Persistence and stability of discrete vortices

Dmitry Pelinovsky
Department of Mathematics, McMaster University, Canada

$$i\dot{u}_{n,m} + \epsilon\left(u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1}\right) + |u_{n,m}|^2 u_{n,m} = 0$$

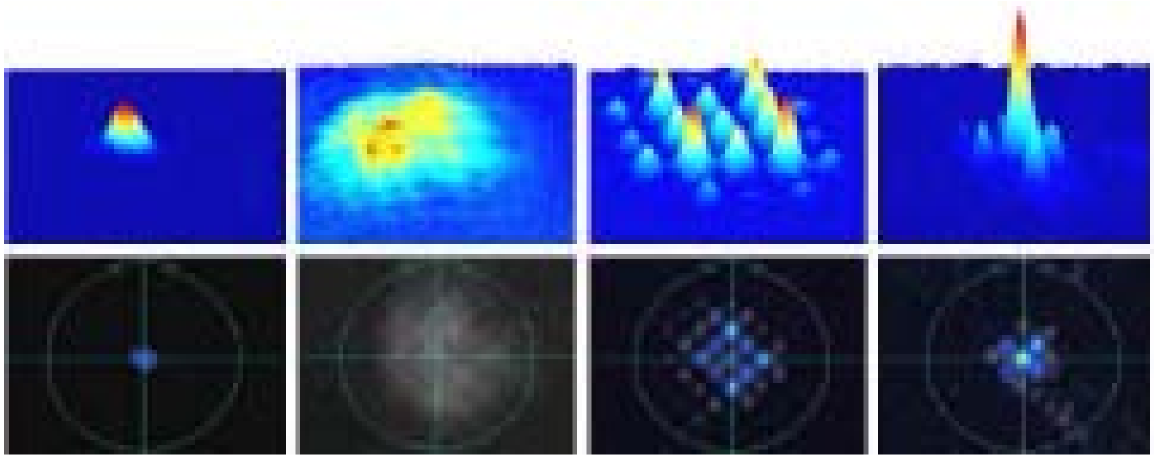
With P. Kevrekidis (University of Massachusetts at Amherst)

Physica D 212, 1–19 (2005)
Physica D 212, 20–53 (2005)
Proc. Roy. Soc. Lond. A 462, 2671–2694 (2006)
Physica D, submitted (2007)

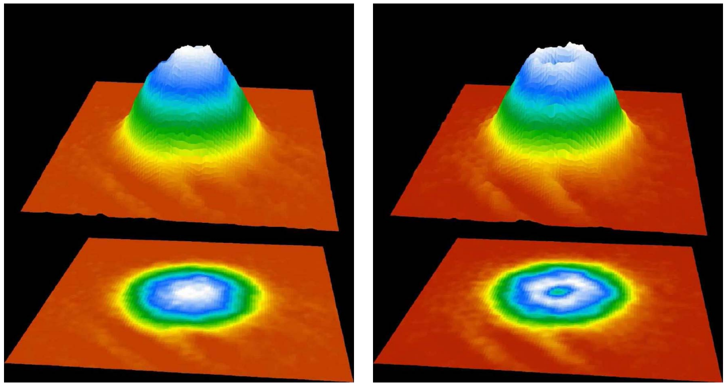
Cuernavaca, Mexico, January 12, 2007

Experimental pictures

- Discrete solitons



- Discrete vortices



Main Formalism

$$\textbf{1D}: \qquad i\dot{u}_n + \epsilon\left(u_{n+1} - 2u_n + u_{n-1}\right) + |u_n|^2 u_n = 0, \qquad n \in \mathbb{Z}$$

- Vector space $\Omega = L^2(\mathbb{Z}, \mathbb{C})$ for $\{u_n\}_{n\in\mathbb{Z}}$:

$$(\mathbf{u}, \mathbf{w})_\Omega = \sum_{n\in\mathbb{Z}} \bar{u}_n w_n, \qquad \|\mathbf{u}\|_\Omega^2 = \sum_{n\in\mathbb{Z}} |u_n|^2 < \infty.$$

- Hamiltonian formulation:

$$i\dot{u}_n = \frac{\partial H}{\partial \bar{u}_n}, \qquad H = \sum_{n\in\mathbb{Z}} \epsilon|u_{n+1} - u_n|^2 - \frac{1}{2}|u_n|^4$$

- Existence problem for time-periodic solutions

$$u_n(t) = \phi_n e^{i(\mu - 2\epsilon)t + i\theta_0}, \qquad \mu \in \mathbb{R}, \quad \theta_0 \in \mathbb{R}$$

such that

$$(\mu - |\phi_n|^2)\phi_n = \epsilon\left(\phi_{n+1} + \phi_{n-1}\right).$$

- Stability problem for time-periodic solutions

$$u_n(t) = e^{i(1-2\epsilon)t + i\theta_0}\left(\phi_n + (u_n + iw_n)e^{\lambda t} + (\bar{u}_n + i\bar{w}_n)e^{\bar{\lambda} t}\right)$$

such that

$$\left(1 - 3\phi_n^2\right)u_n - \epsilon\left(u_{n+1} + u_{n-1}\right) = -\lambda w_n,$$
$$\left(1 - \phi_n^2\right)w_n - \epsilon\left(w_{n+1} + w_{n-1}\right) = \lambda u_n.$$

where $\lambda \in \mathbb{C}$ and $(\mathbf{u}, \mathbf{w}) \in \Omega \times \Omega$

Existence problem in one dimension

$$(\mu - |\phi_n|^2)\phi_n = \epsilon\,(\phi_{n+1} + \phi_{n-1})$$

- All localized solutions for $\epsilon \neq 0$ are real-valued: $\boldsymbol{\phi} \in L^2(\mathbb{Z}, \mathbb{R})$

$$\bar{\phi}_n \phi_{n+1} - \phi_n \bar{\phi}_{n+1} = \text{const} \quad n \in \mathbb{Z}$$

$$\frac{\phi_{n+1}}{\bar{\phi}_{n+1}} = \frac{\phi_n}{\bar{\phi}_n} : \qquad 2\arg(\phi_{n+1}) = 2\arg(\phi_n) = \text{mod}(2\pi)$$

- There exists a transformation from $\epsilon < 0$ to $\epsilon > 0$

$$\phi_n \mapsto (-1)^n \phi_n, \qquad \epsilon \mapsto -\epsilon$$

Existence problem in one dimension

$$(\mu - |\phi_n|^2)\phi_n = \epsilon\,(\phi_{n+1} + \phi_{n-1})$$

- There exists a spectral band for $|\mu| \leq 2\epsilon$:

$$\phi_n = e^{ikn}: \quad \mu = \mu(k) = 2\epsilon\cos k, \qquad k \in \mathbb{R}$$

- Localized solutions do not exist for $\mu < -2\epsilon < 0$:

$$-(|\mu| - 2\epsilon)\sum_{n\in\mathbb{Z}} \phi_n^2 - \sum_{n\in\mathbb{Z}} \phi_n^4 = \epsilon \sum_{n\in\mathbb{Z}} (\phi_{n+1} + \phi_n)^2$$

- Scaling transformation for localized solutions with $\mu > 2\epsilon > 0$:

$$\phi_n = \sqrt{\mu}\hat{\phi}_n, \qquad \epsilon = \mu\hat{\epsilon}$$

Existence problem in one dimension

$$(1 - \phi_n^2)\phi_n = \epsilon\,(\phi_{n+1} + \phi_{n-1})$$

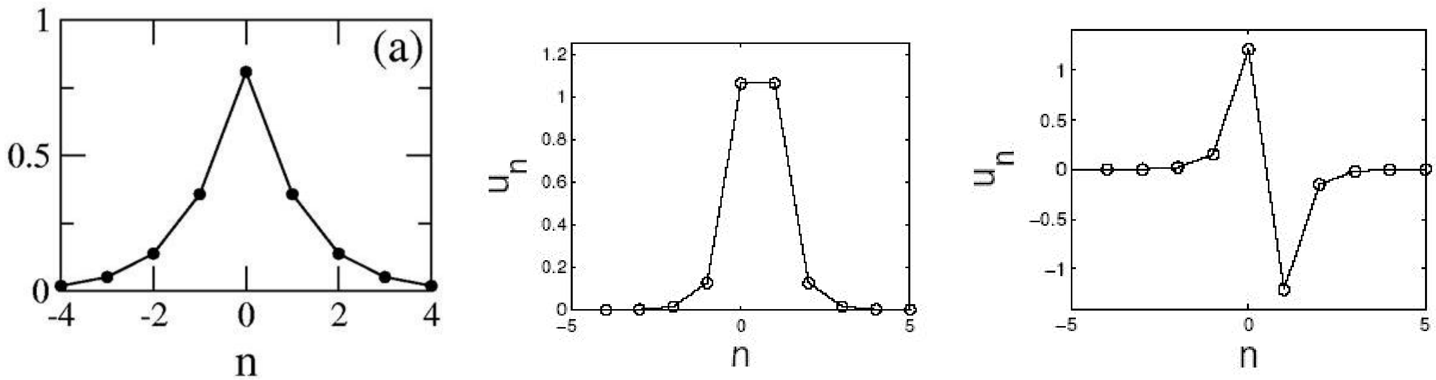
- There exists an analytic function $\boldsymbol{\phi}(\epsilon)$ for $0 < \epsilon < \epsilon_0$:

$$\lim_{\epsilon \to 0} \phi_n = \begin{cases} \pm 1, & n \in S, \\ 0, & n \in \mathbb{Z} \backslash S, \end{cases} \qquad \dim(S) < \infty$$

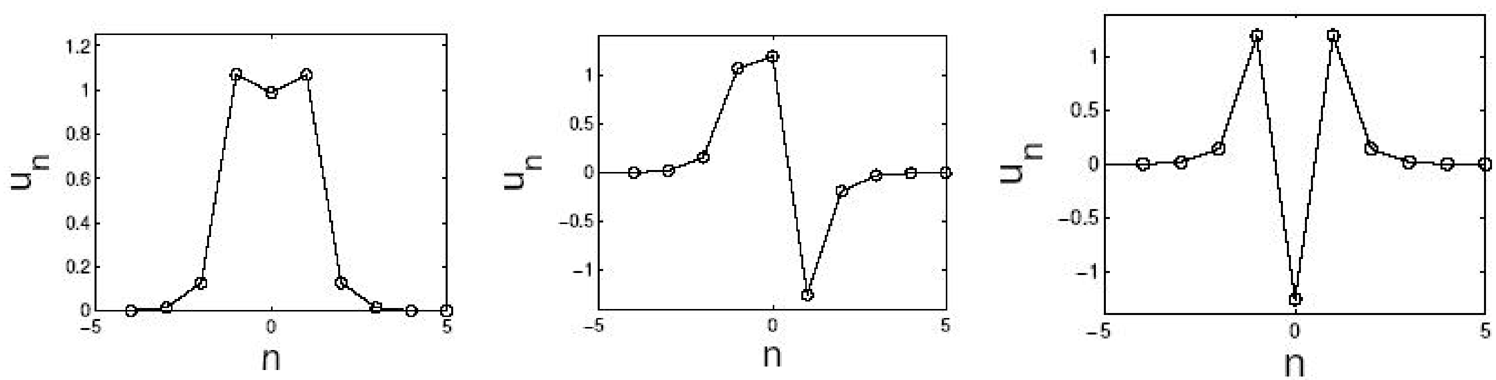
$$\lim_{|n| \to \infty} e^{\kappa |n|} |\phi_n| = \phi_\infty, \qquad \kappa > 0, \quad \phi_\infty > 0.$$

- MacKay, Aubry (1994): inverse function theorem
- Hennig, Tsironis (1999): bounds on ϵ_0
- Bergamin, Bountis (2000): symbolic dynamics for invertible maps
- Alfimov, Konotop (2004): complete classification of localized modes

- **Fundamental and two-node modes**



- **Three-node modes**



Stability problem in one dimension

$$\left(1 - 3\phi_n^2\right) u_n - \epsilon \left(u_{n+1} + u_{n-1}\right) = -\lambda w_n,$$
$$\left(1 - \phi_n^2\right) w_n - \epsilon \left(w_{n+1} + w_{n-1}\right) = \lambda u_n.$$

- Matrix-vector form for $(\mathbf{u}, \mathbf{w}) \in L^2(\mathbb{Z}, \mathbb{C}^2)$

$$\mathcal{L}_+ \mathbf{u} = -\lambda \mathbf{w}, \qquad \mathcal{L}_- \mathbf{w} = \lambda \mathbf{u},$$

- Hamiltonian form for $\boldsymbol{\psi} = (\mathbf{u}, \mathbf{w})$:

$$\mathcal{J}\mathcal{H}\boldsymbol{\psi} = \lambda \boldsymbol{\psi}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}.$$

Splitting of zero eigenvalues

Eigenvalues of $\mathcal{H}$ at $\epsilon = 0$:

- $\gamma = -2$ of multiplicity N
- $\gamma = 0$ of multiplicity N
- $\gamma = +1$ of multiplicity ∞

Eigenvalues of $\mathcal{J}\mathcal{H}$ at $\epsilon = 0$:

- $\lambda = 0$ of multiplicity $2N$
- $\lambda = +i$ of multiplicity ∞
- $\lambda = -i$ of multiplicity ∞

Lemma: Let γ_j be small eigenvalues of $\mathcal{H}$ as $\epsilon \to 0$. There exists N pairs of small eigenvalues λ_j and $-\lambda_j$ of $\mathcal{J}\mathcal{H}$:

$$\lim_{\epsilon \to 0} \gamma_j = 0, \qquad \lim_{\epsilon \to 0} \frac{\lambda_j^2}{\gamma_j} = 2, \qquad 1 \leq j \leq N.$$

Corollary:

When $\gamma_j > 0$, there exists one unstable EV $\lambda_j > 0$.

When $\gamma_j < 0$, there exists one pair $\lambda_j \in i\mathbb{R}$ with negative Krein signature:

$$(\boldsymbol{\psi}, \mathcal{H}\boldsymbol{\psi}) = (\mathbf{u}, \mathcal{L}_+\mathbf{u}) + (\mathbf{w}, \mathcal{L}_-\mathbf{w}) = 2\,(\mathbf{w}, \mathcal{L}_-\mathbf{w}) < 0.$$

Count of small eigenvalues of $\mathcal{H}$

Lemma: Let n_0 be the number of sign-differences in the vector $\boldsymbol{\phi}$ at $\epsilon = 0$. There exists n_0 negative eigenvalues γ_j and $N - n_0 - 1$ positive eigenvalues γ_j for any $\epsilon \neq 0$.

Proof:

- By discrete Sturm Theorem, $\#_{<0}(\mathcal{L}_-) = n_0$, since

$$\mathcal{L}_-\boldsymbol{\phi} = \mathbf{0}.$$

- By theory of difference equations, $\dim(\mathcal{L}_-) = 1$ for any $\epsilon \neq 0$, since

$$\mathcal{L}_-\mathbf{w} = \mathbf{0}, \qquad \mathbf{w} = c_1\boldsymbol{\phi} + c_2\mathbf{w}_2.$$

- By our analysis, the number of sign-differences in the vector $\boldsymbol{\phi}$ is continuous at $\epsilon = 0$.

Count of unstable eigenvalues of $\mathcal{JH}$

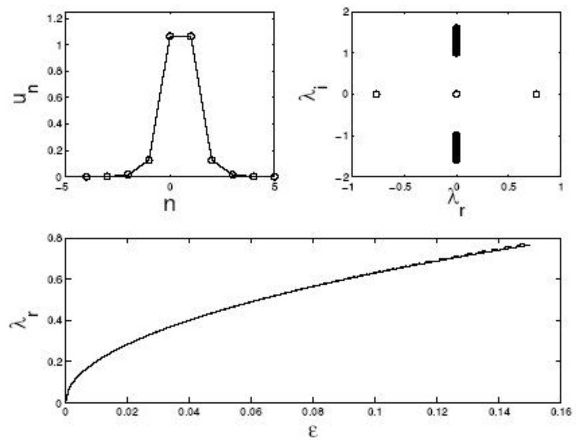
$$N_{\text{real}} = N - 1 - n_0, \qquad N^-_{\text{imag}} = n_0, \qquad N_{\text{comp}} = 0$$

Theorem: The only stable N-pulse discrete soliton near $\epsilon = 0$ is the soliton with an alternating sequence of up and down pulses.

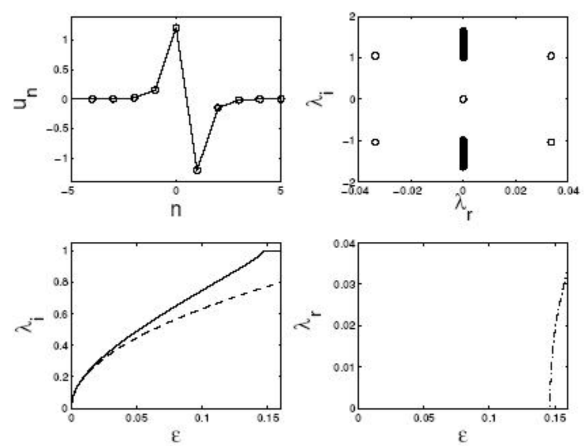
- Weinstein (1999): stability of discrete soliton with $N = 1$
- Kapitula, Kevrekidis, Malomed (2001):
  instabilities of twisted modes and other multi-pulse solitons
- Morgante, Johansson, Kopidakis, Aubry (2002):
  numerical analysis of instabilities of multi-pulse solitons with $N > 1$
- Sandstede, Jones, Alexander (1997): analysis of the orbit-flip bifurcation and multi-pulse homoclinic orbits

Numerical analysis of discrete solitons

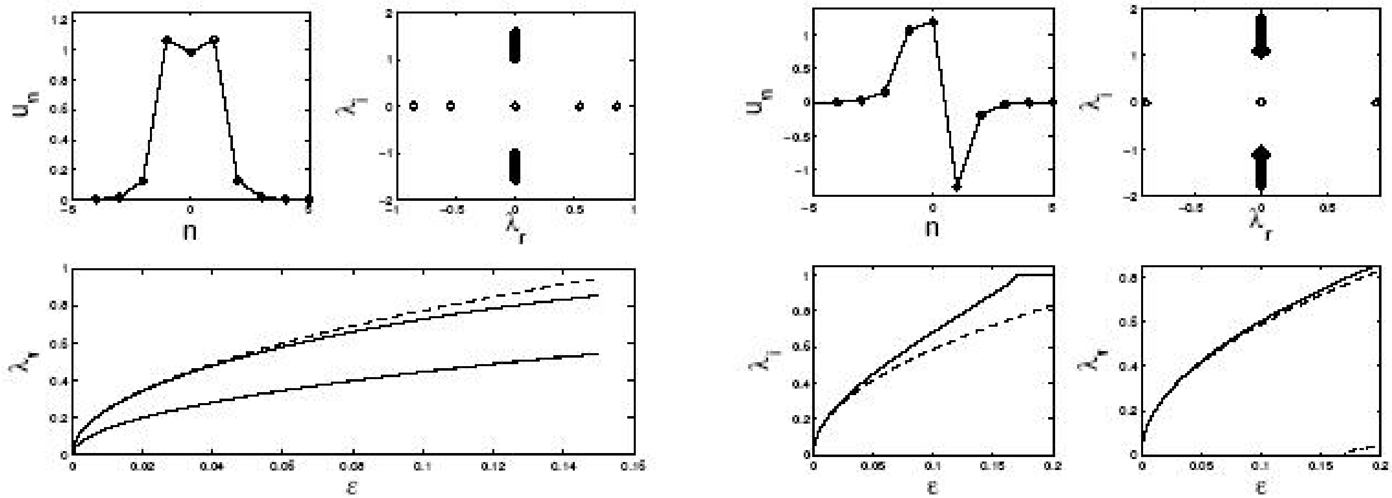
- Page mode

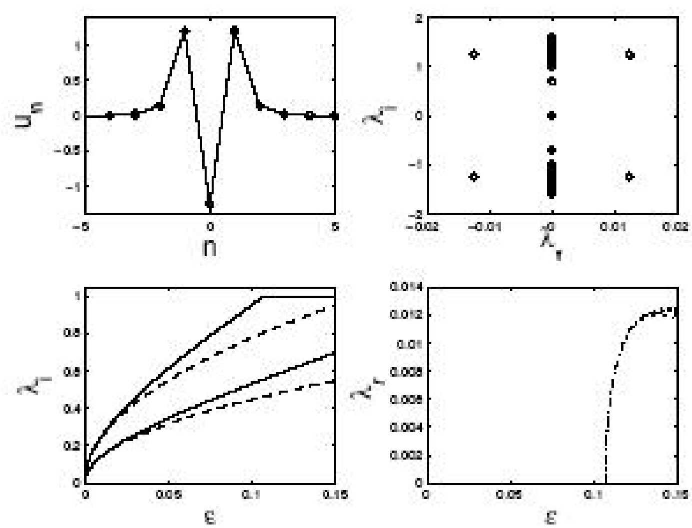


- Twisted mode



• Three-node modes





Discrete NLS equations

- Scalar NLS equation

$$i\dot{u}_{n,m} + \epsilon \left(u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1}\right) + |u_{n,m}|^2 u_{n,m} = 0$$

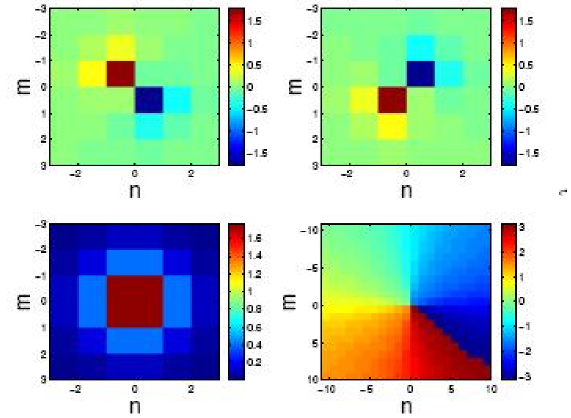
- Vector NLS equation

$$\begin{aligned} i\dot{u}_{n,m} + \epsilon\Delta u_{n,m} + \left(|u_{n,m}|^2 + \beta|v_{n,m}|^2\right) u_{n,m} &= 0, \\ i\dot{v}_{n,m} + \epsilon\Delta v_{n,m} + \left(\beta|u_{n,m}|^2 + |v_{n,m}|^2\right) v_{n,m} &= 0, \end{aligned}$$

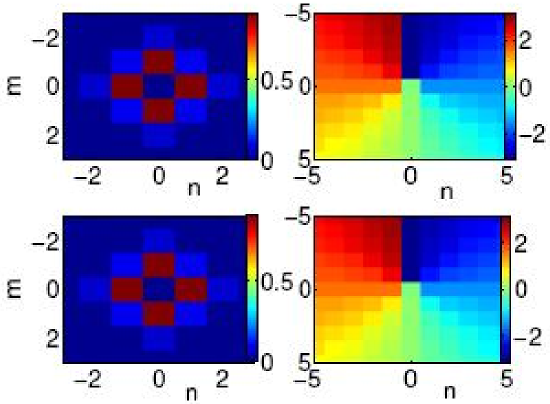
where $\epsilon > 0$ and $\beta > 0$ are parameters.

Numerical pictures

- Off-site vortex (vortex cell) on a square contour



- On-site vector vortex (vortex cross) on a diagonal contour

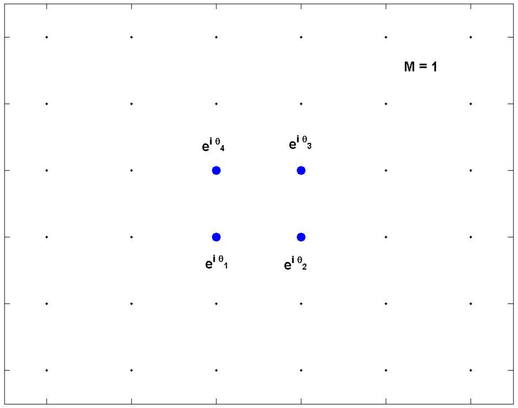


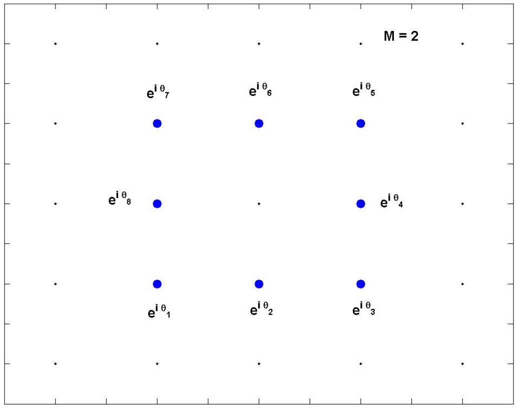
Existence problem

$$(1 - |\phi_{n,m}|^2)\phi_{n,m} = \epsilon \left(\phi_{n+1,m} + \phi_{n-1,m} + \phi_{n,m+1} + \phi_{n,m-1}\right)$$

Limiting solution:

$$\epsilon = 0 : \quad \phi_{n,m}^{(0)} = \begin{cases} e^{i\theta_{n,m}}, & (n,m) \in S, \\ 0, & (n,m) \in \mathbb{Z}^2 \backslash S, \end{cases}$$





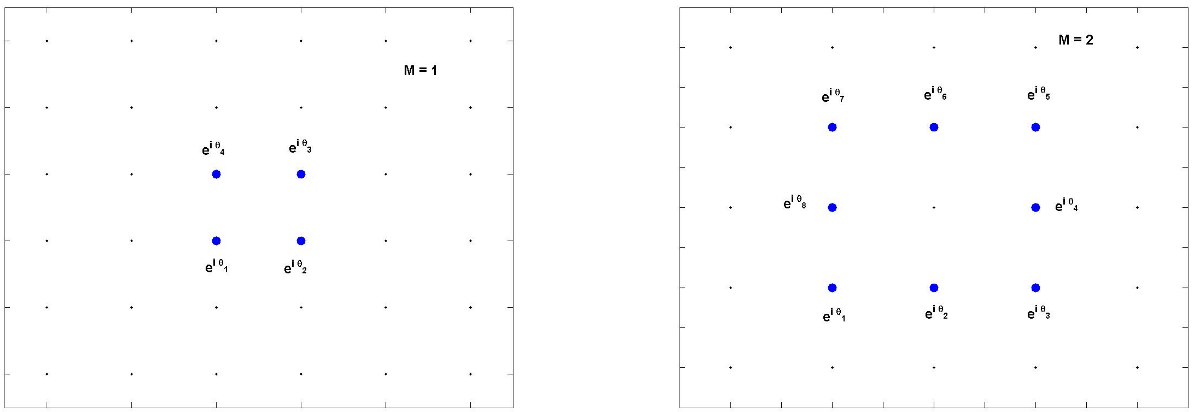
Examples of a square discrete contour S

Existence problem

$$(1 - |\phi_{n,m}|^2)\phi_{n,m} = \epsilon \left(\phi_{n+1,m} + \phi_{n-1,m} + \phi_{n,m+1} + \phi_{n,m-1}\right)$$

Limiting solution:

$$\epsilon = 0: \quad \phi^{(0)}_{n,m} = \begin{cases} e^{i\theta_{n,m}}, & (n,m) \in S, \\ 0, & (n,m) \in \mathbb{Z}^2 \backslash S, \end{cases}$$



Examples of a square discrete contour S

Which configurations $\theta_{n,m}$ can be continued for $\epsilon \neq 0$?

Proposition: Let $N = \dim(S)$ and $\mathcal{T}$ be the torus on $[0, 2\pi]^N$. There exists a vector-valued function $\mathbf{g} : \mathcal{T} \mapsto \mathbb{R}^N$, such that the limiting solution is continued to $\epsilon \neq 0$ if and only if $\boldsymbol{\theta} \in \mathcal{T}$ is a root of $\mathbf{g}(\boldsymbol{\theta}, \epsilon) = \mathbf{0}$.

- The Jacobian of the nonlinear system:

$$\mathcal{H} = \begin{pmatrix} 1 - 2|\phi_{n,m}|^2 & -\phi_{n,m}^2 \\ -\bar{\phi}_{n,m}^2 & 1 - 2|\phi_{n,m}|^2 \end{pmatrix} - \epsilon \left(\delta_{+1,0} + \ldots + \delta_{0,-1}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- $\mathcal{H}$ is a self-adjoint Fredholm operator of index zero:

$$\dim(\ker(\mathcal{H}^{(0)}) = N$$

Properties of $\mathbf{g}(\boldsymbol{\theta})$

- $\mathbf{g}$ is analytic, such that

$$\mathbf{g}(\boldsymbol{\theta}, \epsilon) = \sum_{k=1}^{\infty} \epsilon^k \mathbf{g}^{(k)}(\boldsymbol{\theta})$$

- $\mathbf{g}$ has gauge symmetry, such that

$$\mathbf{g}(\boldsymbol{\theta}_*, \epsilon) = \mathbf{0} \quad \mapsto \quad \mathbf{g}(\boldsymbol{\theta}_* + \theta_0 \mathbf{1}, \epsilon) = \mathbf{0}$$

- If $\mathbf{g}^{(1)}(\boldsymbol{\theta}_*) = \mathbf{0}$ and $\mathcal{M}_1 = \mathcal{D}\mathbf{g}^{(1)}(\boldsymbol{\theta}_*)$ has a kernel with eigenvector $\mathbf{1}$, there exists a unique continuation of the limiting solution for $\epsilon \neq 0$.

- If $\mathbf{g}^{(1)}(\boldsymbol{\theta}_*) = \mathbf{0}$, the kernel of $\mathcal{M}_1$ is $(1 + d)$-dimensional, and

$$\Big(\mathbf{g}^{(2)}(\boldsymbol{\theta}_*), \ker(\mathcal{M}_1)\Big) \neq 0,$$

the limiting solution can not be continued to $\epsilon \neq 0$.

First-order reductions : classification of solutions

$$\mathbf{g}_j^{(1)}(\boldsymbol{\theta}) = \sin(\theta_j - \theta_{j+1}) + \sin(\theta_j - \theta_{j-1}) = 0, \quad 1 \leq j \leq 4M$$

- (1) Discrete solitons

$$\theta_j = \{0, \pi\}, \qquad 1 \leq j \leq 4M$$

- (2) Symmetric vortices of charge L

$$\theta_j = \frac{\pi L(j-1)}{2M}, \qquad 1 \leq j \leq 4M,$$

- (3) One-parameter asymmetric vortices of charge $L = M$

$$\theta_{j+1} - \theta_j = \left\{ \begin{array}{c} \theta \\ \pi - \theta \end{array} \right\} \mathrm{mod}(2\pi), \qquad 1 \leq j \leq 4M$$

where M is number of nodes at each side of the square contour and L is the vortex charge along the discrete contour.

First-order reductions : persistence of solutions

$$\mathcal{M}_1 = \mathcal{D}\mathbf{g}^{(1)}(\boldsymbol{\theta}) = \begin{pmatrix} a_1 + a_2 & -a_2 & 0 & \dots & a_1 \\ -a_2 & a_2 + a_3 & -a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -a_1 & 0 & 0 & \dots & a_{N-1} + a_N \end{pmatrix},$$

where $a_j = \cos(\theta_{j+1} - \theta_j)$

- $\mathcal{M}_1$ has a simple zero eigenvalue if all $a_j \neq 0$ and
$$\left(\prod_{i=1}^{N} a_i\right) \left(\sum_{i=1}^{N} \frac{1}{a_i}\right) \neq 0.$$
Family (1) persists for $\epsilon \neq 0$.

- If all $a_j = a = \cos(\frac{\pi L}{2M})$, eigenvalues of $\mathcal{M}_1$ are:
$$\lambda_n = 4a \sin^2 \frac{\pi n}{4M}, \qquad 1 \leq n \leq 4M$$
Family (2) persists for $\epsilon \neq 0$ and $L \neq M$.

Second-order reductions : termination of solutions

- If all $a_j = \pm a = \cos\theta$, there are $2M-1$ negative eigenvalues of $\mathcal{M}_1$, 2 zero eigenvalues and $2M-1$ positive eigenvalues of $\mathcal{M}_1$.

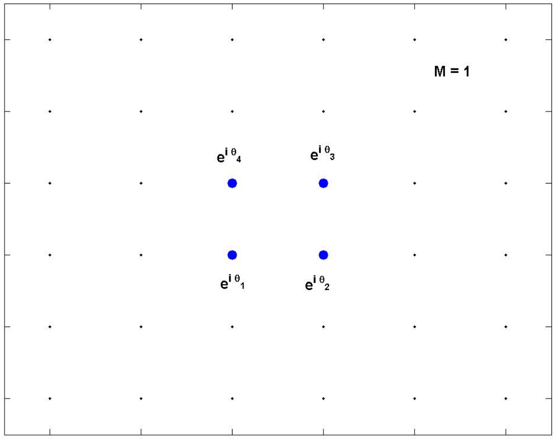
- Persistence of family (3) depends on $\mathbf{g}^{(2)}(\boldsymbol{\theta})$

$$\mathbf{g}_j^{(2)} = \frac{1}{2}\sin(\theta_{j+1}-\theta_j)\left[\cos(\theta_j-\theta_{j+1})+\cos(\theta_{j+2}-\theta_{j+1})\right] + \frac{1}{2}\sin(\theta_{j-1}-\theta_j)\left[\cos(\theta_j-\theta_{j-1})+\cos(\theta_{j-2}-\theta_{j-1})\right]$$

- We have $(\mathbf{g}^{(2)}, \ker(\mathcal{M}_1)) \neq 0$ for all members of family (3) excluding the only configuration:

$$\theta_1 = 0, \quad \theta_2 = \theta, \qquad \theta_3 = \pi, \quad \theta_4 = \pi + \theta.$$

Higher-order reductions : termination of the last family



- Symbolic software algorithm is used on a squared domain of N_0-by-N_0 lattice nodes, where $N_0 = 2K + 2M + 1$, and K is the order of the Lyapunov-Schmidt reductions.
- Super-symmetric family (3) has $\mathbf{g}^{(k)}(\boldsymbol{\theta}) = 0$ for $k = 1, 2, 3, 4, 5$ but $\mathbf{g}^{(6)}(\boldsymbol{\theta}) \neq 0$, unless $\theta_{j+1} - \theta_j = \frac{\pi}{2}$.
- Moreover, $(\mathbf{g}^{(6)}, \ker(\mathcal{M}_1)) \neq 0$, such that *all* asymmetric vortices (3) terminate.

Stability problem and zero eigenvalues

Matrix-vector Hamiltonian form of the stability problem:

$$\mathcal{H}\boldsymbol{\psi} = i\lambda\sigma\boldsymbol{\psi},$$

where

- $\boldsymbol{\psi} \in l^2(\mathbb{Z}^2, \mathbb{C}^2)$
- $\mathcal{H}$ is the Jacobian (energy) operator
- σ is the diagonal matrix of $(1, -1)$

Eigenvalues of $\mathcal{H}$ at $\epsilon = 0$:

- $\gamma = -2$ of multiplicity N
- $\gamma = 0$ of multiplicity N
- $\gamma = +1$ of multiplicity ∞

Eigenvalues of $i\sigma\mathcal{H}$ at $\epsilon = 0$:

- $\lambda = 0$ of multiplicity $2N$
- $\lambda = +i$ of multiplicity ∞
- $\lambda = -i$ of multiplicity ∞

How do zero eigenvalues split?

Stability results of Lyapunov-Schmidt reductions

- First-order splitting of zero eigenvalues of $\mathcal{H}$:

$$\mathcal{M}_1 \mathbf{c} = \gamma \mathbf{c}$$

- First-order splitting of zero eigenvalues of $i\sigma\mathcal{H}$:

$$\mathcal{M}_1 \mathbf{c} = \frac{\lambda^2}{2} \mathbf{c}$$

- Second-order splitting of zero eigenvalues of $\mathcal{H}$:

$$\mathcal{M}_1 = 0, \qquad \mathcal{M}_2 \mathbf{c} = \gamma \mathbf{c}$$

- Second-order splitting of zero eigenvalues of $i\sigma\mathcal{H}$:

$$\mathcal{M}_1 = 0, \qquad \mathcal{M}_2 \mathbf{c} = \frac{\lambda^2}{2} \mathbf{c} + \lambda \mathcal{L}_2 \mathbf{c}$$

where $\mathcal{M}_2^T = \mathcal{M}_2$ and $\mathcal{L}_2^T = -\mathcal{L}_2$.

- Six-order splitting : symbolic software algorithm

Negative index theory

Number of eigenvalues of $\mathcal{H}$:

- $n(\mathcal{H})$ - negative
- $p(\mathcal{H})$ - small positive

Number of eigenvalues of $i\sigma\mathcal{H}$:

- N_r - small real (unstable)
- N_c - small complex (unstable)
- N_i^+ - small imaginary with positive energy
- N_i^- - small imaginary with negative energy

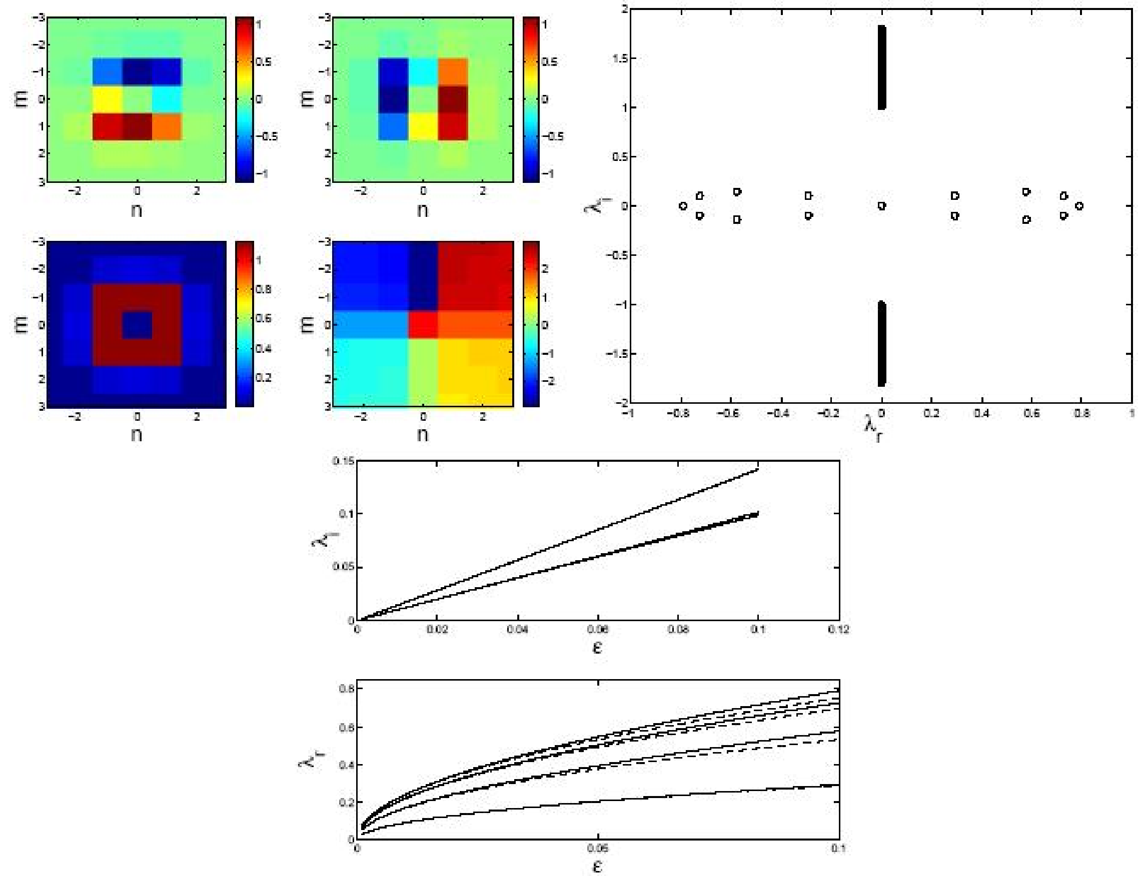
By Lyapunov–Schmidt reductions,

$$n(\mathcal{H}) + p(\mathcal{H}) = 2N - 1, \qquad 2N_r + 2N_c + 2N_i^+ + 2N_i^- = 2N - 2$$

By closure relation for negative index,

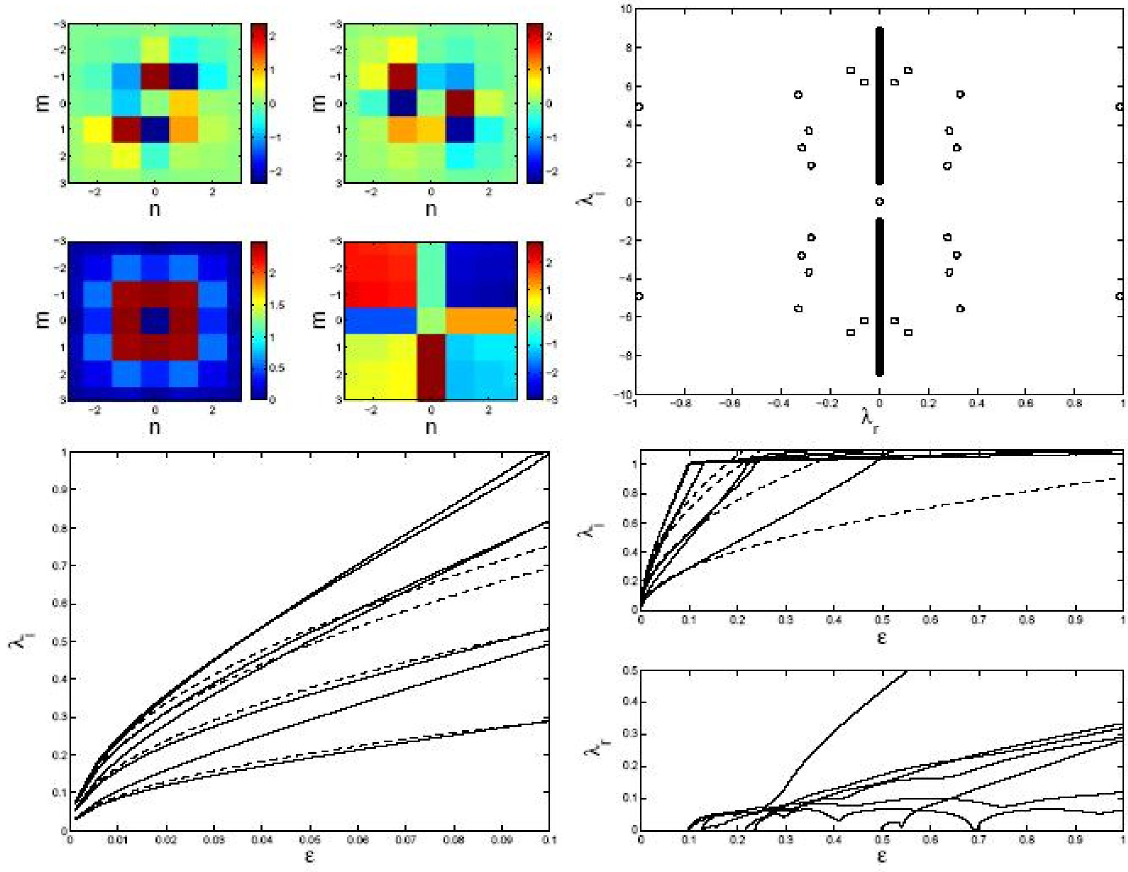
$$N_r + N_c + 2N_i^- = n(\mathcal{H}) - 1$$

Numerical analysis: symmetric vortex with $L = 1$ and $M = 2$



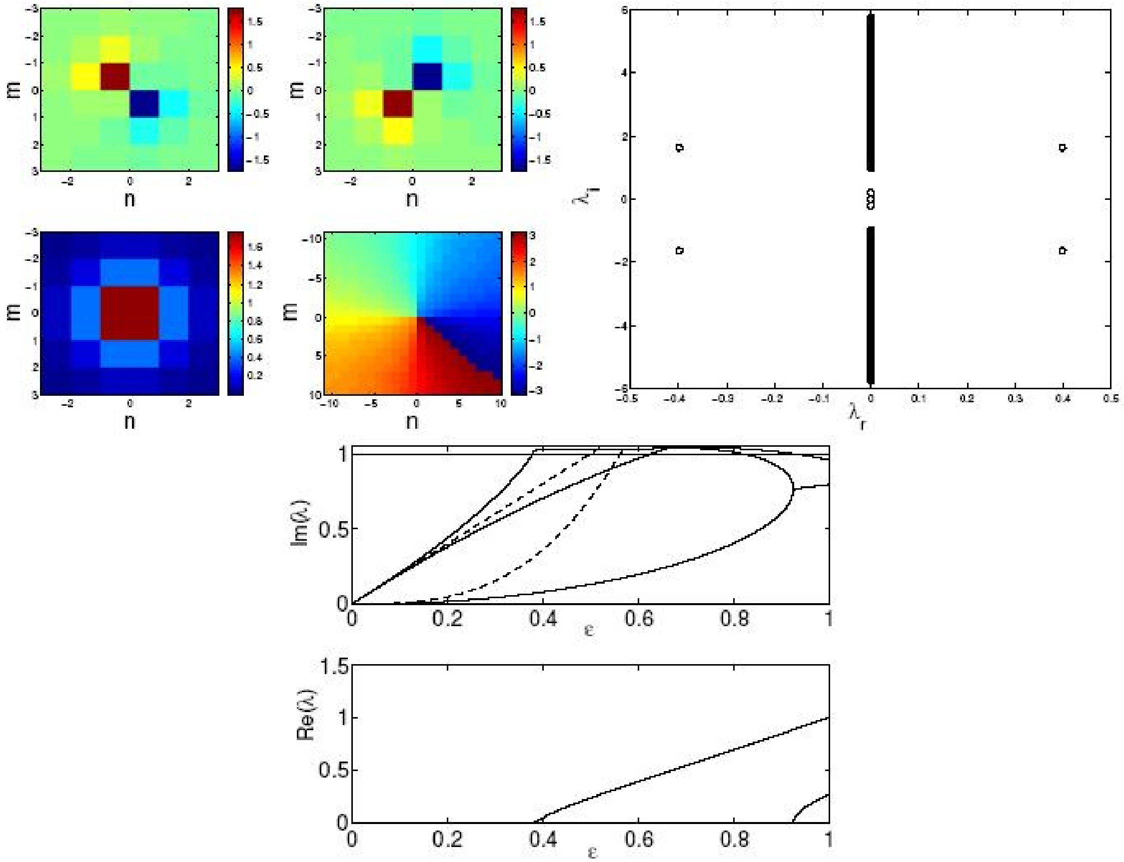
$$\mathcal{M}_1 \mathbf{c} = \gamma \mathbf{c} : \quad n(\mathcal{H}) = 8, p(\mathcal{H}) = 7, N_r = 7$$

Numerical analysis: symmetric vortex with $L = 3$ and $M = 2$



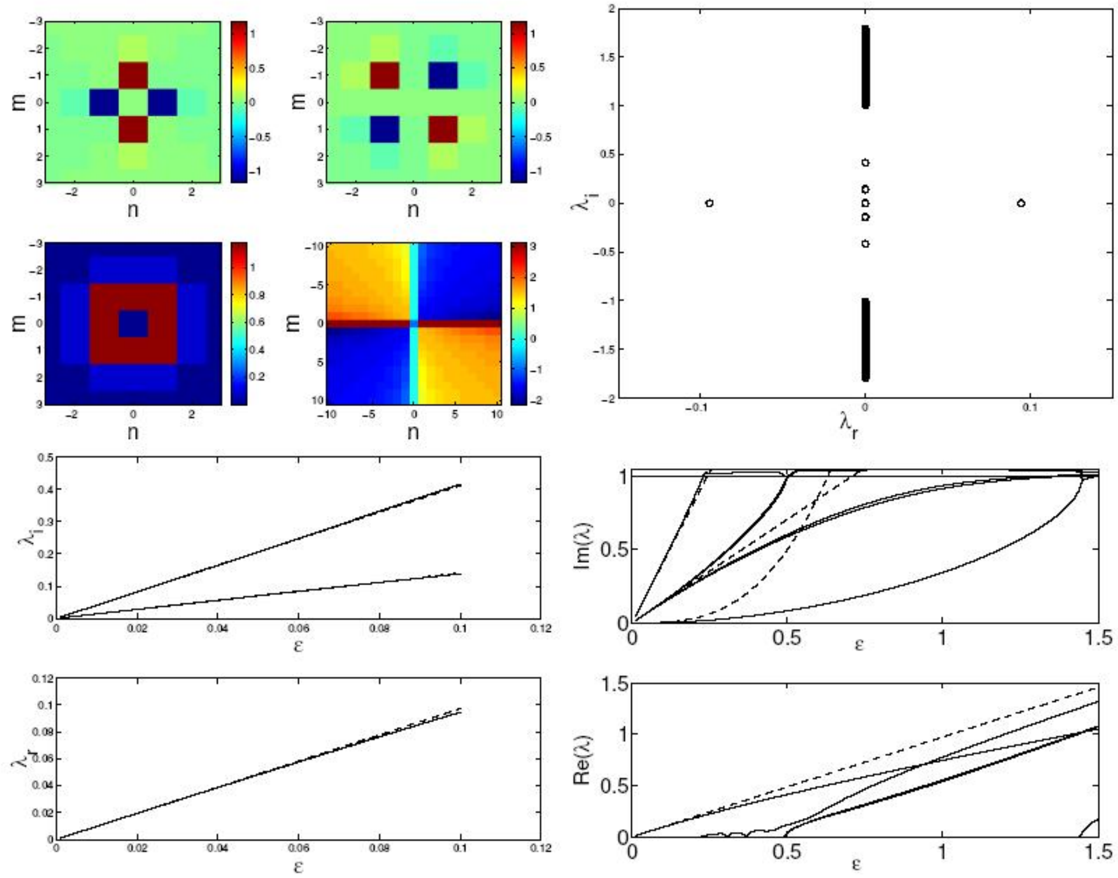
$$\mathcal{M}_1 \mathbf{c} = \gamma \mathbf{c} : \quad n(\mathcal{H}) = 15, p(\mathcal{H}) = 0, N_i^- = 7$$

Numerical analysis: symmetric vortex with $L = M = 1$



$$\mathcal{M}_2 \mathbf{c} = \gamma \mathbf{c} : \quad n(\mathcal{H}) = 5, p(\mathcal{H}) = 2, N_i^+ = 1, N_i^- = 2$$

Numerical analysis: symmetric vortex with $L = M = 2$



$$\mathcal{M}_2 \mathbf{c} = \gamma \mathbf{c} : \quad n(\mathcal{H}) = 10, p(\mathcal{H}) = 5, N_r = 1, N_i^+ = 2, N_i^- = 4$$

Vector on-site vortices on diagonal contours:

$$(1 - |\phi_{n,m}|^2 - \beta|\psi_{n,m}|^2)\phi_{n,m} = \epsilon\,(\phi_{n+1,m} + \phi_{n-1,m} + \phi_{n,m+1} + \phi_{n,m-1})$$
$$(1 - \beta|\phi_{n,m}|^2 - |\psi_{n,m}|^2)\psi_{n,m} = \epsilon\,(\psi_{n+1,m} + \psi_{n-1,m} + \psi_{n,m+1} + \psi_{n,m-1})$$

Diagonal discrete contour

$$S = \{(-1,0);(0,-1);(1,0);(0,1)\} \subset \mathbb{Z}^2$$

Limiting solution

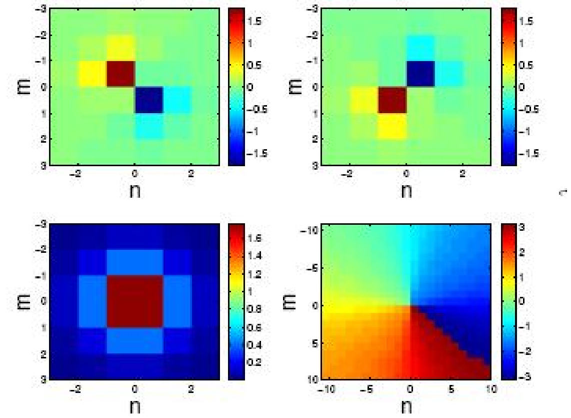
$$\phi^{(0)}_{n,m} = \begin{cases} ae^{i\theta_j}, & (n,m) \in S \\ 0, & (n,m) \notin S \end{cases} \qquad \psi^{(0)}_{n,m} = \begin{cases} be^{i\nu_j}, & (n,m) \in S \\ 0, & (n,m) \notin S \end{cases}$$

where

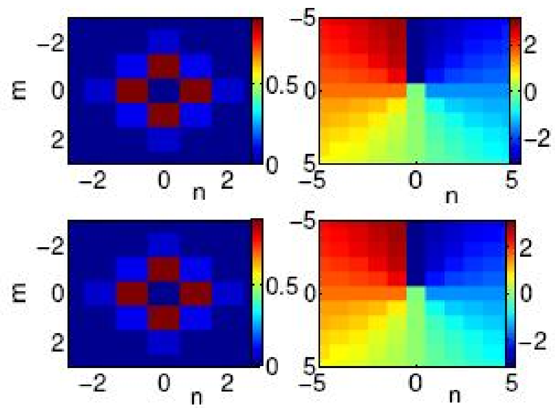
$$a^2 + \beta b^2 = 1, \qquad \beta a^2 + b^2 = 1.$$

Numerical pictures

- Off-site vortex (vortex cell) on a square contour

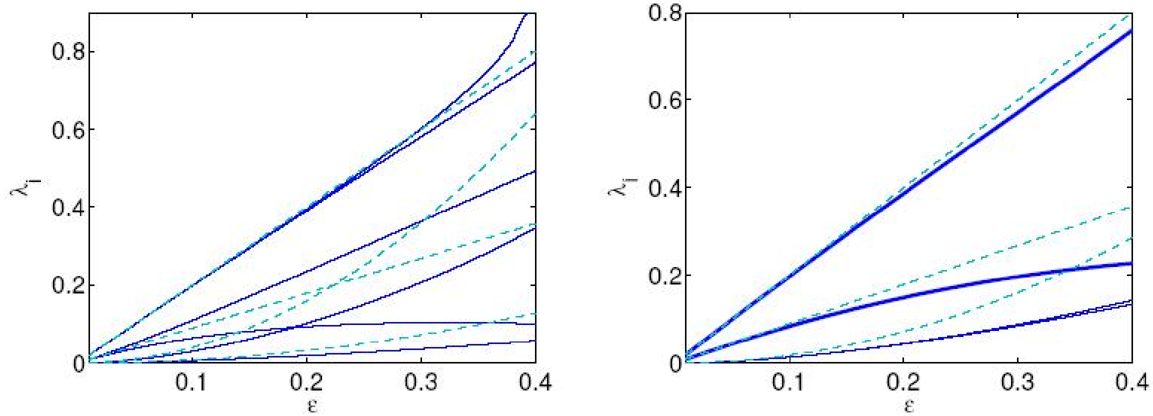


- On-site vector vortex (vortex cross) on a diagonal contour



Numerical and analytical results for vector on-site vortices:

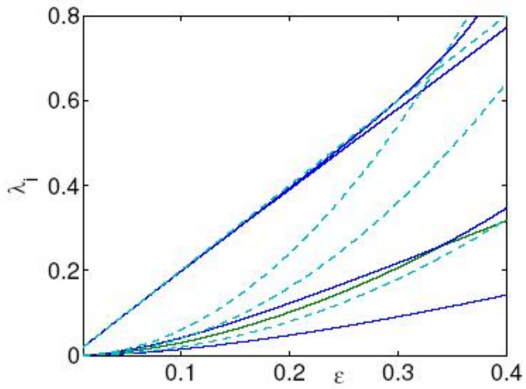
$\beta = \frac{2}{3}$ Left: (1,1) vector vortex. Right: (1,-1) vector vortex

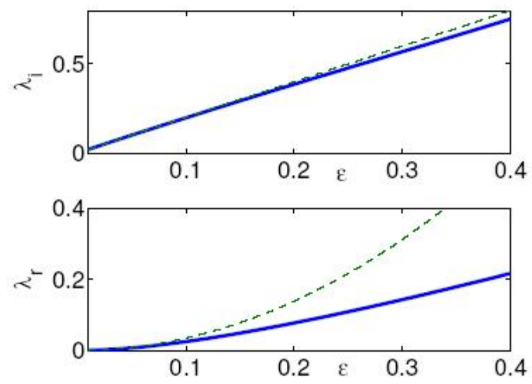


$$(1, \pm 1): \quad n(\mathcal{H}) = 14, p(\mathcal{H}) = 0, N_i^- = 6$$

Numerical and analytical results for vector on-site vortices:

$\beta = 1$ Left: (1,1) vector vortex. Right: (1,-1) vector vortex



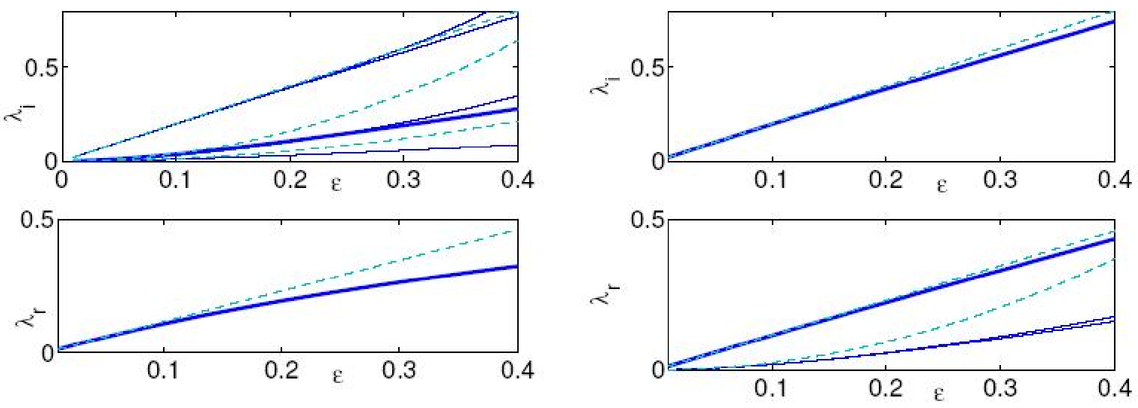


$$(1,1): \quad n(\mathcal{H}) = 9, p(\mathcal{H}) = 2, N_i^+ = 1, N_i^- = 4$$

$$(1,-1): \quad n(\mathcal{H}) = 9, p(\mathcal{H}) = 2, N_r = 2, N_i^- = 3$$

Numerical and analytical results for vector on-site vortices:

$\beta = 2$ Left: (1,1) vector vortex. Right: (1,-1) vector vortex



$$(1,1): \quad n(\mathcal{H}) = 9, p(\mathcal{H}) = 2, N_r = 2, N_i^+ = 1, N_i^- = 3$$

$$(1,-1): \quad n(\mathcal{H}) = 9, p(\mathcal{H}) = 2, N_r = 4, N_i^- = 2$$

Summary:

- Systematic classification of discrete vortices
- Rigorous study of their existence and stability
- Predictions of stable and unstable vortices
- Interplay between analytical and numerical work