Two-pulse solutions in the fifth-order KdV equation

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Background references:

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Department of Mathematics, Loughborough University, February 28, 2007

Background and motivations

Fifth-order KdV equation

$$u_t + u_{xxx} - u_{xxxxx} + 2uu_x = 0$$

has traveling wave solutions $u = \phi(z)$, $z = x - ct$, where $\phi(z)$ solves the fourth-order ODE

$$\phi^{(\mathrm{iv})} - \phi'' + c\phi = \phi^2.$$

Applications:

- capillary-gravity water waves (Craig–Groves, 1994)
- chains of coupled oscillators (Gorshkov–Ostrovsky, 1979)
- magneto–acoustic waves in plasma (Kawahara, 1972)

Solitary waves

Stability of the critical point $(0,0,0,0)$ in the fourth-order ODE:

$$\phi \sim e^{\kappa z} : \qquad \kappa^4 - \kappa^2 + c = 0.$$

Existence of localized solutions:

- $c < 0$ - no pulse solutions (Tovbis, 2000; Lombardi, 2000)
- $0 < c < \frac{1}{4}$ - unique one-pulse solution (Amick–Toland, 1992; Groves, 1998)
- $c > \frac{1}{4}$ - unique one-pulse and infinite countable set of two-pulse solutions (Buffoni–Sere, 1996)

$\Rightarrow$ The domain of our studies is $c > \frac{1}{4}$.

Mathematical problems

(1) Numerical approximations of two-pulse solutions

- numerical shooting method and continuation techniques (Champneys, 1993)
- iterations in Fourier space (Petviashvili's method)

$\Rightarrow$ Petviashvili's iterations diverge for two-pulse solutions!

Mathematical problems

(1) Numerical approximations of two-pulse solutions

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$\Rightarrow$ Petviashvili's iterations diverge for two-pulse solutions!

(2) Spectral stability of two-pulse solutions

- Lyapunov–Schmidt reductions (Sandstede, 1998)
- Count of eigenvalues in Pontryagin space (Krein's signatures)

$\Rightarrow$ The count of eigenvalues is inconclusive for two-pulse solutions!

Part I: Petviashvili's method (1976)

ODE for solitary waves

$$\phi^{(\mathrm{iv})} - \phi'' + c\phi = \phi^2, \qquad z \in \mathbb{R}$$

The ODE becomes the fixed-point problem in $H^2(\mathbb{R})$:

$$\hat{\phi}(k) = \frac{\widehat{\phi^2}(k)}{(c + k^2 + k^4)}, \qquad k \in \mathbb{R}$$

where $c > 0$ and $\hat{\phi}(k)$ is the Fourier transform of $\phi(z)$.

Iterations $\{\hat{u}_n(k)\}_{n=0}^{\infty}$ are defined recursively in $H^2_{\mathrm{ev}}(\mathbb{R})$:

$$\hat{u}_{n+1}(k) = M_n^2 \frac{\widehat{u_n^2}(k)}{(c + k^2 + k^4)}, \qquad M[\hat{u}_n] = \frac{\int_{\mathbb{R}} (c + k^2 + k^4) \left[\hat{u}_n(k)\right]^2 dk}{\int_{\mathbb{R}} \hat{u}_n(k) \widehat{u_n^2}(k) dk}$$

Convergence Theorem (2004)

- Let $\hat{\phi}(k)$ be a solution of the fixed-point problem in $H^2_{\rm ev}(\mathbb{R})$
- Let $\mathcal{H}$ be the Jacobian operator of the ODE at $\phi(z)$:
  $\mathcal{H} = c - \partial_z^2 + \partial_z^4 - 2\phi(z)$

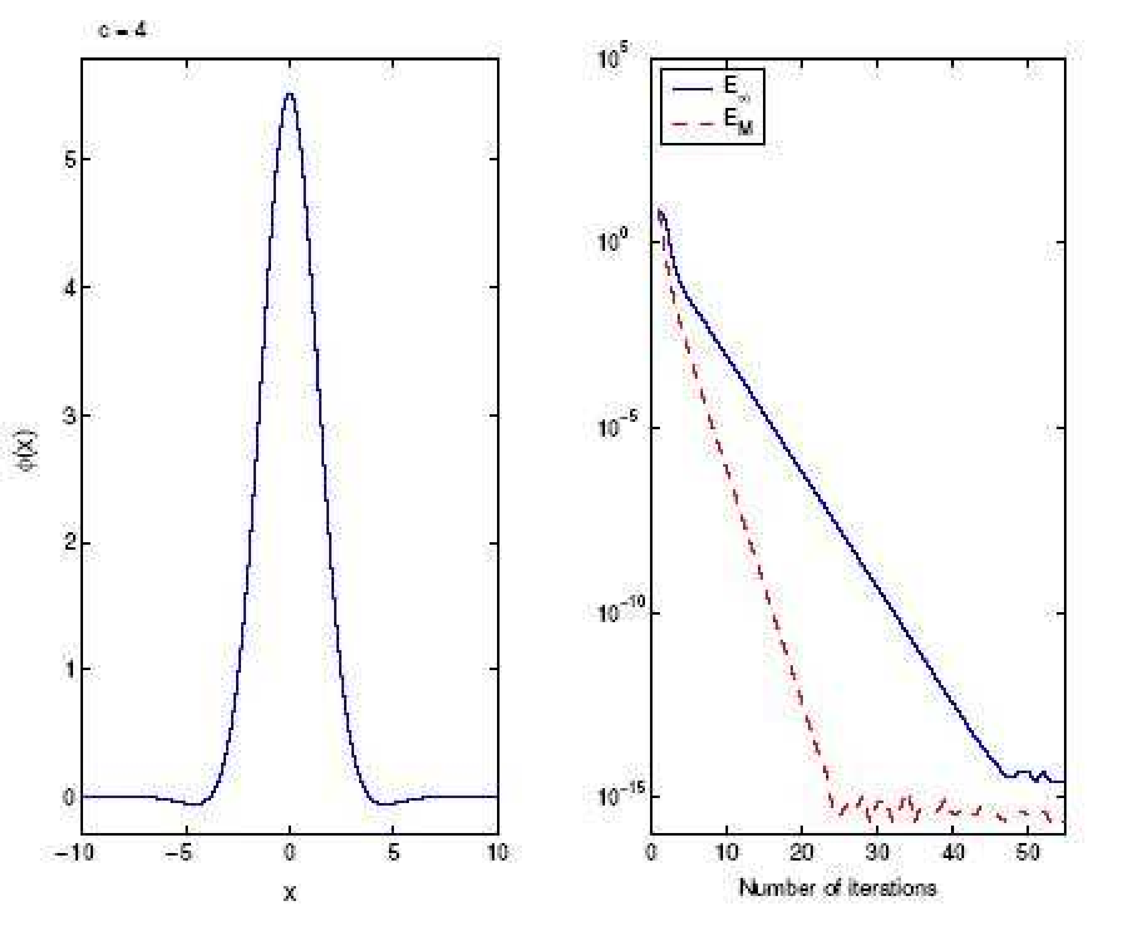
Theorem: *If $\mathcal{H}$ has exactly one negative eigenvalue and a simple zero eigenvalue and if*

$$\text{either} \quad \phi(z) \geq 0 \qquad \text{or} \quad \left| \inf_{z\in\mathbb{R}} \phi(z) \right| < \frac{c}{2},$$

then there exists an open neighborhood of $\hat{\phi}$ in $H^2_{\rm ev}(\mathbb{R})$, in which $\hat{\phi}$ is the unique fixed point and the sequence of iterations $\{\hat{u}_n(k)\}_{n=0}^{\infty}$ converges to $\hat{\phi}$.

One-pulse solutions

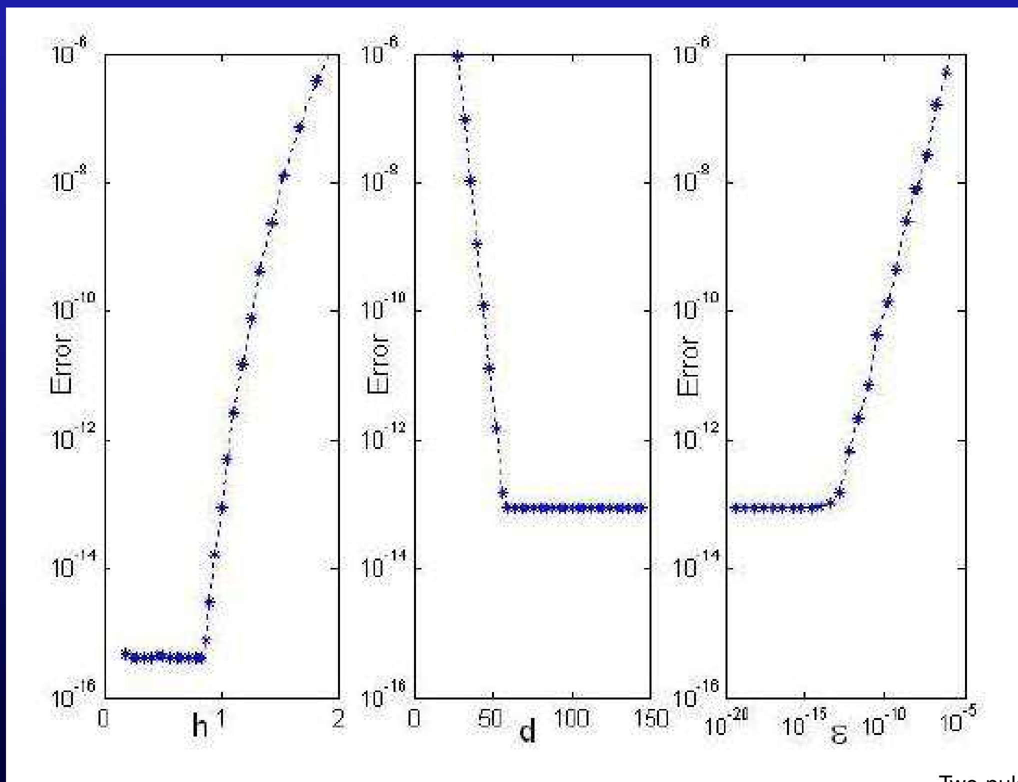
Let $\phi \equiv \Phi(z)$ be a one-pulse solution in $\mathcal{H} = c - \partial_z^2 + \partial_z^4 - 2\Phi(z)$. Then, $\mathcal{H}$ has exactly one negative eigenvalue and a simple kernel with $\Phi'(z)$ in $H^2(\mathbb{R})$.



Analysis of convergence

Numerical factors for numerical error:

- truncation of $z \in \mathbb{R}$ to the interval $z \in [-d, d]$
- truncation of Fourier series by the discrete sum with N terms
- small tolerance ε for $E_M = |M_n - 1|$ and $E_\infty = \|u_{n+1} - u_n\|_{L^\infty}$



Two-pulse solutions

Let $\phi(z)$ be a two-pulse solution. Then,

$$\phi(z) = \Phi(z - s) + \Phi(z + s) + \varphi(z),$$

where $\|\varphi\|_{L^\infty} = \mathrm{O}(e^{-2\kappa s})$ and $|s - s_0| = \mathrm{O}(e^{-2\kappa s})$, where κ is the decay rate of $\Phi(z)$ and s_0 is a non-degenerate extremum point of $W(s)$ in

$$W = \int_{\mathbb{R}} \Phi^2(z)\Phi(z + 2s)dz.$$

The operator $\mathcal{H}$ has two finite negative eigenvalues, a simple kernel with $\phi'(z)$, and a small eigenvalue μ in $H^2(\mathbb{R})$, such that

$$\left| \mu + \frac{2W''(2s_0)}{\|\Phi'\|_{L^2}^2} \right| \leq Ce^{-4\kappa s_0}.$$

Note to the proof

The function $\varphi(z)$ satisfies the ODE

$$\left(c - \partial_z^2 + \partial_z^4 - 2\Phi(z-s) - 2\Phi(z+s)\right)\varphi - \varphi^2 = 2\Phi(z-s)\Phi(z+s),$$

such that $\tilde{\varphi}(z) = \varphi(z+s)$ satisfies

$$\mathcal{H}\tilde{\varphi} = 2\Phi(z+2s)\tilde{\varphi} + \tilde{\varphi}^2 + 2\Phi(z)\Phi(z+2s).$$

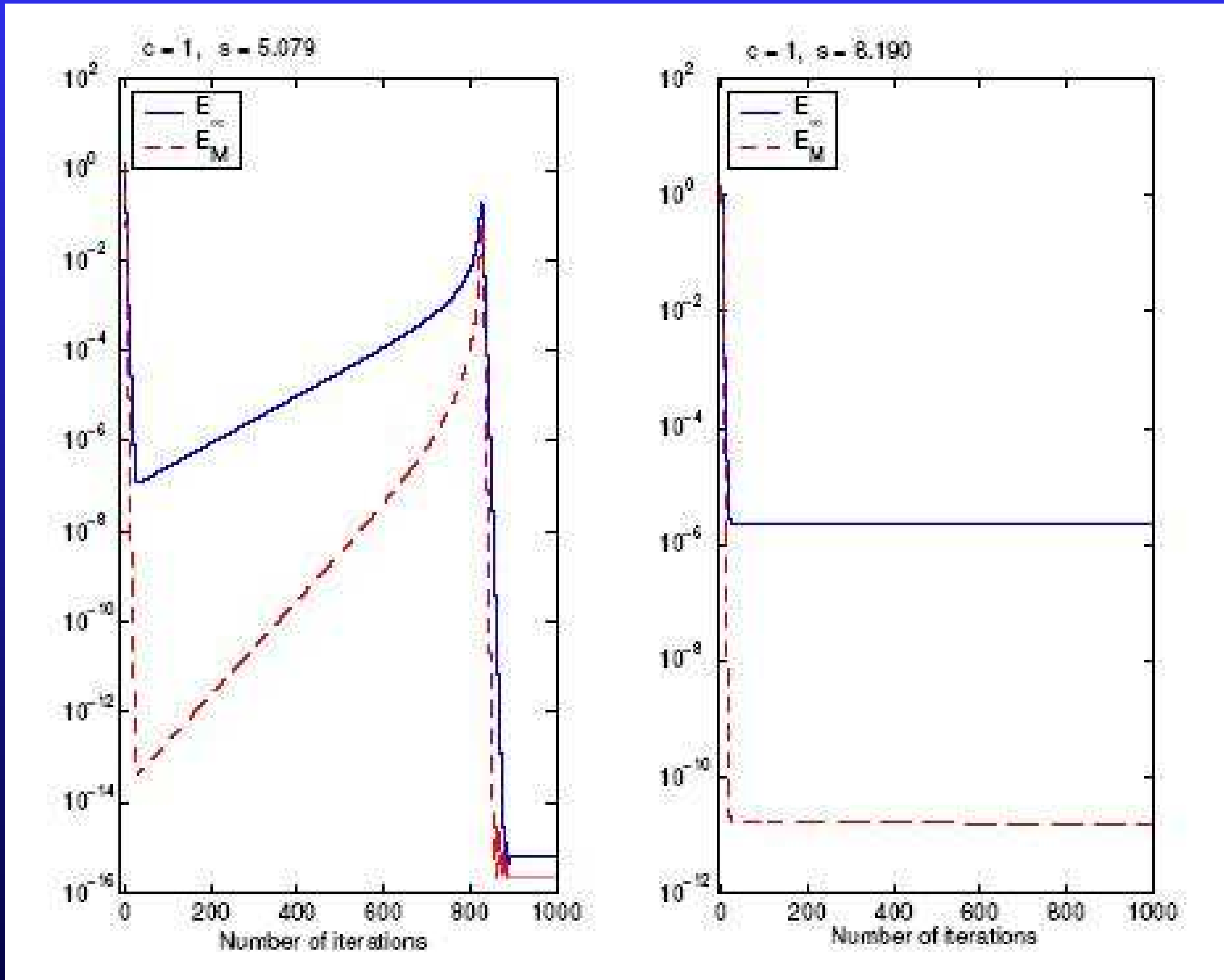
Since s is not yet defined, let $\tilde{\varphi} \in X_c \subset L^2 : \ (\tilde{\varphi}, \Phi') = 0$. By the method of Lyapunov–Schmidt reductions, s must be the root of

$$\begin{aligned} F(s,\epsilon) &= \left(\Phi'(z), 2\Phi(z+2s)\tilde{\varphi} + \tilde{\varphi}^2 + 2\Phi(z)\Phi(z+2s)\right) \\ &= -W'(s) + \mathrm{o}(e^{-2\kappa s}). \end{aligned}$$

If the extremum of $W(s)$ is non-degenerate at $s = s_0$, a unique two-pulse solution persists near $s = s_0$.

Construction of two-pulse solutions

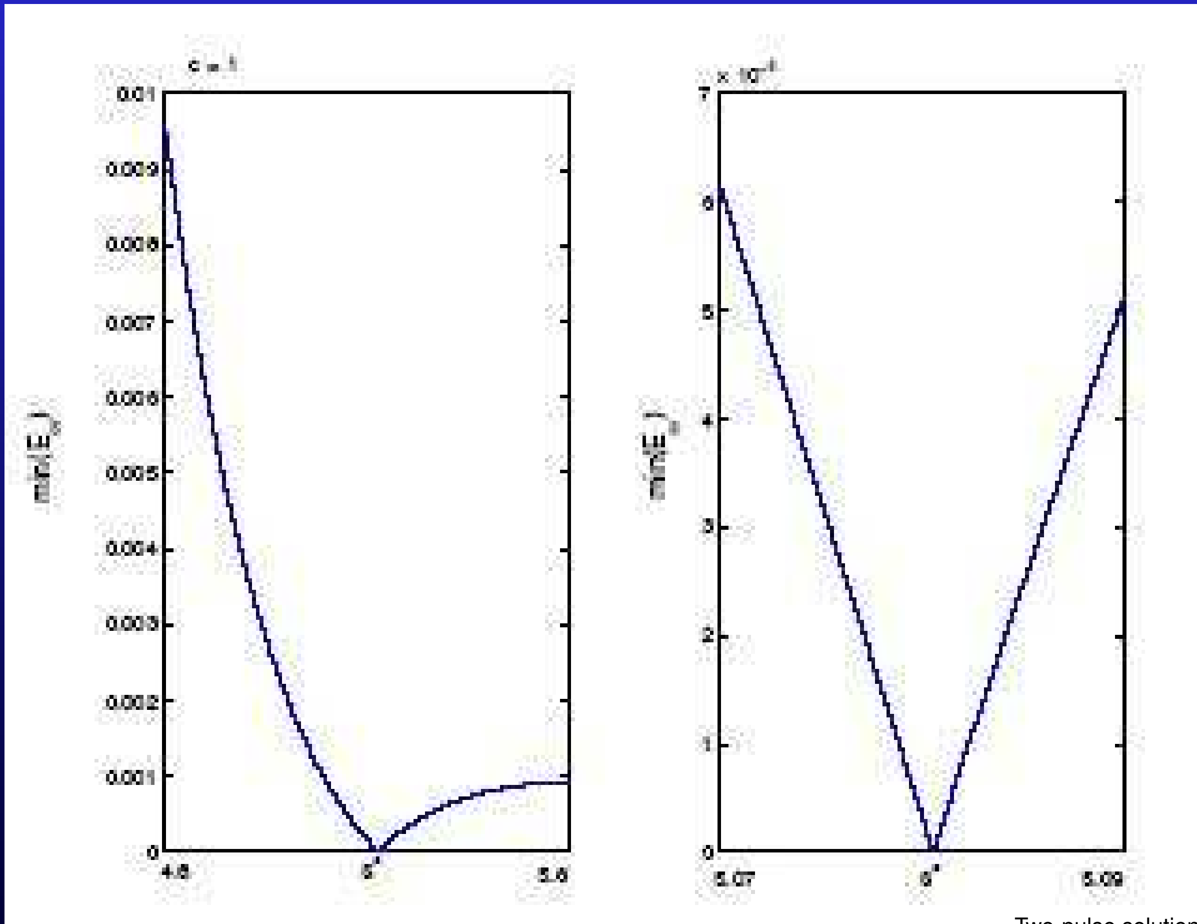
Iterations of the Petviashvili's method:



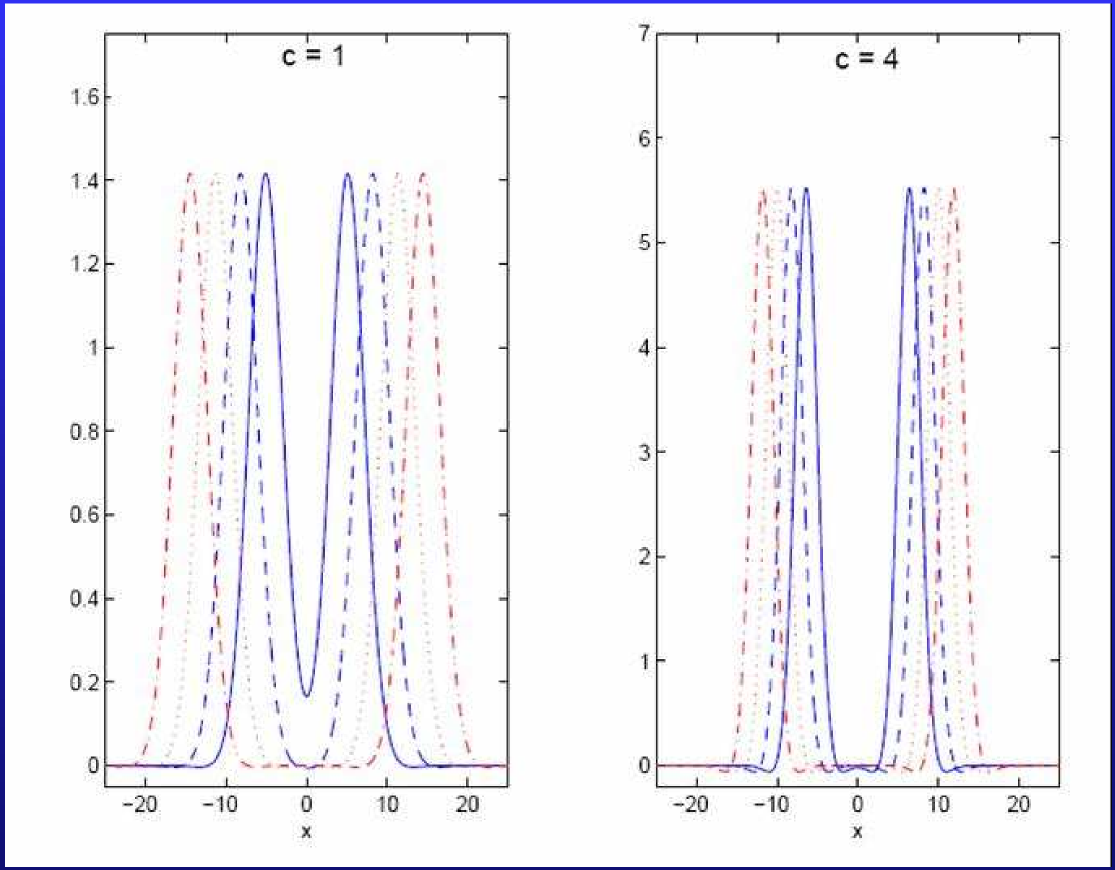
Numerical algorithm

Theorem: *There exists $s = s_*$ near $s = s_0$ such that the iteration method with $u_0 = \Phi(z - s_0) + \Phi(z + s_0)$ converges to a two-pulse solution $\phi(z)$ in a local neighborhood of ϕ in $H^2_{\mathrm{ev}}(\mathbb{R})$.*

Minimum error for root search:



Numerical two-pulse solutions



distance s	effective potential	root finding
s_1	5.058733328146916	5.079717398028492
s_2	8.196800619090793	8.196620796452045

Part II: Spectral stability

Linearized problem for spectral stability

$$\partial_z \mathcal{H} v = \lambda v, \qquad v \in X_c^* \subset L^2 : \quad (v, \phi) = 0.$$

Eigenvalues with $\mathrm{Re}(\lambda) > 0$ result in spectral instability.

Let $\phi(z)$ be a two-pulse solution. There exists a pair of small eigenvalues λ of the linearized operator $\partial_z \mathcal{H}$, such that

$$\left| \lambda^2 + \frac{4W''(2s_0)}{P'(c)} \right| \leq C e^{-4\kappa s_0}, \qquad P'(c) = \frac{d}{dc} \|\Phi\|_{L^2}^2 > 0.$$

- $W''(2s_0) > 0$ - pair of purely imaginary eigenvalues
- $W''(2s_0) < 0$ - pair of real eigenvalues

Spectral stability theorem

Notations:

- $N_{\rm real}$ - the number of real positive eigenvalues
- $N_{\rm comp}$ - the number of complex eigenvalues in the first open quadrant
- $N_{\rm imag}^{-}$ - the number of simple positive imaginary eigenvalues with $(\mathcal{H}v, v) \leq 0$
- The kernel of $\mathcal{H}$ is simple and $P'(c) > 0$

Theorem: Then,

$$N_{\rm real} + 2N_{\rm comp} + 2N_{\rm imag}^{-} = n(\mathcal{H}) - 1,$$

where $n(\mathcal{H})$ is the number of negative eigenvalues of $\mathcal{H}$.

Sylvester–Pontryagin–Grillakis Theorem

Theorem: Let L and M be self-adjoint operators in H with finitely many negative eigenvalues $n(L)$ and $n(M)$ and empty kernels. Then, there are exactly $n(L)$ and $n(M)$ eigenvalues γ of $Lu = \gamma M u$ in $L^2(\mathbb{R})$ such that $(u, Lu) \leq 0$ and $(u, Mu) \leq 0$.

References:

L. Pontryagin, Izv. Acad. Nauk SSSR **8**, 243–280 (1944)

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For the KdV linearization,

$$H = X_c, \quad \gamma = -\lambda^2, \quad L = -\partial_z \mathcal{H} \partial_z, \quad M = \mathcal{H}^{-1}.$$

Applications of the Theorem

Counts of eigenvalues:

- One-pulse solutions

$$n(\mathcal{H}) = 1, \qquad N_{\rm real} = N_{\rm comp} = N^-_{\rm imag} = 0$$

The one-pulse solution is a ground state (Levandosky, 1999)

- Two-pulse solutions with $W''(2s_0) < 0$

$$n(\mathcal{H}) = 2, \qquad N_{\rm real} = 1, \qquad N_{\rm comp} = N^-_{\rm imag} = 0$$

The two-pulse solution with $W''(2s_0) < 0$ is spectrally unstable.

Applications of the Theorem

Counts of eigenvalues:

- Two-pulse solutions with $W''(2s_0) > 0$

$$n(\mathcal{H}) = 3, \qquad N_{\text{real}} = 0, \qquad N_{\text{comp}} + N^{-}_{\text{imag}} = 1$$

The count is inconclusive for these solutions.

Applications of the Theorem

Counts of eigenvalues:

- Two-pulse solutions with $W''(2s_0) > 0$

$$n(\mathcal{H}) = 3, \qquad N_{\rm real} = 0, \qquad N_{\rm comp} + N_{\rm imag}^- = 1$$

The count is inconclusive for these solutions.

Theorem: Let $\lambda \in i\mathbb{R}$ be a simple eigenvalue of $\partial_z \mathcal{H}$ with eigenfunction $v_0 \in X_c$ such that $(\mathcal{H} v_0, v_0) < 0$. Then, it is structurally stable to parameter continuations.

Corollary: The two-pulse solution with $W''(2s_0) > 0$ is spectrally stable with

$$N_{\rm comp} = 0, \qquad N_{\rm imag}^- = 1.$$

Note to the proof

Exponentially weighted space

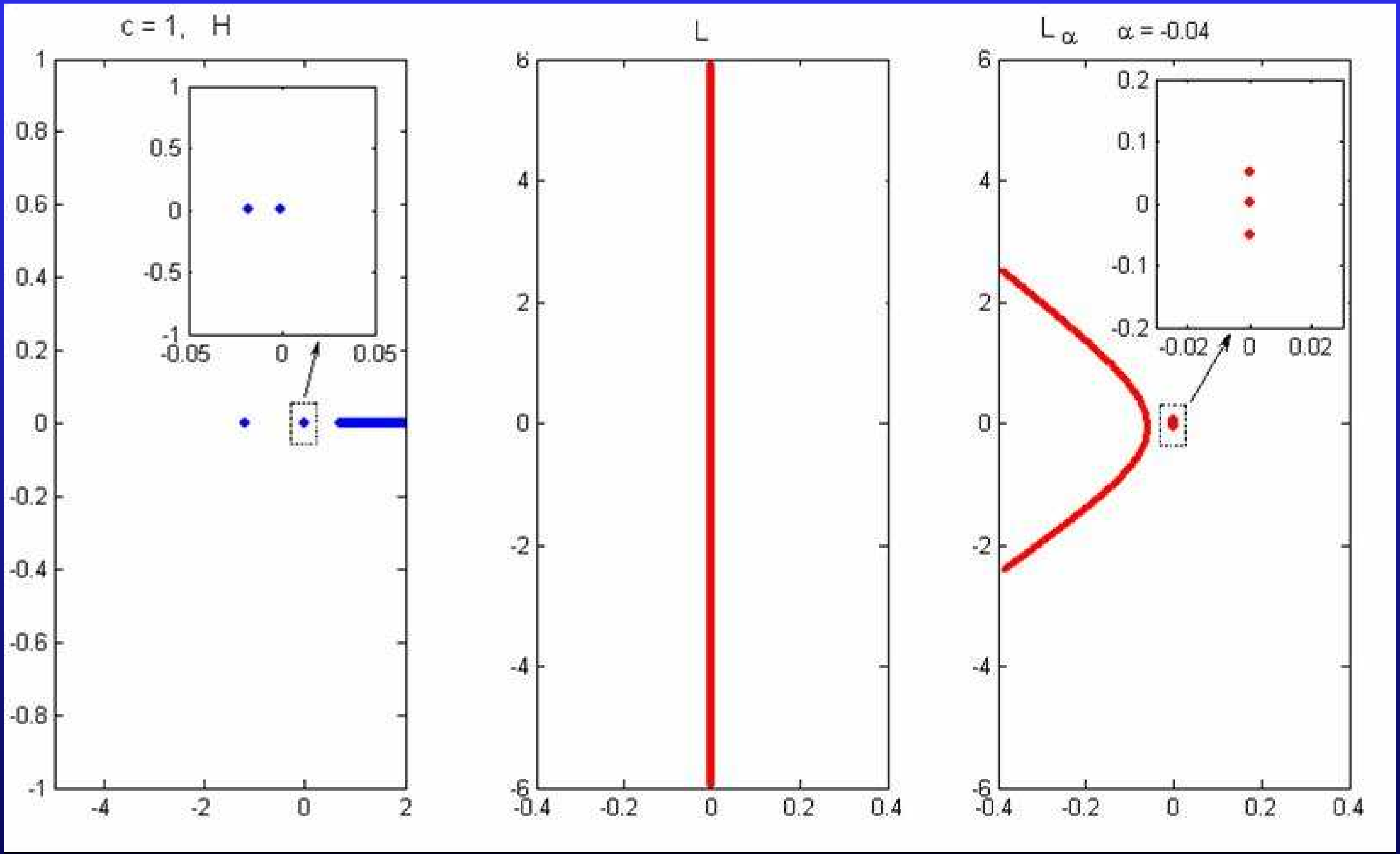
$$H^2_\alpha = \left\{ v \in H^2_{\rm loc}(\mathbb{R}) : \quad e^{\alpha z} v(z) \in H^2(\mathbb{R}) \right\}.$$

The continuous spectrum of $\partial_z \mathcal{H}$ for small $\alpha > 0$ resides at a simply-connected curve in the left half-plane of $\lambda \in \mathbb{C}$.

- If $v_0 \in H^2(\mathbb{R})$ for a simple eigenvalue $\lambda_0 \in i\mathbb{R}_+$, then $v_0 \in H^2_\alpha(\mathbb{R})$ for the same eigenvalue.
- If $(\mathcal{H} v_0, v_0) < 0$, there exists $w_0 \in H^2(\mathbb{R})$, such that $v_0 = w_0'$ and $(w_0, v_0) \in i\mathbb{R}_+$.
- For a parameter continuation in $\partial_z(\mathcal{H} + \delta V(z))$ where $V \in L^\infty(\mathbb{R})$, we show that $v_\delta \in H^2(\mathbb{R})$, $v_\delta = w_\delta'(z)$, and (w_δ, v_δ) are all continuous in δ. Therefore, $(w_\delta, v_\delta) \in i\mathbb{R}_+$ and $\lambda_\delta \in i\mathbb{R}_+$ in the relation $\lambda_\delta (w_\delta, v_\delta) = (\mathcal{H} v_\delta, v_\delta) < 0$.

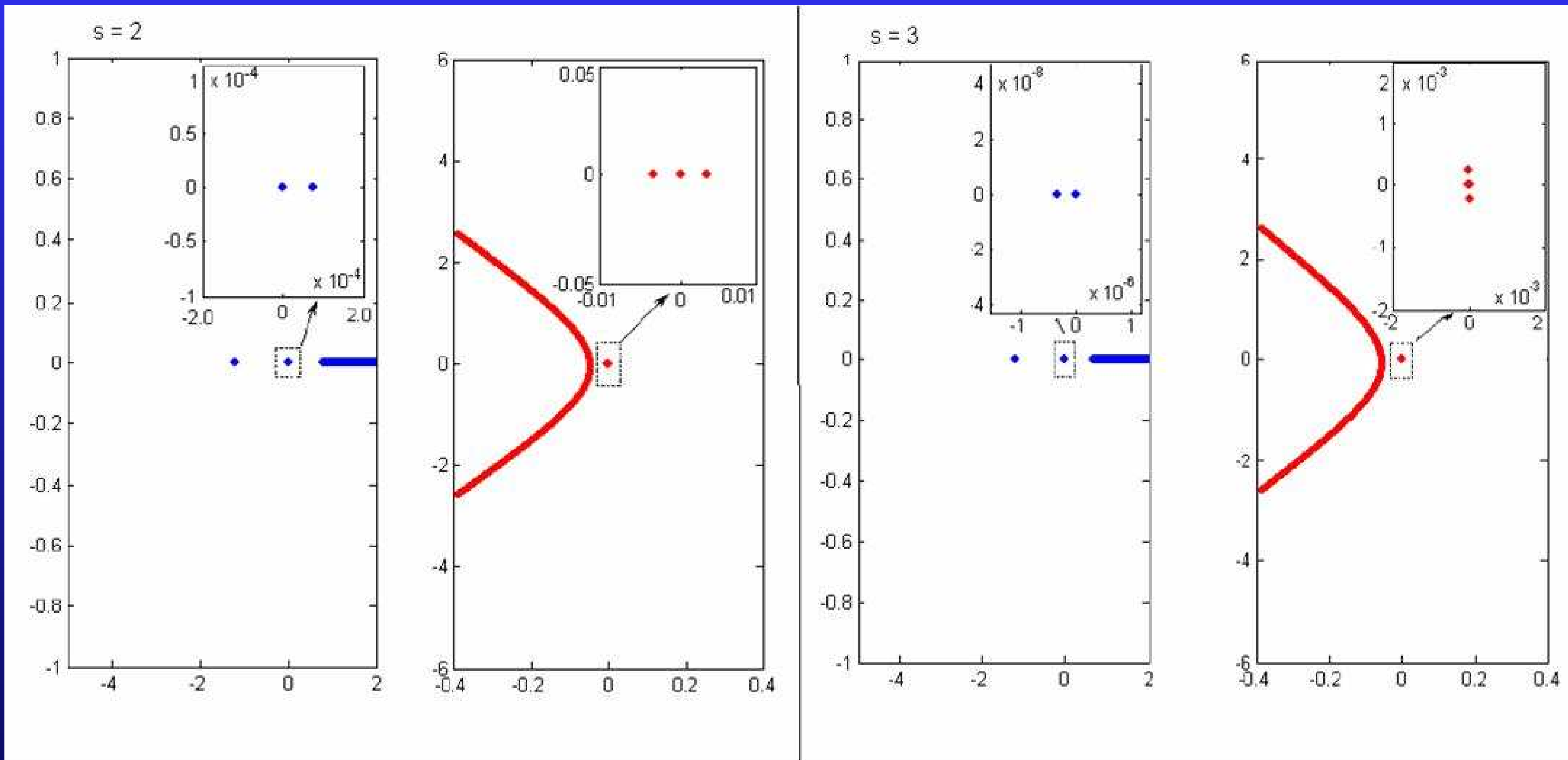
Numerical spectrum

First two-pulse solution:



Numerical spectrum

Second (left) and third (right) two-pulse solutions:

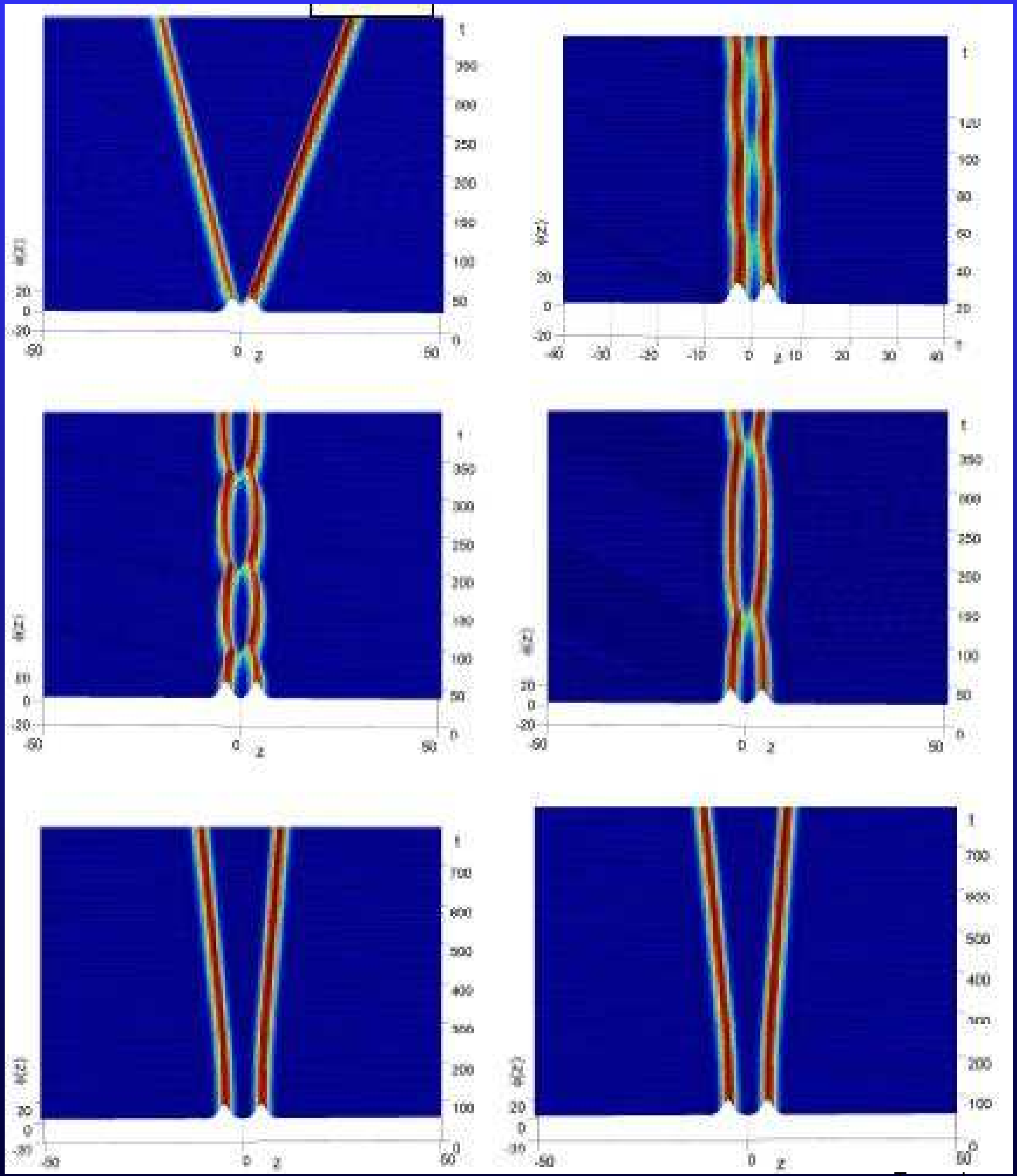


Numerical spectrum

Table of numerical approximations of small eigenvalues:

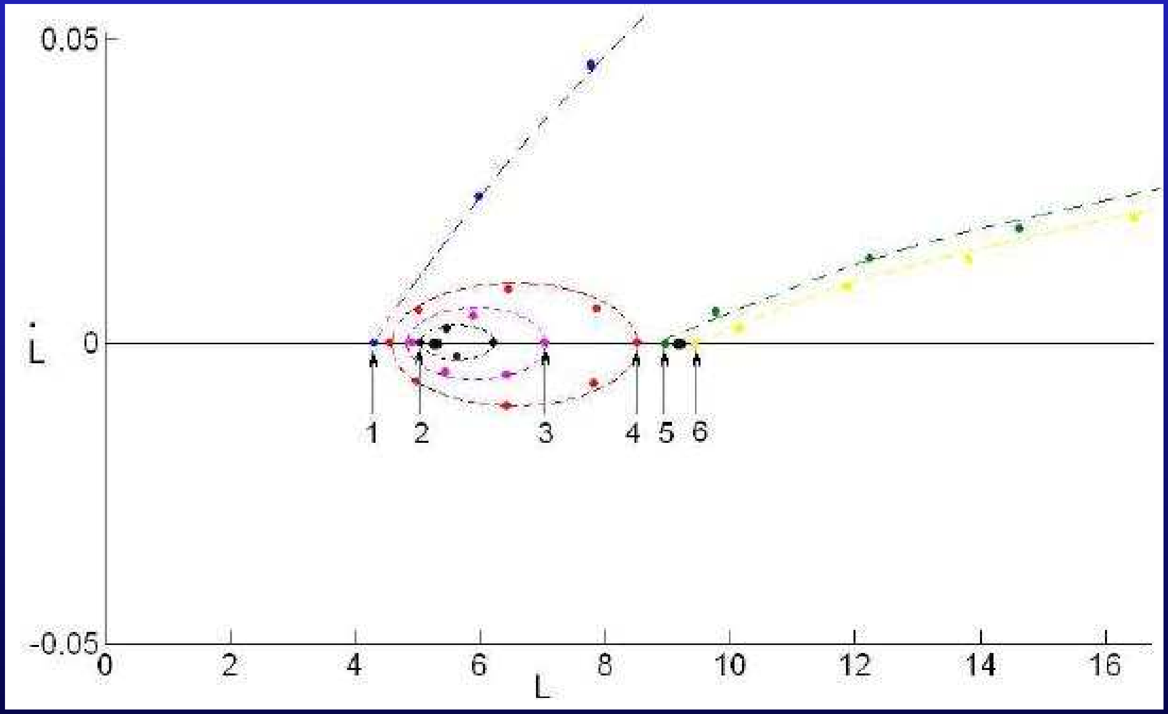
	first solution	second solution
"Zero" EV of $\mathcal{H}$	$1.2156 \cdot 10^{-9}$	$2.6678 \cdot 10^{-9}$
Small EV of $\mathcal{H}$	$1.7845 \cdot 10^{-2}$	$7.6638 \cdot 10^{-5}$
"Zero" EVs of $\mathcal{L}_\alpha$	$0.3653 \cdot 10^{-5}$	$0.5321 \cdot 10^{-5}$
Re of small EVs of $\mathcal{L}_\alpha$	$4.5291 \cdot 10^{-6}$	$3.2845 \cdot 10^{-3}$
Im of small EVs of $\mathcal{L}_\alpha$	$0.5021 \cdot 10^{-1}$	$1.1523 \cdot 10^{-8}$

Time-evolution simulations



mparison with the Newton's particle equa

Conjecture (Gorshkov–Ostrovskii): If the initial value $u(x,0)$ is locally close to the two-pulse solution $\Phi(x-s_0)+\Phi(x+s_0)$, the solution $u(x,t)$ remains close to the two-pulse solution $\Phi(x-ct-s(t))+\Phi(x-ct+s(t))$ for $0 \le t \le Ce^{\kappa_0 s}$, where $s(t)$ satisfies the Newton's particle law $P'(c)\ddot{s} = -W''(2s)$.



Conclusions

Outcomes of our work:

- Application of Pontryagin spaces to KdV equations
- Numerical approximations of two-pulse solutions
- Proof of structural stability of embedded eigenvalues

Open problems:

- Error bounds on validity of the Newton's particle law
- Numerical approximations of three- and multi-pulse solutions
- Proof of asymptotic stability of multi-pulse solutions

Software for relevant computations