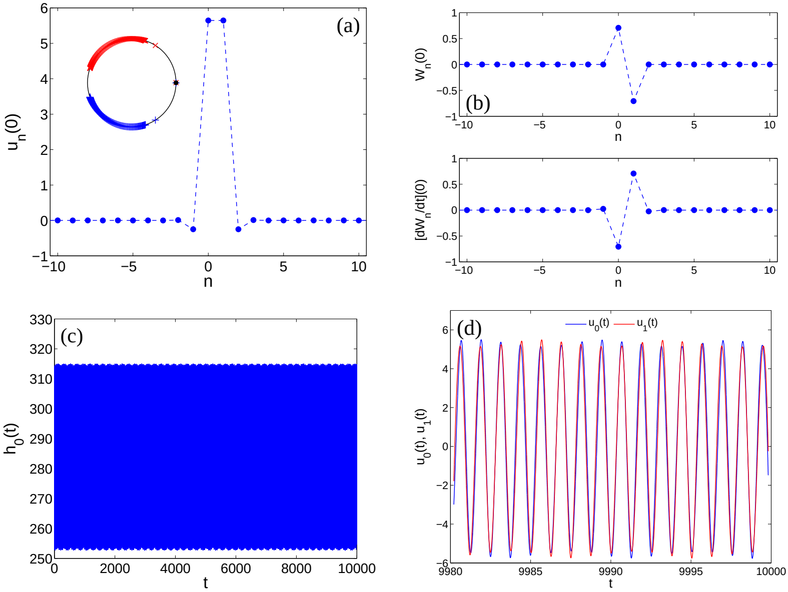

For the hard potential with $T'(E) < 0$ and $T(E) < 2\pi$,

- $0 < T < \pi$: the Krein signatures of the internal mode and the wave spectrum in the upper semi-circle coincide;
- $\pi \leq T < 2\pi$: the Krein signatures of the internal mode and the wave spectrum in the upper semi-circle are opposite to each other.

# Numerical illustration: hard $\phi^4$ potential $T = \pi$



# Numerical illustration: hard $\phi^4$ potential $T < \pi$



# Numerical illustration in 1D

The dNLS equation

$$i\dot{u}_n + \epsilon(u_{n+1} - 2u_n + u_{n-1}) + |u_n|^2 u_n = 0, \quad n \in \mathbb{Z}.$$

For $\epsilon = 0.07$ and $\omega = 1$, we have $\Omega \approx 0.598$, so that $\Omega < \omega$ but $2\Omega > \omega$.

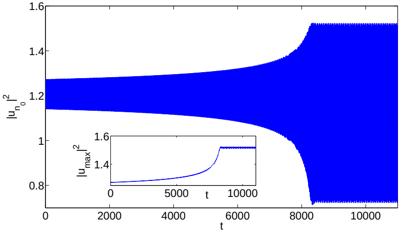



Figure: Evolution of a two-site discrete soliton in 1D dNLS.

Recall the Negative Index Theory:

$$N_{\rm re}(JH''(u_0)) + 2N_{\rm c}(JH''(u_0)) + 2N_{\rm im}^{-}(JH''(u_0)) = N_{\rm neg}(H''(u_0)) = 2$$

# A simple argument for nonlinear instability

Using the asymptotic multi-scale expansion for solutions to the KG lattice,

$$u(t) = U(t) + \delta\left(c(\tau)W(t)e^{i\Omega t} + \bar{c}(\tau)\bar{W}(t)e^{-i\Omega t}\right) + \delta^2 Y(t) + \mathcal{O}(\delta^3),$$

yields

- the breather $U(t+T) = U(t)$,
- the Floquet mode $W(t+T) = W(t)$ for eigenvalues $\lambda = \pm i\Omega$,
- the slowly varying envelope $c(\tau)$, $\tau = \delta^2 t$,
- the correction terms at $\mathcal{O}(\delta^2)$,

$$Y(t) = c(\tau)^2 P(t) e^{2i\Omega t} + |c(\tau)|^2 Q(t) + \bar{c}(\tau)^2 \bar{P}(t) e^{-2i\Omega t},$$

where $P(t) \in H^2_{\rm per}((0,T), \ell^\infty(\mathbb{Z}))$ and $Q(t) \in H^2_{\rm per}((0,T), \ell^2(\mathbb{Z}))$ from the assumptions of the theory.

The correction term $P(t)$ satisfies Sommerfeld radiation boundary conditions at infinity due to coupling with the spectral bands.

# A simple argument for nonlinear instability

Removing secular terms at $\mathcal{O}(\delta^3)$ yields the amplitude equation

$$iK\frac{dc}{d\tau} + \beta|c|^2 c = 0,$$

where $K \in \mathbb{R}$ is the Krein quantity of the eigenvalue $\lambda = i\Omega$ and $\mathrm{Im}(\beta)$ encodes Sommerfeld conditions. By the Fermi Golden Rule, $\mathrm{Im}(\beta) \neq 0$.

For the hard potential with $T'(E) < 0$ and $T(E) < 2\pi$,

- $K > 0$ for eigenvalue $\lambda = i\Omega$;
- $\mathrm{Im}(\beta) > 0$ if $0 < T < \pi$ and $\mathrm{Im}(\beta) < 0$ if $\pi \leq T < 2\pi$.

If $\mathrm{sign}(K) = -\mathrm{sign}(\mathrm{Im}(\beta))$, then $|c|^2$ grows in $\tau$,

$$K\frac{d|c|^2}{d\tau} = -2\mathrm{Im}(\beta)|c|^4,$$

hence, the breather is nonlinearly unstable.

For NLS-type models, see S. Cuccagna (2009); M. Maeda (2014).

# Conclusions

- Spectral stability theory is well-developed for relative equilibria in Hamiltonian systems.
- Negative eigenvalues of the quadratic Hamiltonian show up in the spectral stability problem either as unstable eigenvalues or as stable eigenvalues of negative Krein signature.
- If no negative eigenvalues exist, nonlinear stability holds by Lyapunov method. In the presence of negative eigenvalues, nonlinear instability may destroy stationary states in spite of their spectral stability.

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- Negative eigenvalues of the quadratic Hamiltonian show up in the spectral stability problem either as unstable eigenvalues or as stable eigenvalues of negative Krein signature.
- If no negative eigenvalues exist, nonlinear stability holds by Lyapunov method. In the presence of negative eigenvalues, nonlinear instability may destroy stationary states in spite of their spectral stability.
- Breathers are not relative equilibria of the Hamiltonian system. The generalization of the above results to breathers is not trivial.
- Energy stability criterion is presented for breathers for the first time.
- We have also shown that spectrally stable multi-site breathers may be either nonlinearly stable or unstable, depending on their period $T$.