Variational characterization of periodic waves in the fractional KdV equation

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Fractional Korteweg de Vries Equation

The fractional KdV is a popular model for dynamics of waves in shallow fluids:

$$u_t + 2uu_x = (-\Delta)^{\alpha/2} u_x,$$

where the fractional Laplacian $(-\Delta)^{\alpha/2}$ is defined by

$$\widehat{(-\Delta)^{\alpha/2}} u(\xi) = |\xi|^{\alpha}\, \hat{u}(\xi), \qquad \xi \in \mathbb{R}.$$

Integrable cases: Benjamin–Ono equation ($\alpha = 1$) and KdV equation ($\alpha = 2$).

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Here we consider 2π-periodic solutions on $\mathbb{T} := [-\pi, \pi]$, so that $\xi \in \mathbb{Z}$.

1. New variational formulation for travelling periodic waves.
2. Positivity of periodic travelling wave profiles.
3. Convergence of Petviashvili's method for fixed-point iterations

Background

- Well-posedness in Sobolev spaces:
  - F. Linares, D. Pilod, J.C. Saut (2014)
  - L. Molinet, D. Pilod, S. Vento (2018)

- Existence and modulation stability of periodic waves by using
  - pertubative methods for $\alpha > \frac{1}{2}$ in M. Johnson (2013),
  - variational methods for $\alpha > \frac{1}{3}$ in V.Hur, M. Johnson (2015)
  - fixed-point methods in H. Chen (2004) and H. Chen, J. Bona (2013)

- Existence and stability of solitary waves in J. Angulo (2018):
  - stable for $\frac{1}{2} < \alpha \leq 2$
  - unstable for $\frac{1}{3} < \alpha < \frac{1}{2}$

- Convergence of Petviashvili's method near periodic waves in
  - J. Alvarez, A. Duran (2017)
  - D. Clamond, D. Dutykh (2018)

Particular family of travelling periodic waves

The periodic travelling wave solution takes the form

$$u(x,t) = \psi(x - ct).$$

Integrating the equation with zero constant yields the boundary value problem

$$(c + (-\Delta)^{\alpha/2})\psi = \psi^2, \qquad \psi \in H^{\alpha}_{per}.$$

Advantage: If $c + (-\Delta)^{\alpha/2}$ is positive, this can be used for fixed-point iterations.

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With the transformation

$$\psi(x) = c + \phi(x),$$

the same boundary-value problem can be written as

$$(c - (-\Delta)^{\alpha/2})\phi + \phi^2 = 0, \qquad \phi \in H^{\alpha}_{per}.$$

Advantage: if $c - (-\Delta)^{\alpha/2}$ vanishes, this can be used for local bifurcation theory.

Stokes expansions of small-amplitude waves

Consider the BVP as a bifurcation problem:

$$(c - (-\Delta)^{\alpha/2})\phi + \phi^2 = 0, \qquad \phi \in H^\alpha_{per},$$

with the spectrum $\sigma(c - (-\Delta)^{\alpha/2}) = \{c, c-1, c-2^\alpha, \dots\}$
and Fourier modes $\{1, e^{\pm ix}, e^{\pm 2ix}, \dots\}$.

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Theorem. For every $\alpha > \frac{1}{2}$, there exists a locally unique, even, single-lobe solution $\phi \in H^{\alpha}_{per}$ bifurcating from zero solution. The wave profile ϕ and the wave speed c are real analytic in wave amplitude a and satisfy the following Stokes expansions

$$\phi = a\cos(x) + a^2\phi_2(x) + a^3\phi_3(x) + \mathcal{O}(a^4),$$
$$c = 1 + c_2 a^2 + \mathcal{O}(a^4).$$

with

$$\phi_2(x) = -\frac{1}{2} + \frac{1}{2(2^{\alpha}-1)}\cos(2x) \ \text{ and } \ c_2 = 1 - \frac{1}{2(2^{\alpha}-1)}.$$

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Threshold behavior: $c_2 > 0$ for $\alpha > \alpha_0$ and $c_2 < 0$ for $\alpha < \alpha_0$,
where $\alpha_0 = \frac{\log 3}{\log 2} - 1 \approx 0.585$.

Stationary equation for travelling periodic waves

Periodic travelling wave $u(x,t) = \psi(x - ct)$ satisfies the stationary equation:

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The stationary equation is the Euler–Lagrange equation for the action $G(u) = E(u) + cF(u) + bM(u)$, where

$$E(u) = \frac{1}{2}\oint ((-\Delta)^{\alpha/4}u)^2 - \frac{1}{3}\oint u^3 dx, \quad F(u) = \frac{1}{2}\oint u^2 dx, \quad M(u) = \oint u\, dx.$$

Standard variational method: to find minimizers of energy $E(u)$ subject to the fixed momentum $F(u)$ and mass $M(u)$.

Drawbacks of the standard variational method

Due to Galilean transformation $\psi(x) = a + \varphi(x)$ with $a := \frac{1}{2}(c - \sqrt{c^2 + 4b})$, ψ solves $(c + (-\Delta)^{\alpha/2})\psi - \psi^2 + b = 0$ if and only if φ solves

$$(\omega + (-\Delta)^{\alpha/2})\varphi - \varphi^2 = 0, \quad \omega := \sqrt{c^2 + 4b}.$$

φ is a minimizer of energy $E(u)$ at fixed momentum $F(u) = \mu$.

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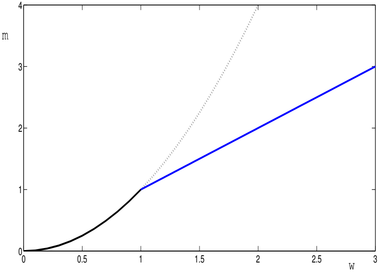
$$\alpha = 1:$$



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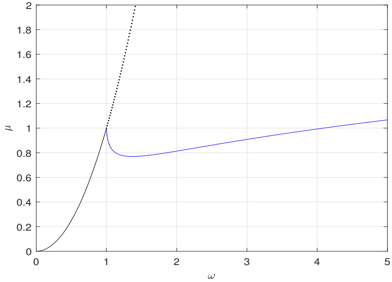

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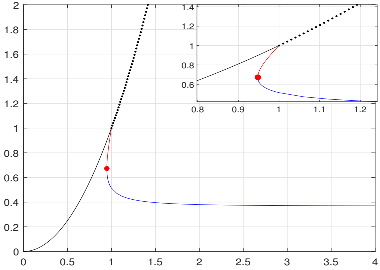

New variational formulation

Find minimizers of the quadratic energy

$$\mathcal{B}_c(u) := \frac{1}{2} \oint \left[((-\Delta)^{\alpha/4} u)^2 + cu^2 \right] dx$$

subject to fixed cubic energy and zero-mean constraint:

$$Y := \left\{ u \in H^{\frac{\alpha}{2}}_{\rm per} : \quad \oint u^3 dx = 1, \quad \oint u dx = 0 \right\}.$$

Theorem (Natali–Le–P., Nonlinearity 33 (2020), 1956)

There exists a constrained minimizer $u_* \in Y$ for every $\alpha > \frac{1}{3}$ and every $c > -1$.

Minimizer u_* yields the periodic wave $\psi \in H^{\frac{\alpha}{2}}_{\rm per}$ of the stationary equation

$$(c + (-\Delta)^{\alpha/2})\psi - \psi^2 + b(c) = 0, \qquad b(c) = \frac{1}{2\pi} \oint \psi^2 dx = \frac{1}{\pi} F(\psi).$$

Advantages of the new variational method

No fold point appears for $\alpha < \alpha_0$:

$$c = -1 + \frac{1}{2(2^\alpha - 1)} a^2 + \mathcal{O}(a^4), \quad b(c) = \frac{1}{2} a^2 + \mathcal{O}(a^4).$$

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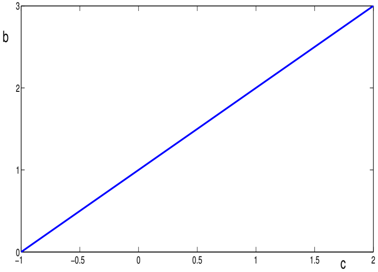
$b(c)$ versus c for $\alpha = 1$:



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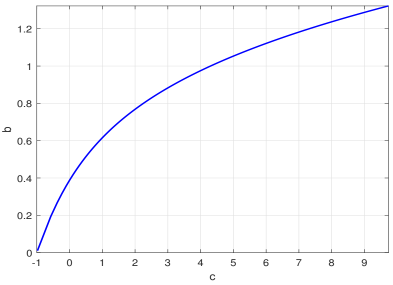
$b(c)$ versus c for $\alpha = 0.6$:



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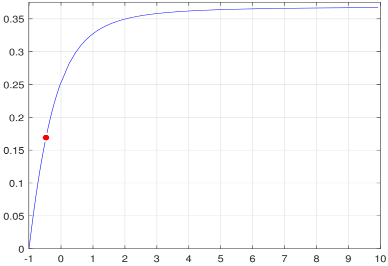
$b(c)$ versus c for $\alpha = 0.5$:



Stability theory

Stability of the periodic wave satisfying $(c + (-\Delta)^{\alpha/2})\psi - \psi^2 + b = 0$ is determined by the linearized operator $\mathcal{L} : H^\alpha_{\rm per} \subset L^2_{\rm per} \mapsto L^2_{\rm per}$ given by

$$\mathcal{L} = (-\Delta)^{\alpha/2} + c - 2\psi.$$

$n(\mathcal{L})$ = number of negative eigenvalues, $z(\mathcal{L})$ = multiplicity of zero eigenvalue.

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The self-adjoint operator enjoys Sturm's oscillation theory.

Lemma (Hur–Johnson, 2015)

Assume $\alpha \in (\frac{1}{3}, 2]$ and that $\psi \in H^\alpha_{\rm per}$ admits only one maximum on $\mathbb{T}$. An eigenfunction of $\mathcal{L}$ for the n-th eigenvalue changes its sign at most $2(n-1)$ times.

This property and the variational formulation implies that

$$1 \leq n(\mathcal{L}) \leq 2, \quad 1 \leq z(\mathcal{L}) \leq 2.$$

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The kernel of $\mathcal{L}$ can be characterized from the following criterion.

Lemma (Hur–Johnson, 2015)

Assume $\alpha \in (\frac{1}{3}, 2]$ *and that* $\psi \in H^\alpha_{\rm per}$ *admits only one maximum on* $\mathbb{T}$. $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\partial_x \psi)$ *if and only if* $\{1, \psi, \psi^2\} \in \mathrm{Range}(\mathcal{L})$.

If ψ is C^1 with respect to (c, b), then

$$\mathcal{L}\partial_b \psi = -1, \qquad \mathcal{L}\partial_c \psi = -\psi, \qquad \mathcal{L}\psi = -\psi^2 - b,$$

so that $z(\mathcal{L}) = 1$. However, ψ is not C^1 at the fold bifurcation!

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Theorem (Haragus-Kapitula, 2008)

The periodic wave with profile $\psi \in H^\alpha_{\rm per}$ is stable in the time evolution of the KdV equation if

$$n(\mathcal{L}|_{\{1,\psi\}^\perp}) = 0, \quad z(\mathcal{L}|_{\{1,\psi\}^\perp}) = 1$$

and unstable if

$$n(\mathcal{L}|_{\{1,\psi\}^\perp}) = 1.$$

It is dificult to compute $n(\mathcal{L}|_{\{1,\psi\}^\perp})$ and $z(\mathcal{L}|_{\{1,\psi\}^\perp})$ if $n(\mathcal{L}) = 2$ or $z(\mathcal{L}) = 2$.

New approach in the stability theory

Assume that the minimizer of

$$\mathcal{B}_c(u) := \frac{1}{2} \oint \left[((-\Delta)^{\alpha/4} u)^2 + c u^2 \right] dx$$

subject to $\oint u^3 dx = 1$ an $\oint u dx = 0$ is non-degenerate.

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$\mathrm{Ker}(\mathcal{L}|_{\{1,\psi^2\}^\perp}) = \mathrm{span}(\partial_x \psi)$ and the mapping $c \mapsto \psi \in H^\alpha_{\rm per}$ is C^1 in c so that

$$\mathcal{L}1 = -2\psi + c, \qquad \mathcal{L}\psi = -\psi^2 - b(c), \qquad \mathcal{L}\partial_c \psi = -\psi - b'(c).$$

and

$$n(\mathcal{L}) = \left\{ \begin{array}{ll} 1, & c + 2b'(c) \geq 0, \\ 2, & c + 2b'(c) < 0, \end{array} \right. \qquad z(\mathcal{L}) = \left\{ \begin{array}{ll} 1, & c + 2b'(c) \neq 0, \\ 2, & c + 2b'(c) = 0, \end{array} \right.$$

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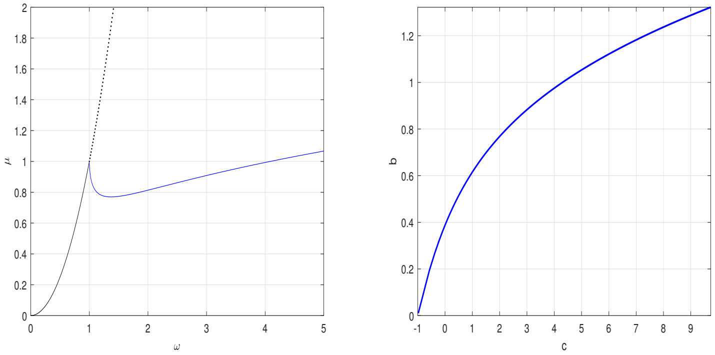
Theorem (Natali–Le-P, 2020)

The periodic wave $\psi \in H^{\alpha}_{\rm per}$ is stable if $b'(c) > 0$ and unstable if $b'(c) < 0$.

Comparison between standard and new methods

$\|\psi\|_{L^2}^2$ versus either ω (left) or c (right) for $\alpha = 0.6$:



The family of periodic waves is stable.

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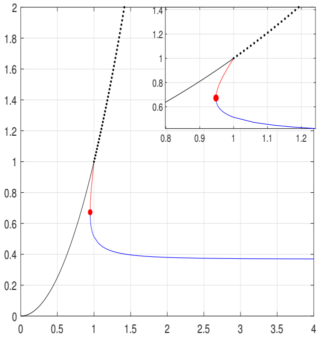
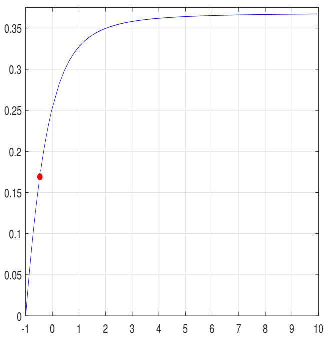
$\|\psi\|_{L^2}^2$ versus either ω (left) or c (right) for $\alpha = 0.5$:





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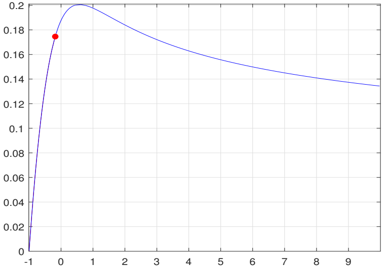
Comparison between standard and new methods

$\|\psi\|_{L^2}^2$ versus c for $\alpha = 0.45$:



For $\alpha < 0.5$, there exists $c_0 = c_0(\alpha)$ such that the family of periodic orbits is stable for $c \in (-1, c_0)$ and unstable for $c \in (c_0, \infty)$.

Positivity of periodic waves

Here we consider positivity of the profile $\psi \in H^{\alpha}_{\rm per}$ satisfying

$$(c + (-\Delta)^{\alpha/2})\psi = \psi^2, \quad c > 1, \quad b = 0.$$

$\psi > 0$ for every $c > 1$ in the integrable cases:

- BO equation with $\alpha = 1$:

$$\psi(x) = \frac{\sinh \gamma}{\cosh \gamma - \cos x}, \quad c = \coth \gamma.$$

- KdV equation $\alpha = 2$:

$$\psi(x) = \frac{2K(k)^2}{\pi^2} \left[\sqrt{1 - k^2 + k^4} + 1 - 2k^2 + 3k^2 {\rm cn}^2 \left(\frac{K(k)}{\pi} x; k \right) \right]$$

with $c = \frac{4K(k)^2}{\pi^2} \sqrt{1 - k^2 + k^4}$.

Question: Is $\psi > 0$ for every $c > 1$ and every α?

Main result

Theorem (Le–P, SIMA 51 (2019) 2850–2883)

For every $c > 1$ and $\alpha \in (\alpha_0, 2]$, $\psi(x) > 0$ on $\mathbb{T}$ as long as $z(\mathcal{L}) = 1$.

The assumption is only true for $\alpha > \alpha_0 \approx 0.585$ because the fold bifurcation point with $z(\mathcal{L}) = 2$ exists for $\alpha < \alpha_0$.

Proof of positivity: Step 1

- Green's function for $c + (-\Delta)^{\alpha/2}$ is obtained from the solution of

$$(c + (-\Delta)^{\alpha/2})\varphi(x) = h, \qquad h \in L^2_{per},$$

in the convolution form

$$\varphi(x) = \int_{-\pi}^{\pi} G(x-s)h(s)ds$$

or in Fourier form,

$$G_{c,\alpha}(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{e^{inx}}{c+|n|^\alpha} \quad \Rightarrow \quad \|G_{c,\alpha}\|_{L^2_{per}} \leq M_{c,\alpha}, \qquad \alpha > \frac{1}{2}.$$

Lemma (Le–P, FCAA 24 (2021), 1507–1534)

If $\alpha \in (0,2]$ and $c \in (0,\infty)$, then there exists $m_{c,\alpha} > 0$ such that $G_{c,\alpha}(x) \geq m_{c,\alpha}$ for every $x \in [-\pi,\pi]$.

Proof of positivity: Step 2

- Operator A in the positive cone

  From the stationary equation

$$(c + (-\Delta)^{\alpha/2})\psi = \psi^2,$$

we define the nonlinear operator

$$A_{c,\alpha}(\psi) := (c + D^\alpha)^{-1}\psi^2 \Rightarrow A_{c,\alpha}(\psi)(x) = \int_{-\pi}^{\pi} G_{c,\alpha}(x-s)\psi(s)^2 ds,$$

and the positive cone in L^2_{per}

$$P_{c,\alpha} := \left\{ \psi \in L^2_{per} : \quad \psi(x) \geq \frac{m_{c,\alpha}}{M_{c,\alpha}} \|\psi\|_{L^2_{per}}, \quad x \in \mathbb{T} \right\}.$$

1. $A_{c,\alpha}$ is bounded and continuous in L^2_{per} (Young's inequality),
2. $A_{c,\alpha}$ is compact as it is a limit of compact operators $A^{(N)}_{c,\alpha}$, where $A^{(N)}_{c,\alpha}$ are gives by $2N+1$ Fourier partial sum.
3. $A_{c,\alpha}(\psi)$ is closed in $P_{c,\alpha}$: $A_{c,\alpha}(\psi) \geq m_{c,\alpha}\|\psi\|^2_{L^2_{per}} \geq \frac{m_{c,\alpha}}{M_{c,\alpha}}\|A_{c,\alpha}(\psi)\|_{L^2_{per}}$.

Proof of positivity: Step 3

3. Existence of fixed point in the cone

Let

$$B_r := \{\psi \in L^2_{per} : \quad \|\psi\|_{L^2_{per}} < r\}$$

By Kranoselskii's fixed point theorem if there exists r_- and r_+ such that

$$\|A_{c,\alpha}(\psi)\|_{L^2_{per}} < \|\psi\|_{L^2_{per}}, \qquad \psi \in P_{c,\alpha} \cap \partial B_{r_-}$$
$$\|A_{c,\alpha}(\psi)\|_{L^2_{per}} > \|\psi\|_{L^2_{per}}, \qquad \psi \in P_{c,\alpha} \cap \partial B_{r_+}$$

then, $A_{c,\alpha}$ has fixed point in $P_{c,\alpha} \cap B_{r_+} \backslash B_{r_-}$.

- r_- is small enough so that $r_- M_{c,\alpha} < 1$
- r_+ is large enough so that $\sqrt{2\pi} r_+ m_{c,\alpha} > 1$
- $r_- < r_+$ because $\sqrt{2\pi} m_{c,\alpha} \leq M_{c,\alpha}$.

By bootstrapping argument, if $\psi \in L^2_{per}$, then $\psi \in H^\infty_{per}$.

However, the positive fixed point may not have single maximum/minimum on $\mathbb{T}$ since the constant solution $\psi = c$ is a fixed point of $A_{c,\alpha}$ in $P_{c,\alpha}$ $\forall c > 0$.

Proof of positivity: Step 4

4. Distinguishing ψ from constant fixed point

Definition (Leray-Schauder index)

The Leray-Schauder index of the fixed point ψ is defined as $(-1)^N$, where N is the number of unstable eigenvalues of $A'_{c,\alpha}(\psi)$ outside the unit disk with the account of the multiplicities.

For the constant solution $\psi = c$, the linearized operator

$$A'_{c,\alpha}(c) = 2c(c + (-\Delta)^{\alpha/2})^{-1} : L^2_{per} \to L^2_{per}$$

in the space of even functions has $N = k + 1$ unstable eigenvalues outside the unit disk for $c \in (k^\alpha, (k+1)^\alpha)$ with $k \in \mathbb{N}$. The index of the constant solution changes sign every time c crosses the eigenvaue of $(-\Delta)^{\alpha/2}$ at k^α, $k \in \mathbb{N}$.

Number of unstable eigenvalues along solution branches

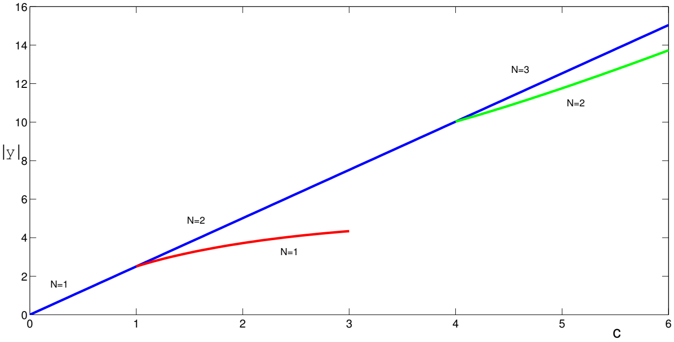


Figure: Schematic representation of bifurcations from the constant fixed point $\psi = c$.

No bifurcations along the single-lobe solutions

Positive fixed point ψ bifurcates for $c > 1$ if $\alpha > \alpha_0$. The linearized operator at ψ is given by

$$A'_{c,\alpha}(\psi) = 2(c + (-\Delta)^{\alpha/2})^{-1}\psi = Id - (c + (-\Delta)^{\alpha/2})^{-1}\mathcal{L},$$

where $\mathcal{L} := c + D^\alpha - 2\psi$ is the linearized operator.

- For $c \gtrsim 1$, $n(\mathcal{L}) = 1$ holds for $\alpha > \alpha_0$ by the perturbation argument.
- For larger $c > 1$, $n(\mathcal{L}) = 1$ remains true as long as $z(\mathcal{L}) = 1$.

Petviashvili method for fixed point iterations

Recall the stationary equation for ψ:

$$(c + (-\Delta)^{\alpha/2})\psi = \psi^2, \quad \Rightarrow \quad \psi = A_{c,\alpha}(\psi) := (c + (-\Delta)^{\alpha/2})^{-1}\psi^2.$$

Recall that the linearized operator

$$A'_{c,\alpha}(\psi) = 2(c + (-\Delta)^{\alpha/2})^{-1}\psi = Id - (c + (-\Delta)^{\alpha/2})^{-1}\mathcal{L},$$

has $N = 1$ unstable eigenvalue outside the unit disk.

$\Rightarrow$ Fixed-point iterations diverge from the periodic wave solution.

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V. Petviashvili (1976) introduced a stabilizing factor in the fixed-point iterations:

$$w_{n+1} = T_{c,\alpha}(w_n) := [M(w_n)]^2 (c + (-\Delta)^{\alpha/2})^{-1}(w_n^2), \qquad n \in \mathbb{N},$$

where

$$M(w) := \frac{\langle (c + (-\Delta)^{\alpha/2})w, w\rangle}{\langle w^2, w\rangle}.$$

If $w = \psi$, then $M(\psi) = 1$ and $T_{c,\alpha}(\psi) = \psi$.

Main result

Theorem (Le–P, SIMA 51 (2019) 2850–2883)

For every $c > 1$ *and* $\alpha \in (\alpha_0, 2]$, *the periodic wave solution* $\psi \in H^{\alpha}_{per}$ *to*

$$(c + (-\Delta)^{\alpha/2})\psi = \psi^2,$$

is an asymptotically stable fixed point of $T_{c,\alpha}$ *as long as* $z(\mathcal{L}) = 1$.

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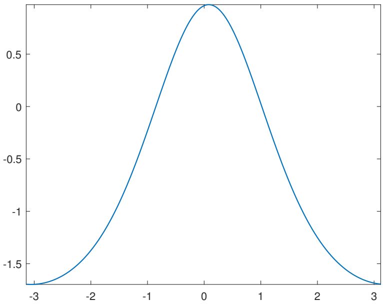
Question: Does the Petviashvili's method converge for sign-indefinite wave such as $\phi = \psi - c$ satisfying $(c - (-\Delta)^{\alpha/2})\phi + \phi^2 = 0$?

Answer:

1. ϕ is an unstable fixed point of $T_{c,\alpha}$ for $\alpha \in (\alpha_0, \alpha_1)$, where $\alpha_1 \approx 1.322$
2. ϕ is an asymptotically stable fixed point for $\alpha \in (\alpha_1, 2]$ if $c \gtrsim 1$ and is unstable if $c \gg 1$.

Iterations of $(c - (-\Delta)^{\alpha/2})\phi + \phi^2 = 0$ with $c = 2$ and $\alpha = 2$



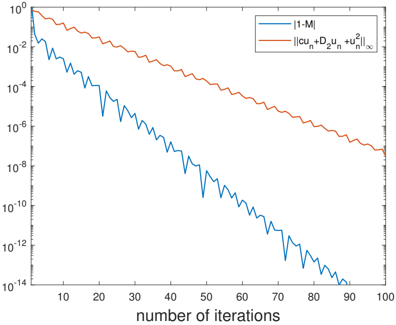


Figure: (Left) The last iteration versus x. (Right) Computational errors versus n.

Iterations of $(c - (-\Delta)^{\alpha/2})\phi + \phi^2 = 0$ with $c = 1.1$ and $\alpha = 1$

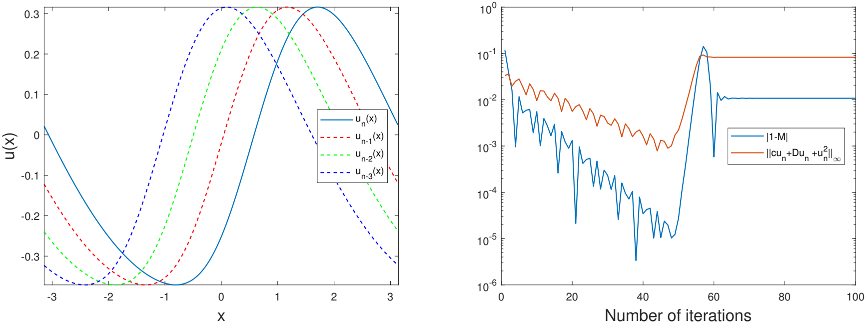


Figure: (Left) The last four iterations versus x. (Right) Computational errors versus n.

Iterations of $(c + (-\Delta)^{\alpha/2})\psi = \psi^2$ with $c = 1.6$ and $\alpha = 1$

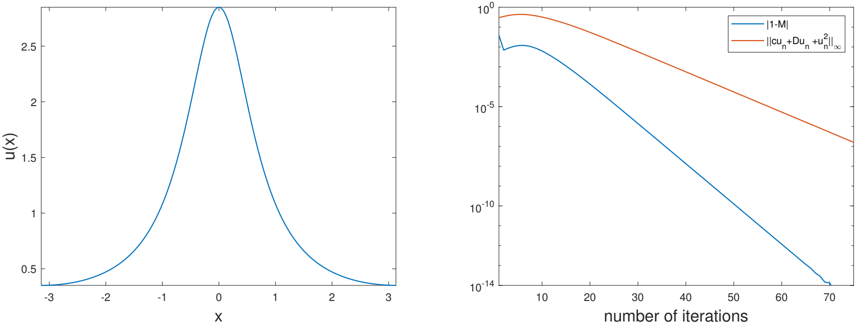


Figure: (Left) The last iteration versus x. (Right) Computational errors versus n.

Summary

For the periodic waves in the fractional KdV equation satisfying

$$(c + (-\Delta)^{\alpha/2})\psi - \psi^2 + b = 0,$$

we have showed the following:

1. Periodic waves with zero-mean profile $\psi \in H^\alpha_{\rm per}$ can be obtained from a new variational problem for every $c \in (-1, \infty)$ and $\alpha \in \left(\frac{1}{3}, 2\right]$.
2. The dependence $b = b(c) = \frac{1}{2\pi} \oint \psi^2 dx$ contains information about the fold bifurcation point and the stability of the periodic waves in the time evolution.
3. For $b = 0$, the profile ψ is positive for every $c > 1$ and $\alpha > \alpha_0 \approx 0.585$ as long as $n(\mathcal{L}) = 1$ and $z(\mathcal{L}) = 1$
4. Petviashvili's method converges for positive ψ and generally diverges for the sign-indefinite ϕ despite the simple connection $\phi = \psi - c$.

Thank you! Questions???