Existence, stability and properties of dark solitons

Dmitry Pelinovsky (McMaster University, Canada)

References:

D.P., Yu. Kivshar, V. Afanasjev, Phys. Rev. E 54, 2015 (1996)

D.P., D. Frantzeskakis, P. Kevrekidis, Phys. Rev. E 72, 016615 (2005)

Wolfgang Pauli Institute, Vienna, June 12-14, 2006

Definitions of dark solitons

Question: What are dark solitons?

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Answers:

- Physicists: Waves in defocusing systems with modulationally stable continuous wave (CW) background

- PDE analysts: Localized solutions of PDEs with non-zero boundary conditions and non-zero phase shift

- Applied mathematicians: A family of traveling waves from KdV solitons (*grey solitons*) to kinks (*black solitons*)

Reasons to dislike dark solitons

Physicists do not like dark solitons as

- dark solitons have infinite energy due to a background
- it is difficult to distinguish experimentally between the soliton and the background
- dark solitons have no direct engineering applications

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Mathematicians do not like dark solitons as

- tricky renormalization of all integral quantities is required
- two-wave radiation is similar to Boussinesq systems
- all results are formal so far and even formal results are too cumbersome in details

Main model for dark solitons

Defocusing one-dimensional NLS equation

$$iu_t = -\frac{1}{2}u_{xx} + f(|u|^2)u,$$

where $f(s)$ is C^∞ with $f'(s) > 0$ for some $0 \leq s \leq s_0$.

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Examples:

- $f = |u|^2$ (integrable by inverse scattering)
- $f = |u|^2 \pm |u|^4$ (cubic-quintic)
- $f = -1/(1+|u|^2)$ (saturable)

Conserved quantities

- Hamiltonian $[u(x,t) \mapsto u(x,t-t_0)]$

$$H = \frac{1}{2}\int_{\mathbb{R}} \left(|u_x|^2 + 2\int_0^{|u|^2} f(s)ds \right) dx$$

- Power $[u \mapsto ue^{i\theta_0}]$ momentum $[u(x,t) \mapsto u(x-x_0,t)]$

$$N = \int_{\mathbb{R}} |u|^2 dx, \qquad P = \frac{i}{2}\int_{\mathbb{R}} \left(\bar{u}u_x - \bar{u}_x u\right) dx$$

- Phase shift

$$S = [\arg(u)]_{x\to-\infty}^{x\to+\infty} = \frac{i}{2}\int_{\mathbb{R}} \left(\frac{\bar{u}_x}{\bar{u}} - \frac{u_x}{u} \right) dx$$

ODE analysis of existence

Traveling stationary solutions

$$u(x,t) = U(x - vt)e^{i\omega t},$$

where (v, ω) are parameters and $U(z)$, $z = x - vt$ satisfies:

$$-\frac{1}{2}U''(z) + \omega U(z) + ivU'(z) + f(|U|^2)U = 0$$

Separation of variables $U(z) = \Phi(z)e^{i\Theta(z)}$ leads to

$$\frac{d}{dz}\left[\Phi^2(\Theta' - v)\right] = 0 \quad \Rightarrow \quad \Theta'(z) = v - \frac{c}{\Phi^2(z)},$$

where c is constant of integration.

Parameters of dark solitons

Recall the Galilei transformation

$$u(x,t) \mapsto u(x-kt,t)e^{ik(x-kt/2)}.$$

The constant c can be chosen from the boundary conditions:

$$\lim_{z\to\pm\infty} \Phi(z) = \sqrt{q}, \qquad \lim_{z\to\pm\infty} \Theta(z) = \Theta_\pm,$$

subject to the sufficient decay of $\Phi(z)$ and $\Theta(z)$ to constant values.

Then,

$$c = vq, \qquad S = \Theta_+ - \Theta_-.$$

Reduction to the second-order ODE

After $\Theta'(z)$ is eliminated from the system, we obtain:

$$\Phi'' - 2(\omega + f(\Phi^2))\Phi + v^2 \frac{\Phi^4 - q^2}{\Phi^3} = 0$$

From existence of the equilibrium state $\Phi = \sqrt{q}$:

$$\omega = -f(q)$$

From the condition that $\Phi = \sqrt{q}$ is a hyperbolic point:

$$v^2 < q f'(q) \equiv c^2,$$

such that the family of dark solitons exist for $-c < v < c$.

Example

Cubic NLS with $f(|u|^2) = |u|^2$:

$$U(z) = \Phi(z)e^{i\Theta(z)} = k\tanh(kz) + iv,$$

where $k = \sqrt{q - v^2}$ and $v^2 < q$.

- When $v \to \sqrt{q}$, the dark soliton approaches the KdV soliton

$$\Phi(z) = \sqrt{q} - \frac{k^2}{2\sqrt{q}}\mathrm{sech}^2(kz) + \mathrm{O}(k^4).$$

- When $v \to 0$, the dark soliton approaches the kink

$$U(z) = \sqrt{q}\tanh(qz)$$

Black soliton

Black soliton corresponds to $v = 0$, when

$$U(z) = \Phi(z)e^{i\Theta(z)} \in \mathbb{R},$$

where $U(z)$ satisfies:

$$U'' + 2(f(q) - f(U^2))U = 0$$

or

$$\frac{1}{2}(U')^2 + 2\int_q^{U^2} (f(q) - f(s))ds = \text{const} = 0$$

Two solutions exist:

- Kink $\Phi(-z) = -\Phi(z)$ and $\Phi(0) = 0$ [$S = \pi$]
- Soliton on background $\Phi(-z) = \Phi(z)$ and $\Phi(0) > 0$ [$S = 0$]

Variational principle for dark solitons

The same ODE for $U(z)$ is obtained from the first variation of

$$\Lambda = H(U) + vP(U) + \omega N(U) + CS(U),$$

where C is arbitrary and $S(U)$ is a Casimir functional. We have seen that $\omega = -f(q)$ and v is a free parameter.

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Let

$$U = \Phi e^{i\Theta}, \qquad \Theta' = v\left(1 - \frac{q}{\Phi^2}\right).$$

Then, the second-order ODE for $\Phi(z)$ is obtained from the first variation of Λ in Φ only if $C = vq$.

Miracle of renormalization

New variational principle for dark solitons:

$$\Lambda = H_r(U) + vP_r(U) : \qquad H_r'(U) + vP_r'(U) = 0,$$

where

$$H_r = \frac{1}{2}\int_{\mathbb{R}} \left(|u_x|^2 + 2\int_q^{|u|^2} (f(s) - f(q))ds \right) dx$$

and

$$P_r = \frac{i}{2}\int_{\mathbb{R}} (\bar{u}u_x - \bar{u}_x u)\left(1 - \frac{q}{|u|^2}\right) dx$$

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Alternative picture on renormalization in

Yu. Kivshar and X. Yang, Phys. Rev. E **49**, 1657 (1994)

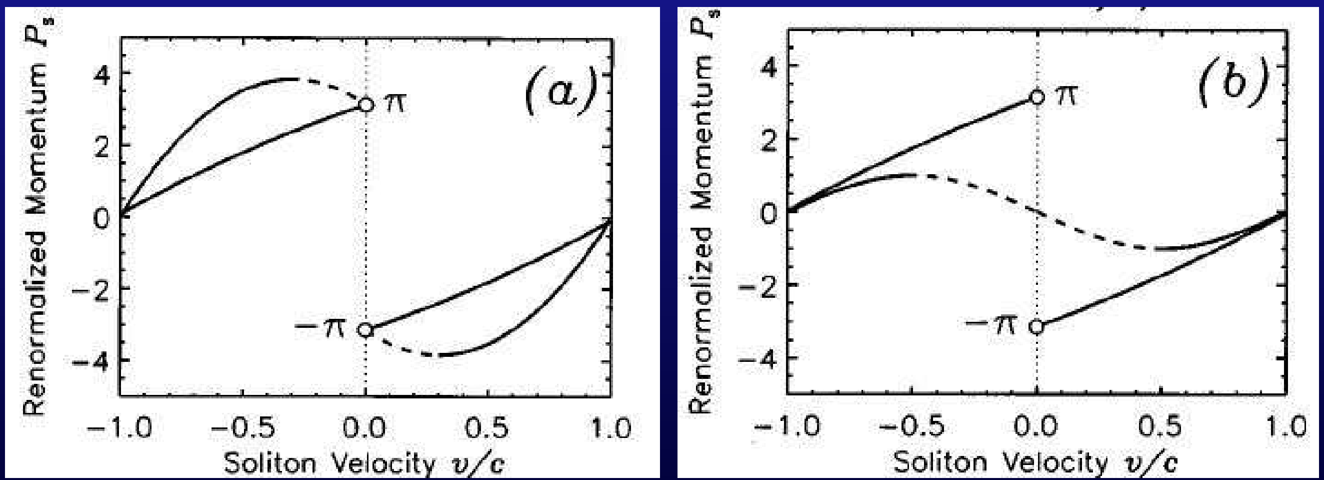
Renormalized momentum

By construction,

$$P_r(v) = -v \int_{\mathbb{R}} \Phi^2 \left(1 - \frac{q}{\Phi^2}\right)^2 dx = -vN(v) + qS(v).$$

Two solutions as $v \longrightarrow 0$:

- Kink with $\lim\limits_{v \longrightarrow 0} P_r(v) = \pi q$
- Soliton on background with $\lim\limits_{v \longrightarrow 0} P_r(v) = 0$



Stability of black solitons

Linearization at the black soliton $U(z)e^{-if(q)t}$ with $v = 0$:

$$u = e^{-if(q)t}\left[U(z) + (u(z) + iw(z))e^{\lambda t} + (\bar{u}(z) + i\bar{w}(z))e^{\bar{\lambda} t}\right]$$

Spectral stability problem:

$$L_+ u = -\lambda w, \qquad L_- w = \lambda u,$$

where

$$\begin{aligned} L_+ &= -\frac{1}{2}\partial_x^2 + f(U^2) - f(q) + 2\Phi^2 f'(\Phi^2), \\ L_- &= -\frac{1}{2}\partial_x^2 + f(U^2) - f(q). \end{aligned}$$

Spectra of $L_\pm$ in $L^2(\mathbb{R})$

Continuous spectra σ_c:

- $\sigma_c(L_+) \geq 2c^2 > 0$
- $\sigma_c(L_-) \geq 0$, with $L_- U(z) = 0$

Kernel and negative eigenvalues in $L^2(\mathbb{R})$:

- Kink with $S = \pi$:
  - $L_+ U'(z) = 0$ and L_+ has no negative eigenvalues
  - L_- has exactly one negative eigenvalue and no kernel
- Soliton on background with $S = 0$:
  - $L_+ U'(z) = 0$ and L_+ has exactly one negative eigenvalue
  - L_- has no negative eigenvalues and no kernel.

Constrained L^2-space

Consider for $|\lambda| \geq \epsilon > 0$:

$$L_+ u = -\lambda w, \qquad L_- w = \lambda u,$$

If $w \in L^2(\mathbb{R})$, then $w(z)$ must be orthogonal to $\ker(L_+) = \{U'(z)\}$.

Define the constrained space

$$X_c = \left\{ w \in L^2(\mathbb{R}) : \quad (U', w) = 0 \right\}$$

For $\lambda \neq 0$, the stability problem is equivalent to the generalized eigenvalue problem in X_c:

$$L_- w = \gamma L_+^{-1} w, \qquad \gamma = -\lambda^2$$

Analysis for kinks only

Theorem: Operator L_- has no negative eigenvalues in X_c if $P_r'(v)|_{v=0} > 0$ and exactly one negative eigenvalue if $P_r'(v)|_{v=0} < 0$.

A delicate detail in the proof: The inhomogeneous equation

$$L_- w = U'(z)$$

have two solutions:

- $w = -zU(z)$ - linearly growing in z
- $w = \partial_v U(z)|_{v\to 0}$ - bounded but non-decaying in z

If the second (bounded) solution is selected, then

$$(U', L_-^{-1} U') = (U', \partial_v U|_{v\to 0}) = P_r'(v)|_{v\to 0}$$

and the statement follows by the variational theory in X_c.

Application of Pontryagin Theorem

Theorem: Eigenvalues of the problem $L_+ L_- w = -\lambda^2 w$ in X_c satisfy:

$$N_{\rm unst}(L_+ L_-) + N_{\rm negKrein}(L_+ L_-) = N_{\rm neg}(L_+) + N_{\rm neg}(L_-)$$

Then,

- Kink with $S = \pi$: stable for $P_r'(0) > 0$ and unstable with one real positive eigenvalue for $P_r'(0) < 0$
- Soliton on background with $S = 0$: always unstable

"Pioneer" results:

- I. Barashenkov, Phys. Rev. Lett. **77**, 1193-1197 (1996)
- D.P., Yu. Kivshar, Phys. Rev. E **54**, 2015-2032 (1996)
- Y. Chen, M. Mitchell (1996) - unpublished

Stability of dark solitons

Stability analysis of black solitons can be extended for the complete family of dark solitons in constrained space associated to $P_r'[u] = 0$, where

$$P_r[u] = \frac{i}{2} \int_{\mathbb{R}} (\bar{u} u_x - \bar{u}_x u) \left(1 - \frac{q}{|u|^2} \right) dx$$

and

$$P_r'[U] = -i(U', u) + i(\bar{U}', \bar{u}) = 0.$$

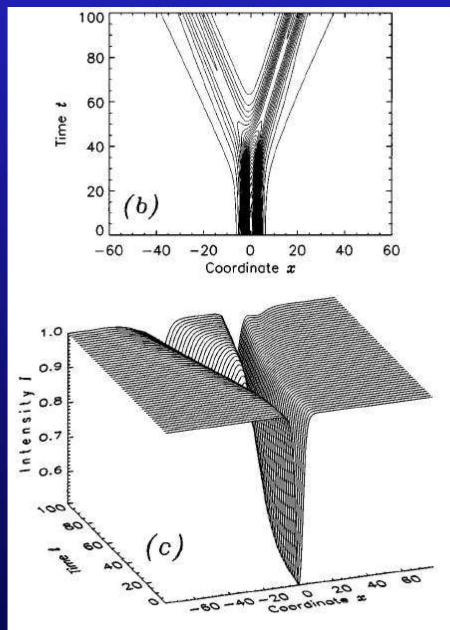
Dark solitons are stable when $P_r'(v) > 0$ and unstable when $P_r'(v) < 0$ at the solution family $P_r(v) = P_r[U]$.

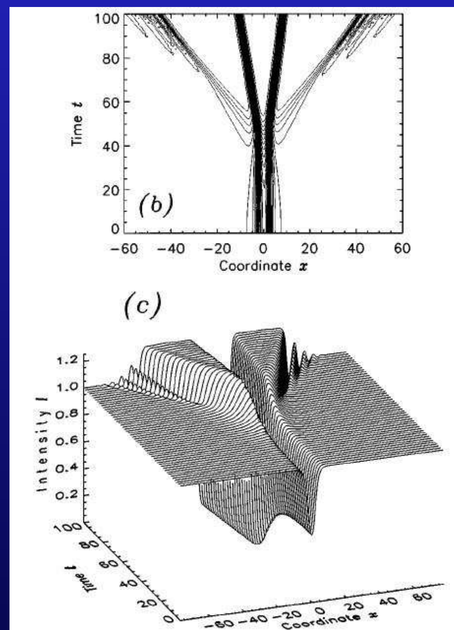
Example

Cubic-quintic NLS:

$$f(|u|^2) = -2|u|^2 + 1.2|u|^4$$

Note that at $q = 1$, $c^2 = qf'(q) = 0.4 > 0$. The limit of black soliton $v = 0$ corresponds to the soliton on background.





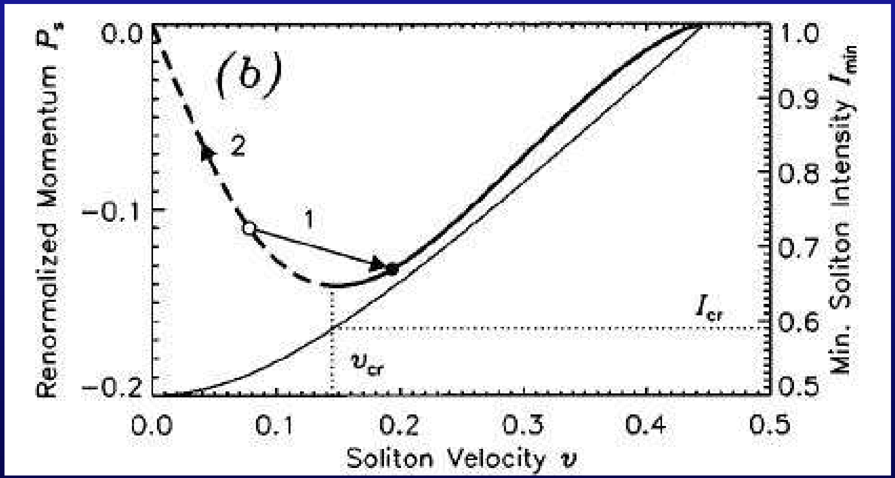
Two scenario of dynamics

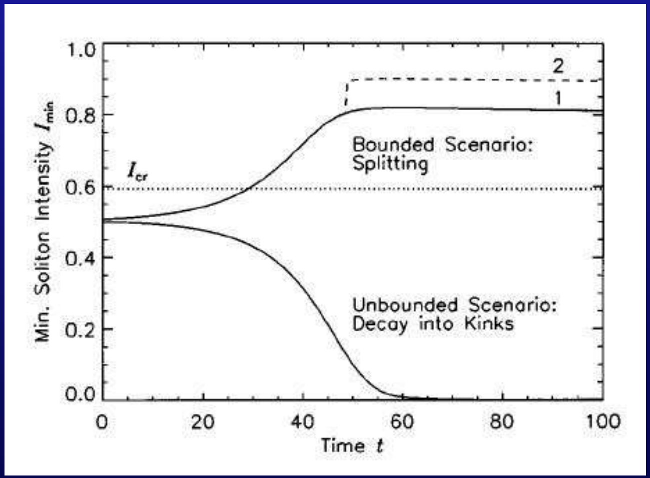
Normal form for instability dynamics (1996):

$$M_r(v_*)\dot{V} + \frac{1}{2}P_r''(v_*)V^2 = 0,$$

where $V = v(t) - v_*$, $P_r'(v_*) = 0$, and $M_r(v_*) > 0$.

When $P_r''(v_*) > 0$, the bounded scenario occurs when $V(0) > 0$ and the unbounded scenario occurs when $V(0) < 0$.



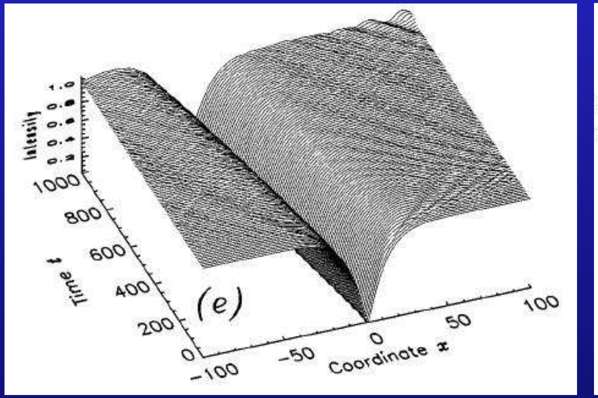


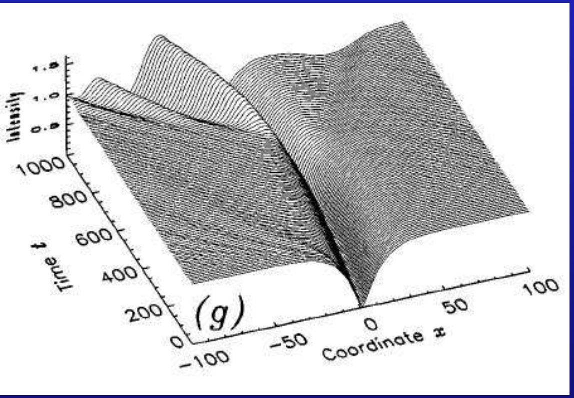
Example

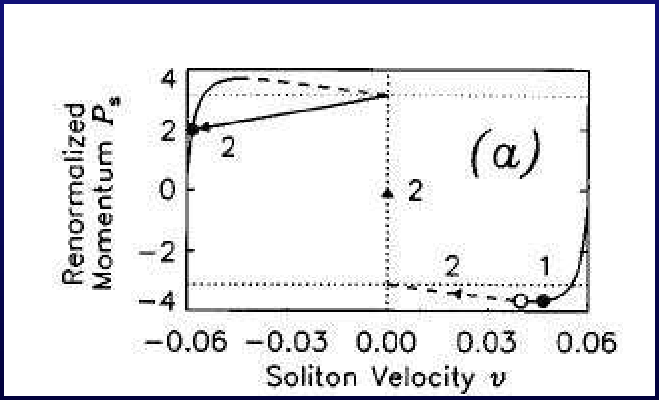
Saturable NLS:

$$f(|u|^2) = -\frac{1}{(1+12|u|^2)^2}$$

The limit of black soliton $v=0$ corresponds to the kink.







Review of other results

- Evans functions for dark solitons

  T. Kapitula and J. Rubin, Nonlinearity **13**, 77 (2000)

- Completeness of eigenfunctions in the cubic NLS equation

  X.Chen, N.Huang, J.Phys.A: Math.Gen. **31**, 6929 (1998)

- Perturbation theory for dark solitons
  - V.Konotop, V.Vekslerchik, Phys. Rev. E **49**, 2397 (1994)
  - Yu. Kivshar and X. Yang, Phys. Rev. E **49**, 1657 (1994)
  - V.Lashkin (2004); N. Bilas and N. Pavloff (2005)

- Transverse instability of dark solitons

  E.A. Kuznetsov and S. Turitsyn, JETP **67**, 1583 (1988)

Work in progress

- Asymptotic stability of dark solitons
- Persistence and dynamics of dark solitons in external potentials

Talk "Oscillations of dark BEC solitons in a parabolic trap"
on Wednesday June 14 at 9:00-10:00

- Normal form analysis of slow dynamics of dark solitons