Periodic waves in discrete MKDV equation: modulational instability and rogue waves

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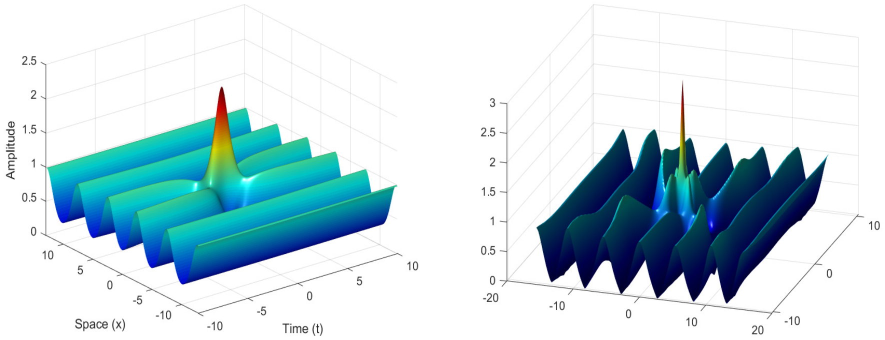
Rogue waves on standing periodic waves

J. Chen, D. Pelinovsky, Proceedings A **474** (2018) 20170814
J. Chen, D. Pelinovsky, R. White, Physica D **405** (2020) 132378

$$(NLS) \quad i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$



Other examples of integrable Hamiltonian systems

- Modified Korteweg–de Vries equation

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

Dnoidal periodic waves are modulationally stable (no rogue waves).
Cnoidal periodic waves are modulationally unstable (rogue waves).
J. Chen & D. Pelinovsky, Nonlinearity **31** (2018) 1955–1980

- Sine–Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0$$

Same conclusion.
D. Pelinovsky & R. White, Proceedings A **476** (2020) 20200490

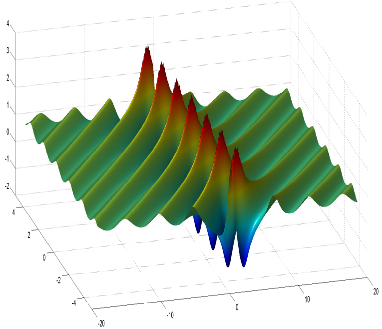
- Derivative NLS equation

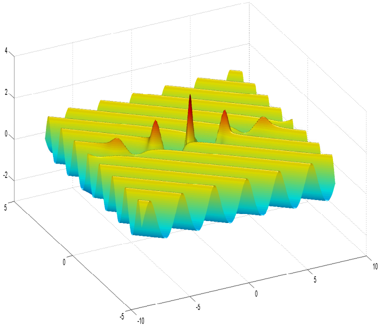
$$i\psi_t + \psi_{xx} + i(|\psi|^2\psi)_x = 0.$$

There exist modulationally stable periodic waves (no rogue waves).
J. Chen, D. Pelinovsky, & J. Upsal, J. Nonlinear Science **31** (2021) 58

Rogue wave for the modified KdV equation

J. Chen & D. Pelinovsky, Journal of Nonlinear Science **29** (2019) 2797–2843





Discrete modified KdV equation

It is considered to be a commuting flow in the Ablowitz–Ladik hierarchy:

$$\dot{u}_n = (1 + u_n^2)(u_{n+1} - u_{n-1}), \quad n \in \mathbb{Z},$$

where $u_n = u_n(t)$ is real.

In the continuum limit, long waves of small amplitudes can be modeled by

$$u_n(t) = \varepsilon \mathfrak{u}(\varepsilon(n + 2t), \frac{1}{3}\varepsilon^3 t),$$

satisfy the continuous the mKdV equation

$$\mathfrak{u}_\tau = 6\mathfrak{u}^2 \mathfrak{u}_\xi + \mathfrak{u}_{\xi\xi\xi},$$

where $\mathfrak{u} = \mathfrak{u}(\xi, \tau)$ with $\xi := \varepsilon(n + 2t)$ and $\tau := \frac{1}{3}\varepsilon^3 t$, and ε is small parameter.

Lax equations

DMKDV is a compatibility condition of the linear Lax system

$$\varphi_{n+1} = \frac{1}{\sqrt{1+u_n^2}} \begin{pmatrix} \lambda & u_n \\ -u_n & \lambda^{-1} \end{pmatrix} \varphi_n$$

and

$$\dot{\varphi}_n = \begin{pmatrix} \frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) & \lambda u_n + \lambda^{-1} u_{n-1} \\ -\lambda u_{n-1} - \lambda^{-1} u_n & -\frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) \end{pmatrix} \varphi_n.$$

There exists another Lax system representation:

$$\varphi_{n+1} = \begin{pmatrix} \lambda & u_n \\ -u_n & \lambda^{-1} \end{pmatrix} \varphi_n$$

and

$$\dot{\varphi}_n = \begin{pmatrix} \frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) + u_n u_{n-1} & \lambda u_n + \lambda^{-1} u_{n-1} \\ -\lambda u_{n-1} - \lambda^{-1} u_n & -\frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) + u_n u_{n-1} \end{pmatrix} \varphi_n.$$

1st Missing point. The method of nonlinearization works for the former system and does not work for the latter system.

Nonlinearization method

If $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ is a solution of Lax system for $\lambda = \lambda_1$,
then $\varphi_n = (-q_n, p_n)^{\mathrm{T}}$ is a solution for $\lambda = \lambda_1^{-1}$.

Assume the relation between solutions of the DMKV and Lax systems:

$$u_n = \lambda_1 p_n^2 + \lambda_1^{-1} q_n^2, \qquad n \in \mathbb{Z}.$$

Then, $\varphi_n = (p_n, q_n)^{\mathrm{T}}$ satisfies the nonlinear symplectic map

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \frac{1}{\sqrt{1 + (\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2)^2}} \begin{pmatrix} \lambda_1 p_n + (\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2) q_n \\ \lambda_1^{-1} q_n - (\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2) p_n \end{pmatrix}$$

and the nonlinear Hamiltonian system

$$\frac{dp_n}{dt} = \frac{\partial H}{\partial q_n}, \quad \frac{dq_n}{dt} = -\frac{\partial H}{\partial p_n},$$

with

$$H(p_n, q_n) = \frac{1}{2}(\lambda_1^2 - \lambda_1^{-2}) p_n q_n + \frac{1}{2}(\lambda_1 p_n^2 + \lambda_1^{-1} q_n^2)(\lambda_1^{-1} p_n^2 + \lambda_1 q_n^2).$$

Restrictions on the class of admissible solutions

In addition to

$$u_n = \lambda_1 p_n^2 + \lambda_1^{-1} q_n^2, \qquad n \in \mathbb{Z},$$

one can easily prove the relation

$$u_{n-1} = \lambda_1^{-1} p_n^2 + \lambda_1 q_n^2, \qquad n \in \mathbb{Z}.$$

By eliminating the squared eigenfunctions, one can show that u_n satisfies the stationary discrete equation

$$(1 + u_n^2)(u_{n+1} + u_{n-1}) = \omega u_n, \quad n \in \mathbb{Z},$$

where $\omega := \lambda_1^2 + \lambda_1^{-2} + 4H$, where $H = H(p_n, q_n)$ is constant.

2[nd] Missing point. This is a standing wave reduction of the Ablowitz–Ladik system, not the traveling wave reducton of DMKDV.

Integrability of the nonlinear Hamiltonian system

The Hamiltonian system for (p_n, q_n) is obtained from the Lax equations

$$W(p_{n+1}, q_{n+1}, \lambda) U(u_n, \lambda_1) - U(u_n, \lambda_1) W(p_n, q_n, \lambda) = 0$$

and

$$\frac{d}{dt} W(p_n, q_n, \lambda) = V(u_n, \lambda_1) W(p_n, q_n, \lambda) - W(p_n, q_n, \lambda) V(u_n, \lambda_1),$$

where

$$W(p_n, q_n, \lambda) = \left(\begin{array}{cc} \frac{1}{2} - \frac{\lambda_1^2 p_n q_n}{\lambda^2 - \lambda_1^2} + \frac{\lambda_1^{-2} p_n q_n}{\lambda^2 - \lambda_1^{-2}} & \lambda \left(\frac{\lambda_1 p_n^2}{\lambda^2 - \lambda_1^2} + \frac{\lambda_1^{-1} q_n^2}{\lambda^2 - \lambda_1^{-2}} \right) \\ -\lambda \left(\frac{\lambda_1 q_n^2}{\lambda^2 - \lambda_1^2} + \frac{\lambda_1^{-1} p_n^2}{\lambda^2 - \lambda_1^{-2}} \right) & -\frac{1}{2} + \frac{\lambda_1^2 p_n q_n}{\lambda^2 - \lambda_1^2} - \frac{\lambda_1^{-2} p_n q_n}{\lambda^2 - \lambda_1^{-2}} \end{array} \right)$$

satisfies

$$\det W(p_n, q_n, \lambda) = -\frac{1}{4} + \frac{\lambda^2 F_1}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_1^{-2})}.$$

Algebraic polynomial for the traveling periodic waves

Due to the squared eigenfunction constraints, we also have

$$W(p_n,q_n,\lambda)=\begin{pmatrix} \frac{1}{2}-\frac{\lambda^2(F_1-u_nu_{n-1})}{(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_1^{-2})} & \frac{\lambda(\lambda^2u_n-u_{n-1})}{(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_1^{-2})} \\ -\frac{\lambda(\lambda^2u_{n-1}-u_n)}{(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_1^{-2})} & -\frac{1}{2}+\frac{\lambda^2(F_1-u_nu_{n-1})}{(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_1^{-2})} \end{pmatrix},$$

which gives

$$\det W(p_n,q_n,\lambda)=-\frac{P(\lambda)}{4(\lambda^2-\lambda_1^2)^2(\lambda^2-\lambda_1^{-2})^2},$$

where

$$P(\lambda):=\lambda^8-2\omega\lambda^6+(2+\omega^2-4F_1^2)\lambda^4-2\omega\lambda^2+1.$$

Thus, λ_1 is selected from two quadruplets of $P(\lambda)$:

$$P(\lambda)=(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_1^{-2})(\lambda^2-\lambda_2^2)(\lambda^2-\lambda_2^{-2}).$$

Dnoidal periodic waves

These are solutions of the form

$$u_n(t) = \frac{\mathrm{sn}(\alpha; k)}{\mathrm{cn}(\alpha; k)} \mathrm{dn}(\alpha n + ct; k), \quad c = \frac{2\mathrm{sn}(\alpha; k)}{\mathrm{cn}(\alpha; k)}, \quad \omega = \frac{2\mathrm{dn}(\alpha; k)}{\mathrm{cn}^2(\alpha; k)},$$

where $\alpha \in (0, K(k))$ and $k \in (0, 1)$ are arbitrary parameters.

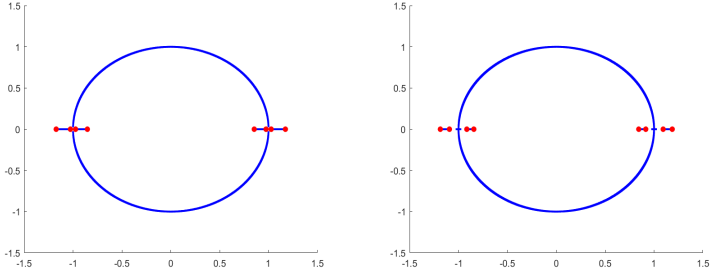
We can find explicitly $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\lambda_1 = \frac{1}{\mathrm{cn}(\alpha; k)} \sqrt{(1 - \mathrm{sn}(\alpha; k)) \left(\mathrm{dn}(\alpha; k) - \sqrt{1 - k^2}\mathrm{sn}(\alpha; k)\right)},$$

$$\lambda_2 = \frac{1}{\mathrm{cn}(\alpha; k)} \sqrt{(1 - \mathrm{sn}(\alpha; k)) \left(\mathrm{dn}(\alpha; k) + \sqrt{1 - k^2}\mathrm{sn}(\alpha; k)\right)},$$

satisfying $0 < \lambda_1 < \lambda_2 < 1 < \lambda_2^{-1} < \lambda_1^{-1}$.

Lax spectrum for dnoidal waves



The spectrum is found numerically for $\alpha = K(k)/M$ with $u_{n+2M} = u_n$:

$$\left\{ \begin{array}{l} \sqrt{1+u_n^2}\, p_{n+1} + \sqrt{1+u_{n-1}^2}\, p_{n-1} - (u_n - u_{n-1}) q_n = z p_n, \\ (u_n - u_{n-1}) p_n + \sqrt{1+u_n^2}\, q_{n+1} + \sqrt{1+u_{n-1}^2}\, q_{n-1} = z q_n, \end{array} \right.$$

where $z := \lambda + \lambda^{-1}$ and (p_n, q_n) is the eigenvector satisfying

$$p_n = \hat{p}_n(\theta) e^{i\theta n}, \quad q_n = \hat{q}_n(\theta) e^{i\theta n}, \quad \hat{p}_{n+2M}(\theta) = \hat{p}_n(\theta), \quad \hat{q}_{n+2M}(\theta) = \hat{q}_n(\theta),$$

Cnoidal periodic waves

These are solutions of the form

$$u_n(t) = \frac{k\,\mathrm{sn}(\alpha;k)}{\mathrm{dn}(\alpha;k)}\mathrm{cn}(\alpha n + ct;k), \quad c = \frac{2\mathrm{sn}(\alpha;k)}{\mathrm{dn}(\alpha;k)}, \quad \omega = \frac{2\mathrm{cn}(\alpha;k)}{\mathrm{dn}^2(\alpha;k)},$$

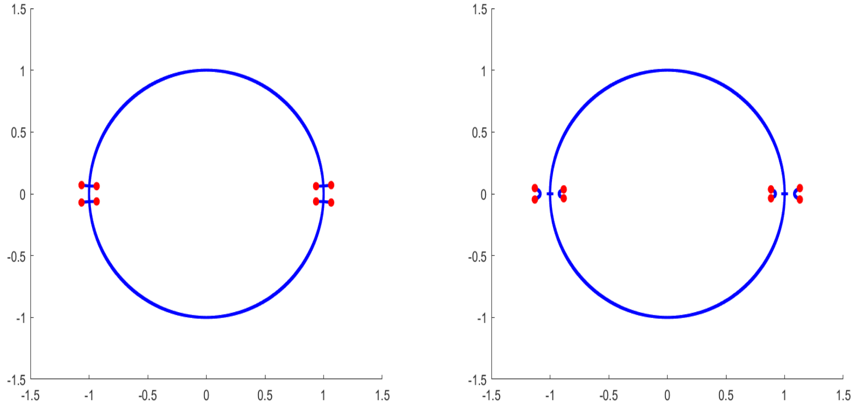
where $\alpha \in (0, K(k))$ and $k \in (0,1)$ are arbitrary parameters.

We can find explicitly $\lambda_1 = \bar{\lambda}_2 \in \mathbb{C}$

$$\lambda_1 = \frac{\sqrt{(1 - k\,\mathrm{sn}(\alpha;k))(\mathrm{cn}(\alpha;k) + i\sqrt{1-k^2}\mathrm{sn}(\alpha;k))}}{\mathrm{dn}(\alpha;k)},$$

satisfying $|\lambda_1| < 1 < |\lambda_1|^{-1}$.

Lax spectrum for cnoidal waves



Stability spectrum for traveling periodic waves

Let $u_n(t) = \phi(\alpha n + ct)$ be a traveling periodic wave of the discrete mKdV equation

$$\dot{u}_n = (1 + u_n^2)(u_{n+1} - u_{n-1}), \quad n \in \mathbb{Z}.$$

Let $\{v_n(t)\}_{n\in\mathbb{Z}}$ be a perturbation of $\{u_n(t)\}_{n\in\mathbb{Z}}$ satisfying the linearized discrete mKdV equation

$$\dot{v}_n = (1 + u_n^2)(v_{n+1} - v_{n-1}) + 2u_n(u_{n+1} - u_{n-1})v_n, \quad n \in \mathbb{Z}.$$

We have the squared eigenfunction relation by brutal computations:

$$v_n = \lambda p_n^2 - \lambda^{-1} q_n^2 + 2u_n p_n q_n,$$

where $\varphi_n = (p_n, q_n)^T$ is a solution of the linear Lax system.

Relation to squared eigenfunctions

Thus, $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ is a solution of the linear Lax system:

$$\varphi_{n+1} = \frac{1}{\sqrt{1+u_n^2}} \left(\begin{array}{cc} \lambda & u_n \\ -u_n & \lambda^{-1} \end{array} \right) \varphi_n$$

and

$$\dot{\varphi}_n = \left(\begin{array}{cc} \frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) & \lambda u_n + \lambda^{-1} u_{n-1} \\ -\lambda u_{n-1} - \lambda^{-1} u_n & -\frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) \end{array} \right) \varphi_n,$$

with the obvious decomposition since $u_n(t) = \phi(\alpha n + ct)$:

$$\varphi_n(t) = \psi(\alpha n + ct) e^{\Omega t}.$$

3rd Missing point. An explicit relation between Ω and $P(\lambda)$ is missing,

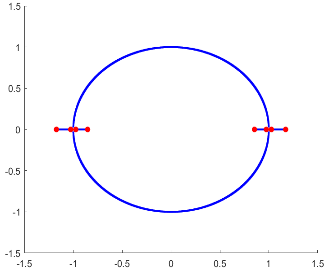
$$P(\lambda) := \lambda^8 - 2\omega\lambda^6 + (2 + \omega^2 - 4F_1^2)\lambda^4 - 2\omega\lambda^2 + 1.$$

As a result, the admissible values of Ω must be found numerically.

Stability of the dnoidal periodic waves

$$u_n(t) = \frac{\mathrm{sn}(\alpha;k)}{\mathrm{cn}(\alpha;k)} \mathrm{dn}(\alpha n + ct; k), \quad c = \frac{2\mathrm{sn}(\alpha;k)}{\mathrm{cn}(\alpha;k)},$$

where $\alpha \in (0, K(k))$ and $k \in (0,1)$ are arbitrary parameters.



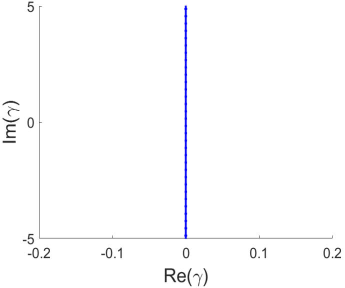


Figure: Lax spectrum (left) and stability spectrum (right) for $k = 0.7$.

Stability of the cnoidal periodic waves

$$u_n(t) = \frac{k\,\mathrm{sn}(\alpha;k)}{\mathrm{dn}(\alpha;k)}\mathrm{cn}(\alpha n + ct;k), \quad c = \frac{2\mathrm{sn}(\alpha;k)}{\mathrm{dn}(\alpha;k)},$$

where $\alpha \in (0, K(k))$ and $k \in (0,1)$ are arbitrary parameters.

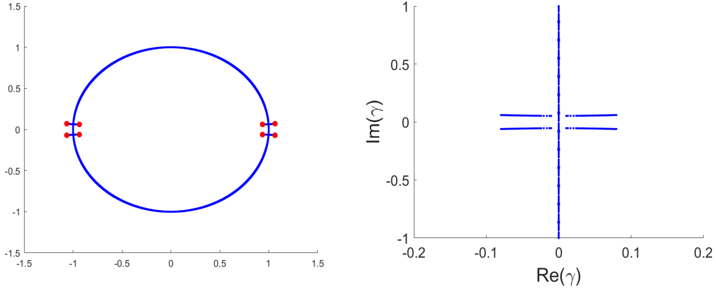


Figure: Lax spectrum (left) and stability spectrum (right) for $k = 0.7$.

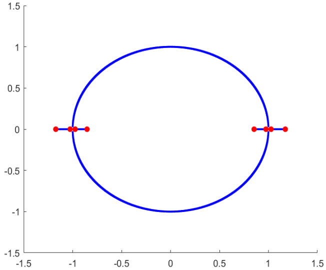
4^{th} Missing point. Numerical accuracy is not very good.

1-fold Darboux transformation

We can use eigenvalues found in the nonlinearization method to define a new solution to the discrete mKdV equation:

$$\hat{u}_n = -\frac{p_n^2 + \lambda_1^2 q_n^2}{\lambda_1^2 p_n^2 + q_n^2} u_n + \frac{(1-\lambda_1^4) p_n q_n}{\lambda_1(\lambda_1^2 p_n^2 + q_n^2)}$$

where $\varphi_n = (p_n, q_n)^T$ is a solution of Lax equations with $\lambda = \lambda_1$.



Trivial new solution

$$\varphi_{n+1} = \frac{1}{\sqrt{1+u_n^2}} \left(\begin{array}{cc} \lambda & u_n \\ -u_n & \lambda^{-1} \end{array} \right) \varphi_n$$

and

$$\dot{\varphi}_n = \left(\begin{array}{cc} \frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) & \lambda u_n + \lambda^{-1} u_{n-1} \\ -\lambda u_{n-1} - \lambda^{-1} u_n & -\frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right) \end{array} \right) \varphi_n.$$

If $\varphi_n = (p_n, q_n)^T$ is obtained from $u_n = \lambda_1 p_n^2 + \lambda_1^{-1} q_n^2$ and λ_1 is a root of $P(\lambda)$, then new solution $\hat{u}_n$ is a half-period translation of the dnoidal wave:

$$\begin{aligned} \hat{u}_n &= -F_1 u_n^{-1} \\ &= -\frac{\sigma_1 \mathrm{sn}(\alpha;k)}{\mathrm{cn}(\alpha;k)} \frac{\sqrt{1-k^2}}{\mathrm{dn}(\xi;k)} \\ &= -\frac{\sigma_1 \mathrm{sn}(\alpha;k)}{\mathrm{cn}(\alpha;k)} \mathrm{dn}(\xi + K(k);k) \\ &= -\sigma_1 u_n(t + c^{-1}K(k)). \end{aligned}$$

Nontrivial new solution

The second, linearly independent solution can be found in the form:

$$\hat{p}_n = p_n \theta_n - \frac{q_n}{p_n^2 + q_n^2}, \quad \hat{q}_n = q_n \theta_n + \frac{p_n}{p_n^2 + q_n^2},$$

where

$$\theta_{n+1} - \theta_n = \frac{(\lambda_1 + \lambda_1^{-1})^2 (u_n^2 - F_1)}{(u_n + u_{n-1})(u_n + u_{n+1})(1 + u_n^2)}$$

and

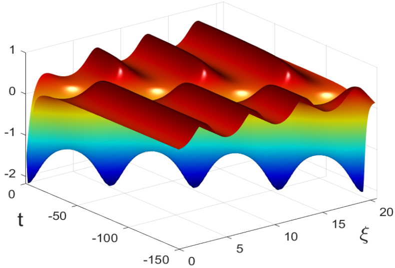
$$\dot{\theta}_n = \frac{(\lambda_1 + \lambda_1^{-1})^2 (u_n^2 + u_{n-1}^2 - 2F_1)}{(u_n + u_{n-1})^2}.$$

If $u_n(t) = \phi(\alpha n + ct)$ is the traveling wave with periodic ϕ, then $\theta_n(t) = an + bt + \chi(\alpha n + ct)$ with periodic χ and uniquely computed parameters a and b.

Algebraic soliton propagating on the dnoidal wave

The new solution is now nontrivial:

$$\hat{u}_n = -\frac{\hat{p}_n^2 + \lambda_1^2 \hat{q}_n^2}{\lambda_1^2 \hat{p}_n^2 + \hat{q}_n^2} u_n + \frac{(1-\lambda_1^4)\hat{p}_n \hat{q}_n}{\lambda_1(\lambda_1^2 \hat{p}_n^2 + \hat{q}_n^2)}$$



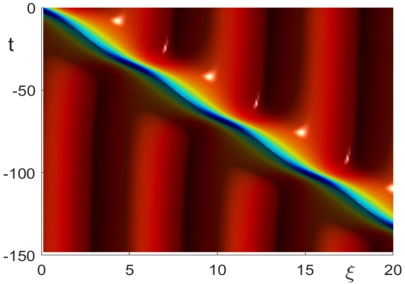


Figure: The solution surface (left: sideview, right: topview) for λ_1.

Algebraic soliton propagating on the dnoidal wave

The new solution is now nontrivial:

$$\hat{u}_n = -\frac{\hat{p}_n^2 + \lambda_1^2 \hat{q}_n^2}{\lambda_1^2 \hat{p}_n^2 + \hat{q}_n^2} u_n + \frac{(1 - \lambda_1^4)\hat{p}_n \hat{q}_n}{\lambda_1(\lambda_1^2 \hat{p}_n^2 + \hat{q}_n^2)}$$

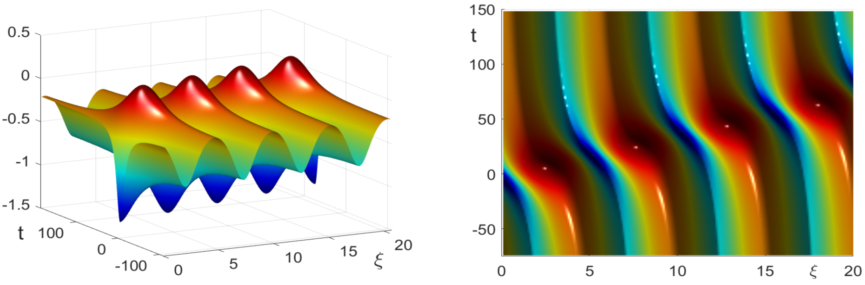


Figure: The solution surface (left: sideview, right: topview) for λ_2.

2-fold Darboux transformation

The 2-fold transformation uses two eigenvalues λ_1 and λ_2:

$$\hat{u}_n = \frac{\Upsilon_n}{\Delta_n} u_n - \frac{\Sigma_n}{\lambda_1 \lambda_2 \Delta_n},$$

where

$$\begin{aligned}
\Upsilon_n &= \lambda_2^2 (q_{2n}^2 + \lambda_2^2 p_{2n}^2)(p_{1n}^2 + \lambda_1^6 q_{1n}^2) + \lambda_1^2 (q_{1n}^2 + \lambda_1^2 p_{1n}^2)(p_{2n}^2 + \lambda_2^6 q_{2n}^2) \\
&\quad - 2\lambda_1^2 \lambda_2^2 (p_{1n}^2 + \lambda_1^2 q_{1n}^2)(p_{2n}^2 + \lambda_2^2 q_{2n}^2) - 2 p_{1n} q_{1n} p_{2n} q_{2n} \lambda_1 \lambda_2 (\lambda_1^4 - 1)(\lambda_2^4 - 1), \\
\Sigma_n &= (\lambda_1^2 - \lambda_2^2)(\lambda_1^2 \lambda_2^2 - 1)[\lambda_1 (\lambda_2^4 - 1) p_{2n} q_{2n} (q_{1n}^2 + \lambda_1^2 p_{1n}^2) \\
&\quad - \lambda_2 (\lambda_1^4 - 1) p_{1n} q_{1n} (q_{2n}^2 + \lambda_2^2 p_{2n}^2)], \\
\Delta_n &= (\lambda_1^2 \lambda_2^2 - 1)^2 (\lambda_1^2 p_{1n}^2 q_{2n}^2 + \lambda_2^2 p_{2n}^2 q_{1n}^2) + (\lambda_1^2 - \lambda_2^2)^2 (\lambda_1^2 \lambda_2^2 p_{1n}^2 p_{2n}^2 + q_{1n}^2 q_{2n}^2) \\
&\quad - 2 p_{1n} q_{1n} p_{2n} q_{2n} \lambda_1 \lambda_2 (\lambda_1^4 - 1)(\lambda_2^4 - 1).
\end{aligned}$$

Two algebraic solitons on the dnoidal wave

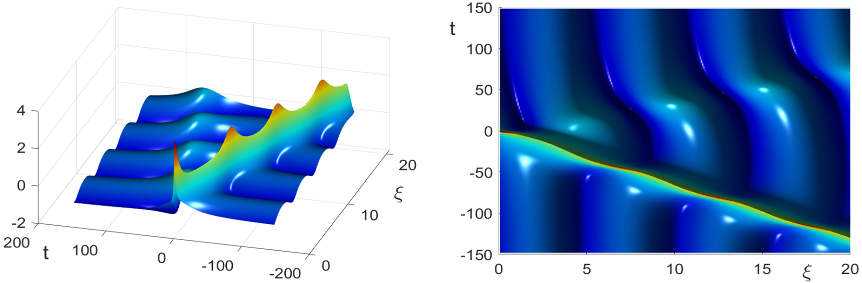


Figure: The solution surface (left: sideview, right: topview) for eigenvalues λ_1 and λ_2.

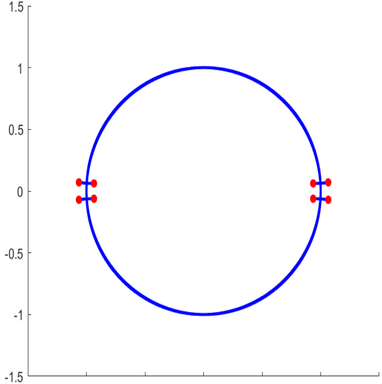
5th Missing point. Is there a completeness result that no more than two algebraic solitons could propagate on the dnoidal wave background?

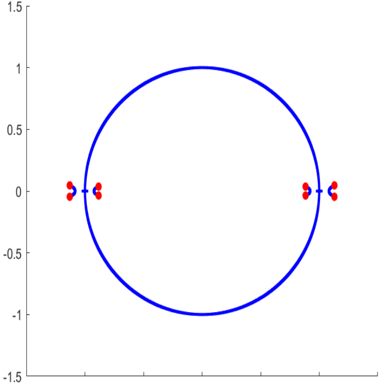
Similar new solutions for the cnoidal wave

For the cnoidal wave, the new solution after 2-fold transformation is real valued if $\lambda_2 = \bar{\lambda}_1$. However, $p_n^2 + q_n^2$ is not sign-definite and the representation

$$\hat{p}_n = p_n\theta_n - \frac{q_n}{p_n^2 + q_n^2}, \quad \hat{q}_n = q_n\theta_n + \frac{p_n}{p_n^2 + q_n^2}$$

cannot be used.





Another representation

The second, linearly independent solution can be found in the form:

$$\hat{p}_n = p_n \theta_n - \frac{1}{2q_n}, \quad \hat{q}_n = q_n \theta_n + \frac{1}{2p_n},$$

where

$$\theta_{n+1} - \theta_n = \frac{(\lambda_1^2 - \lambda_1^{-2})^2 u_n^2}{2(1+u_n^2)(F_1 - u_n u_{n-1})(F_1 - u_{n+1} u_n)}$$

and

$$\dot{\theta}_n = \frac{(\lambda_1^2 - \lambda_1^{-2})^2 u_n u_{n-1}}{(F_1 - u_n u_{n-1})^2},$$

If $u_n(t) = \phi(\alpha n + ct)$ is the traveling wave with periodic ϕ, then $\theta_n(t) = an + bt + \chi(\alpha n + ct)$ with periodic χ and uniquely computed parameters a and b.

Rogue wave on the cnoidal wave

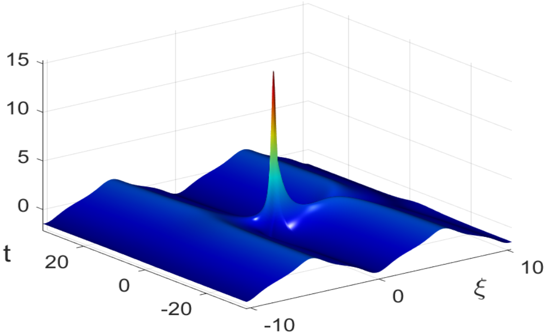


Figure: The solution surface for eigenvalues λ_1 and $\lambda_2 = \bar{\lambda}_1$.

6th Missing point. Do multi-fold transformations exist with higher-order rogue waves on the cnoidal wave background?

Summary

- Traveling periodic waves are recovered from the nonlinearization method based on the constraint $u_n = \lambda_1 p_n^2 + \lambda_1^{-1} q_n^2$ with λ_1 being a root of $P(\lambda)$.
- Dnoidal waves are spectrally (modulationally) stable, whereas cnoidal waves are spectrally (modulationally) unstable.
- Only two distinct algebraic solitons exist on the background of dnoidal waves. A rogue wave exists on the background of cnoidal waves.
- Open questions include
  1. relation between $(1+u_n^2)(u_{n+1}+u_{n-1}) = \omega u_n$ and $\dot{u}_n = (1+u_n^2)(u_{n+1}-u_{n-1})$
  2. connection between $P(\lambda)$ and the stability spectrum Ω.
  3. existence and properties of higher-order rogue waves.

Many thanks for your attention!