Justification of the log-KdV equation in granular chains

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    - R. MacKay, *Phys. Lett. A* **251** (1999), 191 - adaptation of this method to granular chains
    - J. English and R. Pego, *Proc. Amer. Math. Soc.* **133** (2005), 1763 - proof of the double-exponential tails of the solitary waves
    - A. Stefanov and P. Kevrekidis, *J. Nonlinear Sci.* **22** (2012), 327; Nonlinearity **26** (2013), 539 - proof of the bell-shaped profile

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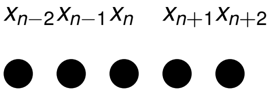
  - A. Stefanov and P. Kevrekidis, *J. Nonlinear Sci.* **22** (2012), 327; Nonlinearity **26** (2013), 539 - proof of the bell-shaped profile
- We consider solitary waves by simplifying the Fermi-Pasta-Ulam lattice to a Korteweg-de Vries equation.

The granular chain

Newton's equations define the FPU (Fermi-Pasta-Ulam) lattice:

$$\frac{d^2 x_n}{dt^2} = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z},$$

where x_n is the displacement of the nth particle from an equilibrium.



The interaction potential for spherical particles is

$$V(x) = \frac{1}{1+\alpha} |x|^{1+\alpha} H(-x), \quad \alpha = \frac{3}{2},$$

where H is the step (Heaviside) function.

H. Hertz, *J. Reine Angewandte Mathematik* **92** (1882), 156

The logarithmic Korteweg–de Vries equation

$1.2 \leq \alpha \leq 1.5$ - for spherical particles of different width and density.

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Consider the FPU lattice for relative displacements $u_n := x_{n+1} - x_n$,

$$\frac{d^2 u_n}{dt^2} - (\Delta u)_n = -(\Delta f_\alpha(u))_n, \quad n \in \mathbb{Z},$$

where $(\Delta u)_n = u_{n+1} - 2u_n + u_{n-1}$ and

$$f_\alpha(u) := u\,(|u|^{\alpha-1} - 1) = (\alpha - 1)\, u \log|u| + \mathcal{O}((\alpha-1)^2).$$

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Set $\alpha = 1 + \varepsilon^2$ and use the asymptotic multi-scale expansion

$$u_n(t) = -v(\xi, \tau) + \text{higher order terms},$$

where $\xi := 2\sqrt{3}\varepsilon\,(n - t)$ and $\tau := \sqrt{3}\varepsilon^3\, t$. At $\mathcal{O}(\varepsilon^4)$, we obtain formally the KdV equation with the logarithmic nonlinearity (log-KdV equation)

$$\partial_\tau v + \partial_\xi^3 v + \partial_\xi (v \log v) = 0.$$

Korteweg–de Vries equation for regular FPU lattices

If $V \in C^3$ with $V''(0) > 0$ and $V'''(0) \neq 0$, the same expansion reduces the FPU lattice to the standard KdV equation

$$\partial_\tau v + v \partial_\xi v + \partial_\xi^3 v = 0.$$

The KdV equation admits the solitary waves $v \sim \mathrm{sech}^2(\xi - c\tau)$.

- The KdV equation has been justified at a time scale of order ε^{-3}: G. Schneider–C.E. Wayne (2000); D. Bambusi–A. Ponno (2006).
- Nonlinear stability of small amplitude FPU solitons was proved: G. Friesecke–R.L. Pego (1999-2004).
- Existence and stability of N-soliton solutions has been proved: A. Hoffman–C.E. Wayne (2009); T. Mizumachi (2009, 2013).

Stationary solutions

Stationary log-KdV equation can be integrated once to get

$$\frac{d^2 v}{d\xi^2} + v \log|v| = 0,$$

which admits the Gaussian solitons

$$v(\xi) = \sqrt{e}\, e^{-\xi^2/4}.$$

A. Chatterjee, PRE **59** (1999), 5912

G. James–D.P., Proc. Roy. Soc. A **470** (2014), 20130465

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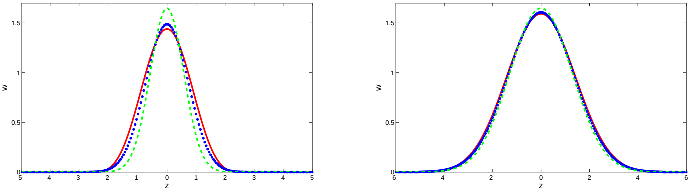

Figure : Solitary waves of the FPU chain (blue), Nesterenko compactons (red) and Gaussian solitons (green) for $\alpha = 1.5$ (left) and $\alpha = 1.1$ (right).

Numerical evidence of convergence of the approximation

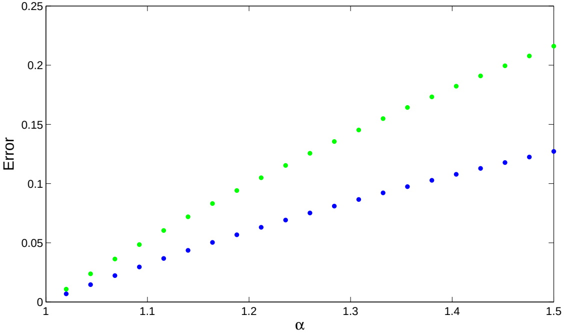

Figure : The L^∞ distance between solitary waves of the FPU chain and either Nesterenko compactons (blue dots) or Gaussian solitons (green dots) vs. α.

Numerical evidence of stability

Lattice of $N = 2000$ particles is excited with the initial impact

$$\dot{x}_n(0) = 0.1\delta_{n,0}, \quad \dot{x}_n(0) = 0 \text{ for all } n \geq 1.$$

A Gaussian solitary wave is formed asymptotically as t evolves.

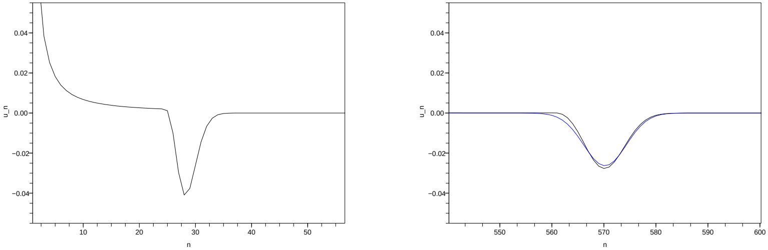

Figure : Formation of a Gaussian wave (blue curve) in the Hertzian FPU lattice with $\alpha = 1.01$: $t \approx 30.5$ (left) and $t \approx 585.6$ (right).

Precompression

Consider again the FPU lattice in the form

$$\frac{d^2 u_n}{dt^2} = -(\Delta |u|^{1+\varepsilon^2} H(-u))_n, \quad n \in \mathbb{Z}.$$

Let $u_n(t) = -v_0 (1 + w_n(t'))$ and $t' = v_0^{\varepsilon^2/2} t$ for a fixed $v_0 > 0$. Then, the FPU lattice is written in the form

$$\frac{d^2 w_n}{dt^2} = (\Delta \tilde{V}_\varepsilon(w))_n, \quad n \in \mathbb{Z}$$

with the regularized potential

$$\tilde{V}_\varepsilon(w) := \frac{1}{2+\varepsilon^2} \left[(1+w)^{2+\varepsilon^2} - 1 \right] - w, \quad w > -1$$

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The KdV scaling $w_n(t) \approx W(\xi, \tau)$ with $\xi := \varepsilon(n-t)$ and $\tau := \varepsilon^3 t$ yields the log–KdV equation

$$2\partial_\tau W + \frac{1}{12}\partial_\xi^3 W + \partial_\xi((1+W)\log(1+W)) = 0.$$

Stationary solutions

Traveling waves $W(\xi - \lambda\tau/2)$ satisfy the stationary log-KdV equation

$$\lambda W(x) = \frac{1}{12} W''(x) + (1+W)\log(1+W), \quad x \in \mathbb{R},$$

where $(1+W)\log(1+W) = W + W^2/2 + \mathcal{O}(W^3)$.

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For any $\lambda > 1$, there exists a unique even solution $W \in H^1(\mathbb{R})$ of the stationary log–KdV equation. Moreover,

- $W(x) > 0$ for all $x \in \mathbb{R}$,
- $W(x) \to 0$ as $|x| \to \infty$ exponentially fast,
- $W \in H^\infty(\mathbb{R})$,
- W' vanishes only at one point on $\mathbb{R}$.

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The travelling solitary wave is orbitally stable in the log–KdV equation
J. Höwing, J. Diff. Eqs. **251** (2011), 2515.

Main results

Theorem 1 (E.Dumas–D.P., 2014)

For every $\lambda > 1$*, there exist positive constants* ε_0 *and* C_0 *s.t. for every* $\varepsilon \in (0, \varepsilon_0)$*, there exists a unique even travelling solution* $w_{\mathrm{stat},\varepsilon}$ *of the FPU lattice in* $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *s.t.*

$$\sup_{z \in \mathbb{R}} |w_{\mathrm{stat},\varepsilon}(z) - W_{\mathrm{stat}}(\varepsilon z)| \le C_0 \varepsilon^{1/6},$$

where W_{stat} *is the unique even solution to the log–KdV equation.*

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where W_{stat} is the unique even solution to the log–KdV equation.

Remarks:

- Moreover, $w_{\text{stat},\varepsilon} \in H^\infty(\mathbb{R})$ and for every $k \in \mathbb{N}$,

$$\sup_{z \in \mathbb{R}} |\partial_z^k w_{\text{stat},\varepsilon}(z) - \varepsilon^k \partial_x^k W_{\text{stat}}(\varepsilon z)| \leq C_k \varepsilon^{k+1/6}.$$

- Moreover, $w_{\text{stat},\varepsilon}$ decays to zero exponentially fast at infinity.
- We have no proof that $w_{\text{stat},\varepsilon}$ is positive.

Main results

Theorem 2 (E.Dumas–D.P., 2014)

For every $\tau_0 > 0$, *there exist positive constants* ε_0, δ_0 *and* C_0 *s.t. for all* $\varepsilon \in (0, \varepsilon_0)$, *when initial data* $w_{\mathrm{ini},\varepsilon} \in l^2(\mathbb{R})$ *satisfy* $\delta := \|w_{\mathrm{ini},\varepsilon} - w_{\mathrm{trav},\varepsilon}(0)\|_{l^2} \leq \delta_0$, *then the unique solution* w_ε *to the FPU lattice belongs to* $C^1([-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}], l^2(\mathbb{Z}))$ *and satisfies*

$$\|w_\varepsilon(t) - w_{\mathrm{trav},\varepsilon}(t)\|_{l^2} \leq C_0\delta, \quad t \in \left[-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}\right].$$

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For every $\tau_0 > 0$, there exist positive constants ε_0, δ_0 and C_0 s.t. for all $\varepsilon \in (0, \varepsilon_0)$, when initial data $w_{\text{ini},\varepsilon} \in l^2(\mathbb{R})$ satisfy $\delta := \|w_{\text{ini},\varepsilon} - w_{\text{trav},\varepsilon}(0)\|_{l^2} \leq \delta_0$, then the unique solution w_ε to the FPU lattice belongs to $C^1([-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}], l^2(\mathbb{Z}))$ and satisfies

$$\|w_\varepsilon(t) - w_{\text{trav},\varepsilon}(t)\|_{l^2} \leq C_0\delta, \quad t \in \left[-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}\right].$$

Remarks:

- The travelling waves of the FPU lattice are stable w.r.t. modulations of any spatial scales, up to the time scale of $\mathcal{O}(\varepsilon^{-3})$.
- The constant C_0 may grow exponentially fast in τ_0.
- The travelling waves of the log-KdV equation are orbitally stable w.r.t. modulations at the spatial scale $\mathcal{O}(\varepsilon^{-1})$ and time $\mathcal{O}(\varepsilon^{-3})$.

Proof of (Approximation) Theorem 1

Travelling wave $w_n(t) = w_{\mathrm{stat},\varepsilon}(z)$, $z = n - ct$ with $c^2 = 1 + \varepsilon^2\lambda$ satisfies the differential advance-delay equation

$$(1+\varepsilon^2\lambda)\frac{d^2 w}{dz^2} = \Delta \tilde{V}_\varepsilon'(w)(z), \quad z \in \mathbb{R}.$$

Adopting the Fourier transform on $L^2(\mathbb{R})$ functions

$$\hat{w}(k) = \mathcal{F}(w)(k) := \int_{-\infty}^{\infty} w(z) e^{-ikz} dz,$$

we can rewrite the problem as the fixed-point equation

$$w(z) = \frac{1}{1+\varepsilon^2\lambda}\int_{-1}^{1} \Lambda(y)\tilde{V}_\varepsilon'(w(z-y))dy, \quad z \in \mathbb{R},$$

where $\Lambda(z) = (1-|z|)_+$ is the hat function.

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- The existence theory (Friesecke–Wattis, 1993) implies that $1+\varepsilon^2\lambda = c^2 > \tilde{V}_\varepsilon''(0) = 1+\varepsilon^2$ (that is, $\lambda > 1$).
- If λ is much larger than 1, the travelling wave has large amplitude.

Partition in the Fourier space

In the equivalent Fourier form, we have

$$\hat{w}(k) = \frac{1}{1+\varepsilon^2\lambda}\hat{\Lambda}(k)\mathcal{F}\left(\tilde{V}'_\varepsilon(w)\right)(k), \quad k \in \mathbb{R},$$

where

$$\hat{\Lambda}(k) := \frac{4}{k^2}\sin^2\left(\frac{k}{2}\right) = 1 - \frac{1}{12}k^2 + \mathcal{O}(k^4)$$

and

$$\tilde{V}'_\varepsilon(w) = w + \varepsilon^2(1+w)\log(1+w) + \mathcal{O}(\varepsilon^4).$$

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Let us divide $\mathbb{R}$ into two sets: $\mathcal{I}_p := [-\varepsilon^p, \varepsilon^p]$ and $\mathcal{J}_p := \mathbb{R}\backslash I$, where $p > 0$ is to be defined. The solution is to be decomposed as $\hat{w} = \hat{u} + \hat{v}$, where

$$\hat{u}(k) := \chi_{\mathcal{I}_p}(k)\hat{w}(k), \quad \hat{v}(k) := \chi_{\mathcal{J}_p}(k)\hat{w}(k).$$

For $\lambda > 1$, $R > 0$ and $r \in (-1, 0)$ (all ε-independent), we define the set

$$B_{R,r} := \{u \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) : \ r \leq \inf_{\mathbb{R}} u, \quad \sup_{\mathbb{R}} u \leq R\},$$

Steps in the proof of Theorem 1

1. For any $u \in B_{R,r}$ and for any small ε, there exists a unique component v such that

$$\|v\|_{L^2 \cap L^\infty} \leq C_{R,r}\varepsilon^{2-2p}\|u\|_{L^2}, \quad p < 1,$$

where the positive constant $C_{R,r}$ is independent of ε and $\|u\|_{L^2}$.

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2. For any small ε, there exists a unique component u in $B_{R,r}$ near the solution $W_{\mathrm{stat}}(\varepsilon\cdot)$ to the stationary log–KdV equation s.t.

$$\|u - W_{\mathrm{stat}}(\varepsilon\cdot)\|_{L^2 \cap L^\infty} \le C_{R,r,\lambda} \max\{\varepsilon^{4p-2}, \varepsilon^{2-2p}\} \|W_{\mathrm{stat}}(\varepsilon\cdot)\|_{L^2},$$

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3. Since $\|W_{\rm stat}(\varepsilon\cdot)\|_{L^2} = O(\varepsilon^{-1/2})$ as $\varepsilon \to 0$, we require

   $$2 - 2p - \frac{1}{2} > 0 \quad \text{and} \quad 4p - 2 - \frac{1}{2} > 0 \quad \Rightarrow \quad p \in \left(\frac{5}{8}, \frac{6}{8}\right).$$

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4. The optimal value $p = 2/3$ yields $\|w_{\rm stat} - W_{\rm stat}(\varepsilon \cdot)\|_{L^\infty} \leq C_0 \varepsilon^{1/6}$.

Proof of (Stability) Theorem 2

The scalar FPU lattice equation can be written in the vector form

$$\left\{ \begin{array}{l} \dot{w}_n = p_{n+1} - p_n, \\ \dot{p}_n = \tilde{V}'_\varepsilon(w_n) - \tilde{V}'_\varepsilon(w_{n-1}), \end{array} \right. \qquad n \in \mathbb{Z}.$$

The energy functional is conserved at any $(w,p) \in C^1(\mathbb{R}, l^2(\mathbb{Z}))$:

$$H := \frac{1}{2} \sum_{n\in\mathbb{Z}} p_n^2 + \sum_{n\in\mathbb{Z}} \tilde{V}_\varepsilon(w_n).$$

Let $(w_{\mathrm{trav}}, p_{\mathrm{trav}}) \in C^1(\mathbb{R}, l^2(\mathbb{Z}))$ denote the travelling wave to the FPU lattice with the speed c. Then, $w_{\mathrm{trav}}(t) = w_{\mathrm{stat}}(n-ct)$ satisfy

$$\left\{ \begin{array}{l} -cw'_{\mathrm{stat}}(z) = p_{\mathrm{stat}}(z+1) - p_{\mathrm{stat}}(z), \\ -cp'_{\mathrm{stat}}(z) = \tilde{V}'_\varepsilon(w_{\mathrm{stat}}(n-ct)) - \tilde{V}'_\varepsilon(w_{\mathrm{stat}}(n-1-ct)), \end{array} \right. \qquad z \in \mathbb{R}.$$

Decomposition and the energy method

For any fixed c, we decompose

$$w(t) = w_{\rm trav}(t) + \mathcal{W}(t), \quad p(t) = p_{\rm trav}(t) + \mathcal{P}(t),$$

such that $H = H_0 + H_1 + H_2 + H_R$ with

$$\begin{aligned} H_0 &= \frac{1}{2}\sum_{n\in\mathbb{Z}} p_{\rm stat}^2(n-ct) + \sum_{n\in\mathbb{Z}} \tilde{V}_\varepsilon(w_{\rm stat}(n-ct)), \\ H_1 &= \sum_{n\in\mathbb{Z}} p_{\rm stat}(n-ct)\mathcal{P}_n + \sum_{n\in\mathbb{Z}} \tilde{V}_\varepsilon'(w_{\rm stat}(n-ct))\mathcal{W}_n, \\ H_2 &= \frac{1}{2}\sum_{n\in\mathbb{Z}} \mathcal{P}_n^2 + \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{V}_\varepsilon''(w_{\rm stat}(n-ct))\mathcal{W}_n^2, \end{aligned}$$

and

$$|H_R| \leq C_\rho \sup_{z\in\mathbb{R}} |\tilde{V}_\varepsilon'''(w_{\rm stat}(z))| \, \|\mathcal{W}\|_{l^2}^3 \leq C_\rho \varepsilon^2 \|\mathcal{W}\|_{l^2}^3,$$

as long as $\|\mathcal{W}\|_{l^2} \leq \rho$. Here we recall that

$$\tilde{V}_\varepsilon''(w) = (1+\varepsilon^2)(1+w)^{\varepsilon^2}, \quad \tilde{V}_\varepsilon'''(w) = \varepsilon^2(1+\varepsilon^2)(1+w)^{\varepsilon^2-1}.$$

Energy estimates

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- H_1 is controlled in terms of H_2:

$$\frac{dH_1}{dt} = \frac{c}{2}\sum_{n\in\mathbb{Z}} w'_{\rm stat}(n-ct)\tilde{V}_\varepsilon'''(w_{\rm stat}(n-ct))\left(\mathcal{W}_n^2 + \mathcal{O}(\mathcal{W}_n^3)\right).$$

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- H_1 is controlled in terms of H_2:

$$\frac{dH_1}{dt} = \frac{c}{2}\sum_{n\in\mathbb{Z}} w'_{\mathrm{stat}}(n-ct)\tilde{V}_\varepsilon'''(w_{\mathrm{stat}}(n-ct))\left(\mathcal{W}_n^2 + \mathcal{O}(\mathcal{W}_n^3)\right).$$

Hence, we have

$$\left|\frac{dH_1}{dt}\right| \leq C_\rho \varepsilon^3 (1+\rho)\|\mathcal{W}\|_{l^2}^2 \leq 2C_\rho \varepsilon^3 (1+\rho) H_2,$$

and

$$H_1(t) - H_1(0) \geq -2C_\rho \varepsilon^3 (1+\rho)\int_0^{|t|} H_2(t')dt'.$$

End of the proof of Theorem 2

By using the energy expansion, we have

$$H - H_0 - H_1(0) \geq -2C_\rho \varepsilon^3 (1+\rho) \int_0^{|t|} H_2(t')dt' + H_2(t)(1 - C_\rho \varepsilon^2 \rho).$$

By Gronwall's inequality, we obtain

$$H_2(t) \leq \frac{H - H_0 - H_1(0)}{1 - C_\rho \varepsilon^2 \rho} e^{\tilde{C}_\rho \varepsilon^3 |t|} \leq \frac{H_2(0) + H_R(0)}{1 - C_\rho \varepsilon^2 \rho} e^{\tilde{C}_\rho \varepsilon^3 |t|} \leq \tilde{\tilde{C}}_\rho^2 \delta^2 e^{\tilde{C}_\rho \tau_0}.$$

Theorem 2 is proved in the ball in $l^2(\mathbb{Z})$ with radius $\rho := C_0 \delta$.

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Remark: The proof of nonlinear stability uses the KdV limit scaling of small ε, but does not rely on the stability of KdV travelling waves.

Justification result

Theorem 3 (Schneider-Wayne, 2000; E.Dumas–D.P., 2014)

Let $W \in C([-\tau_0, \tau_0], H^s(\mathbb{R}))$ be a solution to the log–KdV equation for some integer $s \geq 6$ and some $\tau_0 > 0$. Assume that there exists $r_W > -1$ such that $W \geq r_W$. Then there exist positive constants ε_0 and C_0 s.t. for all $\varepsilon \in (0, \varepsilon_0)$, when initial data $w_{\mathrm{ini},\varepsilon} \in l^2(\mathbb{R})$ are given s.t.

$$\|w_{\mathrm{ini},\varepsilon} - W(\varepsilon\cdot, 0)\|_{l^2} \leq \varepsilon^{3/2},$$

the unique solution w_ε to the FPU lattice belongs to $C^1([-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}], l^2(\mathbb{Z}))$ and satisfies

$$\|w_\varepsilon(t) - W(\varepsilon(\cdot - t), \varepsilon^3 t)\|_{l^2} \leq C_0\varepsilon^{3/2}, \quad t \in \left[-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}\right].$$

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Remarks:

- The proof relies on the energy method as in Theorem 2.
- The result suggests correlation between stability of KdV and FPU travelling waves but C_0 may grow exponentially fast in τ_0.

Discussion

Consider the FPU lattice

$$\frac{d^2 w_n}{dt^2} = (\Delta \tilde{V}_\varepsilon(w))_n, \quad n \in \mathbb{Z},$$

with the nonlinear potential

$$\tilde{V}_\varepsilon(w) = \frac{1}{2} w^2 + \frac{\varepsilon^2}{p+1} w^{p+1}, \quad \text{for an integer } p \geq 2.$$

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- The question of stability of FPU solitons is still opened for $p \geq 3$...