Persistence and stability of discrete vortices

Dmitry Pelinovsky

Department of Mathematics, McMaster University, Canada

$$\mathbf{1D}: \quad i\dot{u}_n + \epsilon\left(u_{n+1} - 2u_n + u_{n-1}\right) + |u_n|^2 u_n = 0, \quad n \in \mathbb{Z}$$
$$\mathbf{2D}: \quad i\dot{u}_{n,m} + \epsilon\Delta_{2d} u_{n,m} + |u_{n,m}|^2 u_{n,m} = 0, \quad (n,m) \in \mathbb{Z}^2$$

Joint work with P. Kevrekidis
(University of Massachusetts at Amherst, USA)

Applied Mathematics Seminar, University of Toronto, April 4 2005

Experimental motivations

- Bose-Einstein condensates in optical lattices
- Light-induced photonic lattices
- Coupled optical waveguides

Persistence of localized solutions

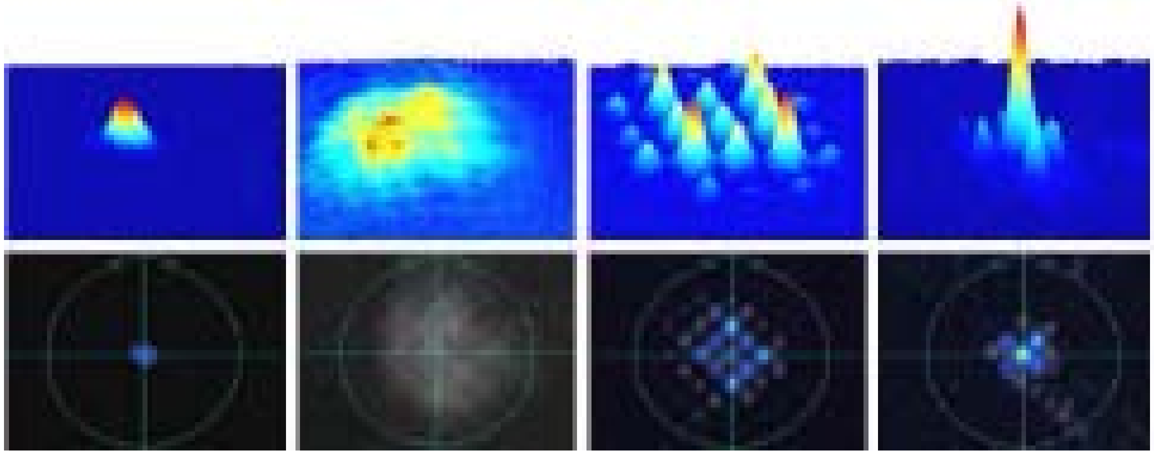
- Implicit Function Theorem
- Lyapunov–Schmidt reductions

Stability of localized solutions

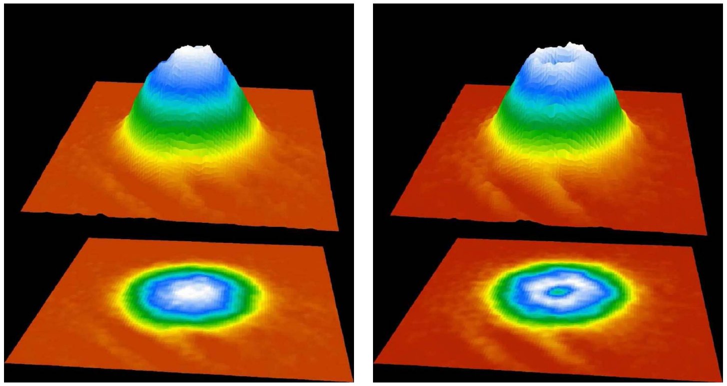
- Splitting of zero eigenvalues
- Negative index theory

Experimental pictures

- Discrete solitons



- Discrete vortices



Main Formalism

$$\mathbf{1D}: \qquad i\dot{u}_n + \epsilon\left(u_{n+1} - 2u_n + u_{n-1}\right) + |u_n|^2 u_n = 0, \qquad n \in \mathbb{Z}$$

- Vector space $\Omega = L^2(\mathbb{Z}, \mathbb{C})$ for $\{u_n\}_{n\in\mathbb{Z}}$:

$$(\mathbf{u}, \mathbf{w})_\Omega = \sum_{n\in\mathbb{Z}} \bar{u}_n w_n, \qquad \|\mathbf{u}\|_\Omega^2 = \sum_{n\in\mathbb{Z}} |u_n|^2 < \infty.$$

- Hamiltonian formulation:

$$i\dot{u}_n = \frac{\partial H}{\partial \bar{u}_n}, \qquad H = \sum_{n\in\mathbb{Z}} \epsilon|u_{n+1} - u_n|^2 - \frac{1}{2}|u_n|^4$$

- Existence problem for time-periodic solutions

$$u_n(t) = \phi_n e^{i(\mu - 2\epsilon)t + i\theta_0}, \qquad \mu \in \mathbb{R}, \ \ \theta_0 \in \mathbb{R}$$

such that

$$(\mu - |\phi_n|^2)\phi_n = \epsilon\left(\phi_{n+1} + \phi_{n-1}\right).$$

- Existence problem for time-periodic solutions

$$u_n(t) = \phi_n e^{i(\mu - 2\epsilon)t + i\theta_0}, \qquad \mu \in \mathbb{R}, \quad \theta_0 \in \mathbb{R}$$

such that

$$(\mu - |\phi_n|^2)\phi_n = \epsilon \left(\phi_{n+1} + \phi_{n-1}\right).$$

- Stability problem for time-periodic solutions

$$u_n(t) = e^{i(1-2\epsilon)t + i\theta_0} \left(\phi_n + (u_n + i w_n) e^{\lambda t} + (\bar{u}_n + i \bar{w}_n) e^{\bar{\lambda} t}\right)$$

such that

$$\left(1 - 3\phi_n^2\right) u_n - \epsilon \left(u_{n+1} + u_{n-1}\right) = -\lambda w_n,$$
$$\left(1 - \phi_n^2\right) w_n - \epsilon \left(w_{n+1} + w_{n-1}\right) = \lambda u_n.$$

where $\lambda \in \mathbb{C}$ and $(\mathbf{u}, \mathbf{w}) \in \Omega \times \Omega$

Existence problem in one dimension

$$(\mu - |\phi_n|^2)\phi_n = \epsilon\,(\phi_{n+1} + \phi_{n-1})$$

Existence problem in one dimension

$$(\mu - |\phi_n|^2)\phi_n = \epsilon\,(\phi_{n+1} + \phi_{n-1})$$

- All localized solutions for $\epsilon \neq 0$ are real-valued: $\boldsymbol{\phi} \in L^2(\mathbb{Z}, \mathbb{R})$

$$\bar{\phi}_n \phi_{n+1} - \phi_n \bar{\phi}_{n+1} = \text{const} \quad n \in \mathbb{Z}$$

$$\frac{\phi_{n+1}}{\bar{\phi}_{n+1}} = \frac{\phi_n}{\bar{\phi}_n} : \qquad 2\arg(\phi_{n+1}) = 2\arg(\phi_n) = \text{mod}(2\pi)$$

Existence problem in one dimension

$$(\mu - |\phi_n|^2)\phi_n = \epsilon\,(\phi_{n+1} + \phi_{n-1})$$

- All localized solutions for $\epsilon \neq 0$ are real-valued: $\boldsymbol{\phi} \in L^2(\mathbb{Z}, \mathbb{R})$

$$\bar{\phi}_n \phi_{n+1} - \phi_n \bar{\phi}_{n+1} = \text{const} \quad n \in \mathbb{Z}$$

$$\frac{\phi_{n+1}}{\bar{\phi}_{n+1}} = \frac{\phi_n}{\bar{\phi}_n} : \qquad 2\arg(\phi_{n+1}) = 2\arg(\phi_n) = \mathrm{mod}(2\pi)$$

- There exists a transformation from $\epsilon < 0$ to $\epsilon > 0$

$$\phi_n \mapsto (-1)^n \phi_n, \qquad \epsilon \mapsto -\epsilon$$

Existence problem in one dimension

$$(\mu - |\phi_n|^2)\phi_n = \epsilon\left(\phi_{n+1} + \phi_{n-1}\right)$$

- There exists a spectral band for $|\mu| \leq 2\epsilon$:

$$\phi_n = e^{ikn}: \quad \mu = \mu(k) = 2\epsilon\cos k, \qquad k \in \mathbb{R}$$

Existence problem in one dimension

$$(\mu - |\phi_n|^2)\phi_n = \epsilon\,(\phi_{n+1} + \phi_{n-1})$$

- There exists a spectral band for $|\mu| \leq 2\epsilon$:

$$\phi_n = e^{ikn} : \quad \mu = \mu(k) = 2\epsilon\cos k, \qquad k \in \mathbb{R}$$

- Localized solutions do not exist for $\mu < -2\epsilon < 0$:

$$-(|\mu| - 2\epsilon)\sum_{n\in\mathbb{Z}} \phi_n^2 - \sum_{n\in\mathbb{Z}} \phi_n^4 = \epsilon \sum_{n\in\mathbb{Z}} (\phi_{n+1} + \phi_n)^2$$

Existence problem in one dimension

$$(\mu - |\phi_n|^2)\phi_n = \epsilon\,(\phi_{n+1} + \phi_{n-1})$$

- There exists a spectral band for $|\mu| \leq 2\epsilon$:

$$\phi_n = e^{ikn}: \quad \mu = \mu(k) = 2\epsilon\cos k, \qquad k \in \mathbb{R}$$

- Localized solutions do not exist for $\mu < -2\epsilon < 0$:

$$-(|\mu| - 2\epsilon)\sum_{n\in\mathbb{Z}}\phi_n^2 - \sum_{n\in\mathbb{Z}}\phi_n^4 = \epsilon\sum_{n\in\mathbb{Z}}(\phi_{n+1} + \phi_n)^2$$

- Scaling transformation for localized solutions with $\mu > 2\epsilon > 0$:

$$\phi_n = \sqrt{\mu}\hat{\phi}_n, \qquad \epsilon = \mu\hat{\epsilon}$$

Existence problem in one dimension

$$(1 - \phi_n^2)\phi_n = \epsilon\left(\phi_{n+1} + \phi_{n-1}\right)$$

- There exists an analytic function $\boldsymbol{\phi}(\epsilon)$ for $0 < \epsilon < \epsilon_0$:

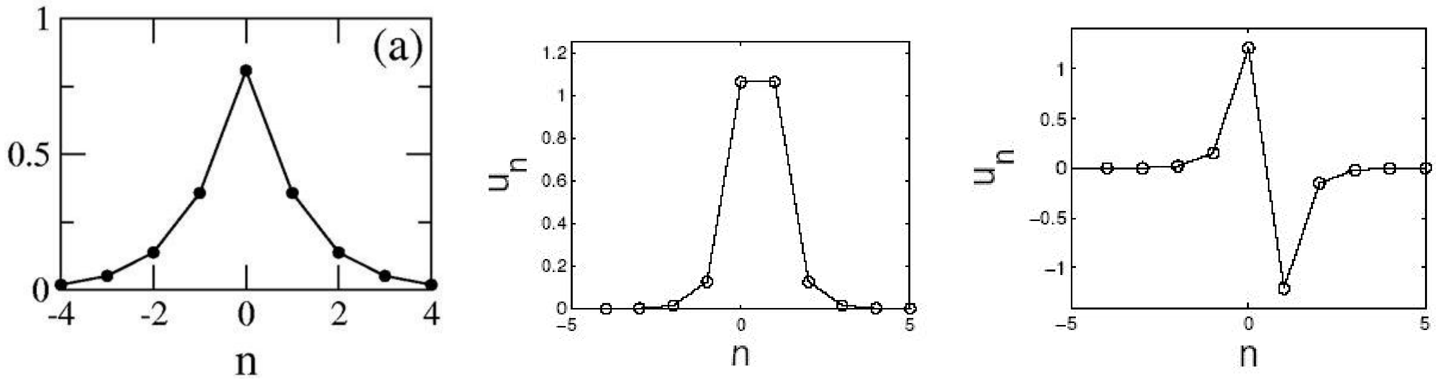
$$\lim_{\epsilon \longrightarrow 0} \phi_n = \begin{cases} \pm 1, & n \in S, \\ 0, & n \in \mathbb{Z} \backslash S, \end{cases} \qquad \dim(S) < \infty$$

$$\lim_{|n| \to \infty} e^{\kappa |n|} |\phi_n| = \phi_\infty, \qquad \kappa > 0, \quad \phi_\infty > 0.$$

- MacKay, Aubry (1994): inverse function theorem
- Hennig, Tsironis (1999): bounds on ϵ_0
- Bergamin, Bountis (2000): symbolic dynamics for invertible maps
- Alfimov, Konotop (2004): complete classification of localized modes

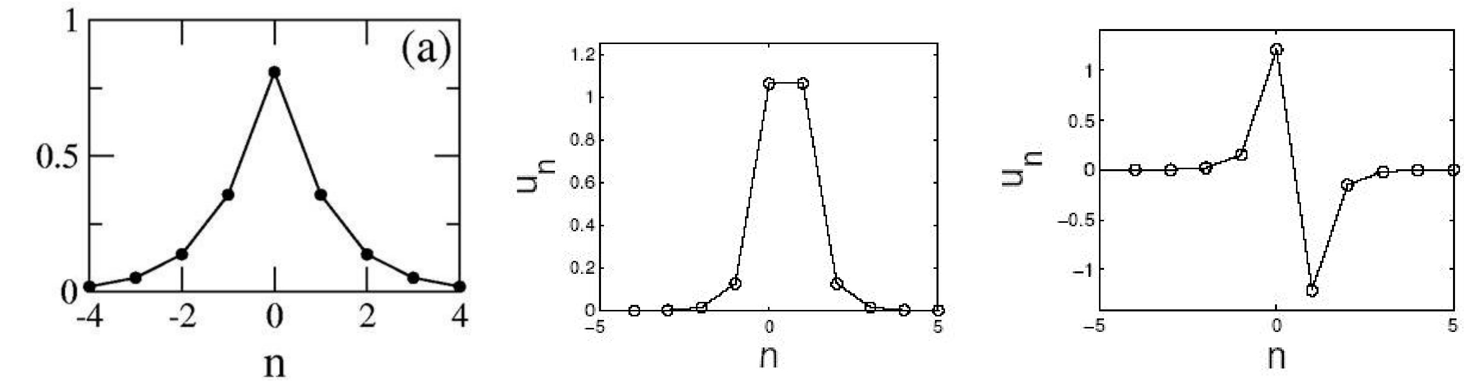
Families of discrete solitons

- **Fundamental and two-node modes**

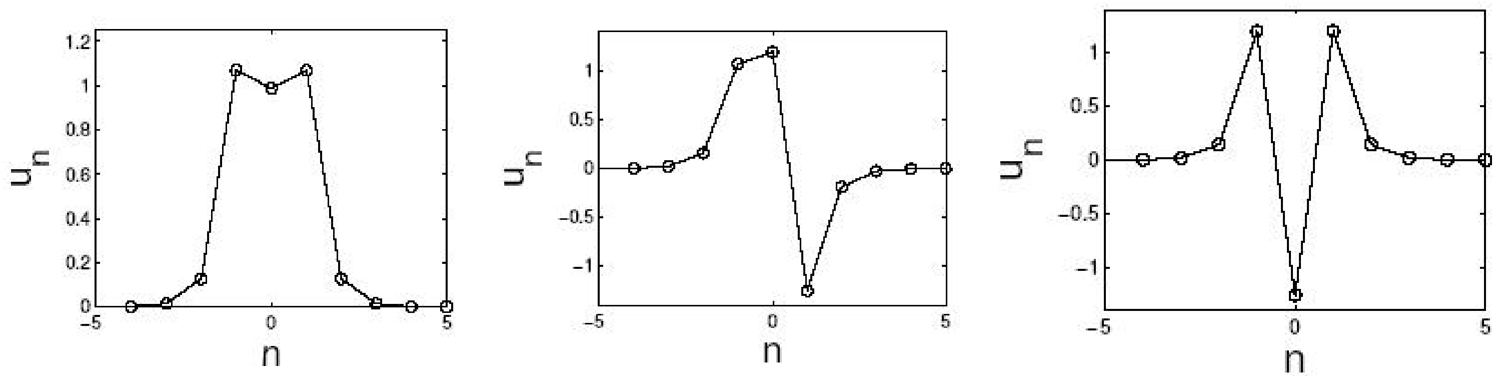


Families of discrete solitons

- **Fundamental and two-node modes**



- **Three-node modes**



Stability problem in one dimension

$$\left(1 - 3\phi_n^2\right) u_n - \epsilon\left(u_{n+1} + u_{n-1}\right) = -\lambda w_n,$$
$$\left(1 - \phi_n^2\right) w_n - \epsilon\left(w_{n+1} + w_{n-1}\right) = \lambda u_n.$$

- Matrix-vector form for $(\mathbf{u}, \mathbf{w}) \in L^2(\mathbb{Z}, \mathbb{C}^2)$

$$\mathcal{L}_+ \mathbf{u} = -\lambda \mathbf{w}, \qquad \mathcal{L}_- \mathbf{w} = \lambda \mathbf{u},$$

- Hamiltonian form for $\boldsymbol{\psi} = (\mathbf{u}, \mathbf{w})$:

$$\mathcal{J}\mathcal{H}\boldsymbol{\psi} = \lambda\boldsymbol{\psi}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}.$$

Splitting of zero eigenvalues

Eigenvalues of $\mathcal{H}$ at $\epsilon = 0$:

- $\gamma = -2$ of multiplicity N
- $\gamma = 0$ of multiplicity N
- $\gamma = +1$ of multiplicity ∞

Eigenvalues of $\mathcal{J}\mathcal{H}$ at $\epsilon = 0$:

- $\lambda = 0$ of multiplicity $2N$
- $\lambda = +i$ of multiplicity ∞
- $\lambda = -i$ of multiplicity ∞

Splitting of zero eigenvalues

Eigenvalues of $\mathcal{H}$ at $\epsilon = 0$:

- $\gamma = -2$ of multiplicity N
- $\gamma = 0$ of multiplicity N
- $\gamma = +1$ of multiplicity ∞

Eigenvalues of $\mathcal{J}\mathcal{H}$ at $\epsilon = 0$:

- $\lambda = 0$ of multiplicity $2N$
- $\lambda = +i$ of multiplicity ∞
- $\lambda = -i$ of multiplicity ∞

Lemma: Let γ_j be small eigenvalues of $\mathcal{H}$ as $\epsilon \longrightarrow 0$. There exists N pairs of small eigenvalues λ_j and $-\lambda_j$ of $\mathcal{J}\mathcal{H}$:

$$\lim_{\epsilon \longrightarrow 0} \gamma_j = 0, \qquad \lim_{\epsilon \longrightarrow 0} \frac{\lambda_j^2}{\gamma_j} = 2, \qquad 1 \leq j \leq N.$$

Splitting of zero eigenvalues

Eigenvalues of $\mathcal{H}$ at $\epsilon = 0$:

- $\gamma = -2$ of multiplicity N
- $\gamma = 0$ of multiplicity N
- $\gamma = +1$ of multiplicity ∞

Eigenvalues of $\mathcal{J}\mathcal{H}$ at $\epsilon = 0$:

- $\lambda = 0$ of multiplicity $2N$
- $\lambda = +i$ of multiplicity ∞
- $\lambda = -i$ of multiplicity ∞

Lemma: Let γ_j be small eigenvalues of $\mathcal{H}$ as $\epsilon \to 0$. There exists N pairs of small eigenvalues λ_j and $-\lambda_j$ of $\mathcal{J}\mathcal{H}$:

$$\lim_{\epsilon \to 0} \gamma_j = 0, \qquad \lim_{\epsilon \to 0} \frac{\lambda_j^2}{\gamma_j} = 2, \qquad 1 \leq j \leq N.$$

Corollary:
When $\gamma_j > 0$, there exists one unstable EV $\lambda_j > 0$.
When $\gamma_j < 0$, there exists one pair $\lambda_j \in i\mathbb{R}$ with negative Krein signature:

$$(\boldsymbol{\psi}, \mathcal{H}\boldsymbol{\psi}) = (\mathbf{u}, \mathcal{L}_+\mathbf{u}) + (\mathbf{w}, \mathcal{L}_-\mathbf{w}) = 2\,(\mathbf{w}, \mathcal{L}_-\mathbf{w}) < 0.$$

Count of small eigenvalues of $\mathcal{H}$

Lemma: Let n_0 be the number of sign-differences in the vector $\boldsymbol{\phi}$ at $\epsilon = 0$. There exists n_0 negative eigenvalues γ_j and $N - n_0 - 1$ positive eigenvalues γ_j for any $\epsilon \neq 0$.

Count of small eigenvalues of $\mathcal{H}$

Lemma: Let n_0 be the number of sign-differences in the vector $\boldsymbol{\phi}$ at $\epsilon = 0$. There exists n_0 negative eigenvalues γ_j and $N - n_0 - 1$ positive eigenvalues γ_j for any $\epsilon \neq 0$.

- By discrete Sturm Theorem, $\#_{<0}(\mathcal{L}_-) = n_0$, since
$$\mathcal{L}_- \boldsymbol{\phi} = \mathbf{0}.$$

- By theory of difference equations, $\dim(\mathcal{L}_-) = 1$ for any $\epsilon \neq 0$, since
$$\mathcal{L}_- \mathbf{w} = \mathbf{0}, \qquad \mathbf{w} = c_1 \boldsymbol{\phi} + c_2 \mathbf{w}_2.$$

- By our analysis, the number of sign-differences in the vector $\boldsymbol{\phi}$ is continuous at $\epsilon = 0$.

Count of unstable eigenvalues of $\mathcal{J}\mathcal{H}$

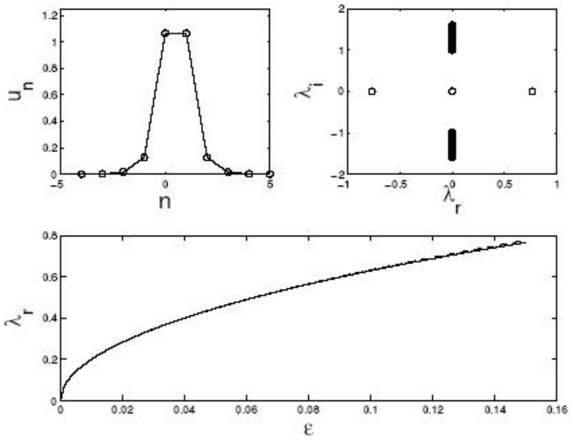
$$N_{\text{real}} = N - 1 - n_0, \qquad N^-_{\text{imag}} = n_0, \qquad N_{\text{comp}} = 0$$

Theorem: The only stable N-pulse discrete soliton near $\epsilon = 0$ is the soliton with an alternating sequence of up and down pulses.

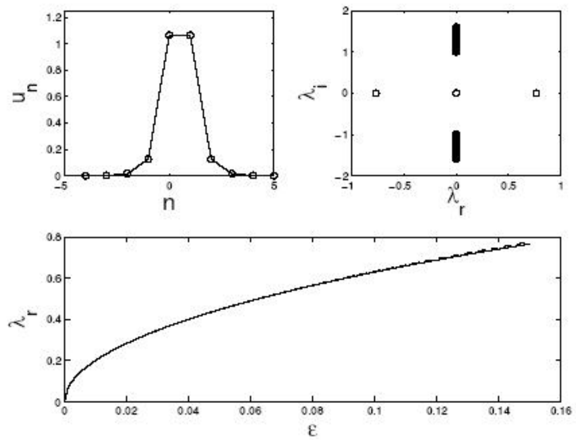
- Weinstein (1999): stability of discrete soliton with $N = 1$
- Kapitula, Kevrekidis, Malomed (2001): instabilities of twisted modes and other multi-pulse solitons
- Morgante, Johansson, Kopidakis, Aubry (2002): numerical analysis of instabilities of multi-pulse solitons with $N > 1$
- Sandstede, Jones, Alexander (1997): analysis of the orbit-flip bifurcation and multi-pulse homoclinic orbits

Numerical analysis of discrete solitons

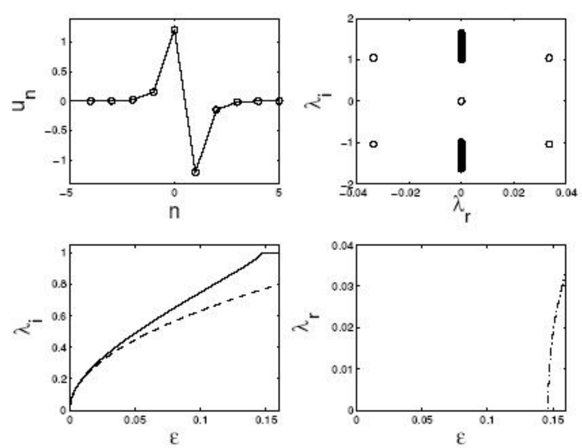
- **Page mode**



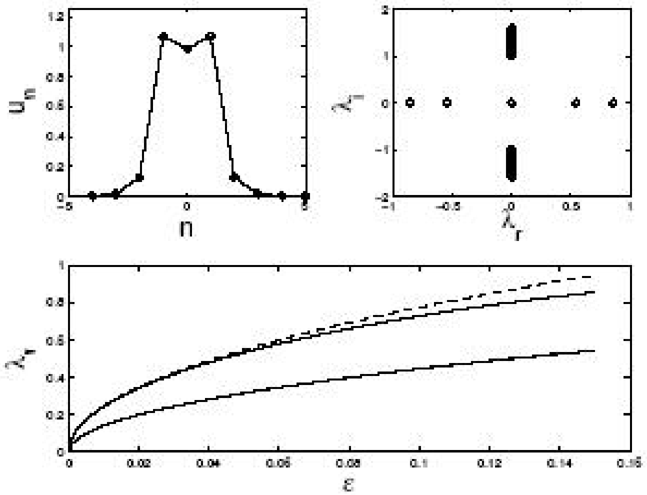
- **Page mode**

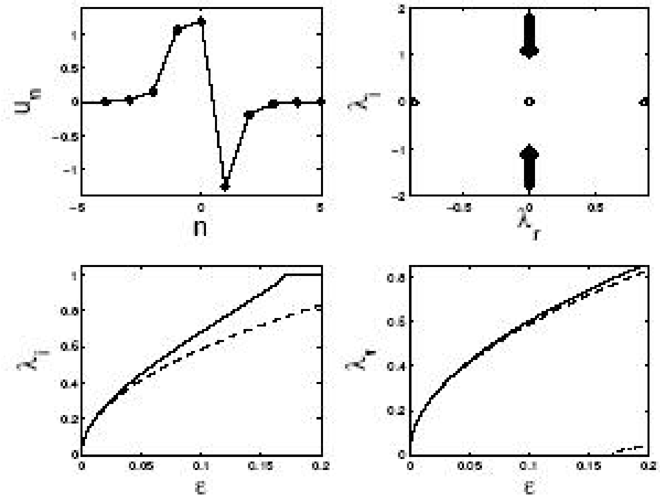


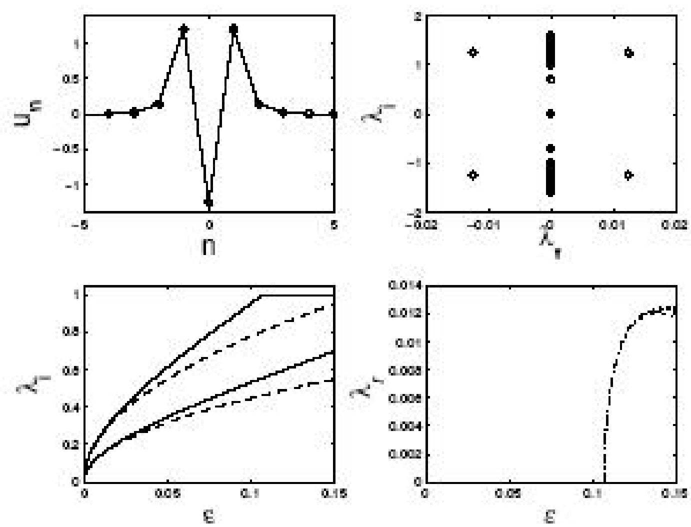
- **Twisted mode**



- **Three-node modes**







Remarks on related results

- Negative Index Theorem

$$N_{\text{real}} + 2N_{\text{imag}}^{-} + 2N_{\text{comp}} = N + n_0 - 1 = n(\mathcal{H}) - 1$$

  - Kapitula, Kevrekidis, Sandstede (2004): Grillakis' Diagonalization
  - Pelinovsky (2005): Sylvester' Inertia Law

Remarks on related results

- Negative Index Theorem

$$N_{\text{real}} + 2N^{-}_{\text{imag}} + 2N_{\text{comp}} = N + n_0 - 1 = n(\mathcal{H}) - 1$$

  - Kapitula, Kevrekidis, Sandstede (2004): Grillakis' Diagonalization
  - Pelinovsky (2005): Sylvester' Inertia Law

- Perturbation Theory

$$\mathcal{M}\mathbf{c} = \gamma\mathbf{c}, \qquad \mathbf{c} \in \mathbb{R}^N, \qquad \mathcal{M} = \mathcal{H}\Big|_{\ker(H^{(0)})}$$

where

$$\mathcal{M} = \begin{pmatrix} a_1 & -a_1 & 0 & \dots & 0 \\ -a_1 & a_1 + a_2 & -a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & a_{N-1} \end{pmatrix}$$

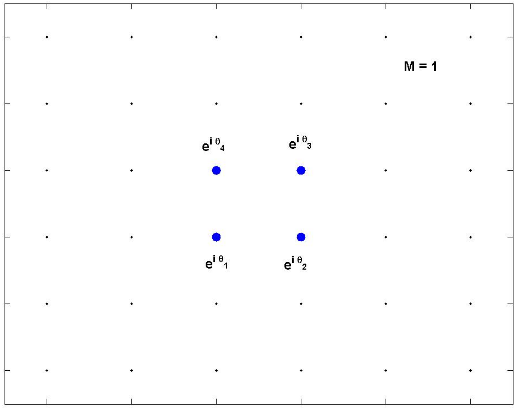
and $a_j = \pm 1$ depending on the sign-difference in $\boldsymbol{\phi}$.

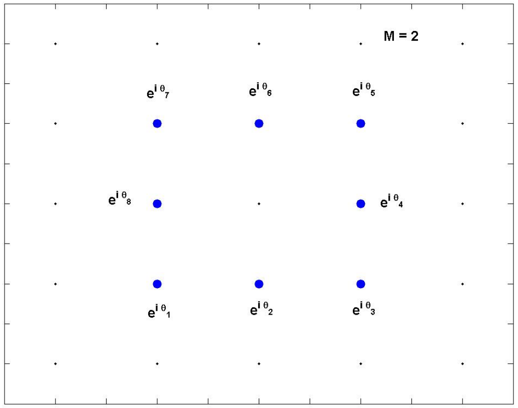
Existence problem in two dimension

$$(1 - |\phi_{n,m}|^2)\phi_{n,m} = \epsilon\left(\phi_{n+1,m} + \phi_{n-1,m} + \phi_{n,m+1} + \phi_{n,m-1}\right)$$

Limiting solution:

$$\epsilon = 0: \qquad \phi^{(0)}_{n,m} = \begin{cases} e^{i\theta_{n,m}}, & (n,m) \in S, \\ 0, & (n,m) \in \mathbb{Z}^2 \backslash S, \end{cases}$$





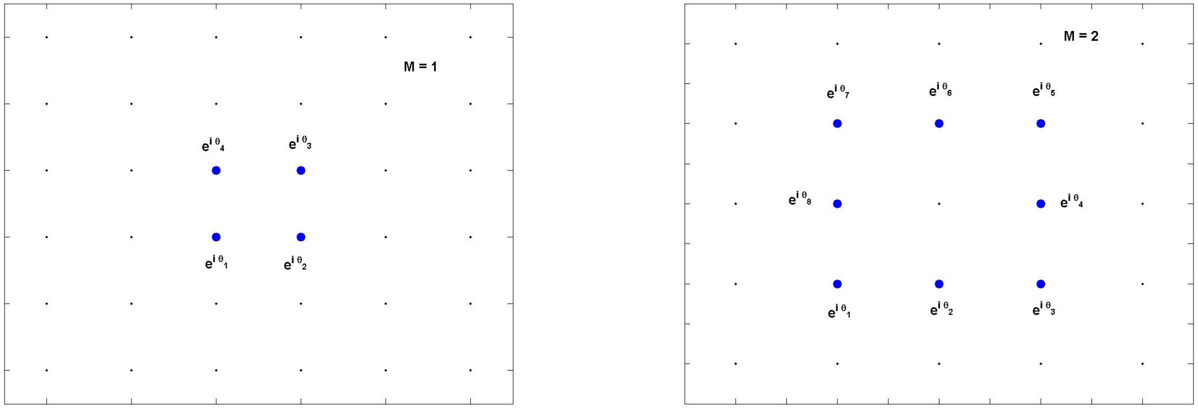
Examples of a square discrete contour S

Existence problem in two dimension

$$(1-|\phi_{n,m}|^2)\phi_{n,m} = \epsilon\left(\phi_{n+1,m}+\phi_{n-1,m}+\phi_{n,m+1}+\phi_{n,m-1}\right)$$

Limiting solution:

$$\epsilon = 0: \qquad \phi^{(0)}_{n,m} = \begin{cases} e^{i\theta_{n,m}}, & (n,m)\in S, \\ 0, & (n,m)\in \mathbb{Z}^2\backslash S, \end{cases}$$



Examples of a square discrete contour S

What phase configurations $\theta_{n,m}$ can be continued for $\epsilon \neq 0$?

Lyapunov-Schmidt reductions

Proposition: Let $N = \dim(S)$ and $\mathcal{T}$ be the torus on $[0, 2\pi]^N$. There exists a vector-valued function $\mathbf{g} : \mathcal{T} \mapsto \mathbb{R}^N$, such that the limiting solution is continued to $\epsilon \neq 0$ if and only if $\boldsymbol{\theta} \in \mathcal{T}$ is a root of $\mathbf{g}(\boldsymbol{\theta}, \epsilon) = \mathbf{0}$.

Lyapunov-Schmidt reductions

Proposition: Let $N = \dim(S)$ and $\mathcal{T}$ be the torus on $[0, 2\pi]^N$. There exists a vector-valued function $\mathbf{g} : \mathcal{T} \mapsto \mathbb{R}^N$, such that the limiting solution is continued to $\epsilon \neq 0$ if and only if $\boldsymbol{\theta} \in \mathcal{T}$ is a root of $\mathbf{g}(\boldsymbol{\theta}, \epsilon) = \mathbf{0}$.

- The Jacobian of the nonlinear system:

$$\mathcal{H} = \begin{pmatrix} 1 - 2|\phi_{n,m}|^2 & -\phi_{n,m}^2 \\ -\bar{\phi}_{n,m}^2 & 1 - 2|\phi_{n,m}|^2 \end{pmatrix} - \epsilon \delta_{\pm 1, \pm 1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- $\mathcal{H}$ is a self-adjoint Fredholm operator of index zero:

$$\dim(\ker(\mathcal{H}^{(0)}) = N$$

- Analytic functions:

$$\mathbf{g}(\boldsymbol{\theta}, \epsilon) = \sum_{k=1}^{\infty} \epsilon^k \mathbf{g}^{(k)}(\boldsymbol{\theta})$$

- Analytic functions:

$$\mathbf{g}(\boldsymbol{\theta}, \epsilon) = \sum_{k=1}^{\infty} \epsilon^k \mathbf{g}^{(k)}(\boldsymbol{\theta})$$

- Gauge symmetry:

$$\mathbf{g}(\boldsymbol{\theta}_*, \epsilon) = \mathbf{0} \quad \mapsto \quad \mathbf{g}(\boldsymbol{\theta}_* + \theta_0 \mathbf{p}_0, \epsilon) = \mathbf{0},$$

where $\mathbf{p}_0 = (1, 1, ..., 1)$.

- Analytic functions:

$$\mathbf{g}(\boldsymbol{\theta}, \epsilon) = \sum_{k=1}^{\infty} \epsilon^k \mathbf{g}^{(k)}(\boldsymbol{\theta})$$

- Gauge symmetry:

$$\mathbf{g}(\boldsymbol{\theta}_*, \epsilon) = \mathbf{0} \quad \mapsto \quad \mathbf{g}(\boldsymbol{\theta}_* + \theta_0 \mathbf{p}_0, \epsilon) = \mathbf{0},$$

  where $\mathbf{p}_0 = (1, 1, ..., 1)$.

- Let $\boldsymbol{\theta}_*$ be the root of $\mathbf{g}^{(1)}(\boldsymbol{\theta}) = \mathbf{0}$ and $\mathcal{M}_1 = \mathcal{D}\mathbf{g}^{(1)}(\boldsymbol{\theta}_*)$. If $\dim(\ker(\mathcal{M}_1)) =$ 1, there exists a unique analytic continuation of the limiting solution for $\epsilon \neq 0$.

- Analytic functions:

$$\mathbf{g}(\boldsymbol{\theta}, \epsilon) = \sum_{k=1}^{\infty} \epsilon^k \mathbf{g}^{(k)}(\boldsymbol{\theta})$$

- Gauge symmetry:

$$\mathbf{g}(\boldsymbol{\theta}_*, \epsilon) = \mathbf{0} \quad \mapsto \quad \mathbf{g}(\boldsymbol{\theta}_* + \theta_0 \mathbf{p}_0, \epsilon) = \mathbf{0},$$

  where $\mathbf{p}_0 = (1, 1, ..., 1)$.

- Let $\boldsymbol{\theta}_*$ be the root of $\mathbf{g}^{(1)}(\boldsymbol{\theta}) = \mathbf{0}$ and $\mathcal{M}_1 = \mathcal{D}\mathbf{g}^{(1)}(\boldsymbol{\theta}_*)$. If $\dim(\ker(\mathcal{M}_1))$ 1, there exists a unique analytic continuation of the limiting solution for $\epsilon \neq 0$.

- Let $\boldsymbol{\theta}_*$ be a $(1 + d)$-parameter solution of $\mathbf{g}^{(1)}(\boldsymbol{\theta}) = \mathbf{0}$. The limiting solution can not be continued to $\epsilon \neq 0$ if $\mathbf{g}^{(2)}(\boldsymbol{\theta}_*)$ is not orthogonal to $\ker(\mathcal{M}_1)$.

First-order reductions : classification of solutions

$$\mathbf{g}_j^{(1)}(\boldsymbol{\theta}) = \sin(\theta_j - \theta_{j+1}) + \sin(\theta_j - \theta_{j-1}) = 0, \quad 1 \leq j \leq 4M$$

First-order reductions : classification of solutions

$$\mathbf{g}_j^{(1)}(\boldsymbol{\theta}) = \sin(\theta_j - \theta_{j+1}) + \sin(\theta_j - \theta_{j-1}) = 0, \quad 1 \le j \le 4M$$

- (1) Discrete solitons

$$\theta_j = \{0, \pi\}, \qquad 1 \le j \le 4M$$

- (2) Symmetric vortices of charge L

$$\theta_j = \frac{\pi L(j-1)}{2M}, \qquad 1 \le j \le 4M,$$

- (3) One-parameter asymmetric vortices of charge $L = M$

$$\theta_{j+1} - \theta_j = \left\{ \begin{array}{c} \theta \\ \pi - \theta \end{array} \right\} \bmod(2\pi), \qquad 1 \le j \le 4M,$$

First-order reductions : persistence of solutions

$$\mathcal{M}_1 = \begin{pmatrix} a_1 + a_2 & -a_2 & 0 & \ldots & a_1 \\ -a_2 & a_2 + a_3 & -a_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ -a_1 & 0 & 0 & \ldots & a_{N-1} + a_N \end{pmatrix}, \quad a_j = \cos(\theta_{j+1} - \theta_j)$$

First-order reductions : persistence of solutions

$$\mathcal{M}_1 = \begin{pmatrix} a_1 + a_2 & -a_2 & 0 & \dots & a_1 \\ -a_2 & a_2 + a_3 & -a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -a_1 & 0 & 0 & \dots & a_{N-1} + a_N \end{pmatrix}, \quad a_j = \cos(\theta_{j+1} - \theta_j)$$

- $\mathcal{M}_1$ has a simple zero eigenvalue if all $a_j \neq 0$ and

$$\left(\prod_{i=1}^{N} a_i \right) \left(\sum_{i=1}^{N} \frac{1}{a_i} \right) \neq 0.$$

  Family (1) persists for $\epsilon \neq 0$

First-order reductions : persistence of solutions

$$\mathcal{M}_1 = \begin{pmatrix} a_1 + a_2 & -a_2 & 0 & \dots & a_1 \\ -a_2 & a_2 + a_3 & -a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -a_1 & 0 & 0 & \dots & a_{N-1} + a_N \end{pmatrix}, \quad a_j = \cos(\theta_{j+1} - \theta_j)$$

- $\mathcal{M}_1$ has a simple zero eigenvalue if all $a_j \neq 0$ and
$$\left(\prod_{i=1}^{N} a_i \right) \left(\sum_{i=1}^{N} \frac{1}{a_i} \right) \neq 0.$$
Family (1) persists for $\epsilon \neq 0$

- If all $a_j = a = \cos(\frac{\pi L}{2M})$, eigenvalues of $\mathcal{M}_1$ are:
$$\lambda_n = 4a \sin^2 \frac{\pi n}{4M}, \qquad 1 \leq n \leq 4M$$
Family (2) persists for $\epsilon \neq 0$ and $L \neq M$

Second-order reductions : termination of solutions

- If all $a_j = \pm a = \cos\theta$, there are $2M-1$ negative eigenvalues of $\mathcal{M}_1$, 2 zero eigenvalues and $2M-1$ positive eigenvalues of $\mathcal{M}_1$.

- Persistence of family (3) depends on $\mathbf{g}^{(2)}(\boldsymbol{\theta})$

$$\begin{aligned} \mathbf{g}_j^{(2)} &= \frac{1}{2}\sin(\theta_{j+1}-\theta_j)\left[\cos(\theta_j-\theta_{j+1})+\cos(\theta_{j+2}-\theta_{j+1})\right] \\ &+ \frac{1}{2}\sin(\theta_{j-1}-\theta_j)\left[\cos(\theta_j-\theta_{j-1})+\cos(\theta_{j-2}-\theta_{j-1})\right] \end{aligned}$$

- If $\ker(\mathcal{M}_1) = \{\mathbf{p}_0, \mathbf{p}_1\}$, then $(\mathbf{g}^{(2)}, \mathbf{p}_1) \neq 0$.

- Family (3) terminates except for one super-symmetric configuration:

$$\theta_1 = 0, \quad \theta_2 = \theta, \qquad \theta_3 = \pi, \quad \theta_4 = \pi + \theta,$$

Higher-order reductions : termination of super-symmetric fam

- Symbolic software algorithm is used on a squared domain of N_0-by-N_0 lattice nodes, where $N_0 = 2K + 2M + 1$, and K is the order of the Lyapunov-Schmidt reductions.

- Super-symmetric family (3) has $\mathbf{g}^{(k)}(\boldsymbol{\theta}) = 0$ for $k = 1, 2, 3, 4, 5$ but $\mathbf{g}^{(6)}(\boldsymbol{\theta}) \neq 0$, unless $\theta_{j+1} - \theta_j = \frac{\pi}{2}$.

- Moreover, $(\mathbf{g}^{(6)}, \mathbf{p}_1) \neq 0$.

- All asymmetric vortices (3) terminate

- All symmetric vortices (2) persist.

Stability of solutions in Lyapunov-Schmidt reductions

- First-order splitting of zero eigenvalues of $\mathcal{H}$:

$$\mathcal{M}_1 \mathbf{c} = \gamma \mathbf{c}$$

- First-order splitting of zero eigenvalues of $\mathcal{J}\mathcal{H}$:

$$\mathcal{M}_1 \mathbf{c} = \frac{\lambda^2}{2} \mathbf{c}$$

Stability of solutions in Lyapunov-Schmidt reductions

- First-order splitting of zero eigenvalues of $\mathcal{H}$:

$$\mathcal{M}_1 \mathbf{c} = \gamma \mathbf{c}$$

- First-order splitting of zero eigenvalues of $\mathcal{J}\mathcal{H}$:

$$\mathcal{M}_1 \mathbf{c} = \frac{\lambda^2}{2} \mathbf{c}$$

- Second-order splitting of zero eigenvalues of $\mathcal{H}$:

$$\mathcal{M}_1 = 0, \qquad \mathcal{M}_2 \mathbf{c} = \gamma \mathbf{c}$$

- Second-order splitting of zero eigenvalues of $\mathcal{J}\mathcal{H}$:

$$\mathcal{M}_1 = 0, \qquad \mathcal{M}_2 \mathbf{c} = \frac{\lambda^2}{2} \mathbf{c} + \lambda \mathcal{L}_2 \mathbf{c}$$

Stability of solutions in Lyapunov-Schmidt reductions

- First-order splitting of zero eigenvalues of $\mathcal{H}$:

$$\mathcal{M}_1 \mathbf{c} = \gamma \mathbf{c}$$

- First-order splitting of zero eigenvalues of $\mathcal{J}\mathcal{H}$:

$$\mathcal{M}_1 \mathbf{c} = \frac{\lambda^2}{2} \mathbf{c}$$

- Second-order splitting of zero eigenvalues of $\mathcal{H}$:

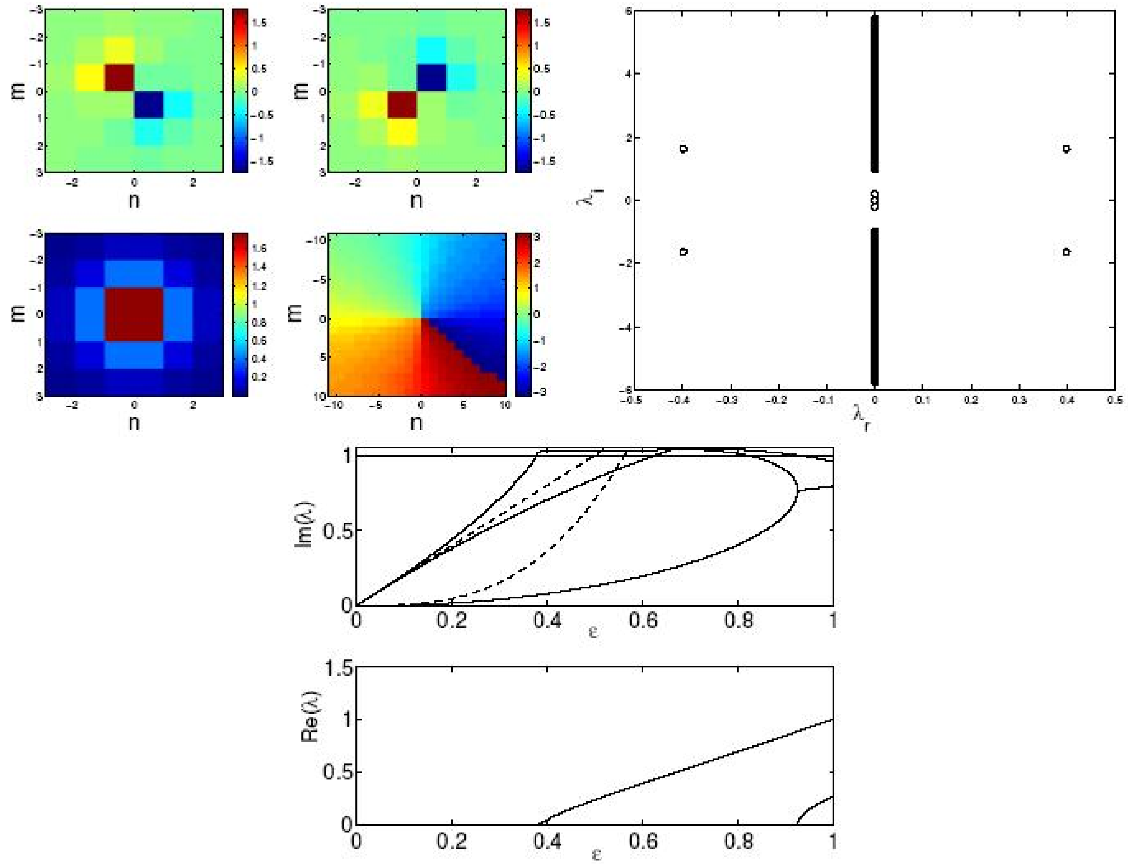
$$\mathcal{M}_1 = 0, \qquad \mathcal{M}_2 \mathbf{c} = \gamma \mathbf{c}$$

- Second-order splitting of zero eigenvalues of $\mathcal{J}\mathcal{H}$:

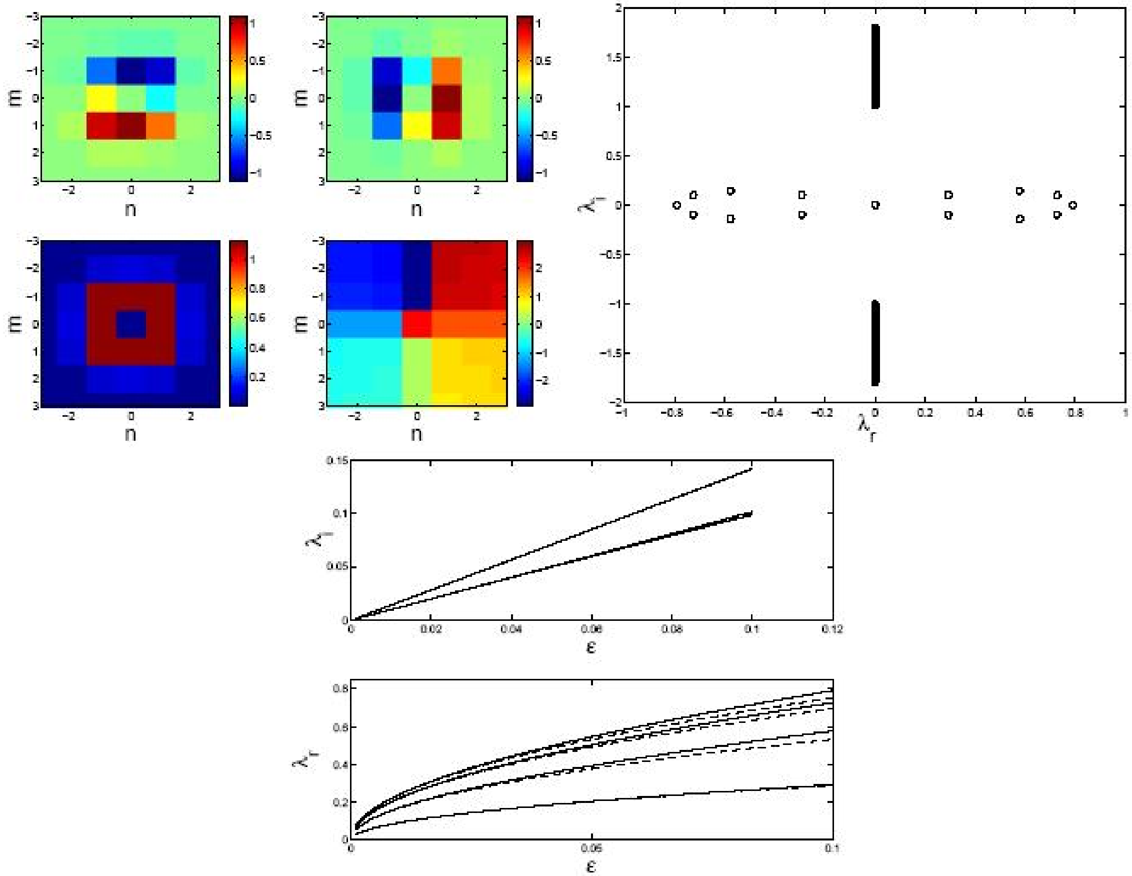
$$\mathcal{M}_1 = 0, \qquad \mathcal{M}_2 \mathbf{c} = \frac{\lambda^2}{2} \mathbf{c} + \lambda \mathcal{L}_2 \mathbf{c}$$

- Six-order splitting : symbolic software algorithm

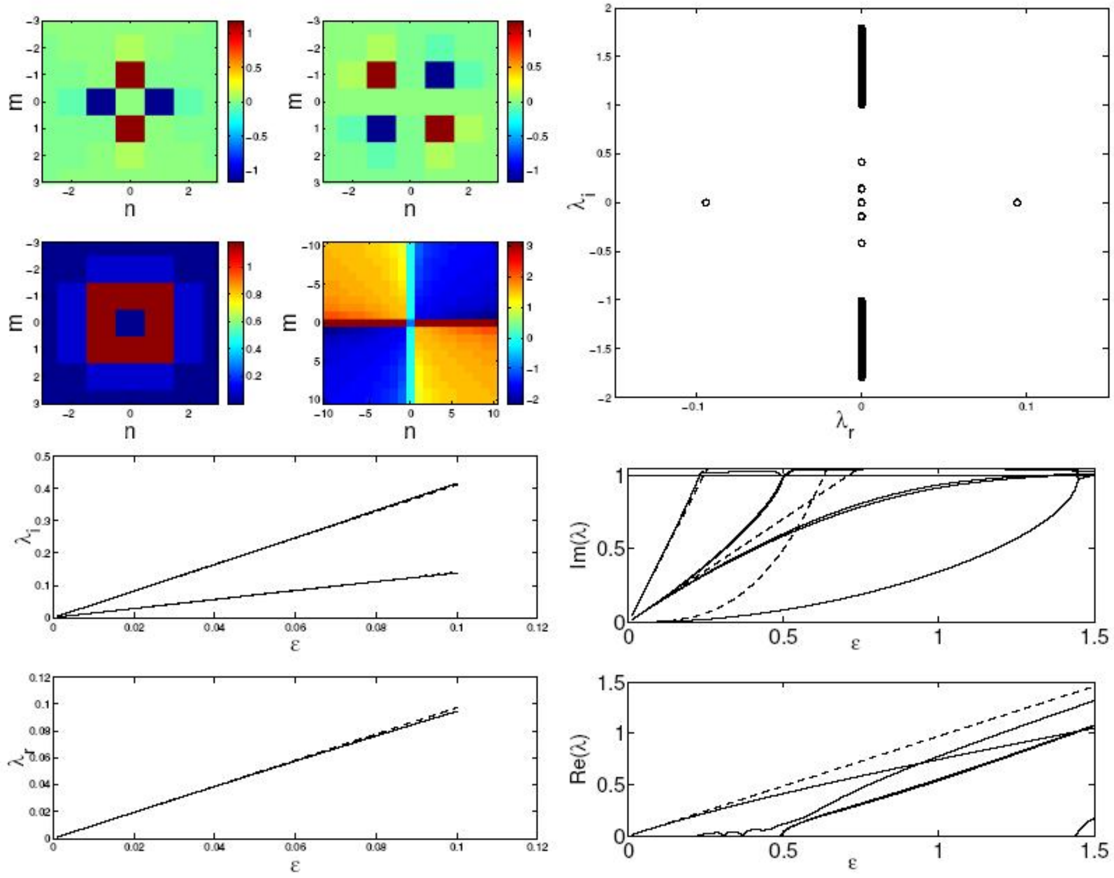
Numerical analysis: symmetric vortex with $L = M = 1$



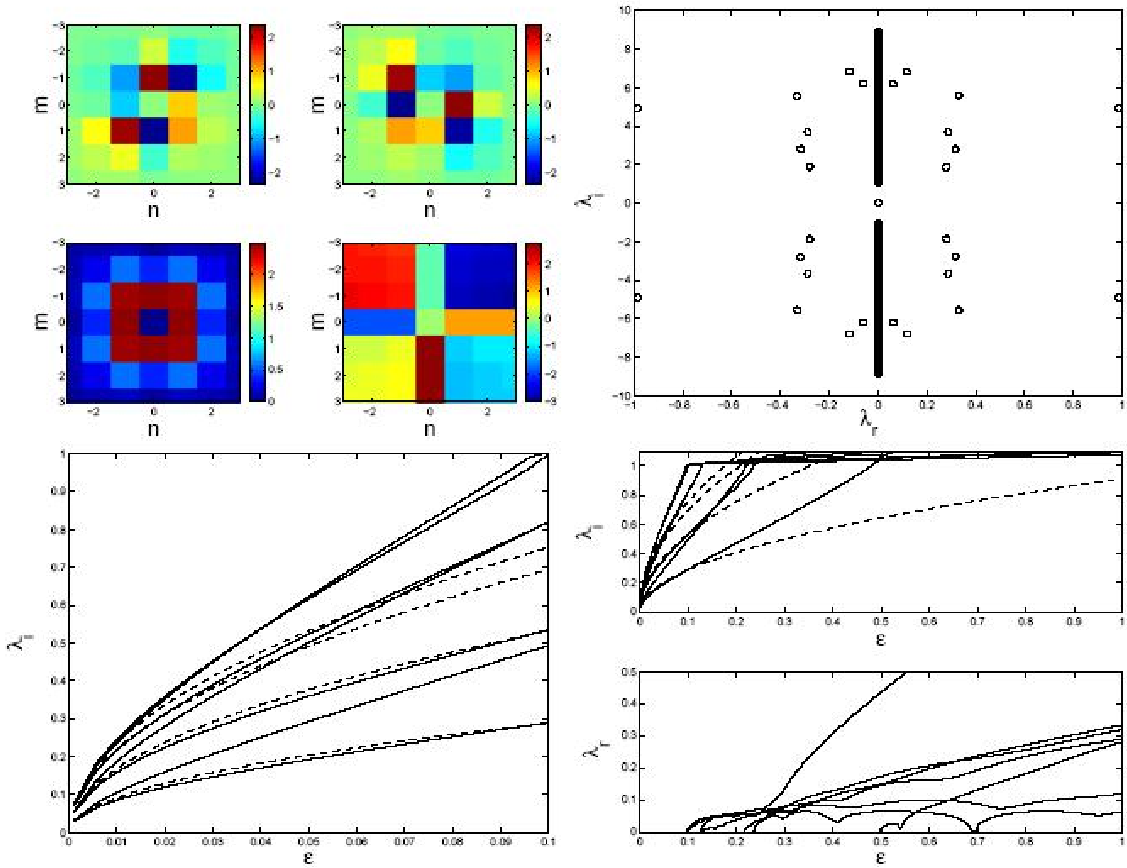
Numerical analysis: symmetric vortex with $L = 1$ and $M = 2$



Numerical analysis: symmetric vortex with $L = M = 2$



Numerical analysis: symmetric vortex with $L = 3$ and $M = 2$



Summary:

- **Systematic classification of discrete vortices**
- **Rigorous study of their existence and stability**
- **Predictions of stable and unstable localized modes**

contour S_M	vortex of charge L	linearized energy H	stable and unstable eig
$M = 1$	symmetric $L = 1$	$n(H) = 5$, $p(H) = 2$	$N_r = 0$, $N_i^+ = 1$, $N_i^- =$
$M = 2$	symmetric $L = 1$	$n(H) = 8$, $p(H) = 7$	$N_r = 1$, $N_i^+ = 0$, $N_i^- =$
$M = 2$	symmetric $L = 2$	$n(H) = 10$, $p(H) = 5$	$N_r = 1$, $N_i^+ = 2$, $N_i^- =$
$M = 2$	symmetric $L = 3$	$n(H) = 15$, $p(H) = 0$	$N_r = 0$, $N_i^+ = 0$, $N_i^- =$
$M = 2$	asymmetric $L = 1$	$n(H) = 9$, $p(H) = 6$	$N_r = 6$, $N_i^+ = 0$, $N_i^- =$
$M = 2$	asymmetric $L = 3$	$n(H) = 14$, $p(H) = 1$	$N_r = 1$, $N_i^+ = 0$, $N_i^- =$