Periodic Waves in Fractional KdV Equation

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Fractional Korteweg de Vries Equation

The fractional KdV is a popular model for dynamics of waves in shallow fluids:

$$u_t + 2uu_x - (D^\alpha u)_x = 0,$$

where the fractional derivative operator D_α is defined by

$$\widehat{D^\alpha u(\xi)} = |\xi|^\alpha \, \hat{u}(\xi), \qquad \xi \in \mathbb{R}.$$

Integrable cases: Benjamin–Ono equation ($\alpha = 1$) and KdV equation ($\alpha = 2$).

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Here we consider 2π-periodic solutions on $\mathbb{T} := (-\pi, \pi)$, so that $\xi \in \mathbb{Z}$.

1. Positivity of periodic travelling wave solution
2. Convergence of Petviashvili method for fixed-point iterations
3. New variational formulation of periodic wave solutions

Background

- Well-posedness in Sobolev spaces:
  - F. Linares, D. Pilod, J.C. Saut (2014)
  - L. Molinet, D. Pilod, S. Vento (2018)

- Existence and modulation stability of periodic waves by using
  - pertubative methods in M. Johnson (2013),
  - variational methods in H. Chen, J. Bona (2013), V.Hur, M. Johnson (2015)
  - fixed-point methods in H. Chen (2004)

- Existence and stability of solitary waves in J. Angulo (2018):
  - stable for $\frac{1}{2} < \alpha \leq 2$
  - unstable for $\frac{1}{3} < \alpha < \frac{1}{2}$

- Convergence of Petviashvili's method near periodic waves in
  - J. Alvarez, A. Duran (2017)
  - D. Clamond, D. Dutykh (2018)

Stationary equations for periodic waves

The right propagating, periodic travelling wave solution takes the form

$$u(x,t) = \psi(x - ct), \qquad c > 0.$$

Integrating the equation with zero constant yields the boundary value problem

$$(c + D^\alpha)\psi = \psi^2, \qquad \psi \in H^\alpha_{per}(-\pi,\pi).$$

Advantage: $c + D^\alpha$ is positive (useful for fixed-point iterations).

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The left propagating wave $u(x,t) = \phi(x + ct)$ with same $c > 0$ is related to ψ by

$$\phi(x) = \psi(x) - c,$$

and satisfies the boundary value problem

$$(c - D^\alpha)\phi + \phi^2 = 0, \qquad \phi \in H^\alpha_{per}(-\pi,\pi).$$

Advantage: $c - D^\alpha$ may vanish (useful for local bifurcation theory).

Stokes expansions of small-amplitude waves

Consider the BVP as a bifurcation problem:

$$(c - D^\alpha)\phi + \phi^2 = 0, \qquad \phi \in H^\alpha_{per}(-\pi, \pi),$$

with $\sigma(c - D^\alpha) = \{c, c - 1, c - 2^\alpha, c - 3^\alpha, \dots\}$.

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Theorem. For every $\alpha > \frac{1}{2}$, there exists a locally unique even solution ϕ bifurcating from zero solution. The wave profile ϕ and the wave speed c are real analytic in wave amplitude a and satisfy the following Stokes expansions

$$\phi = a\cos(x) + a^2\phi_2(x) + a^3\phi_3(x) + \mathcal{O}(a^4),$$
$$c = 1 + c_2 a^2 + \mathcal{O}(a^4).$$

with

$$\phi_2(x) = -\frac{1}{2} + \frac{1}{2(2^\alpha - 1)}\cos(2x) \text{ and } c_2 = 1 - \frac{1}{2(2^\alpha - 1)}.$$

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Note the threshold behavior: $c_2 > 0$ for $\alpha > \alpha_0$ and $c_2 < 0$ for $\alpha < \alpha_0$, where $\alpha_0 \approx 0.585$.

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Positivity of ψ

ϕ is not positive. Recall the relation between ψ and ϕ

$$\psi(x) = \phi(x) + c, \qquad x \in [-\pi, \pi].$$

Since $c \approx 1$ and $\phi \approx 0$, then ψ is positive.

In the integrable cases, ψ remains positive for every $c > 1$, e.g.

$$\alpha = 1: \quad \psi(x) = \frac{\sinh\gamma}{\cosh\gamma - \cos x}, \quad c = \coth\gamma.$$

Question: Is ψ positive for every $c > 1$ if $\alpha > \alpha_0$?

Main result on positivity of ψ

Theorem (Le–P, 2019)

For every $c > 1$ and $\alpha \in (\alpha_0, 2]$, there exists an even single-lobe solution $\psi \in H^{\alpha}_{per}(-\pi, \pi)$ to the BVP

$$(c + D_\alpha)\psi = \psi^2.$$

such that $\psi(x) > 0$ on $[-\pi, \pi]$.

We say the periodic wave has single-lobe profile if there is only one maximum and minimum of ψ on the period.

Small-amplitude waves bifurcating from zero at $c = 1$ are single-lobe solutions.

Proof of positivity of ψ: Step 1

Green's function for $c + D^\alpha$ is obtained from the solution of

$$(c + D^\alpha)\varphi(x) = h, \qquad h \in L^2_{per}(-\pi, \pi),$$

in the convolution form

$$\varphi(x) = \int_{-\pi}^{\pi} G(x - s)h(s)ds$$

or in Fourier form,

$$G_{c,\alpha}(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{e^{inx}}{c + |n|^\alpha} \quad \Rightarrow \quad \|G_{c,\alpha}\|_{L^2_{per}} \leq M_{c,\alpha}, \qquad \alpha > 1/2.$$

For $\alpha \leq 2$, the Greens function is strictly positive:

$$G_{c,\alpha}(x) \geq m_{c,\alpha},$$

Nieto (2010) for $\alpha \in (0, 1)$; Bai-Lu (2005) for $\alpha \in (1, 2)$.

Proof of positivity of ψ: Step 2

Operator A in the positive cone

From the BVP

$$(c + D^\alpha)\psi = \psi^2,$$

we define the nonlinear operator

$$A_{c,\alpha}(\psi) := (c + D^\alpha)^{-1}\psi^2 \Rightarrow A_{c,\alpha}(\psi)(x) = \int_{-\pi}^{\pi} G_{c,\alpha}(x - s)\psi(s)^2 ds,$$

and the positive cone in $L^2_{per}(-\pi, \pi)$

$$P_{c,\alpha} := \left\{ \psi \in L^2_{per}(-\pi, \pi) : \psi(x) \geq \frac{m_{c,\alpha}}{M_{c,\alpha}} \|\psi\|_{L^2_{per}}, x \in [-\pi, \pi] \right\}.$$

- i) $A_{c,\alpha}$ is bounded and continuous in $L^2_{per}(-\pi, \pi)$ (Young's inequality),
- ii) $A_{c,\alpha}$ is compact as it is a limit of compact operators $A^{(N)}_{c,\alpha}$, where $A^{(N)}_{c,\alpha}$ are gives by $2N + 1$ Fourier coefficients.
- iii) $A_{c,\alpha}(\psi)$ is closed in $P_{c,\alpha}$: $A_{c,\alpha}(\psi) \geq m_{c,\alpha}\|\psi\|^2_{L^2_{per}} \geq \frac{m_{c,\alpha}}{M_{c,\alpha}} \|A_{c,\alpha}(\psi)\|_{L^2_{per}}$.

Proof of positivity of ψ: Step 3

3) Existence of fixed point in the cone

Let

$$B_r := \{\psi \in L^2_{per}(-\pi,\pi) : \|\psi\|_{L^2_{per}} < r\}$$

By Kranoselskii's fixed point theorem if there exists r_- and r_+ such that

$$\|A_{c,\alpha}(\psi)\|_{L^2_{per}} < \|\psi\|_{L^2_{per}}, \qquad \psi \in P_{c,\alpha} \cap \partial B_{r_-}$$
$$\|A_{c,\alpha}(\psi)\|_{L^2_{per}} > \|\psi\|_{L^2_{per}}, \qquad \psi \in P_{c,\alpha} \cap \partial B_{r_+}$$

then, $A_{c,\alpha}$ has fixed point in $P_{c,\alpha}$.

- r_- is small enough so that $r_- M_{c,\alpha} < 1$
- r_+ is large enough so that $\sqrt{2\pi} r_+ m_{c,\alpha} > 1$
- $r_- < r_+$ because $\sqrt{2\pi} m_{c,\alpha} \leq M_{c,\alpha}$.

By bootstrapping argument, if $\psi \in L^2_{per}$, then $\psi \in H^\infty_{per}$.

Remark: The positive fixed point may not be single-lobe since the constant solution $\psi = c$ is always a positive fixed point of $A_{c,\alpha}$ in $P_{c,\alpha}$ for every $c > 0$.

Proof of positivity of ψ: Step 4

4) Distinguishing ψ from constant fixed point

Definition (Leray-Schauder index)

The Leray-Schauder index of the fixed point ψ is defined as $(-1)^N$, where N is the number of unstable eigenvalues of $A'_{c,\alpha}(\psi)$ outside the unit disk with the account of the multiplicities.

For the constant solution $\psi = c$, the linearized operator

$$A'_{c,\alpha}(c) = 2c(c + D^\alpha)^{-1} : L^2_{per} \to L^2_{per}$$

in the space of even functions has $N = k + 1$ unstable eigenvalues outside the unit disk for $c \in (k^\alpha, (k+1)^\alpha)$ with $k \in \mathbb{N}$. The index of the constant solution changes sign every time c crosses the resonance at k^α, $k \in \mathbb{N}$.

Number of unstable eigenvalues along solution branches

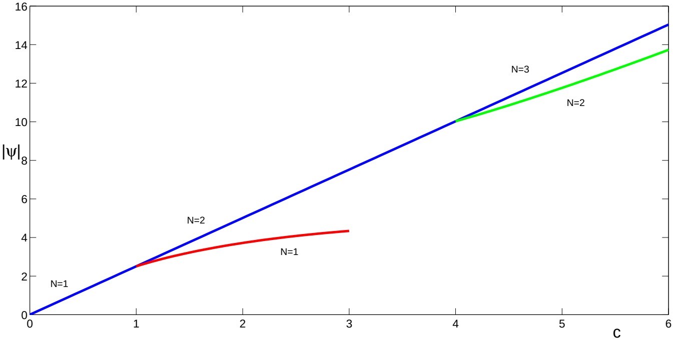

Figure: Schematic representation of bifurcations from the constant fixed point $\psi = c$. Here $\alpha = 2$.

No bifurcations along the single-lobe solutions

Positive single-lobe fixed point ψ bifurcates for $c > 1$ if $\alpha > \alpha_0$. The linearized operator at ψ is given by

$$A'_{c,\alpha}(\psi) = 2(c + D^\alpha)^{-1}\psi = Id - (c + D^\alpha)^{-1}\mathcal{H}_{c,\alpha}.$$

where $\mathcal{H}_{c,\alpha} := c + D^\alpha - 2\psi$ is the linearization of the fractional KdV.

No bifurcations along the single-lobe solutions

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Lemma

$N = 1$ is true for every $c > 1$ along the branch of single-lobe solutions.

- For $c \gtrsim 1$, this can be shown by the perturbative argument (if $\alpha > \alpha_0$).
- For other $c > 1$, we rely on the result of V.Hur and M.Johson (2015), $\mathrm{Ker}(\mathcal{H}_{c,\alpha}) = \mathrm{span}(\partial_x \psi) \Rightarrow$ if $N = 1$ for $c \gtrsim 1$, then $N = 1$ for $c > 1$.

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Petviashvili method for fixed point iterations

Recall the stationary equation for ψ:

$$(c + D^\alpha)\psi = \psi^2, \quad \Rightarrow \quad \psi = A_{c,\alpha}(\psi) := (c + D^\alpha)^{-1}\psi^2.$$

However, the linearized operator

$$A'_{c,\alpha}(\psi) = 2(c - D_\alpha)^{-1}\psi$$

always has $N = 1$ unstable eigenvalue outside the unit disk.
$\Rightarrow$ Fixed-point iterations diverge from the single-lobe periodic waves.

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V. Petviashvili (1976) introduced a stabilizing factor in the fixed-point iterations:

$$w_{n+1} = T_{c,\alpha}(w_n) := [M_{c,\alpha}(w_n)]^2 (c + D^\alpha)^{-1}(w_n^2), \qquad n \in \mathbb{N},$$

where

$$M_{c,\alpha}(w) := \frac{\langle (c + D^\alpha)w, w\rangle}{\langle w^2, w\rangle}, \qquad w \in H^\alpha_{per}(-\pi, \pi).$$

If $w = \psi$, then $M_{c,\alpha}(\psi) = 1$ and $T_{c,\alpha}(\psi) = \psi$.

Main results on convergence of fixed-point iterations

Theorem (Le–P, 2019)

For every $c > 1$ *and* $\alpha \in (\alpha_0, 2]$, *the single-lobe solution* $\psi \in H^\alpha_{per}$ *to*

$$(c + D^\alpha)\psi = \psi^2,$$

is an asymptotically stable fixed point of $T_{c,\alpha}$.

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Question: Does the Petviashvili's method converge for sign-indefinite wave such as ϕ satisfying $(c - D^\alpha)\phi + \phi^2 = 0$?

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Answer:

i) ϕ is an unstable fixed point of $T_{c,\alpha}$ for $\alpha \in (\alpha_0, \alpha_1)$, where $\alpha_1 \approx 1.322$

ii) ϕ is an asymptotically stable fixed point for $\alpha \in (\alpha_1, 2]$ if $c \gtrsim 1$ and is unstable if $c > 1$ is large enough.

Proof of convergence

Consider again the linearized fixed-point iterations:

$$A'_{c,\alpha}(\phi) := 2(-c + D^{\alpha})^{-1}\phi = Id - (-c + D^{\alpha})^{-1}\mathcal{H}_{c,\alpha},$$
$$\mathcal{H}_{c,\alpha} := -c + D^{\alpha} - 2\phi.$$

Spectrum of $A'_{c,\alpha}(\phi)$ is related to the spectrum of $(-c + D^{\alpha})^{-1}\mathcal{H}_{c,\alpha}$:

$$\mathcal{H}_{c,\alpha} v = \lambda(-c + D^{\alpha})v, \qquad v \in H^{\alpha}_{per}(-\pi, \pi),$$

where both $\mathcal{H}_{c,\alpha}$ and $(-c + D^{\alpha})$ are sign-indefinite.

Eigenvalues of $(-c + D^{\alpha})^{-1}\mathcal{H}_{c,\alpha}$ are divided for $c \gtrsim 1$ into two sets $\{\sigma_1, \sigma_2\}$:

1) σ_1 contains sequence of eigenvalues near 1 and converging to 1, related to the subspace $L^2_{per}(-\pi, \pi)\backslash\{e^{ix}, e^{-ix}\}$,
2) σ_2 contains finite number of eigenvalues related to the subspace $\{e^{ix}, e^{-ix}\}$.

Small-amplitude periodic wave: $c \gtrsim 1$

Related to the subspace $\{e^{ix}, e^{-ix}\}$, we find $\sigma_2 = \{-1, 0, \lambda_1, \lambda_2\}$, where

$$\lambda_1 \to \frac{2^{\alpha+1} - 5}{2^{\alpha+1} - 3}, \qquad \lambda_2 < 2, \;\; \lambda_2 \to 2 \text{ as } c \to 1.$$

The eigenvalues $\{-1, 0\}$ are due to exact solutions:

$$\begin{array}{l} (-c + D^\alpha)^{-1} \mathcal{H}_{c,\alpha} \phi = -\phi, \\ (-c + D^\alpha)^{-1} \mathcal{H}_{c,\alpha} \phi' = 0, \end{array} \qquad \Rightarrow \qquad \{-1, 0\} \subset \sigma_2.$$

for which

$$A'_{c,\alpha}(\phi) = Id - (-c + D^\alpha)^{-1} \mathcal{H}_{c,\alpha} \Rightarrow \{2, 1\} \subset \sigma(A'_{c,\alpha}(\phi)).$$

For $\alpha_0 \approx 0.585$ and $\alpha_1 \approx 1.322$

$$\lambda_1 < 0 \text{ if } \; \alpha \in (\alpha_0, \alpha_1), \qquad \lambda_1 \in (0, 1) \text{ if } \; \alpha \in (\alpha_1, 2]$$

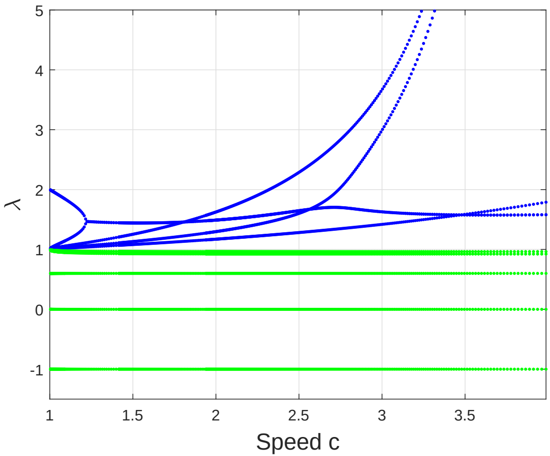


Figure: Eigenvalues of the operator $(-c + D^{\alpha})^{-1}\mathcal{H}_{c,\alpha}$ for $\alpha = 2$ (KdV equation)

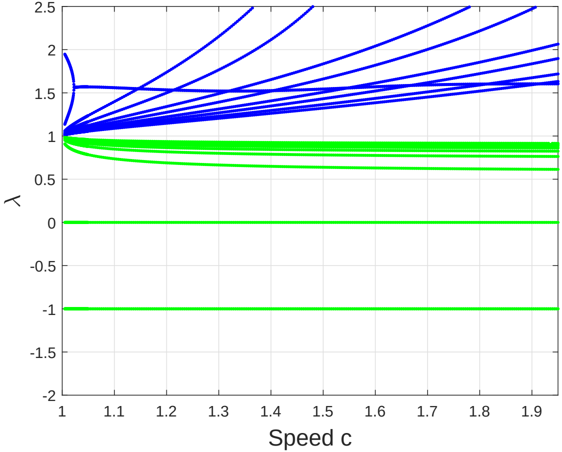


Figure: Eigenvalues of the operator $(-c + D^{\alpha})^{-1}\mathcal{H}_{c,\alpha}$ for $\alpha = 1$ (Benjamin-Ono equation): Here $\lambda = -1$ is a double eigenvalue!

Convergence case for $(-c + D^\alpha)\phi = \phi^2$, $\alpha = 2$, $c = 2$

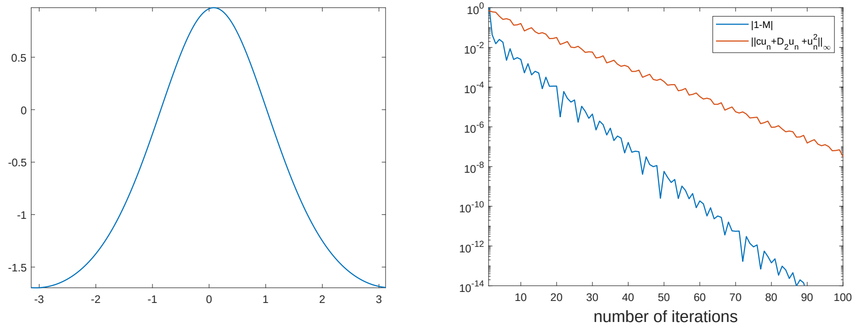
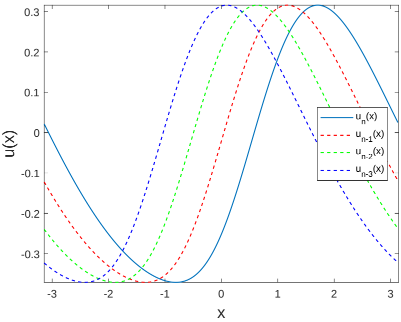


Figure: Iterations for $c = 2$ and $\alpha = 2$. Left) The last iteration versus x. (Right) Computational errors versus n.

Divergence case for $(-c + D^\alpha)\phi = \phi^2$, $\alpha = 1$, $c = 1.1$



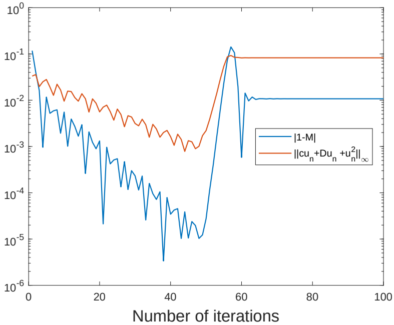


Figure: Iterations for $c = 1.1$ and $\alpha = 1$. (Left) The last four iterations versus x. (Right) Computational errors versus n.

Summary on convergence of Petviashvili's method

- Petviashvili's method does not converge well for left-propagating sign-indefinite periodic waves satisfying

$$(c - D^\alpha)\phi + \phi^2 = 0, \qquad \phi \in H^\alpha_{per}(-\pi,\pi).$$

- Petviashvili's method converge unconditionally for right-propagating positive periodic waves satisfying

$$(c + D^\alpha)\psi = \psi^2, \qquad \psi \in H^\alpha_{per}(-\pi,\pi).$$

where $\psi(x) = c + \phi(x)$. This is related to the fact that

$$A'_{c,\alpha} = 2(c + D^\alpha)^{-1}\psi$$

has only $N = 1$ unstable eigenvalue lying outside the unit disk.

Convergence case for $(c + D^{\alpha})\psi = \psi^2$, $\alpha = 1$, $c = 1.6$

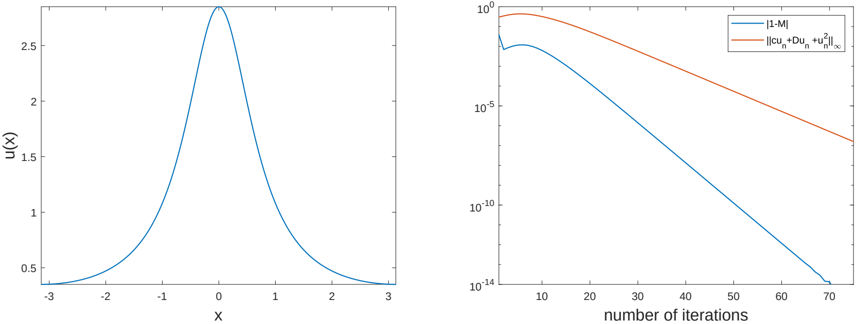


Figure: Iterations for $c = 1.6$ and $\alpha = 1$. (a) The last iteration versus x. (b) Computational errors versus n.

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where b is an integration constant.

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where b is an integration constant.

Thanks to the Galilean transformation

$$\psi(x) = \frac{1}{2}\left(c - \sqrt{c^2 + 4b}\right) + \varphi(x),$$

the periodic wave φ is a solution to the stationary equation

$$(\omega + D^\alpha)\varphi - \varphi^2 = 0,$$

with only one parameter $\omega := \sqrt{c^2 + 4b}$.

Different formulations

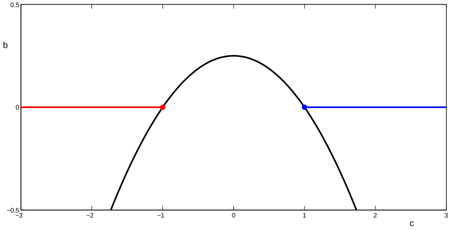
Stationary equation

$$(c + D^\alpha)\psi - \psi^2 + b = 0 \quad \Rightarrow (\omega + D^\alpha)\varphi - \varphi^2 = 0.$$

admit two families of periodic wave solutions:

- ψ is obtained for $c > 1$ and $b = 0$
- ϕ is obtained for $c < -1$ and $b = 0$.

Obstacle on existence: When $\alpha < \alpha_0 \approx 0.585$, Stokes waves ψ bifurcate to $c < 1$ instead of $c > 1$ because $c = 1 + c_2 a^2 + \mathcal{O}(a^4)$ with $c_2 < 0$.



New formulation

Stationary equation

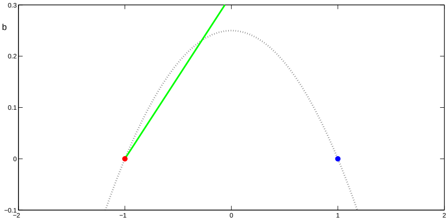
$$(c + D^\alpha)\psi - \psi^2 + b = 0 \quad \Rightarrow (\omega + D^\alpha)\varphi - \varphi^2 = 0.$$

Let ψ has zero mean on $\mathcal{T}$ so that $\psi = \varphi - \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi dx$. Then, b is defined by

$$b(c) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\psi^2 dx.$$

No fold point appears for $\alpha < \alpha_0$:

$$c = -1 + \frac{1}{2(2^\alpha - 1)}a^2 + \mathcal{O}(a^4), \quad b(c) = \frac{1}{2}a^2 + \mathcal{O}(a^4).$$



Existence of periodic waves

Standard variational method: find minimizers of energy

$$E(u) = \frac{1}{2}\int_{-\pi}^{\pi}(D^{\frac{\alpha}{2}}u)^2 - \frac{1}{3}\int_{-\pi}^{\pi}u^3dx,$$

subject to fixed momentum and mass

$$F(u) = \frac{1}{2}\int_{-\pi}^{\pi}u^2dx, \quad M(u) = \int_{-\pi}^{\pi}u\,dx.$$

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$$F(u) = \frac{1}{2}\int_{-\pi}^{\pi} u^2 dx, \quad M(u) = \int_{-\pi}^{\pi} u \, dx.$$

New variational method: find minimizer of the quadratic energy

$$\mathcal{B}_c(u) := \frac{1}{2}\int_{-\pi}^{\pi} \left[(D^{\frac{\alpha}{2}} u)^2 + cu^2 \right] dx$$

subject to fixed cubic energy and zero-mean constraint:

$$Y := \left\{ u \in H^{\frac{\alpha}{2}}_{\rm per}(\mathbb{T}) : \quad \int_{-\pi}^{\pi} u^3 dx = 1, \quad \int_{-\pi}^{\pi} u dx = 0 \right\}.$$

There exists a constrained minimizer $u_* \in Y$ for every $\alpha > 1/3$ and every $c > -1$.

Continuation of periodic waves: standard approach

Hessian operator for both variational problems is the same operator:

$$\mathcal{L} = D^\alpha + c - 2\psi : H^\alpha_{\rm per}(\mathbb{T}) \subset L^2_{\rm per}(\mathbb{T}) \to L^2_{\rm per}(\mathbb{T}),$$

This operator enjoys Sturm's oscillation theory [Hur–Johnson, 2015] which yields

Lemma (Hur–Johnson, 2015)

Assume $\alpha \in (\frac{1}{3}, 2]$ *and that* $\psi \in H^\infty_{\rm per}(\mathbb{T})$ *be an even single-lobe periodic wave. If* $\{1, \psi, \psi^2\} \in {\rm Range}(\mathcal{L})$*, then* ${\rm Ker}(\mathcal{L}) = {\rm span}(\partial_x \psi)$.

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At the fold point for $\alpha < \alpha_0 \approx 0.585$, $\{1, \psi, \psi^2\} \in {\rm Range}(\mathcal{L})$ is false.
As a result, ${\rm dim Ker}(\mathcal{L}) = 2$ at the fold point.
Since c and b are Lagrange multipliers in $G(u) = E(u) + cF(u) + bM(u)$, the periodic wave ψ may not be differentiable in c and b. As a result,

$$\mathcal{L}\partial_c \psi = -\psi, \quad \mathcal{L}\partial_b \psi = -1$$

may not follow from $(c + D^\alpha)\psi - \psi^2 + b = 0$.

Continuation of periodic waves: new approach

For the zero-mean periodic wave ψ with $b(c) = \frac{1}{2\pi} F(\psi)$, we verify:

$$\mathcal{L}\psi = -\psi^2 - b(c),$$
$$\mathcal{L}1 = -2\psi + c.$$

Theorem (Natali,Le,P, 2019)

If $\mathrm{Ker}(\mathcal{L}|_{1^\perp}) = \mathrm{span}(\partial_x \psi)$ *at* $c = c_0$*, then the mapping* $c \mapsto \psi$ *is* C^1 *at* $c = c_0$*.*

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Hence, we add the third equation:

$$\mathcal{L}\partial_c \psi = -\psi - b'(c), \quad \Rightarrow \quad \mathcal{L}\left(1 - 2\partial_c \psi\right) = c + 2b'(c).$$

Corollary

If $c + 2b'(c) \neq 0$, *then* $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x \psi)$. *If* $c + 2b'(c) = 0$, *then* $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x \psi, 1 - 2\partial_c \psi)$.

Stability of periodic waves: new approach

Since ψ is a minimizer of the new variational problem, we have:

$$\mathcal{L}\Big|_{\{1,\psi^2\}^\perp} \geq 0,$$

which yields the exact formula for the number of negative eigenvalues of $\mathcal{L}$:

$$n(\mathcal{L}) = \begin{cases} 1, & c + 2b'(c) \geq 0, \\ 2, & c + 2b'(c) < 0. \end{cases}$$

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and the number of negative eigenvalues in the old variational problem:

$$n(\mathcal{L}\big|_{\{1,\psi\}^\perp}) = \begin{cases} 0, & b'(c) \geq 0, \\ 1, & b'(c) < 0. \end{cases}$$

Theorem (Natali,Le,P, 2019)

Assume $\mathrm{Ker}(\mathcal{L}|_{1^\perp}) = \mathrm{span}(\partial_x \psi)$ *for* $c \in (-1, \infty)$. *The zero-mean periodic wave* ψ *is spectrally stable if* $b'(c) > 0$ *and is spectrally unstable if* $b'(c) < 0$.

Summary

For the periodic waves in the fractional KdV equation satisfying

$$(c + D^\alpha)\psi - \psi^2 + b = 0,$$

we have showed the following:

1. $\psi > 0$ for $c > 1$, $b = 0$, and $\alpha > \alpha_0 \approx 0.585$
2. Petviashvili method diverges for ψ for $c < -1$, $b = 0$, and $\alpha > \alpha_0$
3. Periodic waves ψ with zero mean are obtained with a new variational problem with $b \neq 0$ for both $\alpha > \alpha_0$ and $\alpha < \alpha_0$.

Thank you! Questions???

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