Hamiltonian PT-symmetric chains of coupled pendula

Alexander Chernyavsky and Dmitry Pelinovsky

Department of Mathematics, McMaster University, Ontario, Canada

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PT-symmetric quantum mechanics

Consider the evolution problem

$$i\frac{du}{dt} = Hu, \quad u(t,\cdot) \in L^2, \quad t \in \mathbb{R},$$

where H is a linear operator with a domain in L^2. If H is Hermitian, then $\sigma(H) \subset \mathbb{R}$ and e^{-itH} is unitary on L^2.

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Let us now assume that H is not Hermitian but PT-symmetric, where

- P stands for parity transformation
- T stands for time reversion and complex conjugation,

$$P^2 u(t) = u(t), \quad Tu(t) = \bar{u}(-t).$$

Therefore, operators H and PT commute:

$$PTH = HPT.$$

[C.M. Bender, 2007]

Properties of PT-symmetric systems

If $u(t)$ is a solution of the evolution equation, then

$$v(t) = PTu(t) = P\bar{u}(-t)$$

is also a solution of the same system

$$iv'(t) = Hv \;\Rightarrow\; -iP\bar{u}'(-t) = HP\bar{u}(-t) \;\Rightarrow\; iu'(t) = Hu.$$

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If E is an eigenvalue and U is an eigenfunction, then $\bar{E}$ is also an eigenvalue with the eigenfunction $P\bar{U}$, because

$$u(t) = Ue^{-iEt} \quad \Rightarrow \quad v(t) = P\bar{U}e^{-i\bar{E}t}.$$

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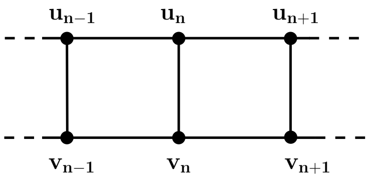
Bender's Conjecture: For many physically relevant PT-symmetric operators H, all eigenvalues are real and all eigenfunctions are PT-symmetric.

Examples of a PT-symmetric lattice

Dimer lattices in nonlinear optics (coupled waveguides):

$$\left\{ \begin{array}{l} i\dot{u}_n + v_n = \epsilon(u_{n+1} - 2u_n + u_{n-1}) + i\gamma u_n + |u_n|^2 u_n, \\ i\dot{v}_n + u_n = \epsilon(v_{n+1} - 2v_n + v_{n-1}) - i\gamma v_n + |v_n|^2 v_n, \end{array} \right.$$

where $\gamma > 0$ is the gain-damping parameter and $\epsilon > 0$ is lattice coupling.



The PT symmetry is

$$P \left[\begin{array}{c} u \\ v \end{array} \right] = \left[\begin{array}{c} v \\ u \end{array} \right], \qquad T \left[\begin{array}{c} u(t) \\ v(t) \end{array} \right] = \left[\begin{array}{c} \bar{u}(-t) \\ \bar{v}(-t) \end{array} \right].$$

Relevant questions

For a single site (say, $\epsilon = 0$), the coupled system is referred to as a dimer.
Linear stability analysis yields that the dimer is stable if $\gamma \in (0, 1)$.
Therefore, the linear system for $\gamma \in (0, 1)$ satisfies Bender's conjecture.
The threshold $\gamma = 1$ is referred to as the PT phase transition.

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Relevant questions:

1. Do the solutions stay bounded for all times if $\gamma \in (0, 1)$?
2. Do there exist linearly stable localized modes on the lattice for $\gamma \in (0, 1)$?
3. Are linearly stable localized modes also stable in the nonlinear dynamics of the lattice?

Unfortunately, many PT-symmetric systems are typically non-Hamiltonian.

Hamiltonian PT-symmetric dimer

A Hamiltonian example of a PT-symmetric dimer is

$$\left\{ \begin{array}{l} i\dot{u}_n + v_n = i\gamma u_n + (|u_n|^2 + 2|v_n|^2)u_n + v_n^2 \bar{u}_n, \\ i\dot{v}_n + u_n = -i\gamma v_n + (2|u_n|^2 + |v_n|^2)v_n + u_n^2 \bar{v}_n. \end{array} \right.$$

where $\gamma > 0$ is the gain-damping parameter and $n = 0$ (a single site).

The Hamiltonian function

$$i\frac{du_n}{dt} = \frac{\partial H}{\partial \bar{v}_n}, \quad i\frac{dv_n}{dt} = \frac{\partial H}{\partial \bar{u}_n},$$

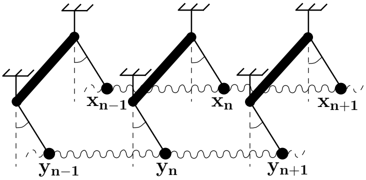
with

$$H = |u_n|^2 + |v_n|^2 + i\gamma(u_n\bar{v}_n - \bar{u}_n v_n) + (u_n\bar{v}_n + \bar{u}_n v_n)(|u_n|^2 + |v_n|^2).$$

Jørgensen–Christiansen (1998); Barashenkov–Gianfreda (2014); Barashenkov–Pelinovsky–Dubard (2015)

Physical context - coupled pendula

A. Pikovsky, M. Rosenblum, J. Kurth *Synchronization* (2001)
M. Bennett, M. Schatz, Rockwood, K. Wiesenfeld (2002)
C. Bender, B. Berntson, D. Parker, E. Samuel (2013)



Newton's equations of motion:

$$\begin{cases} \ddot{x}_n + \sin(x_n) = C\,(x_{n+1} - 2x_n + x_{n-1}) + Dy_n, \\ \ddot{y}_n + \sin(y_n) = C\,(y_{n+1} - 2y_n + y_{n-1}) + Dx_n, \end{cases} \quad n \in \mathbb{Z}, \quad t \in \mathbb{R},$$

where C is the coupling constant for torsional springs and D is the coupling constant for rope tension of the common string. The model is Hamiltonian:

$$H = \sum_{n\in\mathbb{Z}} E(x_n) + E(y_n) + \frac{1}{2}C(x_{n+1} - x_n)^2 + \frac{1}{2}C(y_{n+1} - y_n)^2 - Dx_n y_n.$$

Reduction in the limit of small oscillations

Small coupling constants and periodic movement of the common strings with nearly resonant frequency:

$$C = \epsilon\mu^2, \quad D(t) = 2\gamma\mu^2\cos(2\omega t), \quad \omega^2 = 1 - \mu^2\Omega,$$

where μ is a formal small parameter.

Using expansions like

$$\begin{cases} x_n(t) = \mu\left[A_n(\mu^2 t)e^{i\omega t} + \bar{A}_n(\mu^2 t)e^{-i\omega t}\right] + \mathcal{O}(\mu^3) \\ y_n(t) = \mu\left[B_n(\mu^2 t)e^{i\omega t} + \bar{B}_n(\mu^2 t)e^{-i\omega t}\right] + \mathcal{O}(\mu^3), \end{cases}$$

we obtain the reduced system

$$\begin{cases} 2i\dot{A}_n + \Omega A_n = \epsilon\left(A_{n+1} - 2A_n + A_{n-1}\right) + \gamma\bar{B}_n + \frac{1}{2}|A_n|^2 A_n, \\ 2i\dot{B}_n + \Omega B_n = \epsilon\left(B_{n+1} - 2B_n + B_{n-1}\right) + \gamma\bar{A}_n + \frac{1}{2}|B_n|^2 B_n. \end{cases}$$

The model is Hamiltonian and autonomous.

Reduction to the PT-symmetric dNLS equation

Using the choice

$$u_n := \frac{1}{4}\left(A_n - i\bar{B}_n\right), \quad v_n := \frac{1}{4}\left(A_n + i\bar{B}_n\right),$$

we obtain the coupled PT-dNLS equation

$$\left\{ \begin{array}{l} i\dot{u}_n + \Omega v_n = \epsilon\left(v_{n+1} - 2v_n + v_{n-1}\right) + i\gamma u_n + \left(2|u_n|^2 + |v_n|^2\right) v_n + u_n^2 \bar{v}_n, \\ i\dot{v}_n + \Omega u_n = \epsilon\left(u_{n+1} - 2u_n + u_{n-1}\right) - i\gamma v_n + \left(|u_n|^2 + 2|v_n|^2\right) u_n + \bar{u}_n v_n^2, \end{array} \right.$$

The model is Hamiltonian and PT-symmetric with the energy function

$$\begin{aligned} H = & \sum_{n\in\mathbb{Z}} (|u_n|^2 + |v_n|^2)^2 + (u_n\bar{v}_n + \bar{u}_n v_n)^2 - \Omega(|u_n|^2 + |v_n|^2) \\ & -\epsilon|u_{n+1} - u_n|^2 - \epsilon|v_{n+1} - v_n|^2 + i\gamma(u_n\bar{v}_n - \bar{u}_n v_n). \end{aligned}$$

Another conserved quantity is related to gauge symmetry:

$$Q = \sum_{n\in\mathbb{Z}} (u_n\bar{v}_n + \bar{u}_n v_n).$$

Relevant questions

Let us reiterate the same questions for the main model:

$$\left\{ \begin{array}{l} i\dot{u}_n + \Omega v_n = \epsilon\left(v_{n+1} - 2v_n + v_{n-1}\right) + i\gamma u_n + \left(2|u_n|^2 + |v_n|^2\right) v_n + u_n^2 \bar{v}_n, \\ i\dot{v}_n + \Omega u_n = \epsilon\left(u_{n+1} - 2u_n + u_{n-1}\right) - i\gamma v_n + \left(|u_n|^2 + 2|v_n|^2\right) u_n + \bar{u}_n v_n^2, \end{array} \right.$$

The linear system at zero equilibrium is stable for $\gamma \in (0, |\Omega|)$ (at $\epsilon = 0$).

1. Do the solutions stay bounded for all times if $\gamma \in (0, |\Omega|)$?
2. Do there exist linearly stable localized modes on the lattice for $\gamma \in (0, |\Omega|)$?
3. Are linearly stable localized modes also stable in the nonlinear dynamics of the lattice?

Now we can explore the Hamiltonian structure of the PT lattice to give answers to these questions.

1. Do the solutions stay bounded for all times?

Consider the Hamiltonian function

$$\begin{aligned} H &= \sum_{n\in\mathbb{Z}} (|u_n|^2 + |v_n|^2)^2 + (u_n\bar{v}_n + \bar{u}_n v_n)^2 - \Omega(|u_n|^2 + |v_n|^2) \\ &\quad -\epsilon|u_{n+1} - u_n|^2 - \epsilon|v_{n+1} - v_n|^2 + i\gamma(u_n\bar{v}_n - \bar{u}_n v_n). \end{aligned}$$

If $\Omega < 0$ and $|\gamma| < |\Omega| - 4\epsilon$, then

$$H \geq (|\Omega| - |\gamma| - 4\epsilon)\left(\|u\|^2_{\ell^2} + \|v\|^2_{\ell^2}\right).$$

Therefore, there is a positive constant C that depends on γ, ϵ, Ω and the initial data in $\ell^2(\mathbb{Z})$ such that

$$\|u(t)\|^2_{\ell^2} + \|v(t)\|^2_{\ell^2} \leq C, \quad \text{for every} \quad t \in \mathbb{R}.$$

The condition $|\gamma| < |\Omega| - 4\epsilon$ for $\Omega < 0$ coincides with the condition of linear stability of the zero equilibrium.

What if $\Omega > 0$?

Consider the Hamiltonian function

$$
\begin{aligned}
-H &= \sum_{n\in\mathbb{Z}} -(|u_n|^2 + |v_n|^2)^2 - (u_n\bar{v}_n + \bar{u}_n v_n)^2 + \Omega(|u_n|^2 + |v_n|^2) \\
&\quad + \epsilon|u_{n+1} - u_n|^2 + \epsilon|v_{n+1} - v_n|^2 - i\gamma(u_n\bar{v}_n - \bar{u}_n v_n).
\end{aligned}
$$

If $\Omega > 0$ and $|\gamma| < \Omega$, then

$$
-H \geq (\Omega - |\gamma|)\left(\|u\|_{\ell^2}^2 + \|v\|_{\ell^2}^2\right) - \left(\|u\|_{\ell^2}^2 + \|v\|_{\ell^2}^2\right)^2,
$$

where we have used $\|u\|_{\ell^4} \leq \|u\|_{\ell^2}$. For sufficiently small initial data in $\ell^2(\mathbb{Z})$, we still have

$$
\|u(t)\|_{\ell^2}^2 + \|v(t)\|_{\ell^2}^2 \leq C, \quad \text{for every} \quad t \in \mathbb{R}.
$$

The condition $|\gamma| < \Omega$ for $\Omega > 0$ coincides with the condition of linear stability of the zero equilibrium.

2. Do there exist linearly stable localized modes?

Stationary PT-symmetric localized modes:

$$u(t) = Ue^{-iEt}, \quad v(t) = Ve^{-iEt}, \quad V = \bar{U},$$

where U satisfies the stationary PT-symmetric DNLS equation

$$EU_n + \Omega \bar{U}_n = \epsilon \left(\bar{U}_{n+1} - 2\bar{U}_n + \bar{U}_{n-1} \right) + i\gamma U_n + 3|U_n|^2 \bar{U}_n + U_n^3.$$

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Local bifurcation from the central dimer at $\epsilon = 0$:

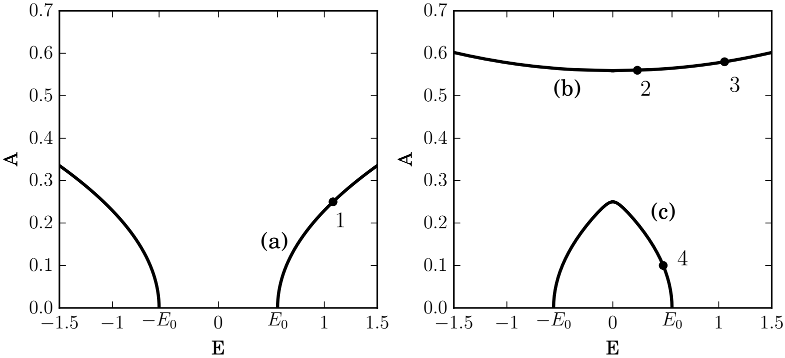
$$(E - i\gamma)U_0 + \Omega \bar{U}_0 = 3|U_0|^2 \bar{U}_0 + U_0^3.$$

In the polar form $U_0 = Ae^{i\theta}$, we obtain the parameterization

$$E^2 = (\Omega - 4A^2)^2 \left[1 - \frac{\gamma^2}{(\Omega - 2A^2)^2} \right].$$

If $A = 0$, then $E = \pm E_0$ with $E_0 := \sqrt{\Omega^2 - \gamma^2} > 0$.

Stationary modes of the central dimer for $|\gamma| < |\Omega|$



Assume $\gamma \neq 0$. Then,

(a) $\Omega < -|\gamma|$ - two symmetric unbounded branches exist for $\pm E > E_0$,

(b) $\Omega > |\gamma|$ - an unbounded branch exists for every $E \in \mathbb{R}$,

(c) $\Omega > |\gamma|$ - a bounded branch exists for $-E_0 < E < E_0$,

Continuation of the localized mode in ϵ

The stationary PT-symmetric localized mode with spatial symmetry

$$U_{-n}(\epsilon) = U_n(\epsilon), \quad n \in \mathbb{Z}, \quad \epsilon \in \mathbb{R},$$

such that $U_n(\epsilon) \to 0$ as $\epsilon \to 0$ for $n \neq 0$.

Theorem

Fix $\gamma \neq 0$, $\Omega \neq 2|\gamma|$, and $E \neq \pm E_0$, where $E_0 := \sqrt{\Omega^2 - \gamma^2} > 0$ and $|\gamma| < |\Omega|$. There exists $\epsilon_0 > 0$ sufficiently small and $C_0 > 0$ such that for every $\epsilon \in (-\epsilon_0, \epsilon_0)$, there exists a unique localized mode $U(\epsilon) \in l^2(\mathbb{Z})$ such that

$$\left| U_0(\epsilon) - Ae^{i\theta} \right| + \sup_{n \in \mathbb{N}} |U_n(\epsilon)| \leq C_0 |\epsilon|.$$

Moreover, the solution U is smooth in ϵ.

Variational characterization of localized modes

From the two conserved quantities H and Q, let us define

$$H_E := H - EQ.$$

Then, the stationary PT-symmetric mode (U, V) with $V = \bar{U}$ is a critical point of the energy function H_E.

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Using expansion

$$H_E(U+u) - H_E(U) = \frac{1}{2}\left\langle \mathcal{H}_E'' u, u \right\rangle_{\ell^2} + \mathcal{O}(\|u\|_{\ell^2}^3),$$

we obtain the Hessian (self-adjoint) operator defined on $\ell^2(\mathbb{Z})$ by

$$\mathcal{H}_E'' = \mathcal{L} + \epsilon\Delta,$$

where $\mathcal{L}$ is block-diagonal into 4-by-4 blocks at each lattice site $n \in \mathbb{Z}$.

Count of eigenvalues of $\mathcal{H}_E''$

Lemma

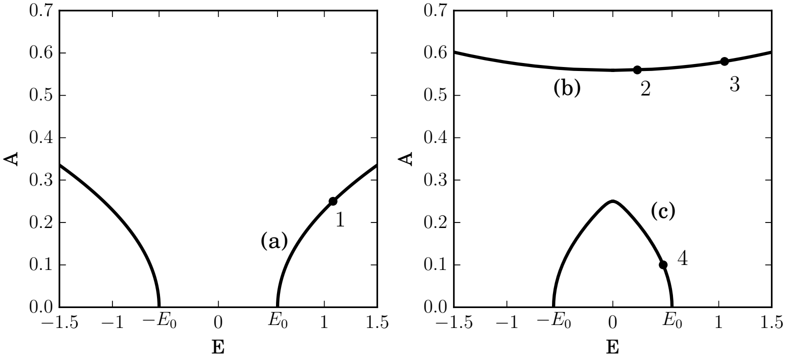
Fix $\gamma \neq 0$, $|\Omega| > |\gamma|$, *and* $E \neq \pm E_0$. *For every* $\epsilon > 0$ *sufficiently small, the operator* $\mathcal{H}_E''$ *admits a one-dimensional kernel in* $\ell^2(\mathbb{Z})$ *spanned by the eigenvector* $i\sigma\Phi$ *due to the gauge invariance,*

$$(i\sigma\Phi)_n := (iU_n, -i\bar{U}_n, iV_n, -i\bar{V}_n).$$

In addition,

- *If* $|E| > E_0$, *the spectrum of* $\mathcal{H}_E''$ *in* $\ell^2(\mathbb{Z})$ *includes infinite-dimensional positive and negative parts.*
- *If* $|E| < E_0$ *and* $\Omega > |\gamma|$, *the spectrum of* $\mathcal{H}_E''$ *in* $\ell^2(\mathbb{Z})$ *includes an infinite-dimensional negative part and either three [branch (b)] or one [branch (c)] simple positive eigenvalues.*

Vakhitov-Kolokolov criterion for branch (c)



The slope criterion

$$\left.\frac{dQ}{dE}\right|_{\epsilon=0} = 4(4A^2 - \Omega)\frac{dA^2}{dE^2}\left[1 + \frac{\Omega\gamma^2}{(2A^2-\Omega)^3}\right].$$

For branch (c), $Q'(E) > 0$ for every $E \in (0, E_0)$ if $\Omega > 2\sqrt{2}|\gamma|$.

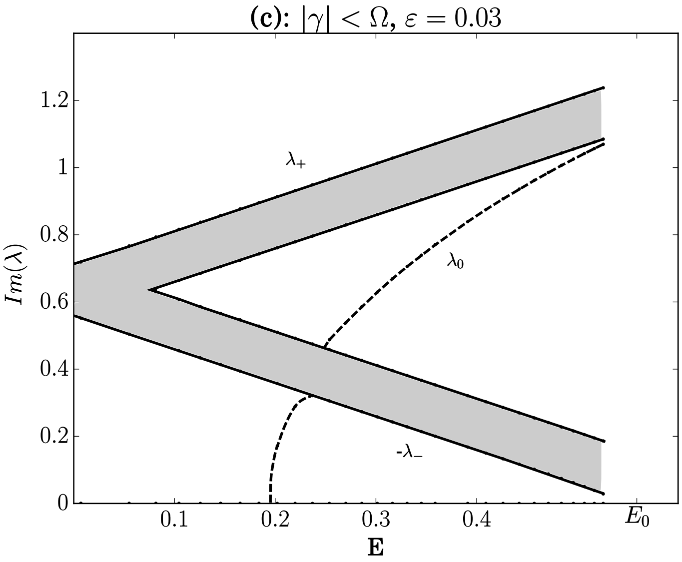
Orbital stability of branch (c)

Theorem

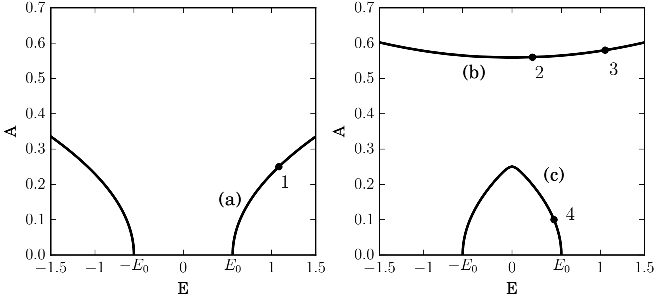
Fix $\gamma \neq 0$, $\Omega > |\gamma|$, and $E \in (-E_0, E_0)$. For every $\epsilon > 0$ sufficiently small, the stationary state (U, V) is orbitally stable in $\ell^2(\mathbb{Z})$ if $\Omega > 2\sqrt{2}|\gamma|$. For every $\Omega \in (|\gamma|, 2\sqrt{2}|\gamma|)$, there exists a value $E_s \in (0, E_0)$ such that the stationary state (U, V) is orbitally stable in $\ell^2(\mathbb{Z})$ if $E_s < |E| < E_0$ and unstable if $|E| < E_s$.

Orbital stability of a localized mode is understood in the following sense: If the initial data is close to (U, V) in $\ell^2(\mathbb{Z})$, then the solution remains close to $\{(U, V)e^{i\theta}\}_{\theta \in \mathbb{R}}$ for all times.

Numerical results on spectral stability - branch (c)



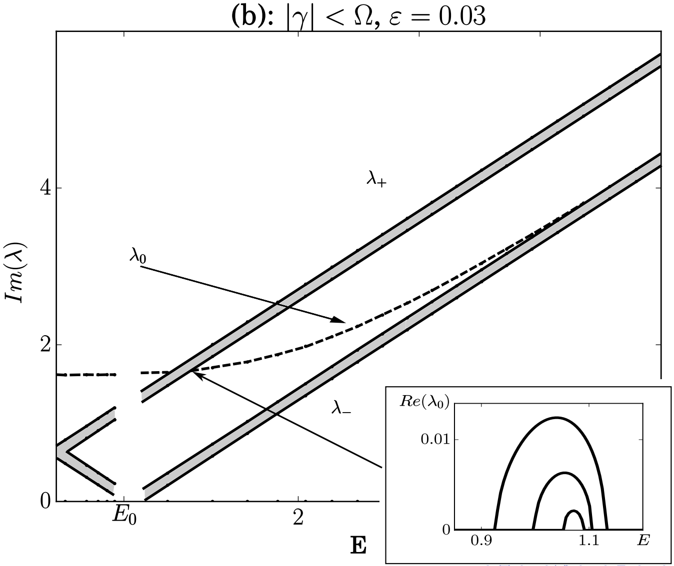
Negative index theory for branch (b)



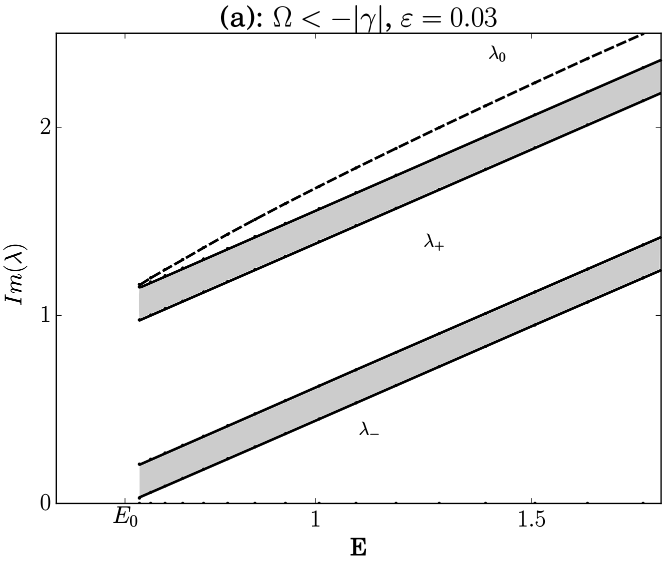
For branch (b), $Q'(E) > 0$ for every $E \in (0, E_0)$, whereas the spectrum of $\mathcal{H}_E''$ in $\ell^2(\mathbb{Z})$ includes only three positive eigenvalues. Then,

- Either the localized mode is spectrally stable with exactly one pair of stable eigenvalues of positive Krein signature;
- Or the localized mode is spectrally unstable either with a quartet of complex eigenvalues or two pairs of real eigenvalues.

Numerical results on spectral stability - branch (b)



Numerical results on spectral stability - branch (a)



Long-time stability result

Branch (a) for $\gamma \neq 0$, $\Omega < -|\gamma|$, and $E \in (-\infty, -E_0) \cup (E_0, \infty)$.

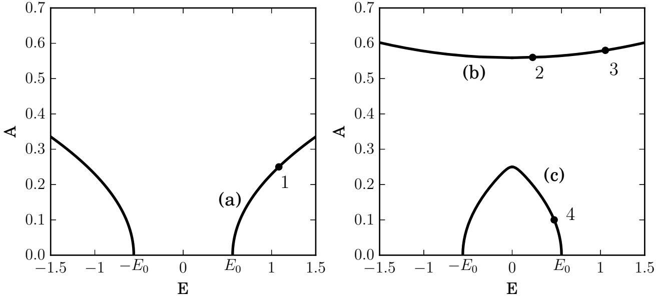
Theorem

For every $\nu > 0$ sufficiently small, there exists $\epsilon_0 > 0$ and $\delta > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ the following is true. If $\psi(0) \in \ell^2(\mathbb{Z})$ satisfies $\|\psi(0) - \Phi\|_{\ell^2} \leq \delta$, then there exist a positive time $t_0 \lesssim \epsilon^{-1/2}$ and a C^1 function $\alpha(t) : [0, t_0] \to \mathbb{R}/(2\pi\mathbb{Z})$ such that the unique solution $\psi(t) : [0, t_0] \to \ell^2(\mathbb{Z})$ satisfies the bound

$$\|e^{i\alpha(t)\sigma}\psi(t) - \Phi\|_{\ell^2} \leq \nu, \quad \text{for every} \quad t \in [0, t_0].$$

Moreover, there exists a positive constant C such that $|\dot{\alpha} - E| \leq C\nu$, for every $t \in [0, t_0]$.

3. Are linearly stable localized modes also stable in the nonlinear dynamics of the lattice

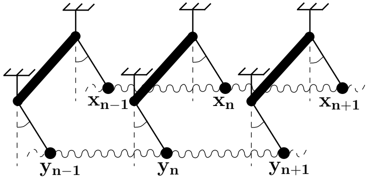


(c) Yes, from standard orbital stability theory.

(b) No, generally.

(a) Yes, for long but finite times.

Another system of coupled oscillators



Newton's equations of motion:

$$\begin{cases} \ddot{x}_n + \sin(x_n) = C\,(x_{n+1} - 2x_n + x_{n-1}) + D(t)(y_n - x_n), \\ \ddot{y}_n + \sin(y_n) = C\,(y_{n+1} - 2y_n + y_{n-1}) + D(t)(x_n - y_n), \end{cases}$$

where C and D are the coupling constant for torsional springs.

Small coupling constants and periodic movement of the common strings with nearly resonant frequency:

$$C = \epsilon\mu^2, \quad D(t) = 2\gamma\mu^2\cos(2\omega t), \quad \omega^2 = 1 - \mu^2\Omega, \quad \mu \ll 1.$$

Reduction to the PT-symmetric dNLS equation

Asymptotic expansions yield the system

$$\left\{ \begin{array}{l} 2i\dot{A}_n + \Omega A_n = \epsilon\left(A_{n+1} - 2A_n + A_{n-1}\right) + \gamma(\bar{B}_n - \bar{A}_n) + \frac{1}{2}|A_n|^2 A_n, \\ 2i\dot{B}_n + \Omega B_n = \epsilon\left(B_{n+1} - 2B_n + B_{n-1}\right) + \gamma(\bar{A}_n - \bar{B}_n) + \frac{1}{2}|B_n|^2 B_n. \end{array} \right.$$

Using the choice

$$u_n := \frac{1}{4}\left(A_n - i\bar{B}_n\right), \quad v_n := \frac{1}{4}\left(A_n + i\bar{B}_n\right),$$

we obtain the coupled PT-dNLS equation

$$\left\{ \begin{array}{l} i\dot{u}_n + \Omega v_n = \epsilon\left(v_{n+1} - 2v_n + v_{n-1}\right) + i\gamma u_n - \gamma\bar{u}_n + \left(2|u_n|^2 + |v_n|^2\right) v_n + u_n^2 \bar{v}_n, \\ i\dot{v}_n + \Omega u_n = \epsilon\left(u_{n+1} - 2u_n + u_{n-1}\right) - i\gamma v_n - \gamma\bar{v}_n + \left(|u_n|^2 + 2|v_n|^2\right) u_n + \bar{u}_n v_n^2, \end{array} \right.$$

The model is Hamiltonian, PT-symmetric, but it is not gauge-invariant.